

Distribution of blackouts in the power grid and the Motter and Lai model.

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Carreras, Dobson and colleagues have studied empirical data on the sizes of the blackouts in real grids and modeled them by computer simulations using the direct current approximation. They have found that the resulting blackout sizes are distributed as a power law and suggested that this is because the grids are driven to the self-organized critical state. In contrast, more recent studies found that the distribution of cascades is bimodal as in a first order phase transition, resulting in either a very small blackout or a very large blackout, engulfing a finite fraction of the system. Here we reconcile the two approaches and investigate how the distribution of the blackouts changes with model parameters, including the tolerance criteria and the dynamic rules of failure of the overloaded lines during the cascade. In addition, we study the same problem for the Motter and Lai model and find similar results, suggesting that the physical laws of flow on the network are not as important as network topology, overload conditions, and dynamic rules of failure.

I. INTRODUCTION

Cascading failures in the power grids continue to happen in spite of efforts to make power grids more resilient [1–3]. The standard criterion of resiliency is the $N - 1$ criterion [4]: the grid must safely operate in the event of the failure of any single line. Carreras et al. have studied empirical data on the sizes of the blackouts in real grids [5] and modeled them by computer simulations using the direct current (DC) approximation [6–8]. They have found that the resulting blackout sizes are distributed as a power law and suggested that this is because the grids are driven to the self-organized critical (SOC) state [9–11]. In their model, they assume that at any stage of the cascade, one of the lines with loads exceeding the maximum values imposed by the $N - 1$ condition fails and immediately all the currents in the grids are redistributed adjusting to the new network topology. The motivation for this “one-by-one” failure rule of the cascade propagation comes from investigation of real blackouts. It is documented [1, 12] that the failures of overloaded lines do not happen instantaneously but require a certain period of time, during which overloaded lines undergo heating expansion. When the expanded line touches the ground or foliage, the current in the line dramatically increases and the line breaks, after which the current in other lines changes almost instantaneously, so that if the current in a previously overloaded line reduces to normal, that line’s length reduces and it may eventually survive the cascade of failures. Ren *et al.* [4] suggested that the power grid is driven to the SOC state by recursive upgrading of the power grid with constantly growing power demand, applying the $N - 1$ condition, or by upgrading lines involved in the recent cascade of failures. They simulated this model of self-organization and found that the system converges in the infinite time limit to a steady state characterized by an exponential

distribution of the blackout sizes, while the SOC models typically display power law distribution of avalanche lengths associated with a second-order critical point as in percolation theory [18, 19, 37].

In contrast, more recent studies [13, 14] have suggested that the distribution of cascades is bimodal, with a first order phase transition, resulting in either a very small blackout or a very large one. In all these studies, the cascades of failures were started by a random failure of a single line, and the currents were computed from a given distribution of loads and generators using the DC current approximation. The difference between these studies was that the maximal loads in [13] were computed not by using $N - 1$ criterion, but by a uniform tolerance algorithm as in [15, 16] and that the overloaded lines during each stage of the cascade were eliminated all-at-once [17–19], not one-by-one.

The difference in the outcome between the “one-by-one” rule and the “all-at-once” rule suggests that the cascade propagation in the overload models inherently depends on the dynamics of the cascade, which makes these models very different from the simple topological models of cascading failures in which the final outcome depends only the network topology and initial failures and is the same for any sequence of removals of the unfunctional elements [21].

Here we reconcile the two approaches and show that the power law distribution of cascades emerges for high protection level, and also in cases where the network topology is close the percolation point. We also show that the “one-by-one” removal rule significantly reduces the sizes of large blackouts and their probabilities, replacing the bimodal distribution of blackouts by an approximate power-law for intermediate protection levels when the “all-at-once” rule still leads to a bimodal distribution. We show that these features are held for both the DC model of a power grid and for the much simpler Motter and Lai model, suggesting that the exact physical

laws of flow on the network (Kirchhoff's laws vs. minimal path rule) are not as relevant as the protection level, network topology and the cascade dynamics rules.

II. POWER-LAW VERSUS BIMODAL DISTRIBUTIONS

In this section we will review the conditions under which the distribution of cascade sizes in overload models are power law or bimodal, and compare these overload models with the topological models [22–29], in which the outcome of the cascade depends only on the initial topology of the network and location of the initially damaged elements. Topological or overload models can be investigated for two different classes of initial attacks.

In the first well studied class [26, 27, 30, 32, 33], the cascade is started by a massive attack, after which only a fraction p of elements survive. For this class, an interesting problem is to study the behavior of the order parameter $S(p)$, the fraction of the surviving nodes at the end of the cascade, as a function of the initially surviving elements p . Many topological and overload models exhibit a first order phase transition at $p = p_t$, at which the order parameter $S(p)$ exhibits a step discontinuity, or a second order transition at $p = p_c$, at which the order parameter is continuous but its derivative with respect to p exhibits a step discontinuity. For finite systems, near $p = p_t$ the distribution of the order parameter becomes bimodal, and the transition point p_c is defined as the point at which the population of the two peaks is equal. For the second order transition the distribution of the order parameter is always unimodal. For some topological and overload models, the line of the first order transitions, p_t , in a plane of two parameters may transform into the line of second order phase transitions, p_c at a certain value of the second parameter [27, 30, 31, 33].

In the second class of initial attacks, such as in the above mentioned discussion of power grids, the cascade is started by the failure of a single element. Here, it is informative to study the distribution of the “blackout” sizes, i.e., the number of nodes that failed during the cascade and how such a probability distribution depends on the model parameters. This type of initial attacks may exist in two flavors.

In the first flavor, the system first undergoes a cascade of failures caused by a massive initial attack with $p > p_t$ and the order parameter stabilizes at $S(p)$. This is analogous to the first class of initial attacks. Afterwards, the avalanche is started by the removal of one additional element. In the well-studied topological models, it is known [32, 33], that

$$S(p) - S(p_t + 0) \sim (p - p_t)^{1/2}, \quad (1)$$

where $S(p_t + 0)$ is the limit of $S(p)$ for $p \rightarrow p_t$ above the step discontinuity. Because in the topological models the order of removal elements does not play any role, the removal of one node after the system is stabilized at p is

identical to the initial removal of $pN + 1$ nodes, where N is the total number of elements in the system. In either situation, the average avalanche size under this initial conditions is $s = N[S(p + 1/N) - S(p)] \approx \partial S(p)/\partial p \sim (p - p_t)^{-1/2}$. Hence, the avalanche size diverges above the transition point p_t , with the divergence characterized by the critical exponent $\gamma = -1/2$. In network theory this transition is called a hybrid transition, but in fact this behavior is completely equivalent to the mean field behavior near the spinodal of the first order phase transition, such as the behavior of the isothermal compressibility near the spinodal of the gas-liquid phase transition. This follows from the fact that the isothermal compressibility is equivalent to the susceptibility in the Ising model, which in turn is equivalent to the average cluster size at the percolation transition. The divergence takes place only for $p > p_t$, while for $p < p_t$ the avalanches remain of finite size or do not exist at all, since $S(p) = 0$ for $p < p_t$. For $p \rightarrow p_t + 0$, the distribution of the avalanche sizes develops a power law behavior with an exponential cutoff S^* : $P(s) \sim s^{-\tau} \exp(-s/S^*)$ with the mean field exponent $\tau = 3/2$ [32, 33], and $S^* \sim (p - p_t)^{-1/\sigma}$ with $\sigma = -1$; they, together with $\gamma = -1/2$, satisfy a usual percolation scaling relation $\gamma = (2 - \tau)/\sigma$, but with different γ and σ than in the percolation theory [36, 37].

For the second flavor of the one-element-failure initial attack, which is typical for most power grid simulations, the system exists in a state characterized by some set of parameters, but the parameter p describing the size of the initial attack is not applicable at all. In this case, both in topological and overload models, the cascading failures evolve as a branching process in which the failure of one element leads to the failure of b elements, which depend on the failed element in the topological models, or get overloaded due to the removal of the failed element in the overload models. The critical point of the simplest branching process with a fixed distribution of the branching factor $P(b)$ is defined by $\langle b \rangle = 1$ [34]. For the mean-field variant of the fixed distribution model, the distribution of the avalanche sizes is a power law with an exponential cutoff $P(s) \sim s^{-\tau} \exp(-s/s^*)$, where $s^* \sim (\langle b \rangle - 1)^{-1/\sigma}$. Here $\sigma = 1/2$ and $\tau = 3/2$ are the critical exponents [36, 37]. These in turn give the values of the two other critical exponents $\gamma = (2 - \tau)/\sigma = 1$ and $\beta = (\tau - 1)/\sigma = 1$, where γ governs the divergence of the average finite avalanche size for $\langle b \rangle \rightarrow 1$ and β governs the probability of an infinite avalanche (i.e., failed network) for $\langle b \rangle > 1$. These infinite avalanches, which are approximated in the finite system by finite avalanches that disintegrate the network, and whose size fluctuates around μN , form the second peak of the bimodal distribution. The finite avalanches, which do not scale as a finite fraction of the network, form a first peak of the distributions, which is separated from the second peak by a huge gap because there are no avalanches of size s , such that $s^* \ll s \ll \mu$ (Fig. 1). Thus, this simple branching process model predicts the existence of regimes with bimodal ($\langle b \rangle > 1$) and power-law distributions with

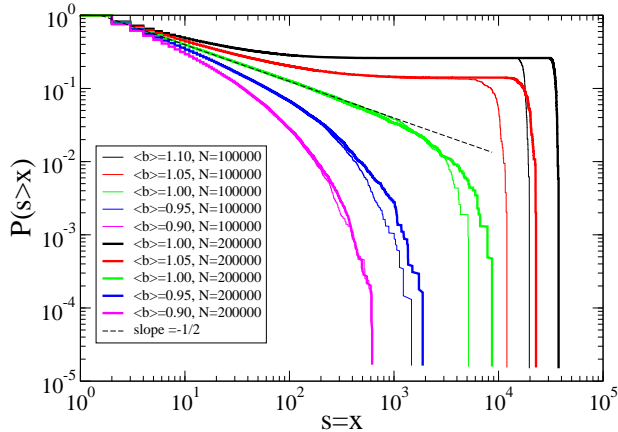


FIG. 1. Cumulative distributions of the avalanche sizes in the failure branching process with the Poisson branching degree distributions, with the average branching factors $\langle b \rangle = 0.9, 0.95, 1, 1.05, 1.1$ for finite system of $N = 100000$ and $N = 200000$ nodes. One can see that the bimodal distribution of avalanches, characterized by a wide horizontal plateau of the cumulative distribution, emerges for $\langle b \rangle > 1$. For $\langle b \rangle < 1$, the distribution has an exponential cutoff. Exactly at the critical point $\langle b \rangle = 1$, the distribution is a power law with a finite-system cutoff. The power law region with the exponent $\tau - 1 = 0.5$ increases with the size of the system.

exponential cutoff ($\langle b \rangle \leq 1$), depending on the parameters of the model. Carreras et al. conjecture [5] that the power grid is a SOC system which somehow drives itself to the critical point of this branching process, at which the distribution of blackouts is a power law. It is not clear how this SOC process works. It may be created by the long-term effect of the application of the $N - 1$ criterion [4] or, more likely, by some empirical compromise between safety and price of keeping $\langle b \rangle$ small: Electric companies want to eliminate large catastrophic blackouts which exist for $\langle b \rangle > 1$, but trying to save money on the infrastructure, are willing to accept finite blackouts. Obviously, this compromise is achieved for $\langle b \rangle = 1 - \epsilon$. In the following sections we will check this failure-branching process model by computer simulations.

III. SIMULATIONS OF THE POWER GRIDS

In order to verify that the $N - 1$ condition in addition to the one-by-one failure rule is responsible for the emergence of the power-law distributions of the blackouts, we perform the DC simulations of several grid topologies including a simplified model of the US Western Interconnect (USWI) [38], Learning Based Synthetic Power Grid (LBSPG) [39, 40], Degree and Distance attachment model (DADA) [13], and two models of artificial topologies: a random 2-d graph with given maximal line length

(RML) and random regular 2-d graph with given degree of a node (RR). In both RML and RR grids there are n_p generator nodes which produce power, n_c loads which consume power and n_t transmitter nodes, which do not produce or consume power but redistribute the flow between several power lines. We assume that all generators or loads produce or consume equal power. We use the model, setting $n_p = n_c = 1000$ and $n_t = 8000$. All $N = n_p + n_c + n_t$ nodes are randomly placed on a square of edge L and periodic boundaries. In the RR model, each node is connected to its k nearest neighbors. In order to make sure that each line connected to exactly k nodes, we make a list of all nearest neighbor pairs and start to select pairs from this list in ascending order of length, creating a link between them if both of the nodes in the pair have less than k links. At the end of the process, the majority of nodes have k links with a few exceptions which have only $k - 1$ links. In the RML model, each node is surrounded with a circle of radius $r = L\sqrt{\langle k \rangle}/(\pi N)$ and is connected to all the nodes within this circle. Since the nodes are randomly distributed geographically, the degree distribution of nodes becomes a Poisson distribution with average degree $\langle k \rangle$. We used $k = 4$ for RR and $\langle k \rangle = 5.0$ for RML. If the grid does not form a single connected cluster, we discard all the nodes which do not belong to the largest connected cluster.

To implement the $N - 1$ condition for all of our models, we proceed as follows: Assuming that the grid contains n_L links, we obtain the current in a given link ij after the removal of each one of the other $(n_L - 1)$ links, and we find the maximal current I_{ij}^* through that link ij obtained over the set of those $(n_L - 1)$ currents. We use this value I_{ij}^* as the maximal possible load that the link ij may sustain. We start each simulation with the removal of a random pair of links, $i_a j_a$ and $i_b j_b$, and recompute the currents in all remaining links, I_{ij}^1 . If in some links $I_{ij}^1 > I_{ij}^*$, we find the link with maximal overload I_{ij}^1/I_{ij}^* , remove it and find the new currents I_{ij}^2 (one-by-one update rule). We continue this process until, after removal of n links, for all remaining links, $I_{ij}^n \leq I_{ij}^*$. If at a certain stage, the grid splits into several disconnected clusters, we apply the power production-consumption equalization for each cluster using the minimal production-consumption rule described in [13]. We compute the blackout size as a fraction of the consumed power lost in the cascade.

In order to carefully assess the importance of the $N - 1$ condition and the update rule, we also perform simulations for networks with uniform tolerance protection in which $I_{ij}^* = (\alpha + 1)I_{i,j}$, where $I_{i,j}$ is the current in line i, j in the unperturbed network. Thus, for each network model, we study four cases: $N - 1$ condition together with one-by-one or all-at-once update rules, and uniform tolerance together with one-by-one or all-at-once update rules.

First, we consider the $N - 1$ condition, along with the one-by-one rule. Implementing the $N - 1$ condition significantly improves the robustness of the grid, as compared to the uniform tolerance model with small α . This

is not surprising because for all four models of the power grid, the effective tolerance of line ij under the $N - 1$ rule, $\alpha_{ij} \equiv I_{ij}^*/I_{ij} - 1$ (where I_{ij} is the initial current in the line ij) is a wide distribution with almost 5% of lines having $\alpha_{i,j} > 1$. Thus, the fraction of runs that result in any additional line overload is about 0.03; for most pairs of initially attacked lines, no further lines are overloaded. For the tolerance model with a uniform α , this level of protection is achieved only for $\alpha > 1$. Thus, implementation of the $N - 1$ condition and the one-by-one rule reduces the sizes of blackouts and create many small cascades that end after few failures [Fig. 2(a,b)]. For all four models of the power grid we observe the emergence of the approximate power law part in the distribution of the size of the small cascades. However, for each model except RML with $\langle k \rangle = 5.0$ (which is close to the RML percolation threshold for $\langle k \rangle = 4.5$), in addition to small blackouts characterized by an approximate power law distribution, we still find some large blackouts with a significant fraction of power loss. Note also that the power-law part of this distribution is still practically absent in the case for the USWI and LBSPG models. Thus, the entire distribution remains bimodal with two peaks for small and large cascades and practically no cascades of an intermediate size. This absence can be observed on the cumulative distribution graph as a large plateau [Fig. 2(a)]. The value of the exponent characterizing the power law part of the cumulative distribution of small cascades varies for different models but remains in the range between 0 and 1.

We next consider the $N - 1$ condition, but with the all-at-once update rule. The power law part of the blackout distribution disappears [Fig. 2(c)]. The distribution becomes strictly bimodal even for the case of RML model with $\langle k \rangle = 5.0$, for which the $N - 1$ condition together with the one-by-one rule produce a power-law distribution of blackouts. Additionally, large blackouts become much larger with the all-at-once rule than with the one-by-one rule; the system is more prone to failure.

Next, we systematically explore how the $N - 1$ condition, uniform tolerance, and different dynamic rules affect the blackout distributions in the USWI and LBSPG models (Fig. 3). First of all, we see that the behaviors of the two models are very similar. This is not surprising because the LBSPG has been designed to reproduce the topological features of USWI. Fig. 3(a) shows that the $N - 1$ condition with all-at-once rule or with one-by-one rule leads to bimodal distributions. Although the chance to obtain a cascade of any length bigger than one is very small, once a cascade starts the chance of obtaining a blackout of a significant fraction of the system size is quite large. The difference between one-by-one and all-at-once rules is that for the latter, the typical maximal blackout corresponds to approximately 40% of all lines, while for the former it is only 6% of all the lines.

If, instead, we return to the uniform tolerance condition of [13], and still use the all-at-once rule for line removal as in [13], the shape of the blackout distribution

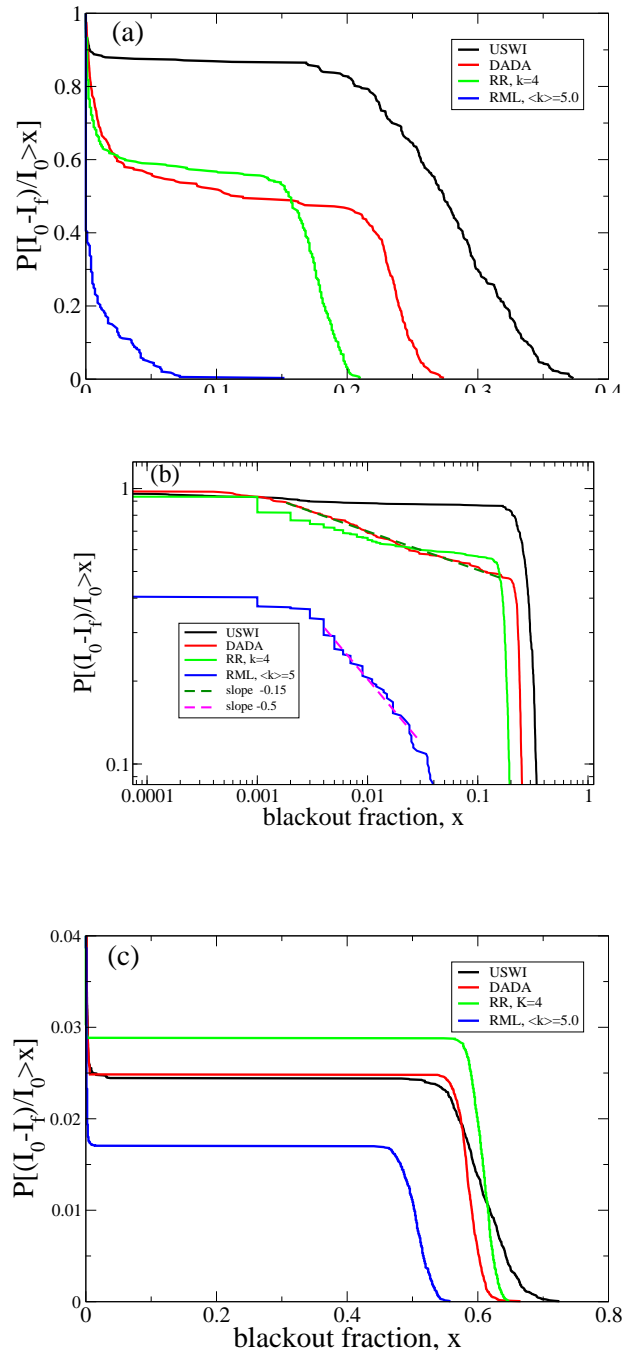


FIG. 2. Cumulative distributions of blackout sizes, measured as a function of the fraction of the consumed power lost in the blackout for different grid topologies described in the text, with the DC current approximation and the $N - 1$ condition implemented. (a) One-by-one update rule. All distributions remain bimodal except for the RML model with $\langle k \rangle = 5.0$. (b) The same as (a) in double logarithmic scale. A power-law distribution with $\tau - 1 = 0.5$ emerges for the case of the RML model. (c) All-at-once update rule. All distributions remain bimodal.

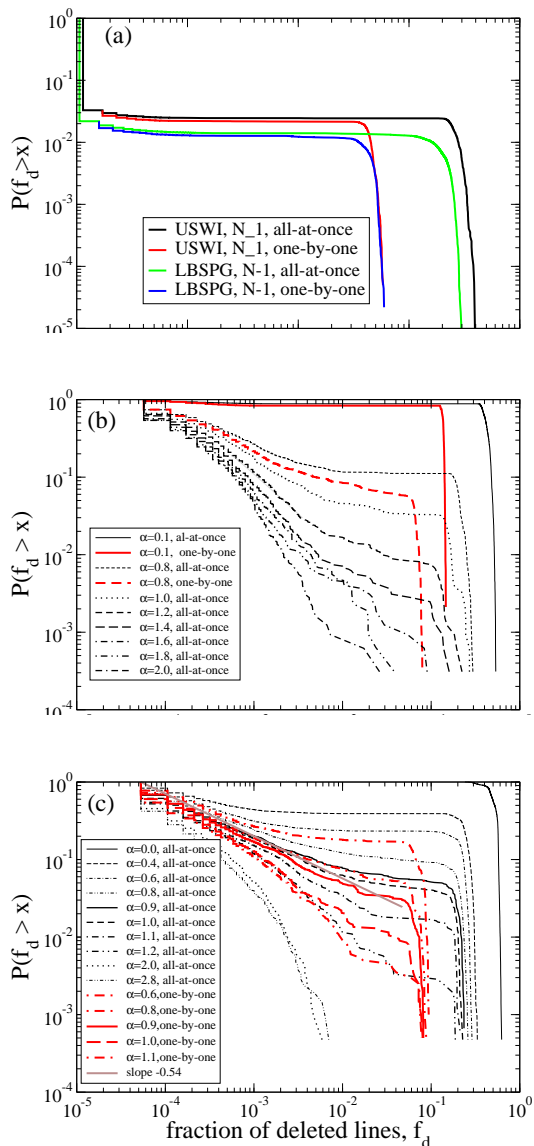


FIG. 3. Cumulative distributions of blackout sizes, measured as a function of the fraction of failed lines for the USWI and LBSPG models. (a) Both models for the $N-1$ condition with all-at-once and one-by-one update rules. (b) USWI model, and (c) LBSPG for different uniform tolerance α and both update rules. One can see that for sufficiently large values of the tolerance, $\alpha > 1.2$, the distribution becomes approximately power-law with large values of $\tau - 1 \geq 1$. The implementation of the one-by-one update rule with the uniform tolerance model shows that the power law distribution emerges for smaller value of the tolerance, $\alpha = 0.8$.

depends only on the tolerance level α [Fig. 3(b),(c)]. As in [13], we start a cascade with an initial removal of a single line. In agreement with the original results of [13], the bimodality disappears for $\alpha > 1$ and a power-law distribution appears instead. The introduction of the one-by-one update rule together with the uniform tolerance condition shows the emergence of the power law for lower values of the tolerance, $\alpha = 0.8$, and a

significant reduction of the sizes of the largest blackouts (Fig. 3(b)).

All in all, our simulations suggest that neither the $N-1$ condition nor the one-by-one update rule are necessary nor sufficient for the emergence of the power-law distribution of the blackouts. The important condition is an overall level of line protection which can be efficiently achieved by increasing enough the uniform tolerance α . The $N-1$ condition alone does not eliminate large catastrophic blackouts in the bimodal distribution; instead it significantly reduces the chance of any blackout caused by initial removal of two lines. The one-by-one update rule also reduces the sizes of largest blackouts and removes the bimodality at smaller values of α , but is not strictly necessary. Another important aspect is the network topology. For sparse homogeneous networks whose failures are close to the percolation threshold such as our RML model, the power-law distribution of blackouts emerges for smaller levels of protection, while for the other models similar failure rules still yield a bimodal distribution.

IV. BETWEENNESS CENTRALITY OVERLOADS WITH THE $N-1$ CONDITION

As one can see, the models of the power grids we study are relatively complex and have many different parameters, the role of which is difficult to clarify. Therefore, it is desirable to study a simpler model of overloads in which the role of each parameter becomes clear. It is also interesting to see if the emergence of the power law distribution is a universal phenomenon for different overload models. A good candidate for such a model is the betweenness centrality model, which displays a clear first order transition for the all-at-once removal rule and uniform tolerance condition employed by Motter and Lai [15]. However, we can expect that, similarly to the power grid models, the implementation of the $N-1$ condition and one-by-one removal rule in the betweenness centrality model will also lead to the emergence of a power-law distribution of the blackout sizes. Testing of this hypothesis will allow us to better understand the general mechanism of the emergence of the power law distribution of the sizes of blackouts in the overload models.

We build the betweenness centrality model as in [30]. Namely, we create a randomly connected graph with a given degree distribution $P(k)$ and compute the shortest paths between each pair of nodes. The length of each edge is assumed to be equal to $1 + \epsilon$, where ϵ is a normally distributed random variable with a small standard deviation $\sigma_\epsilon \ll 1$. This precaution is taken in order to make sure that each pair of nodes ij has a unique shortest path connecting them. The betweenness centrality $b(k)$ of each node k is computed as the number of all shortest paths ij passing through node k , such that $k \neq i$, $k \neq j$. The $N-1$ condition in the betweenness-centrality model should be understood in the following way: If any

node i is deleted, the rest of the nodes remain connected and the betweenness centrality of each remaining node j , $b_i(j)$ does not exceed its maximal possible load $b^*(j)$. Thus

$$b^*(j) = \max_{i \neq j} b_i(j) \quad (2)$$

and the original graph must be biconnected. Note that in the original Motter and Lai model with a uniform tolerance $\alpha > 0$,

$$b^*(j) = (1 + \alpha)b(j), \quad (3)$$

where $b(j)$ is the initial betweenness of node j in a completely intact network.

To implement this model, we first construct a randomly connected graph of N nodes with a given degree distribution $P(k)$ by the Molloy-Read algorithm [41] and find its largest biconnected component with N_b nodes using the Hopcroft-Tarjan algorithm [42]. After that, we compute the betweenness centrality of each node and repeat N_b simulations with each node removed in turn to find $b^*(i)$ for each node. To find the distribution of cascades of failures we remove a pair of random nodes and find the nodes i with overloads $z = b(i)/b^*(i) > 1$. In the first model of all-at-once removal, at each stage of the cascade we remove all the nodes with overloads, recompute the betweenness centrality of the remaining nodes and repeat this process until no overloaded nodes are left. In the second model of one-by-one removal, at each stage of the cascades we remove only one of the overloaded nodes, the one with the largest overload z , we then recompute all the centralities, and repeat this process until no overloaded nodes are left. We study several cases of Erdős Rényi (ER) networks with different average degrees $\langle k \rangle$ and different number of nodes in the biconnected component N_b . The size of the blackout is computed as the fraction of removed nodes at the end of the cascade. We also study the distribution of the number of nodes disconnected from the giant component.

Again, the $N - 1$ condition is not necessary for the emergence of the power-law distribution of the cascades in the Motter and Lai model. For example, if we take an ER network with $\langle k \rangle = 1.5$ and $0.1 < \alpha < 0.2$, we see the transition from a bimodal distribution of the cascades to a power-law distribution with an exponential cut-off [Fig. 4(a)] as α increases. This is somewhat similar to the effect observed for large α and small fraction p of nodes that survive the initial attack in Ref. [30], where the bimodal distribution of the cascades ceases to exist. In fact, small p effectively brings the network to the percolation transition equivalent to small $\langle k \rangle \rightarrow 1$. Replacing the all-at-once update rule by the one-by-one rule shifts the transition from bimodal to a power-law distribution at a smaller α [Fig. 4(b)], but both behaviors can still be seen.

Simultaneous application of the $N - 1$ condition and the one-by-one removal rule leads to the emergence of the power-law distribution of the small cascades for small

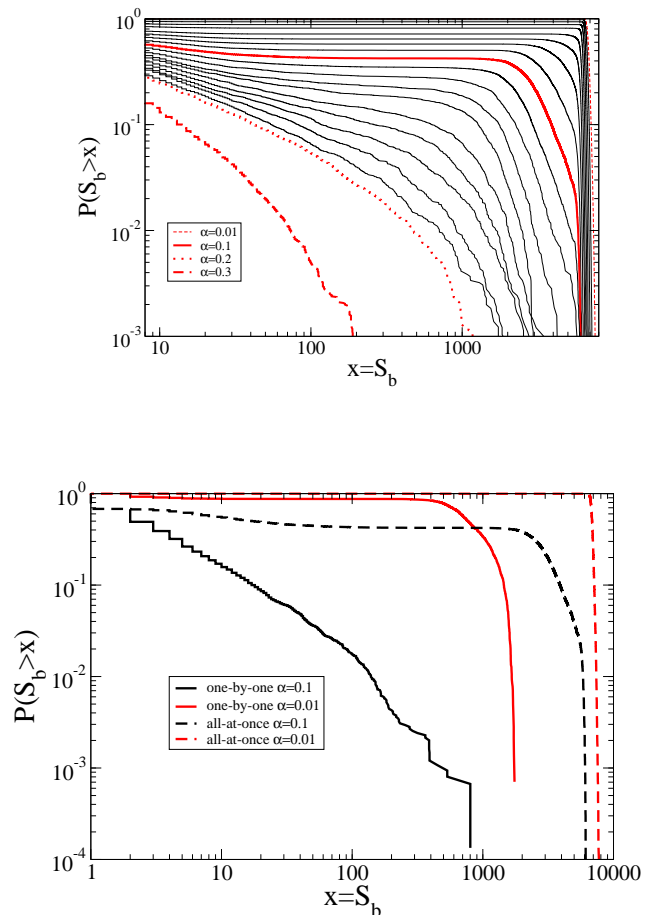


FIG. 4. (a) Cumulative distributions of blackout sizes, measured as the fraction of failed nodes in the Motter and Lai model with different values of tolerance α and all-at-once update rule. Black lines indicate graphs for values of $\alpha = 0.02, 0.03, \dots, 0.19$. (b) Comparison of all-at-once and one-by-one update rules for selected values of the tolerance. One can see the emergence of the power law distribution in the one-by-one removal case for much smaller values of tolerance

values of $\langle k \rangle$ (Fig. 5). For small $\langle k \rangle$, close to the percolation threshold and small system sizes, the bimodality of the blackout distribution disappears and the distribution of blackouts becomes an approximate power law with an exponential cutoff. As the size of the network increases, the exponential cutoff disappears and the bimodality emerges again, with a peak for large blackouts emerging. The beginning of the cumulative distribution for all values of $\langle k \rangle$ and all sizes of the system exhibits the same slope of $-1/2$, which is the mean-field value for the SOC models [11]. For large $\langle k \rangle$, we observe only a small fraction of blackouts distributed as a power law, while the entire distribution remains bimodal; many networks completely disintegrate. However, the increase of the system size leads to the relative increase of the range of the power law behavior. As the system size increases, we observe similar changes in the distribution of the blackouts

for all values of $\langle k \rangle$ studied (Fig. 5). For small system sizes, we observe a smaller initial negative slope ($> -1/2$) and a sharp exponential cutoff. For larger system sizes the slope increases to almost exactly $-1/2$, and the cutoff shifts to larger values. For even larger sizes, the slope of the initial part of the cumulative distribution increases even further ($< -1/2$) but an inflection point appears in the distribution, indicating the start of bimodality. As the system size continues to increase the negative initial slope continues to grow and the plateau region of the cumulative distribution separating large blackouts from small becomes longer and longer.

Interestingly, the behavior of the cumulative distribution of the blackouts for fixed $\langle k \rangle$ and increasing system size is analogous to the behavior of the cumulative distribution of the sizes of avalanches in the failure-branching process on a system of fixed size when the branching factor $\langle b \rangle$ increases (Fig. 1). The cumulative distribution for the avalanche distribution for $\langle b \rangle < 1$ has a negative slope smaller than $\tau - 1$, and has a strong exponential cutoff. As $\langle b \rangle$ increases, the slope increases and the exponential cutoff shifts to larger values. When $\langle b \rangle$ approaches the critical point, $\langle b \rangle = 1$, the distribution develops an inflection point and a plateau that separates finite clusters and the giant component. For $\langle b \rangle > 1$, the initial slope becomes larger than τ . In contrast to this, the behavior of the blackout size distribution, for fixed size, as $\langle k \rangle$ approaches the critical value of the percolation transition is analogous to increase of the system size in the distribution of avalanches in the failure-branching process. As $\langle k \rangle \rightarrow 1$, the range over which we see the power behavior with the mean field critical exponent $\tau = -0.5$ increases. Similarly, in a failure-branching process, the length of the failure avalanche increases, even as its behavior and exponent do not change.

As we saw, Carreras et al. [5] suggested that the power law distribution of the blackouts is due to the long-term evolution of the power-grids, driven by simultaneous growth of power consumption and upgrades of the most vulnerable power lines [4]. However the propagation of cascades in a power grid of fixed size, when applying the $N - 1$ condition and the one-by-one update rule, resembles SOC models, in which the most vulnerable element is removed, and the vulnerability of other elements changes instantaneously. To test this hypothesis, we perform a detailed study of the propagation of the cascade in a biconnected component with $N - 1$ protection and one-by-one update rule. One can see that during the cascade the number of overloaded nodes fluctuate near zero and never gets too large. In the mean-field variant of SOC, the simplest of the SOC models [11], an array of N real numbers, “fitnesses”, uniformly distributed between 0 and 1 is created. At each time step the smallest element in the array and m randomly selected elements are replaced by new values selected from the same uniform distribution. As a result, after certain transition period, the majority of elements have fitness above the critical value $f_c = 1/(m + 1)$. The number of active elements

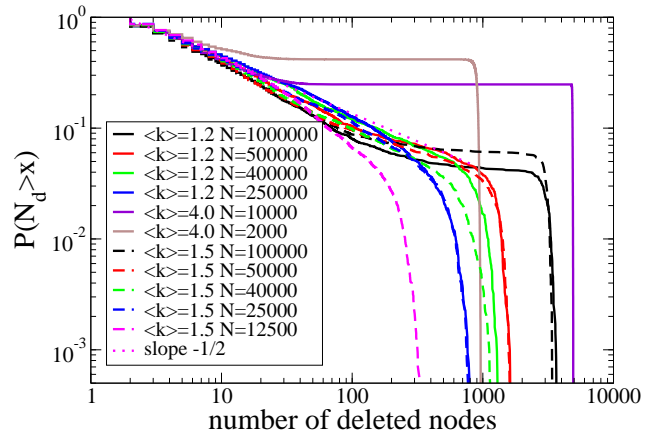


FIG. 5. Cumulative distributions of blackout sizes, measured as the fraction of failed nodes in the Motter and Lai model with $N - 1$ conditions and one-by-one update rule for different system sizes and different average degrees.

(elements with fitness less than a certain value f) performs a random walk, and an avalanche ends when the number of active elements returns to zero. When the value of the selected fitness is equal to the critical value $f = f_c$, the random walk is unbiased, and the distribution of avalanche lengths coincides with the distribution of returns to the origin of a random walker, which has the exponent $\tau - 1 = 1/2$ [35]. The number of active elements for $f = f_c$ scales as N^{d_f} with $d_f = 1/2$. We see [Fig. 6(a)] that in our model the behavior of the number of overloaded nodes is similar to the behavior of the random walk, which is consistent with the observed value of $\tau \approx 0.5$. However, the DFA analysis [43, 44] of the number of overloaded nodes during the cascade gives the value of the Hurst exponent ≈ 0.1 , which is much smaller than the exponent for the uncorrelated random walk of the mean-field SOC model. The number of overloaded nodes scales as $\ln N_b$, or a very small power law. In our model, the fitness of a node is its relative overload z_i , and the level of z_i is not arbitrary as in the SOC models, but is defined initially as $z_i = 1$. When, after implementing the $N - 1$ condition, we initially remove two nodes, the system is placed near the critical point, because the number of overloaded nodes after the initial attack is usually very small. However, this is done not automatically, as in the SOC models, but by implementing the $N - 1$ condition and the initial removal of two nodes. This action results in the overload of very few nodes, because the combined effect of the removal of two nodes rarely creates overloads larger than the maximal overloads created by the separate removal of each of these nodes. This is due to a long-tail effect; the removal of a single node will greatly increase the load of a few nodes, but barely affect most of the other nodes. However, when more nodes are removed in the process of the cascade, the topology

of the network changes and the system is slowly driven away from the critical point.

We also see that the disintegration of the biconnected component is approximately a random removal of nodes due to subsequent overloads, with slight preference for the removal of nodes with $k = 2$ over nodes with $k = 3$. At the beginning of the cascade the removals take place in the biconnected component, reducing it by the length of a chain of nodes with $k = 2$ in between a pair of nodes with $k \geq 3$. Each such removal only reduces the size of the giant component by one. But as more and more nodes are removed, further removals may happen in the singly connected part of the network. Figure 6 (b) shows the size of the biconnected component to which the removed node belongs as function of the cascade stage. If it belongs to a singly connected part of the remaining network, we plot zero instead of the size of the biconnected component. When a node that is part of the singly connected part fails, the network separates into disconnected parts and the size of the giant component drops significantly. At the beginning, the smaller part is usually a dangling end and the drop in size is relatively small, but eventually, two approximately equal parts get separated. At this point the betweenness of all nodes decreases dramatically (approximately by factor of 4) and the cascade ends. Thus, although the cascade dynamics has some similarities with SOC, (at each time step the node with the largest overload is removed and the overloads of the rest of the nodes are changed), our model does not stay at the critical state but moves away from it.

The comparison between a mean-field variant of SOC and our network overload dynamics is also instructive to help us understand when a bimodal distribution occurs and when a power law distribution does. A power law distribution, such as in a branching process, occurs when there is a series of failures, each of which has a chance of precipitating $b = 1$ additional failures. Since there is the possibility, at each stage of the cascade, of no new nodes failing and the process stopping, the chance that exactly one more node will fail decreases exponentially. At the critical point, the cascade begins with $b = 1$, but as the network disintegrates b changes. The all-at-once update rule causes the network to disintegrate in fewer steps, since multiple nodes are removed at each step, and thus b increases during the cascade of failures without many chances for the cascade to end prematurely. Thus, after a few steps $b > 1$, and the process will not stop itself with an intermediate level of destruction. However, there are two different conditions that will prevent b from increasing drastically as the network begins to disintegrate. One is a high level of overall protection, whether through the implementation of the $N - 1$ condition or through a high uniform α . High overall protection means that minor network destruction will not cause widespread overload and drive b away from one; only severe network destruction can do that. Thus, the overload process has a chance to stop spontaneously before it reaches that stage. The other condition is if the network is near the percolation

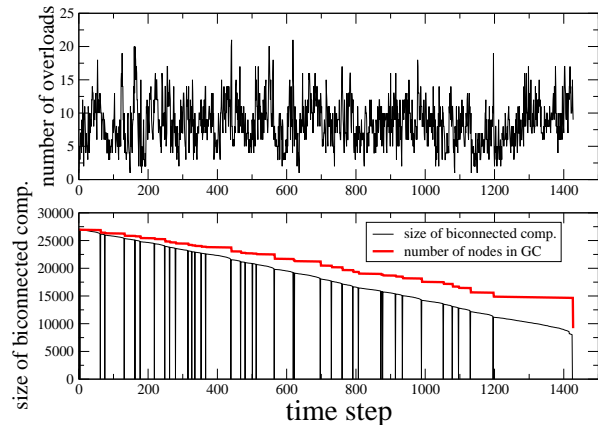


FIG. 6. (a) Dependence of the number of overloaded nodes as function of the cascade step for the Motter and Lai model for $\langle k \rangle = 1.2$ and $N = 500000$, with the $N - 1$ condition and one-by-one update rule. (b) Dependence of the size of the biconnected component and the giant component as a function of the cascade step.

point, where the failure of a single node has a high chance of separating a small, but sizable component from the network. This will drastically decrease the overall load and thus stabilize the network, arresting the cascade of failures [30]. In conclusion, a one-by-one update rule and a high overall protection both contribute to the formation of a power law. The effect of the system size discussed above can also be understood in light of the comparison to a branching process. The larger the system, the more nodes (as an absolute number, not a fraction) fail following the initial attack. Thus, the chance that none of the overloaded nodes will cause another node to overload (i.e. a spontaneous end to the cascade) becomes increasingly unlikely as the number of overloaded nodes increases, even if b remains close to one for many steps of the cascade. Thus, the chance that the cascade of failures will end with an intermediate amount of damage becomes more unlikely when the size of the original system increases.

V. DISCUSSION AND CONCLUSION

We have shown that the DC model of the power grid has remarkable similarities to the Motter and Lai model of betweenness centrality, suggesting that the exact physical laws governing the flux do not play as big a role as the network topology and the dynamics of the cascade propagation. We have shown that both overload models have similarities and differences with the topological models with dependency links [27]. In the topological models the outcome of the cascade depends only on the topology of the network and the location of the initial

damage, while in the overload models the outcome significantly depends on the dynamics of the cascade, which itself depends on the order of removal of overloaded elements and the relative speed of failure of overloaded elements and redistribution of flux after elements failure. In general, if the failure is slow compared to the redistribution of flux (one-by-one removal rule) the cascades become smaller and the distribution of blackouts becomes unimodal, while for fast failures of overloaded elements (the all-at-once removal rule) the distribution of blackouts is bimodal. The shape of the distribution of the blackout sizes becomes approximately a power law as the topology of the network approaches the percolation transition for large level of protection. In summary, the general picture of the behavior of the overload models is consistent with the simple model of the failure branching process. For large protection levels (small $\langle b \rangle$) the distribution of blackout sizes is a power law with exponential cutoff, while for low protection levels (large $\langle b \rangle$) the distribution of blackout sizes is bimodal, with a fraction of large avalanches which decreases as the protection level increases.

The empirical observation of the power law distribution of the blackout sizes in real power grids indicates that these grids are near the critical point of the correspondent failure branching process. However, this is unlikely to be caused by a SOC mechanism in which the model is driven to the critical point by a certain dynamic rule. One hypothesis would be that it is the $N - 1$ cri-

terion and the one-by-one update rule that brings the grid to the SOC state. However, we observe here that these two conditions are neither necessary nor sufficient for the existence of a power law distribution of blackout sizes. Most likely the power grids are brought to the vicinity of the critical point by a risk-price compromise, in which the utilities, with the firm goal of avoiding the catastrophic blackouts that engulf a significant fraction of the entire system, increase the level of protection up to the critical point of the branching process $\langle b \rangle < 1$; but in order to minimize costs under this condition, allow $\langle b \rangle$ to be as close as possible to the critical point from below, so that limited but occasionally very large blackouts are still possible. It is still desirable to develop a model of long time evolution of power grids similar to [4], which reproduces the power-law distribution of avalanches. The observation of power law distribution of the blackouts in the sparse networks close to the percolation point also emphasizes the role of islanding for preventing large blackouts.

VI. ACKNOWLEDGEMENTS

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