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SOLUTION OF THE DIRICHLET PROBLEM IN PARALLEL SLIT DOMAINS

by

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ABSTRACT

In this paper we develop a constructive method of solving the Dirichlet problem for a plane domain, Ω_n, whose boundary consists of a finite number of linear slits distributed along n+1 parallel lines, $\ell_m = \{(x,y) | y = k_m\}$, $(m = 0,1,\ldots,n)$. We call such a domain a parallel slit domain. Our procedure will be an extension of that used by Epstein for the case Ω_o, [Quart. Appl. Math. 6: No. 3, 301-317 (Oct., 1948)].

We seek the function u(x,y), harmonic and bounded in Ω_n, having the boundary values h(x) on the boundary slits, C, of Ω_n. To determine u(x,y), we first determine its values, f(x), on the complementary intervals, D, of its boundary slits along each line. We thus obtain the values of u(x,y) along the boundaries $C \bigcup D$, of n+2 regions - two half-planes and n strips. (See Fig. (1).) From these values, we may determine u(x,y) in each such region by using the appropriate Poisson integral formula.

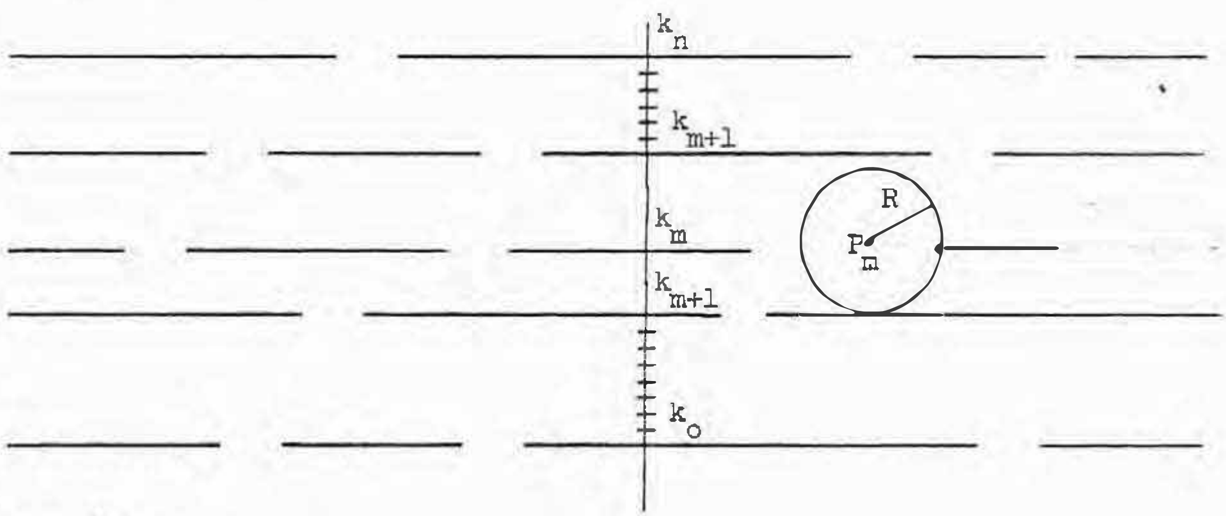

Fig. (1):

Our procedure for finding f(x) is to apply the mean value property to u(x,y) at each point $P_m = (x,k_m)$ in every complementary interval along the line $y = k_m$ for all m. Then, if we average u(x,y) over $C_R(P_m)$, the circle of radius R and center P_m, (see Fig. (1)), we obtain (for m = 0,1, ...,n) the system of integral equations:

$$(1) \qquad f(x) = g_R(x) + \int_{\Delta_m} f(\xi)K_R(x,\xi)d\xi, \quad P_m = (x,k_m) \in D_m$$

where

$$(2) \quad (a) \quad g_R(x) = \int_{\Gamma_m} h(\xi)K_R(x,\xi)d\xi, \quad P_m = (x,k_m) \in D_m$$

$$(b) \quad \Gamma_m = C_{m-1} \bigcup C_m \bigcup C_{m+1}$$

$$(c) \quad \Delta_m = D_{m-1} \bigcup D_m \bigcup D_{m+1}$$

$$(d) \quad C_m = C \bigcap \ell_m$$

$$(\ell_{-1} = \ell_{n+1} = \emptyset)$$

$$(e) \quad D_m = D \bigcap \ell_m$$

The kernel $K_R(x,\xi)$, given in terms of the Poisson kernels of the respective regions determined by $C \bigcup D$, is discontinuous at $\xi = x \pm R$. Thus the applicability of the Fredholm alternative to the integral equation (1) over the Banach space of functions continuous and bounded on D is questionable. However, by choosing R to be the largest radius such that the disc bounded by $C_R(P_m)$ contains no points of C and lies between the lines ℓ_{m-1} and ℓ_{m+1}, (see Fig. (1)), we show that for all P_m in D_m there exists a positive p less than one such that

$$(3) \qquad 0 < \int_{\Delta_m} K_R(x,\xi)d\xi \leq p < 1$$

which is the main result of the paper.

Under this condition it is known that the integral equation (1) has a unique, bounded solution obtainable by the method of successive approximations -- or iteration.

To prove the inequality (3), we temporarily assume that Ω_n is "bounded" -- i.e., that D_m is bounded for each m. We then extend our result to the case that one or more of the D_m is unbounded, but D_{m-1} or D_{m+1} is bounded.

In the case of arbitrary, unbounded D, we form a sequence of related Dirichlet problems for the domains $\Omega_{n\alpha}$ having as boundary along each line, ℓ_m, the slits, C_α, from $\pm\alpha$, $(\alpha > 0)$, to infinity in addition to the boundary, C, of Ω_n. We assign $\Omega_{n\alpha}$ the same boundary values, h(x), as those assigned Ω_n along C and the boundary values zero along C_α. If we denote the solutions of these problems -- obtainable by iteration -- as $u_\alpha(x,y)$, then

$$(4) \qquad u_\alpha(x,y) \longrightarrow u(x,y) \text{ as } \alpha \longrightarrow \infty$$

uniformly on compact subsets of Ω_n.

In the paper, we precede the discussion of the general case (described above) by an outline of Epstein's solution for the domain Ω_0 followed by a treatment of the case of Ω_1 with boundary $C_0 \cup C_1$ where $C_0 = \{(x,-\pi) \mid -a \leq x \leq a\}$ and $C_1 = \{(x,\pi) \mid -a \leq x \leq a\}$ are assigned the respective boundary values -1 and +1. Several asymptotic properties of $f(x)$ and of its iterative approximations are developed, and the results obtained for this case of Ω_1 are then applied towards the determination of its conformal modulus.

We conclude by applying the results obtained for the general case towards the determination of the periods of the harmonic functions conjugate to the harmonic measures of the boundary slits of Ω_n.

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I. Introduction[1]

In this paper we develop a constructive method of solving the Dirichlet problem for a plane domain Ω_n whose boundary consists of a finite number of linear slits distributed along n+1 parallel lines. We call such a domain a parallel slit domain. Our procedure will be an extension of that used by Epstein [1,p. 310] for slits distributed along a single line. The object is to reduce the problem to the solution of a certain integral equation and to prove that the solution of this integral equation is obtainable by the method of successive approximations. We now briefly describe this procedure.

Fig. (1.1):

Let n slits be given which lie on the x-axis and extend to infinity both on the left and on the right. Let the portion of the x-axis consisting of these slits be denoted by C and the remainder by D (consisting of the n intervals $(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)$). Let Ω_0 be the domain bounded by C. The Dirichlet problem for Ω_0 is:

Given: The function h(x) defined on C such that it is bounded on each slit and continuous except perhaps at a finite number of points.

Find: The function u(x,y), harmonic and bounded in Ω_0, which approaches h(x) at every point (x,0) of continuity of h(x) along C.

(The case of n slits lying along a different line or contained in a finite interval, or having distinct boundary values prescribed on their respective upper and lower edges, may be reduced to consideration of the above problem -- see [1, pp. 310, 311]. Furthermore, the

[1] This thesis was prepared under the supervision and guidance of my adviser, Dr. Bernard Epstein, to whom I express my deepest gratitude for his conscientious efforts on my behalf, both as a teacher and as a friend.

existence and uniqueness of $u(x,y)$ -- both here and in subsequent sections -- is assured by Nevanlinna [4, p. 22].)

Epstein's method of solving this problem is to determine the values $u(x,0)$ of the function $u(x,y)$ in the intervals $(a_1,b_1),\ldots,(a_n,b_n)$ of D. Then the values of $u(x,y)$ would be known on the entire x-axis, and for points (x,y) not on the x-axis, they could be determined by the Poisson integral formula:

$$(1.1) \qquad u(x,y) = \frac{|y|}{\pi}\int_{-\infty}^{\infty} \frac{u(\xi,0)d\xi}{(\xi - x)^2 + y^2}$$

$$(1.2) \qquad \text{Letting } u(x,0) = \begin{matrix} h(x), & (x,0) \in C \\ f(x), & (x,0) \in D \end{matrix}$$

we may rewrite (1.1) as

$$(1.3) \qquad u(x,y) = \frac{|y|}{\pi}\int_C \frac{h(\xi)d\xi}{(\xi-x)^2 + y^2} + \frac{|y|}{\pi}\int_D \frac{f(\xi)d\xi}{(\xi-x)^2 + y^2}$$

To determine the values $f(x)$ at the points $(x,0)$ in D, we temporarily assume $(x,0)$ to be fixed in one of the intervals -- say (a_k,b_k) -- of D and then apply the mean value property to $u(x,y)$ at the point $(x,0)$. This will lead us to an integral equation which we may solve for $f(x)$.

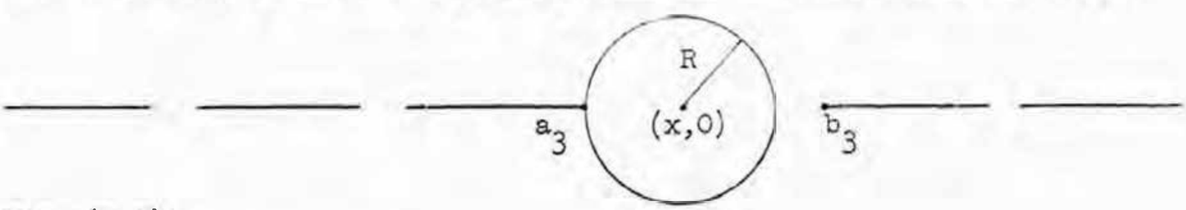

Fig. (1.2):

(1.4) Let (a) $P = (x,0)$

(b) $R = R(x) = \min(x-a_k,\ b_k-x)$

(c) $C_R(P)$ = The circle of radius R centered at P.

By the evenness of $u(x,y)$ in y -- as seen in (1.1) -- the mean value property of $u(x,y)$ at P may be expressed (by using polar coordinates with origin at $(x,0)$) as

$$(1.5) \qquad f(x) = \frac{1}{\pi}\int_0^{\pi} u(x + R\cos\theta, R\sin\theta)d\theta =$$

$$\frac{1}{\pi}\int_0^{\pi} \frac{R\sin\theta}{\pi}\left[\int_C \frac{h(\xi)d\xi}{(\xi-x-R\cos\theta)^2 + (R\sin\theta)^2}\right] d\theta =$$

$$+ \frac{1}{\pi}\int_0^{\pi} \frac{R\sin\theta}{\pi}\left[\int_D \frac{f(\xi)d\xi}{(\xi-x-R\cos\theta)^2 + (R\sin\theta)^2}\right] d\theta =$$

$$= \int_C h(\xi)\left[\frac{R}{\pi^2}\int_0^{\pi} \frac{\sin\theta\, d\theta}{(\xi-x-R\cos\theta)^2 + (R\sin\theta)^2}\right] d\xi$$

$$+ \int_D f(\xi)\left[\frac{R}{\pi^2}\int_0^{\pi} \frac{\sin\theta\, d\theta}{(\xi-x-R\cos\theta)^2 + (R\sin\theta)^2}\right] d\xi$$

where the inversion of order of integration may be justified by Fubini's theorem (which will serve as the justification for a number of such future inversions).

$$(1.6) \qquad \text{Let } \frac{R}{\pi^2}\int_0^{\pi} \frac{\sin\theta\, d\theta}{(\xi-x-R\cos\theta)^2 + (R\sin\theta)^2} = K_R(x,\xi)$$

$$(1.7) \qquad \therefore K_R(x,\xi) = \left[\pi^2(\xi-x)\right]^{-1} \log\left|\frac{\xi - x + R}{\xi - x - R}\right|$$

$$(1.8) \qquad \text{Let } \int_C h(\xi)K_R(x,\xi)d\xi = g_R(x)$$

Thus $g_R(x)$ is a known function, and the mean value property, (1.5), yields the following integral equation for $f(x)$:

$$(1.9) \qquad f(x) = g_R(x) + \int_D f(\xi)K_R(x,\xi)d\xi$$

Since $K_R(x,\xi)$ is discontinuous at $\xi = x \pm R$, it is not at all clear that the integral operator generated by $K_R(x,\xi)$ is completely continuous as an operator on the Banach space of functions continuous and bounded on D. Thus, the applicability of the Fredholm alternative to (1.9) over this space is questionable. Furthermore, even if $K_R(x,\xi)$ belonged to $L^2(D \times D)$, the Fredholm alternative would remain inapplicable in our case since the solution of (1.9) might merely be a solution in norm, not necessarily satisfying (1.9) pointwise.

Nevertheless, the existence theory of the Dirichlet problem tells us that (1.9) possesses a bounded solution and it has been shown [1, p. 305] that this solution is unique and may be obtained by the method of successive approximations providing that there exists $0 < p < 1$ such that for all (x,0) in D

$$(1.10) \qquad 0 < \int_D K_R(x,\xi)d\xi \leq p < 1.$$ [2]

In fact, if the integral equation (1.9) were considered independently of any potential-theoretic motivation and if $K_R(x,\xi)$ were any singular kernel whose absolute value satisfied the inequality (1.10), the same conclusion would hold.

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[2] Actually, the uniqueness of this bounded solution also follows from the uniqueness theory of the Dirichlet problem -- see Nevanlinna [4, p. 22]. However, the solvability of (1.9) by successive approximations relies on the equality (1.10).

Since the inequality (1.10) was proved for $K_R(x,\xi) = [\pi^2(\xi-x)]^{-1} \log \left|\frac{\xi - x + R}{\xi - x - R}\right|$, [1, p. 314], it was concluded that given any measurable $f_o(x)$ bounded on D, the sequence $\{f_n(x)\}$ defined recursively for (x,0) in D by

$$(1.11) \qquad f_n(x) = g_R(x) + \int_D f_{n-1}(\xi)K_R(x,\xi)d\xi$$

converges uniformly to the unique, bounded function $f(x)$ of (1.9).

Before proceeding to Ω_n, we note that $K_R(x,\xi)$ is intimately related to the Poisson kernel of the domains bounded by $C \bigcup D$. In fact, its integrand (here) is simply $1/\pi$ times the Poisson kernel evaluated at the pair of points $(x + R\cos\theta, R\sin\theta)$, $(\xi,0)$ [or $P + Re^{i\theta}$, Q, where $Q = (\xi,0)$] respectively. If the integration had been extended over the entire circle, the factor of $1/\pi$ would have been placed by $1/2\pi$.

Thus, if $\mathcal{P}$ denotes the Poisson kernel, then

$$(1.12) \qquad K_R(x,\xi) = K_R(P,Q) = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{P}(P + Re^{i\theta}, Q)\, d\theta$$

Thus it will be the goal of our extension to determine the equivalent kernel, $K_R(x,\xi)$, and corresponding integral equation, (1.9), for Ω_n and to prove that $K_R(x,\xi)$ satisfies the inequality (1.10). Then, as in the case of Ω_o, it will follow that the integral equation is solvable by the method of successive approximations -- or iteration.

We begin by considering a particular case of Ω_1 and an application of this case. We then proceed to a slightly restricted version of Ω_n which we generalize to arbitrary Ω_n, and we finally treat of several applications.

II. A Particular Case of Ω_1.

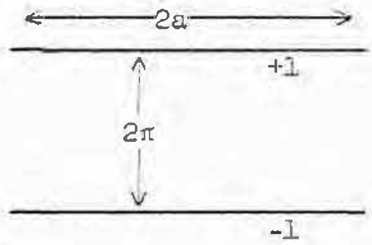

Fig. (2.1):

Let Ω_1 be the domain consisting of the entire plane minus two parallel line segments each of length $2a (a > 0)$ with the values ± 1 prescribed on the upper and lower segments respectively. Suppose further that these segments are so situated that, were their respective end points to be connected by straight line segments, the resulting quadrilateral would be a rectangle. (See Fig. (2.1).) We shall solve the Dirichlet problem for this domain with the given boundary values.

We first note that without loss of generality we may consider these lower and upper line segments to be given respectively as

$$(2.1) \qquad \begin{aligned} &(a) \quad C_o = \{(x,-\pi) \mid -a \le x \le a\} \\ &(b) \quad C_1 = \{(x,\pi) \mid -a \le x \le a\} \end{aligned}$$

The symmetry of the problem, together with the Schwarz reflection principle, tells us that the harmonic function, $u(x,y)$, we seek must have the value zero all along the x-axis; that is, $u(x,0) = 0$. Therefore, we may replace the above problem by the problem in which C_o' = the x-axis, $C_1' = C_1$ and the boundary values are zero on C_o' and unity on C_1' and Ω_1 is replaced by Ω_1' as in Fig. (2.2).

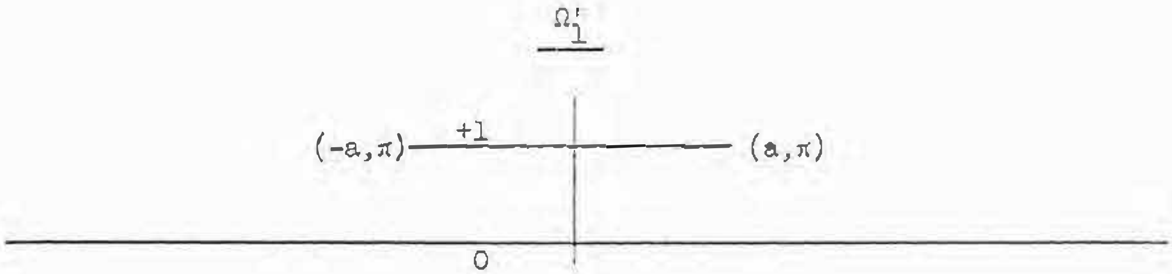

Fig. (2.2):

As in Section I, we note that the problem would be solved if we could determine the values $f(x)$ of $u(x,y)$ along the rest of the line $y = \pi$ (i.e., on D_1'). For simplicity of notation, we will at times use $f(x)$ to denote the value of $u(x,y)$ along all of $y = \pi, (C_1' \bigcup D_1')$.

$$(2.2) \quad u(x,y) = \begin{cases} u_o(x,y) + u_1(x,y), & y \neq 0,\pi \\ 0 & , \ y = 0 \\ f(x) & , \ y = \pi \end{cases}$$

where

$$(2.3) \quad (a) \quad u_o(x,y) = \frac{1}{\pi} e^x \text{Sin}y \int_{-\infty}^{\infty} \frac{f(\xi) e^{\xi} d\xi}{(e^{\xi}+e^x \text{Cos}y)^2 + (e^x \text{Sin}y)^2}$$

$$= \frac{\text{Sin}y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)d\xi}{e^{\xi-x} + 2\text{Cos}y + e^{-(\xi-x)}}, \qquad 0 < y < \pi$$

$$u_o(x,y) = 0, \qquad y \notin [0,\pi]$$

$$(b) \quad u_1(x,y) = \frac{y-\pi}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)d\xi}{(\xi-x)^2 + (y-\pi)^2}, \qquad y > \pi$$

$$u_1(x,y) = 0, \qquad y < \pi$$

Let us label the x-axis ℓ_o, and the line $y = \pi$ as ℓ_1. Then ℓ_o and ℓ_1 divide the upper half-plane into two regions S_o and S_1 respectively where S_o is the strip $\{(x,y)|0 < y < \pi\}$ and S_1 is the half-plane $\{(x,y)|y > \pi\}$. These two regions have the respective Poisson kernels $\wp_o$ and $\wp_1$, each of which depends on the points $P' = (x,y)$ in the interior and $Q = \begin{pmatrix}(\xi,0)\\(\xi,\pi)\end{pmatrix}$ [3] on the boundary of their respective domains. Let $Q_o = (\xi,0)$, $Q_1 = (\xi,\pi)$; therefore

(2.4) (a) $\wp_1 = \wp_1(P',Q_1)$

$$\text{(b)}\quad \wp_o = \begin{cases} \wp_o(P',Q_o) \\ \wp_o(P',Q_1) \end{cases}$$

depending upon whether the boundary point Q lies in ℓ_o or in ℓ_1.

Thus (2.3) could be rewritten as

$$(2.5)\quad \text{(a)}\quad u_o(P') = \int_{Q_o\in\ell_o} 0\cdot\wp_o(P',Q_o)d\xi + \int_{Q_1\in\ell_1} f(\xi)\wp_o(P',Q_1)d\xi$$

$$= \int_{Q_1\in\ell_1} f(\xi)\wp_o(P',Q_1)d\xi\ , \qquad P'\in S_o$$

$$u_o(P') = 0\ , \qquad P'\notin S_o$$

$$\text{(b)}\quad u_1(P') = \int_{Q_1\in\ell_1} f(\xi)\wp_1(P',Q_1)d\xi\ , \qquad P'\in S_1$$

$$u_1(P') = 0\ , \qquad P'\notin S_1$$

---

[3] i.e., $Q = (\xi,0)$ or $Q = (\xi,\pi)$. We will at times use this type of notation to infer that either one or both of the indicated possibilities may hold.

We now apply the mean value property to $u(P')$ at the point $P' = P_1 = (x,\pi)$ in one of the intervals -- say the right interval -- of D_1' and average over the circle $C_R(P_1)$ where

(2.6) $\quad R = R(x) = \min(x-a,\pi).$

(See Fig. (2.3).)

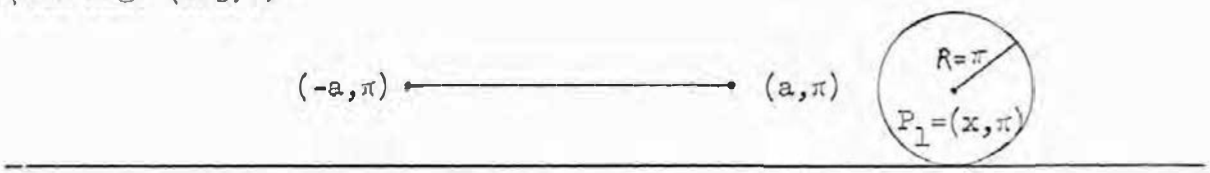

Fig. (2.3):

Therefore, along $C_R(P_1), P' = P_1 + Re^{i\theta}$ and the mean value property of $u(P')$ at P_1 may be expressed as

$$(2.7) \quad f(x) = \frac{1}{2\pi}\int_0^{\pi} u(P_1 + Re^{i\theta})\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{\pi} u_1(P_1 + Re^{i\theta})d\theta + \frac{1}{2\pi}\int_{\pi}^{2\pi} u_o(P_1 + Re^{i\theta})d\theta$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\int_{\ell_1} f(\xi)\wp_1(P_1+Re^{i\theta},Q_1)d\xi\right]d\theta$$

$$+ \frac{1}{2\pi}\int_{\pi}^{2\pi}\left[\int_{\ell_1} f(\xi)\wp_o(P_1+Re^{i\theta},Q_1)d\xi\right]d\theta$$

$$= \int_{\ell_1} f(\xi)\left[\frac{1}{2\pi}\int_0^{\pi}\wp_1(P_1+Re^{i\theta},Q_1)d\theta\right]$$

$$+ \int_{\ell_1} f(\xi)\left[\frac{1}{2\pi}\int_{\pi}^{2\pi}\wp_o(P_1+Re^{i\theta},Q_1)d\theta\right]d\xi$$

(2.8) Let $K_R(x,\xi) = \frac{1}{2\pi}\int_0^{\pi} \varphi_1(P_1+Re^{i\theta},Q_1)\,d\theta + \frac{1}{2\pi}\int_{\pi}^{2\pi}\varphi_o(P_1+Re^{i\theta},Q_1)\,d\theta$

(2.9) $\therefore\ f(x) = g_R(x) + \int_{|\xi|>a} f(\xi)K_R(x,\xi)\,d\xi$

where

(2.10) $g_R(x) = \int_{-a}^{a} K_R(x,\xi)\,d\xi$

To prove the solvability of (2.9) by iteration, we must first show that there exists $0 < p < 1$ such that for all x satisfying $|x| > a$

(2.11) $0 < \int_{|\xi|>a} K_R(x,\xi)\,d\xi \leq p < 1$

(Since $K_R(x,\xi)$ is positive, the left hand inequality is trivial.)

Without loss of generality, we may assume $x > a$. By (1.7) and Fubini's theorem, we have

(2.12) $$\int_{|\xi|>a} K_R(x,\xi)\,d\xi = \frac{1}{2\pi^2}\int_{|\xi|>a} (\xi-x)^{-1}\log\left|\frac{\xi-x+R}{\xi-x-R}\right| d\xi$$

$$+\int_{\pi}^{2\pi}\left[\int_{|\xi|>a}\varphi_o(P_1+Re^{i\theta},Q_1)\,d\xi\right]d\theta$$

$$\leq \frac{1}{2\pi^2}\int_{|\xi|>a}(\xi-x)^{-1}\log\left|\frac{\xi-x+R}{\xi-x-R}\right| d\xi + \int_{\pi}^{2\pi}\left[\int_{-\infty}^{\infty}\varphi_o(P_1+Re^{i\theta},Q_1)\,d\xi\right]d\theta$$

$$= \frac{1}{2\pi^2}\int_{|\xi|>a}(\xi-x)^{-1}\log\left|\frac{\xi-x+R}{\xi-x-R}\right| d\xi + 1/2 - R(x)/\pi^2$$

as may be seen by straightforward computation.

Let $a < x < a+\pi$. Therefore, $R(x) = x-a$, and

$$(2.13)\quad \frac{1}{2\pi^2}\int_{|\xi|>a}(\xi-x)^{-1}\log\left|\frac{\xi-x+R}{\xi-x-R}\right|d\xi = \frac{1}{2\pi^2}\int_{|x+Ru|>a}u^{-1}\log\left|\frac{1+u}{1-u}\right|du$$

$$= \frac{1}{2} - \frac{1}{2\pi^2}\int_{-(a+x)/(x-a)}^{-1}u^{-1}\log\left|\frac{1+u}{1-u}\right|du \le$$

$$\le \frac{1}{2} - \frac{1}{2\pi^2}\int_{-(1+2a/\pi)}^{-1}u^{-1}\log\left|\frac{1+u}{1-u}\right|du$$ [4]

$$(2.14)\qquad \therefore\ a < x < a+\pi \implies$$

$$\int_{|\xi|>a}K_R(x,\xi)d\xi \le 1-(x-a)/\pi^2 - \frac{1}{2\pi^2}\int_{-(1+2a/\pi)}^{-1}u^{-1}\log\left|\frac{1+u}{1-u}\right|du$$

$$\le 1 - \frac{1}{2\pi^2}\int_{-(1+2a/\pi)}^{-1}u^{-1}\log\left|\frac{1+u}{1-u}\right|du < 1$$

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[4] It has been shown [1, p.314] without computation that

$$\frac{1}{\pi^2}\int_{-\infty}^{\infty}(\xi-x)^{-1}\log\left|\frac{\xi-x+R}{\xi-x-R}\right|d\xi = \frac{1}{\pi^2}\int_{-\infty}^{\infty}u^{-1}\log\left|\frac{1+u}{1-u}\right|du = 1$$

Alternatively, the following computational argument may be used:

$$\frac{1}{\pi^2}\int_{-\infty}^{\infty}u^{-1}\log\left|\frac{1+u}{1-u}\right|du = \frac{4}{\pi^2}\int_0^1 u^{-1}\log\left(\frac{1+u}{1-u}\right)du = \frac{8}{\pi^2}\int_0^1\left(\sum_{n=0}^{\infty}\frac{u^{2n}}{2n+1}\right)du.$$

$$= \frac{8}{\pi^2}\sum_{n=0}^{\infty}\int_0^1\frac{u^{2n}}{2n+1}du = \frac{8}{\pi^2}\sum_{n=0}^{\infty}(2n+1)^{-2} = (8/\pi^2)\cdot(\pi^2/8) = 1$$

(2.15) $\quad x > a + \pi \implies R(x) = \pi \implies$

$$\int_{|\xi| > a} K_R(x,\xi)\,d\xi \leq \frac{1}{2\pi^2} \int_{-\infty}^{\infty} (\xi - x)^{-1} \log\left|\frac{\xi - x + R}{\xi - x - R}\right| d\xi + 1/2 - R(x)/\pi^2$$

$$= 1 - 1/\pi < 1.$$

III. Asymptotic Behavior of $f_n(x)$ and of $f(x)$

In this section we shall study the behavior of $f_n(x)$ at infinity and at $\pm a$ in terms of the behavior of $f_{n-1}(x)$ at these points. We will also show that

$$(3.1) \qquad f(x) \sim A/x^2 \quad \text{as} \quad |x| \to \infty$$

(where A is a certain positive constant).

Theorem (3.1): <u>If</u>: $f_{n-1}(x) \to \alpha$ as $x \to \infty$

<u>Then</u>: $f_n(x) \to (1 - \frac{1}{\pi})\alpha$ as $x \to \infty$

(A similar statement holds for $x \to -\infty$.)

<u>Proof</u>:

$$(3.2) \qquad f_n(x) = \int_{-a}^{a} K_R(x,\xi)\,d\xi + \int_{|\xi|>u} f_{n-1}(\xi) K_R(x,\xi)\,d\xi$$

(Henceforth in this proof we take $R = \pi$ since the fact that x approaches infinity implies that eventually $x > a + \pi$ and therefore $R = R(x) = \pi$.)

A comparison of (2.3) and (2.8) tells us that

$$(3.3) \quad K_\pi(x,\xi) = \left[2\pi^2(\xi-x)\right]^{-1} \log \left|\frac{\xi - x + \pi}{\xi - x - \pi}\right|$$

$$+ \frac{e^\xi}{2\pi^2} \int_0^\pi \frac{\exp(x-\pi \mathrm{Cos}\Theta)\,\mathrm{Sin}(\pi \mathrm{Sin}\Theta)\,d\Theta}{(e^\xi - \exp(x-\pi\mathrm{Cos}\Theta)\mathrm{Cos}(\pi\mathrm{Sin}\Theta))^2 + (\exp(x-\pi\mathrm{Cos}\Theta)\mathrm{Sin}(\pi\mathrm{Sin}\Theta))^2}$$

Therefore

$$(3.4) \qquad \int_{-a}^{a} K_\pi(x,\xi)\,d\xi = \frac{1}{2\pi^2} \int_{-(a+x)/\pi}^{(a-x)/\pi} u^{-1} \log \left|\frac{1+u}{1-u}\right| du$$

$$+\frac{1}{2\pi^2}\int_0^{\pi} \mathrm{Tan}^{-1}\left[\frac{(e^{a}-e^{-a})\exp(x-\pi\mathrm{Cos}\theta)\mathrm{Sin}(\pi\mathrm{Sin}\theta)}{\exp\left[2(x-\pi\mathrm{Cos}\theta)\right]-(e^{a}+e^{-a})\exp(x-\pi\mathrm{Cos}\theta)\mathrm{Cos}(\pi\mathrm{Sin}\theta)+1}\right]d\theta$$

and

$$(3.5)\quad \int_{|\xi|>a} f_{n-1}(\xi)K_{\pi}(x,\xi)d\xi = \frac{1}{2\pi^2}\int_{-\infty}^{-(a+x)/\pi} u^{-1} f_{n-1}(x+\pi u)\log\left|\frac{1+u}{1-u}\right|du$$

$$+\frac{1}{2\pi^2}\int_{(a-x)/\pi}^{\infty} u^{-1}f_{n-1}(x+\pi u)\log\left|\frac{1+u}{1-u}\right|du$$

$$+\frac{1}{2\pi^2}\int_0^{\pi}\int_{\pi(\mathrm{Sin}\theta-\frac{1}{2})}^{\mathrm{Tan}^{-1}\left[\frac{e^{-a}-\exp(x-\pi\mathrm{Cos}\theta)\mathrm{Cos}(\pi\mathrm{Sin}\theta)}{\exp(x-\pi\mathrm{Cos}\theta)\mathrm{Sin}(\pi\mathrm{Sin}\theta)}\right]} f_{n-1}\left[x-\pi\mathrm{Cos}\theta\right.$$

$$\left.+\log\mathrm{Cos}(\pi\mathrm{Sin}\theta-\phi)-\log\mathrm{Cos}\phi\right]d\phi d\theta$$

$$+\frac{1}{2\pi^2}\int_0^{\pi}\int_{\mathrm{Tan}^{-1}\left[\frac{e^{a}-\exp(x-\pi\mathrm{Cos}\theta)\mathrm{Cos}(\pi\mathrm{Sin}\theta)}{\exp(x-\pi\mathrm{Cos}\theta)\mathrm{Sin}(\pi\mathrm{Sin}\theta)}\right]}^{\pi/2} f_{n-1}\left[x-\pi\mathrm{Cos}\theta\right.$$

$$\left.+\log\mathrm{Cos}(\pi\mathrm{Sin}\theta-\phi)-\log\mathrm{Cos}\phi\right]d\phi d\theta$$

Let the integrals appearing on the right hand sides of (3.4) and (3.5) be labeled according to their order of appearance as $F_{n_i}(x)$; $i = 1,2,\ldots,6$.

It is immediately obvious (by inspection) that $F_{n_1}(x)$ and $F_{n_2}(x)$ approach zero as x approaches infinity. Furthermore the boundedness of $f_{n-1}(\xi)$ -- which follows from the boundedness of $f_o(\xi)$ -- implies that

$F_{n_3}(x)$ and $F_{n_5}(x)$ each approach zero as x approaches infinity, since the limits of the respective u and ϕ integrations become the same as x approaches infinity. Therefore we need only examine $F_{n_4}(x)$ and $F_{n_6}(x)$ as x approaches infinity.

Lemma (3.1a): If: $f_{n-1}(x) \to \alpha$ as $x \to \infty$

Then: $F_{n_4}(x) \to \alpha/2$ as $x \to \infty$

Proof:

$$(3.6) \qquad \alpha/2 = \frac{\alpha}{2\pi^2} \int_{-\infty}^{\infty} u^{-1} \log\left|\frac{1+u}{1-u}\right| du$$

(See Footnote No. 4, p. 11)

$$(3.7) \qquad \therefore F_{n_4}(x) - \alpha/2 = \frac{-\alpha}{2\pi^2} \int_{-\infty}^{(a-x)/\pi} u^{-1} \log\left|\frac{1+u}{1-u}\right| du$$

$$+ \frac{1}{2\pi^2} \int_{(a-x)/\pi}^{\infty} u^{-1} [f_{n-1}(x+\pi u) - \alpha] \log\left|\frac{1+u}{1-u}\right| du$$

$$(3.8) \qquad - \frac{\alpha}{2\pi^2} \int_{-\infty}^{(a-x)/\pi} u^{-1} \log\left|\frac{1+u}{1-u}\right| du \to 0 \qquad \text{as } x \to \infty$$

$$(3.9) \qquad \frac{1}{2\pi^2} \int_{(a-x)/\pi}^{\infty} u^{-1} [f_{n-1}(x+\pi u) - \alpha] \log\left|\frac{1+u}{1-u}\right| du$$

$$= \frac{1}{2\pi^2} \int_{(a-x)/\pi}^{(a-x)/2\pi} u^{-1} [f_{n-1}(x+\pi u) - \alpha] \log\left|\frac{1+u}{1-u}\right| du$$

$$+ \frac{1}{2\pi^2} \int_{(a-x)/2\pi}^{\infty} u^{-1} [f_{n-1}(x+\pi u) - \alpha] \log\left|\frac{1+u}{1-u}\right| du$$

Let M be an upper bound on $|f_{n-1}(x)|$. Therefore

$$(3.10)\qquad \frac{1}{2\pi^2}\left|\int_{(a-x)/\pi}^{(a-x)/2\pi} u^{-1}[f_{n-1}(x+\pi u)-\alpha]\log\left|\frac{1+u}{1-u}\right|du\right|$$

$$\le \frac{2M}{2\pi^2}\int_{(a-x)/\pi}^{(a-x)/2\pi} u^{-1}\log\left|\frac{1+u}{1-u}\right|du \to 0 \qquad \text{as } x\to\infty.$$

$$(3.11)\qquad \text{Let}\quad F_{n_7}(x) = \frac{1}{2\pi^2}\int_{(a-x)/2\pi}^{\infty} u^{-1}[f_{n-1}(x+\pi u)-\alpha]\log\left|\frac{1+u}{1-u}\right|du$$

By (3.6)-(3.9), $F_{n_4}(x) - \alpha/2 \to 0$ as $x\to\infty \Longleftrightarrow F_{n_7}(x)\to 0$ as $x\to\infty$. But $f_{n-1}(x)\to\alpha$ as $x\to\infty$. Therefore, for a given $\epsilon > 0$,

$$(3.12)\qquad |f_{n-1}(x+\pi u)-\alpha| < \epsilon$$

if $x+\pi u$ is large enough. But in $F_{n_7}(x)$

$$(3.13)\qquad u \ge (a-x)/2\pi \Longrightarrow x+\pi u \ge (x+a)/2$$

Threfore, for $(x+a)/2$ -- and hence x -- large enough,

$$(3.14)\qquad |F_{n_7}(x)| < \frac{\epsilon}{2\pi^2}\int_{(a-x)/2\pi}^{\infty} u^{-1}\log\left|\frac{1+u}{1-u}\right|du$$

$$\le \frac{\epsilon}{2\pi^2}\int_{-\infty}^{\infty} u^{-1}\log\left|\frac{1+u}{1-u}\right|du = \epsilon/2$$

<u>Lemma (3.1b):</u> <u>If</u>: $f_{n-1}(x) \to \alpha$ as $x\to\infty$

<u>Then:</u> $F_{n_6}(x) \to (\frac{1}{2} - \frac{1}{\pi})\alpha$

Proof:

It suffices to show that the inner integral of $F_{n_6}(x)$ -- which we call $G_{n_6}(x,\theta)$ -- satisfies

(3.15) $$G_{n_6}(x,\theta) \to (1-\text{Sin}\theta)\alpha\pi \qquad \text{as } x \to \infty.$$

For then

(3.16) $$F_{n_6}(x) \to \frac{1}{2\pi^2} \int_{00}^{\pi} (1-\text{Sin}\theta)\alpha\pi d\theta = (\frac{1}{2} - \frac{1}{\pi})\alpha \qquad \text{as } x \to \infty.$$

(3.17) Let (a) $$\text{Tan}^{-1}\left[\frac{e^{a}-\exp(x-\pi\text{Cos}\theta)\text{Cos}(\pi\text{Sin}\theta)}{\exp(x-\pi\text{Cos}\theta)\text{Sin}(\pi\text{Sin}\theta)}\right] = G(x,\theta)$$

(b) $$f_{n-1}\left[x-\pi\text{Cos}\theta+\log\text{Cos}(\pi\text{Sin}\theta-\phi)-\log\text{Cos}\phi\right] = F_{n-1}(x,\theta,\phi)$$

(Note: Under the change of variable performed in (3.5) to yield $F_{n_5}(x)$ and $F_{n_6}(x)$,

$$\xi = x-\pi\text{Cos}\theta+\log\text{Cos}(\pi\text{Sin}\theta-\phi)-\log\text{Cos}\phi .$$

$$\therefore f_{n-1}(\xi) = F_{n-1}(x,\theta,\phi).$$)

(3.18) $\therefore$ (a) $$\lim_{x \to \infty} G(x,\theta) = \pi(\text{Sin}\theta - \tfrac{1}{2}) = \lim_{x \to \infty} G(x/2,\theta)$$

(b) $$\lim_{x \to \infty} F_{n-1}(x,\theta,\phi) = \alpha \quad \text{(for each pair } (\theta,\phi))$$

(c) $$|F_{n-1}(x,\theta,\phi)| \leq M \text{ (uniformly)}; \quad |\alpha| \leq M$$

(3.19) $$G_{n_6}(x,\theta)-(1-\text{Sin}\theta)\alpha\pi = \int_{G(x,\theta)}^{\pi/2} F_{n-1}(x,\theta,\phi)d\phi - \int_{\pi(\text{Sin}\theta-\frac{1}{2})}^{\pi/2} \alpha\, d\phi$$

$$= \int_{G(x,\theta)}^{G(x/2,\theta)} \left[F_{n-1}(x,\theta,\phi)-\alpha\right] d\phi + \int_{G(x/2,\theta)}^{\pi/2} \left[F_{n-1}(x,\theta,\phi)-\alpha\right] d\phi$$

$$- \int_{\pi(\mathrm{Sin}\theta - \frac{1}{2})}^{G(x,\theta)} \alpha \, d\phi$$

(3.20) (a) $$\int_{\pi(\mathrm{Sin}\theta - \frac{1}{2})}^{G(x,\theta)} \alpha \, d\phi \to 0 \quad \text{as } x \to \infty$$

(by (3.17)(a))

(b) $$\int_{G(x,\theta)}^{G(x/2,\theta)} \left[F_{n-1}(x,\theta,\phi)-\alpha\right] d\phi \to 0 \quad \text{as } x \to \infty$$

(by (3.17)(a) and (3.17)(c))

(3.21) Let $$G_{n_7}(x,\theta) = \int_{G(x/2,\theta)}^{\pi/2} \left[F_{n-1}(x,\theta,\phi)-\alpha\right] d\phi$$

$$\therefore G_{n_6}(x,\theta)-(1-\mathrm{Sin}\theta)\alpha\pi \to 0 \quad \text{as } x \to \infty \iff G_{n_7}(x,\theta) \to 0$$

as $x \to \infty$. But $F_{n-1}(x,\theta,\phi) = f_{n-1}(\xi) \to \alpha$ as $\xi \to \infty$ (or as $x \to \infty$) for each pair (θ,ϕ).

Therefore, for a given $\epsilon > 0$,

(3.22) $$|f_{n-1}(\xi)-\alpha| < \epsilon$$

providing ξ is large enough. But in $G_{n_7}(x,\theta)$ -- where x and θ are fixed and ϕ ranges over $[G(x/2,\theta),\pi/2]$,

(3.23) $$\xi \geq x/2 + a$$

(as may be verified by a straightforward calculation).

Therefore, for $x/2 + a$ -- and hence x -- large enough,

$$(3.24) \qquad |G_{n_7}(x,\theta)| \leq \int_{G(x/2,\theta)}^{\pi/2} |F_{n-1}(x,\theta,\phi)-\alpha|\,d\phi < \epsilon\int_{G(x/2,\theta)}^{\pi/2} d\phi$$

$$< \epsilon\int_{\pi(\mathrm{Sin}\theta-\frac{1}{2})}^{\pi/2} d\phi = (1-\mathrm{Sin}\theta)\pi\epsilon \leq 2\pi\epsilon$$

(Since $G(x/2,\theta) \geq \pi(\mathrm{Sin}\theta - \frac{1}{2})$)

$$(3.25) \qquad \therefore \lim_{x\to\infty} f_n(x) = \lim_{x\to\infty}\left[F_{n_4}(x) + F_{n_6}(x)\right] = \alpha/2+(\tfrac{1}{2}-\tfrac{1}{\pi})\alpha = (1-\tfrac{1}{\pi})\alpha$$

QED.

<u>Theorem (3.2):</u> <u>If:</u> $f_{n-1}(x) \to \beta$ as $x \to a^+$

<u>Then:</u> $f_n(x) \to 1/4 + (3/4)\beta$ as $x \to a^+$

(A similar statement holds for $x \to -a^-$.)

<u>Proof:</u>

In this proof we take $R = R(x) = x-a$ since the fact that x approaches a^+ implies that eventually $a < x < a + \pi$ and therefore $R(x) = x-a$. We will also have recourse to equations (3.2), (3.4), and (3.5) from which the functions $F_{n_i}(x)$; $i = 1,2,\ldots,6$, $G_{n_5}(x,\theta)$, $G_{n_6}(x,\theta)$, and $F_{n-1}(x,\theta,\phi)$ shall again be selected. However, we now replace every occurrence of π in the respective integrands (except in the fraction $\pi/2$) by $x-a$, the value of R now under consideration, and study these functions as x approaches a^+.

$$(3.26) \qquad \therefore f_n(x) = \sum_{i=1}^{6} F_{n_i}(x)$$

and

$$(3.27)\qquad F_{n_1}(x) = \frac{1}{2\pi^2}\int_{-(a+x)/(x-a)}^{-1} u^{-1}\log\left|\frac{1+u}{1-u}\right| du$$

$$\to \frac{1}{2\pi^2}\int_{-\infty}^{-1} u^{-1}\log\left|\frac{1+u}{1-u}\right| du = 1/8 \qquad \text{as } x \to a^+$$

By L'Hospital's rule, we find that

$$(3.28)\qquad F_{n_2}(x) =$$

$$\frac{1}{2\pi^2}\int_0^{\pi} \mathrm{Tan}^{-1}\left[\frac{(e^a-e^{-a})\exp[x-(x-a)\mathrm{Cos}\theta]\,\mathrm{Sin}[(x-a)\mathrm{Sin}\theta]}{\exp(2[x-(x-a)\mathrm{Cos}\theta])-(e^a+e^{-a})\exp[x-(x-a)\mathrm{Cos}\theta]\,\mathrm{Cos}[(x-a)\mathrm{Sin}\theta]+1}\right] d\theta$$

$$\to 1/8 \qquad \text{as } x \to a^+$$

$$(3.29)\qquad \therefore\ g_R(x) = F_{n_1}(x) + F_{n_2}(x) \to 1/4 \qquad \text{as } x \to a^+$$

Again we use the fact that $|f_{n-1}(\xi)| \leq M$. Thus we see (by inspection) that

$$(3.30)\qquad F_{n_3}(x) \to 0 \qquad \text{as } x \to a^+$$

(since the limits of integration become the same as $x \to a^+$).

Furthermore, letting

$$(3.31)\qquad L(t,x,\theta) = \frac{e^t-\exp[x-(x-a)\mathrm{Cos}\theta]\,\mathrm{Cos}[(x-a)\mathrm{Sin}\theta]}{\exp[x-(x-a)\mathrm{Cos}\theta]\,\mathrm{Sin}[(x-a)\mathrm{Sin}\theta]}$$

we find that

$$(3.32)\qquad F_{n_5}(x) \to 0 \qquad \text{as } x \to a^+$$

since its inner integral

(3.33) $\quad G_{n_5}(x,\theta) \to 0 \quad \text{as } x \to a^+$

by the boundedness of f_{n-1} and by the fact that both limits of integration in $G_{n_5}(x,\theta)$ approach $-\pi/2$ as x approaches a^+.

($\mathrm{Tan}^{-1}[L(-a,x,\theta)]$ approaches $-\pi/2$ as x approaches a^+ since $L(-a,x,\theta)$ is negative for x near a^+ and thus approaches $-\infty$ as x approaches a^+.)

(3.34) $$F_{n_4}(x) = \frac{1}{2\pi^2}\int_{-1}^{\infty} u^{-1} f_{n-1}[x+(x-a)u] \log\left|\frac{1+u}{1-u}\right| du$$

$$\to \frac{1}{2\pi^2}\int_{-1}^{\infty} u^{-1}\,\beta \log\left|\frac{1+u}{1-u}\right| du = (3/8)\beta \qquad \text{as } x \to a^+$$

(by Lebesgue's theorem of dominated convergence).

We now prove that $F_{n_6}(x)$ approaches $(3/8)\beta$ as x approaches a^+ by showing that

(3.35) $\quad G_{n_6}(x,\theta) \to [(\pi+\theta)/2]\beta \qquad \text{as } x \to a^+.$

Then

(3.36) $$F_{n_6}(x) \to \frac{1}{2\pi^2}\int_0^{\pi} [(\pi+\theta)/2]\beta \, d\theta = (3/8)\beta \quad \text{as } x \to a^+$$

<u>Lemma (3.2)</u>: <u>If</u>: $f_{n-1}(x) \to \beta \quad \text{as } x \to a^+$

<u>Then:</u> $G_{n_6}(x,\theta) \to [(\pi+\theta)/2]\beta \quad \text{as } x \to a^+$

<u>Proof:</u>

By L'Hospital's rule

(3.37) $\quad L(a,x,\theta) \to -\mathrm{Tan}\,\theta/2 \quad \text{as } x \to a^+$

$$(3.38)\qquad \therefore\ \mathrm{Tan}^{-1}[L(a,x,\theta)] \to -\theta/2 \quad \text{as } x \to a^+$$

$$(3.39)\qquad [(\pi+\theta)/2]\beta = \int_{-\theta/2}^{\pi/2} \beta\, d\phi$$

$$(3.40)\qquad \therefore\ G_{n_6}(x,\theta) - [(\pi+\theta)/2]\beta = \int_{\mathrm{Tan}^{-1}[L(a,x,\theta)]}^{-\theta/2} F_{n-1}(x,\theta,\phi)\,d\phi$$

$$+ \int_{-\theta/2}^{\pi/2} [F_{n-1}(x,\theta,\phi) - \beta]\,d\phi$$

$$(3.41)\qquad \int_{\mathrm{Tan}^{-1}[L(a,x,\theta)]}^{-\theta/2} F_{n-1}(x,\theta,\phi)\,d\phi \to 0 \quad \text{as } x \to a^+$$

(by (3.36) and (3.17)(b))

Since $F_{n-1}(x,\theta,\phi) = f_{n-1}(\xi)$ approaches β as x (and therefore ξ) approaches a^+ for each pair (θ,ϕ), we conclude that

$$(3.42)\qquad \int_{-\theta/2}^{\pi/2} [F_{n-1}(x,\theta,\phi) - \beta]\,d\phi \to \int_{-\theta/2}^{\pi/2} 0\,d\phi = 0 \qquad \text{as } x \to a$$

(by Lebesgue's theorem of dominated convergence).

$$(3.43)\qquad \therefore\ G_{n_6}(x,\theta) - [(\pi+\theta)/2]\beta \to 0 \qquad \text{as } x \to a^+.$$

$$(3.44)\qquad \therefore\ \lim_{x\to a^+} f_n(x) = \lim_{x\to a^+} [g_R(x) + F_{n_4}(x) + F_{n_6}(x)]$$

$$= 1/4 + (3/8)\beta + (3/8)\beta = 1/4 + (3/4)\beta \qquad \text{QED.}$$

For simplicity of notation in what follows, we let

(3.45) (a) $g_R(x) = g(x)$

(b) $$\int_{|\xi| > a} w(\xi) K_R(x,\xi) d\xi = (Kw)(x)$$

Thus the operator K as defined in (3.45)(b) satisfies

(3.46) (a) $(K^o w)(x) = w(x)$

(b) $(K^n w)(x) = [K(K^{n-1} w)](x)$

K is a positive, additive, and bounded operator since

(3.47) $$0 < \int_{|\xi| > a} K_R(x,\xi) d\xi \leq p < 1$$

Let $f_n^{[h]}(x)$ be the n^{th} approximation resulting from the initial approximation h(x). Thus

(3.48) (a) $f_1^{[h]}(x) = g(x) + (Kh)(x)$

(b) $f_2^{[h]}(x) = g(x) + (Kg)(x) + (K^2 h)(x)$

Therefore, by a trivial induction, we obtain

(3.49) $$f_n^{[h]}(x) = \sum_{m=o}^{n-1} (K^m g)(x) + (K^n h)(x) = f_n^{[0]}(x) + (K^n h)(x)$$

If M is an upper bound on $|h(x)|$, then by (3.47) and (3.46)(b) we have

(3.50) $|(K^n h)(x)| \leq Mp^n \to 0$ as $n \to \infty$

Therefore, we confine our attention to $f_n^{[0]}(x)$ in applying the above theorems.

<u>Corollary (3.1):</u> (a) $f_n^{[0]}(x) \to 0$ as $|x| \to \infty$ (for all n)

(b) $\therefore f(x) \to 0$ as $|x| \to \infty$

<u>Corollary (3.2):</u> (a) $f_n^{[0]}(x) \to 1 - (3/4)^n$ as $x \to \begin{Bmatrix} a^+ \\ -a^- \end{Bmatrix}$

(b) $\therefore f(x) \to 1$ as $x \to \begin{Bmatrix} a^+ \\ -a^- \end{Bmatrix}$

(<u>Note:</u> In each of the above corollaries, the second result follows without computation from the theory since infinity and $(\pm a, \pi)$ are points of continuity of the boundary values zero and unity given on the x-axis and C_1' respectively.)

<u>Proof:</u> (a)

Let us label the common limit of $f_n^{[0]}(x)$ as x approaches $\begin{Bmatrix} a^+ \\ -a^- \end{Bmatrix}$ as β_n.

(3.51) $\therefore \beta_o=0,\ \beta_1=1/4,\ \beta_2=1/4+(1/4)(3/4),\ \beta_3=1/4+(1/4)(3/4)+(1/4)(3/4)^2$

and by induction we obtain

$$(3.52)\qquad \beta_n = \begin{Bmatrix} 0, & n = 0 \\ 1/4 \sum_{m=o}^{n-1} (3/4)^m = 1-(3/4)^n, & n \neq 0 \end{Bmatrix} = 1 - (3/4)^n$$

<u>Theorem (3.3):</u> $f(x) \sim A/x^2$ as $|x| \to \infty$ (where A is a certain positive constant).

<u>Proof:</u>

By the Schwarz reflection principle, the Dirichlet problems stated for Ω_1 and Ω_1' are not only equivalent, they are the same. Thus the common solution, $u(x,y)$, of both problems may be developed in a Fourier series outside the circle of radius $\sigma, \sigma^2 = a^2 + \pi^2$, centered at the origin. The Fourier series will be uniformly convergent outside compact sets which properly contain the closure of the above disc. (See Fig. (3.1).)

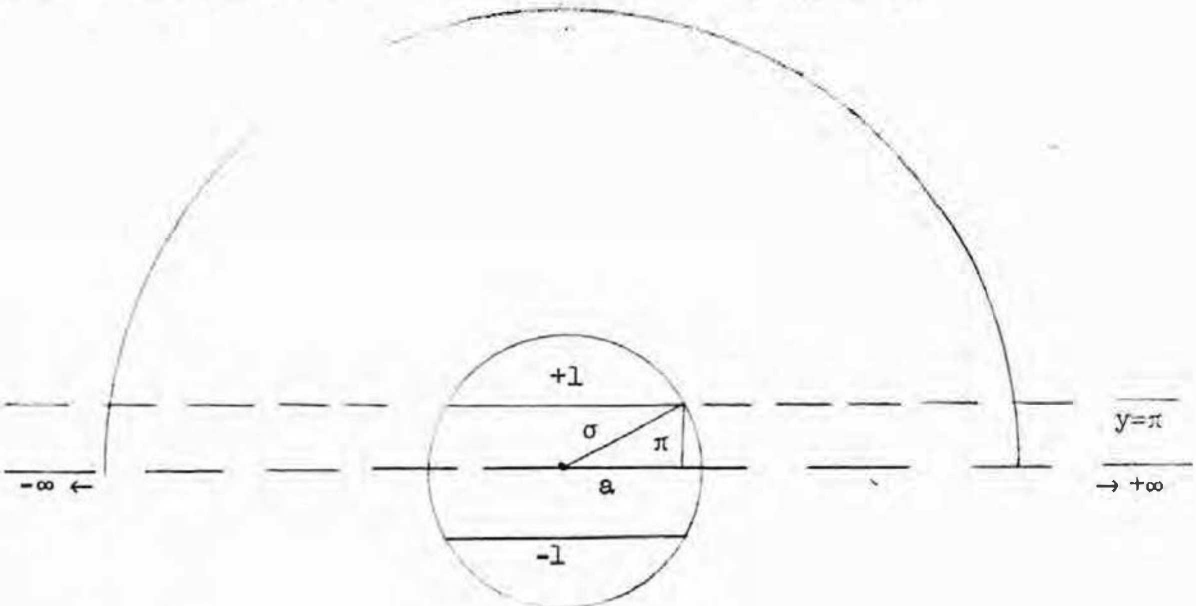

Fig. (3.1):

We determine the behavior of f(x) for large x by studying the behavior of the Fourier series of u(x,y) along $y = \pi$ for large x.

(3.53) $$\therefore\ u(x,y) = u(r,\theta) = \sum_{n=0}^{\infty} (\sigma/r)^n \left[a_n \mathrm{Cos}\, n\theta + b_n \mathrm{Sin}\, n\theta\right]$$

$$(\text{for } r = (x^2 + y^2)^{1/2} > \sigma, \quad \theta = \mathrm{Tan}^{-1}(y/x))$$

But the fact that u(x,y) is odd in y tells us that $u(r,\theta)$ is odd in θ. Therefore the cosine terms vanish; i.e., $a_n = 0$ for all n and

(3.54) $$u(x,y) = \sum_{n=1}^{\infty} b_n (\sigma/r)^n \,\mathrm{Sin}\, n\theta$$

But along $y = \pi$,

(3.55) (a) $\text{Sin}\theta = \pi/r$

(b) $\text{Sin}2\theta = 2(\pi/r)\left[1-(\pi/r)^2\right]^{1/2}$

(3.56) $\therefore\ u(x,\pi) = b_1\sigma\pi/r^2 + O(1/r^3) + \sum_{n=3}^{\infty} b_n(\sigma/r)^n \text{ Sin}\, n\theta$

and

Lemma (3.3a): $\sum_{n=3}^{\infty} b_n(\sigma/r)^n \text{Sin}\, n\theta = O(1/r^3)$ for r large enough

Proof:

(3.57) $\left|\sum_{n=3}^{\infty} b_n(\sigma/r)^n \text{ Sin}\, n\theta\right| \leq \sum_{n=3}^{\infty} |b_n(\sigma/r)^n|$

But the fact that $\sum_{n=3}^{\infty} b_n(\sigma/r)^n \text{ Sin}\, n\theta$ is the Fourier series of a function of one variable for each fixed $r > \sigma$ implies that its coefficients approach zero as n approaches infinity. Thus they are certainly bounded as n approaches infinity. Thus if we let $r = r_1 > \sigma$, there exists $M > 0$ such that

(3.58) $|b_n(\sigma/r_1)^n| \leq M$

Let $r \geq 2r_1 > \sigma$.

(3.59) $\therefore\ \sum_{n=3}^{\infty} |b_n(\sigma/r)^n| = \sum_{n=3}^{\infty} |b_n(\sigma/r_1)^n(r_1/r)^n| \leq M \sum_{n=3}^{\infty} (r_1/r)^n$

$\leq M(r_1/r)^3 \sum_{n=0}^{\infty} (1/2)^n = 2M(r_1/r)^3 = O(1/r^3)$ QED.

(3.60) $\therefore\ f(x) = u(x,\pi) = b_1\sigma\pi/r^2 + O(1/r^3) \sim b_1\sigma\pi/r^2$

$= b_1\sigma\pi/(x^2+\pi^2) \sim b_1\sigma\pi/x^2 = A/x^2.$ QED.

Lemma (3.3b): $A = b_1 \sigma\pi > 0$

Proof:

We need only show that $b_1 > 0$ and this fact follows from

$$(3.61)\quad b_1 = \frac{1}{\pi}\int_{00}^{2\pi} u(\sigma,\Theta)\,\mathrm{Sin}\Theta d\Theta$$

since

$$(3.62)\quad (a)\quad \Theta \in (0,\pi) \Longrightarrow \text{both } u(\sigma,\Theta),\ \mathrm{Sin}\Theta > 0$$

$$(b)\quad \Theta \in (\pi,2\pi) \Longrightarrow \text{both } u(\sigma,\Theta),\ \mathrm{Sin}\Theta < 0$$

Inequality (3.62)(a) follows from the maximum and minimum principles. For, since $u(\sigma,\Theta)$ is not constant on the boundary of Ω_1', it is not constant in the interior of Ω_1'. Therefore its values in the interior must lie strictly between their maximum and minimum on the boundary. Therefore, except at the points $(\pm a,\pi)$,

$$(3.63)\quad \Theta \in (0,\pi) \Longrightarrow 0 < u(x,y) = u(\sigma,\Theta) < 1$$

Similarly, except at the points $(\pm a,-\pi)$,

$$(3.64)\quad \Theta \in (\pi,2\pi) \Longrightarrow -1 < u(x,y) = u(\sigma,\Theta) < 0 \qquad \text{QED.}$$

Corollary (3.3): $f_n^{[0]}(x) = O(1/x^2)$ for all n

Proof:

The fact that $f(x) \sim A/x^2$ implies that $f(x) = O(1/x^2)$. Furthermore, since

$$(3.65)\quad f(x) = \lim_{n\to\infty} f_n^{[0]}(x) = \sum_{m=o}^{\infty} (K^m g)(x), \quad g(x) \geq 0$$

$$(3.66)\quad 0 \leq f_n^{[0]}(x) \leq f(x) \quad \text{for all n}$$

$$(3.67)\quad \therefore\ f_n^{[0]}(x) = O(1/x^2) \quad \text{for all n}$$

In the next section we shall need a bound on $\partial u(r,\theta)/\partial r$ for large r. Therefore we state and prove

Theorem (3.4): $\partial u(r,\theta)/\partial r = O(1/r^2)$ for large r.

Proof:

Formally, we obtain from (3.54)

$$(3.68)\quad \partial u(r,\theta)/\partial r = -\sum_{n=1}^{\infty} (nb_n/\sigma)(\sigma/r)^{n+1}\operatorname{Sin} n\theta$$

which we justify by showing that the derived series is uniformly convergent for $r \geq 2r_1 > \sigma$. The proof of its uniform convergence will also prove that it is $O(1/r^2)$.

$$(3.69)\quad |(nb_n/\sigma)(\sigma/r_1)^{n+1}| \leq Mn/r_1$$

(by (3.58))

$$(3.70)\quad \therefore\ r \geq 2r_1 > \sigma \Longrightarrow |-\sum_{n=1}^{\infty} (nb_n/\sigma)(\sigma/r)^{n+1}\operatorname{Sin} n\theta|$$

$$\leq \sum_{n=1}^{\infty} |(nb_n/\sigma)(\sigma/r)^{n+1}| = \sum_{n=1}^{\infty} |(nb_n/\sigma)(\sigma/r_1)^{n+1}(r_1/r)^{n+1}|$$

$$\leq M/r_1 \sum_{n=1}^{\infty} n(r_1/r)^{n+1} \leq Mr_1/r^2 \sum_{n=1}^{\infty} n(1/2)^{n-1} = 4Mr_1/r^2 = O(1/r^2)$$

Thus (3.69) and (3.70) imply that $\partial u(r,\theta)/\partial r$ may be obtained by term-by-term differentiation of $u(r,\theta)$ for all $r > \sigma$ and that for all $r > \sigma$

$$(3.71)\quad \partial u(r,\theta)/\partial r = O(1/r^2).$$

IV. Application

Consider the conformal map of Ω_1 onto an annulus centered at the origin with outer radius unity and inner radius ρ. (See Fig. (4.1).) Our application will be the determination of ρ -- which is essentially the conformal modulus of Ω_1 -- (actually the conformal modulus of Ω_1 is defined as $-2\pi/\log \rho$) purely in terms of the values $f(x)$ of $u(x,y)$ along $y = \pi$.

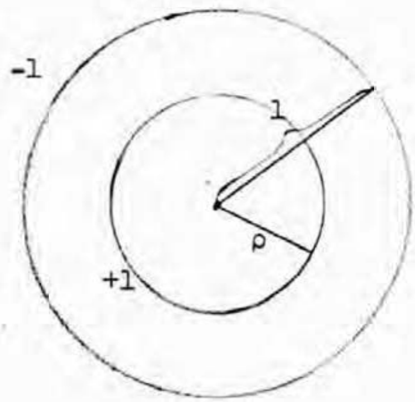

Fig. (4.1):

The slits C_o and C_1 are carried into the circles centered at the origin of radii 1 and ρ respectively. The boundary values -1 and +1 are then assumed respectively on these circles. The Dirichlet problem for this annulus is then solved to yield

(4.1) $\quad j(r,\theta) = 2\log r/\log \rho - 1$

If $\ell(r,\theta)$ is a harmonic conjugate of $j(r,\theta)$, then

(4.2) $\quad j(r,\theta) + i\ell(r,\theta) = 2\log z/\log \rho - 1 + i\delta$

where $z = re^{i\theta}$, and δ is a real constant. Thus

(4.3) $\quad \ell(r,\theta) = 2\theta/\log \rho + \delta$

and its period P around the inner circle is

(4.4) $P = -4\pi/\log \rho$

(4.5) $\therefore \quad \rho = \exp(-4\pi/P)$

Thus, to find ρ we must find P. But the period of a harmonic function is a conformal invariant. Therefore, if $v(x,y)$ is a harmonic conjugate of $u(x,y)$, then the period of $v(x,y)$ as the point (x,y) performs a circuit -- say B -- about C_1 (the preimage of the inner circle) is P.

Therefore, by the Cauchy-Riemann equations,

(4.6) $$P = \oint_B \partial v/\partial s \, ds = \oint_B \partial u/\partial n \, ds$$

where s is the parameter of arc length along B and $\partial/\partial n$ indicates differentation with respect to the outward normal along B.

We take the circuit B to be a semicircle centered at the origin, based on the x-axis, and surrounding only C_1 (of the boundary components of Ω_1). -- See Fig. (3.1).

(4.7) $$\therefore \quad \oint_B \partial u/\partial n \, ds = \int_{\curvearrowleft} \partial u/\partial n \, ds + \int_{\rightarrow} \partial u/\partial n \, ds$$

We then let the radius of B expand to infinity and show that

(4.8) $$\int_{\curvearrowleft} \partial u/\partial n \, ds \to 0$$

The period, P, of $v(x,y)$ around C_1 will then be

(4.9) $$P = \int_{\rightarrow} \partial u/\partial n \, ds = + \int_{-\infty}^{\infty} [\partial u(x,0)/\partial y] ds$$

Along the large semicircle $\partial u/\partial n = -\partial u/\partial r$ (by the same reasoning as above). Therefore, by Theorem (3.4), we have

$$(4.10)\quad \oint \partial u/\partial n\, ds = -\int_0^{\pi} \left[\partial u(R,\Theta)/\partial r\right]\cdot Rd\Theta = \int_0^{\pi} O(1/R^2)\cdot Rd\Theta$$

$$= O(1/R) \to 0 \qquad \text{as } R \to \infty$$

$$(4.11)\quad \partial u(x,0)/\partial y = \lim_{y \to 0} y^{-1}\left[u(x,y)-u(x,0)\right] = \lim_{y\to 0} y^{-1}u(x,y)$$

($u(x,0) \equiv 0$, as indicated in the remark following (2.1).)

Thus, in evaluating $\partial u(x,0)/\partial y$, we need deal only with the values of $u(x,y)$ in the strip

$$S = \{(x,y)\,|\,0 < y < \pi\}.$$

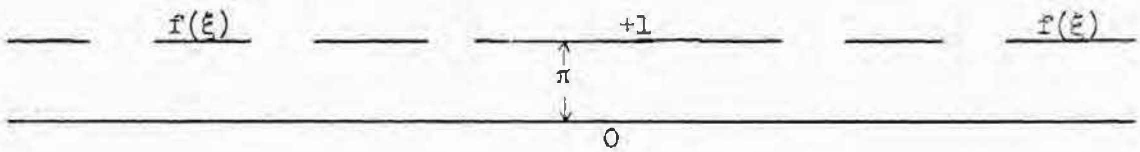

Fig. (4.2):

Therefore, using (4.11) and (2.3) (a), we obtain

$$(4.12)\quad \partial u(x,0)/\partial y = \lim_{y\to 0^+} y^{-1}u(x,y) = \lim_{y\to 0^+} \frac{\operatorname{Sin} y}{\pi y}\int_{-\infty}^{\infty} \frac{f(\xi)\,d\xi}{e^{\xi-x}+2\operatorname{Cos} y+e^{-(\xi-x)}}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)\,d\xi}{e^{\xi-x}+2+e^{-(\xi-x)}}$$

$$(4.13)\quad \therefore\; P = \int_{-\infty}^{\infty} \left[\partial u(x,0)/\partial y\right]dx = \int_{-\infty}^{\infty}\left[\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)\,d\xi}{e^{\xi-x}+2+e^{-(\xi-x)}}\right]dx$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\left[\int_{-\infty}^{\infty} \frac{dx}{e^{\xi-x}+2+e^{-(\xi-x)}}\right]d\xi = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\,d\xi \quad [5]$$

---

[5] Note: P may be approximated by $P_n = \frac{1}{\pi}\int_{-\infty}^{\infty} f_n^{[0]}(\xi)\,d\xi$

since $f_n^{[0]}(\xi)$ converges uniformly to $f(\xi)$ and since $f_n^{[0]}(\xi) = O(1/\xi^2)$ as shown in Corollary (3.3).

To justify interchanging the order of integration in (4.13), we need only show that $f(\xi)$ belongs to $L^1(-\infty,\infty)$. But this follows from the fact that $f(\xi) = O(1/\xi^2)$ for $|\xi| > a$ -- as shown in Theorem (3.3) -- and that $f(\xi) = 1$ for $-a \leq \xi \leq a$.

$$(4.14) \quad \therefore \quad \rho = \exp\left[-4\pi^2\left(\int_{-\infty}^{\infty} f(\xi)\,d\xi\right)^{-1}\right]$$

V. Ω_n: Derivation of the Integral Equation

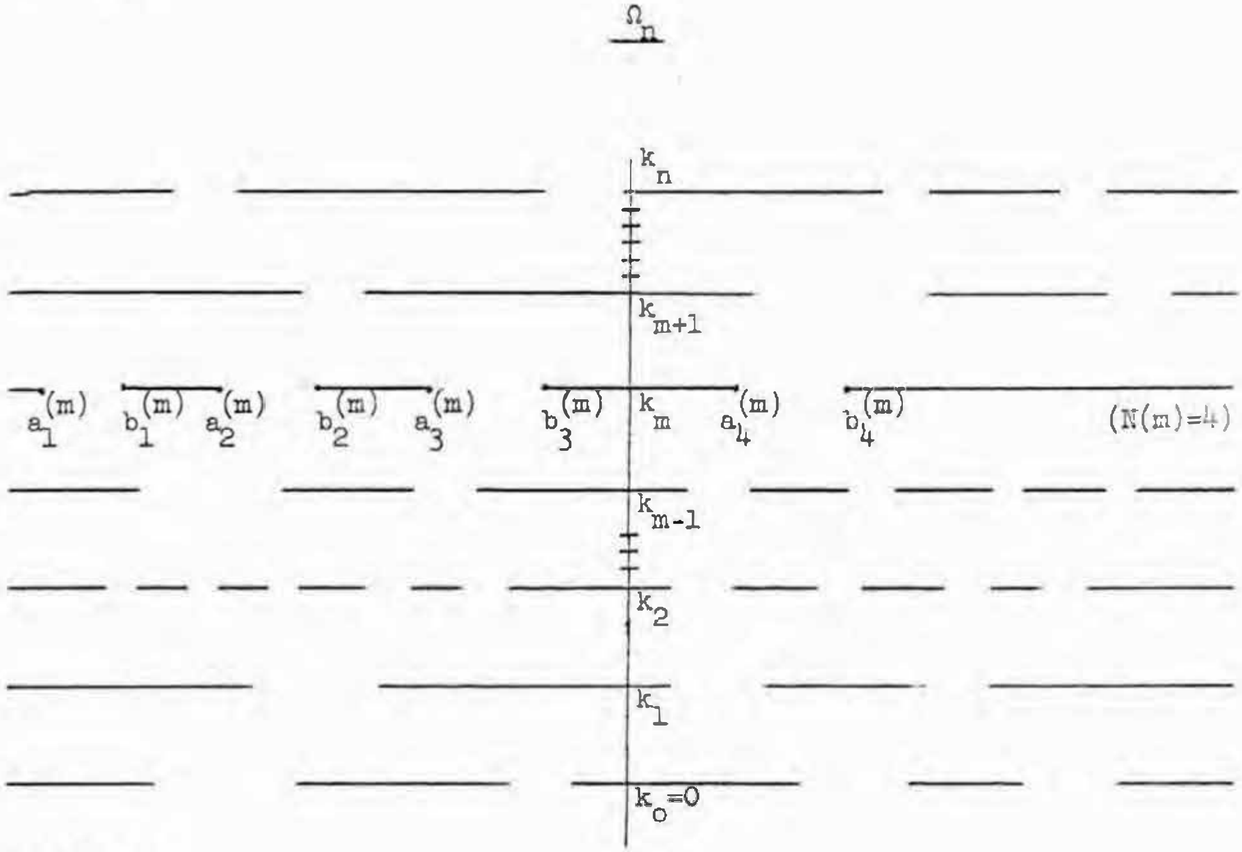

Fig (5.1):

In passing from our particular case of Ω_1 to Ω_n, we confine our attention at first to the case of a domain whose boundary slits along each line extend to infinity both on the left and on the right. Thus the complementary intervals along each line will be bounded. Such Ω_n will be termed "bounded" and once the result is proved for "bounded" Ω_n we will be able to extend it to arbitrary Ω_n.

Let the n+1 lines containing the boundary slits be labeled ℓ_m with corresponding equations $y = k_m$ $(m = 0,1,2,\dots,n)$. Let $d_m = k_m - k_{m-1}$ $(m = 1,2,\dots,n)$. Without loss of generality we may assume that $k_o = 0$ and that one of the $d_m = \pi$.

We label the complementary intervals along ℓ_m as $(a_q^{(m)}, b_q^{(m)})$; $q = 1,2,\dots,N(m)$ where $N(m)$ is the number of complementary intervals along ℓ_m. (See Fig. (5.1).)

(5.1) Let (a) $D_m = \bigcup_{q=1}^{N(m)} (a_q^{(m)}, b_q^{(m)})$

(b) $C_m = \ell_m - D_m$

(c) $C = \bigcup_{m=o}^{n} C_m$ [5]

(d) $D = \bigcup_{m=o}^{n} D_m$

Let the functions $h_m(x)$, bounded and possessing at most a finite number of discontinuities, be prescribed on the components of C_m for each m. Find the function $u(x,y)$ harmonic and bounded in the interior of Ω_n which approaches (for each m) the boundary values $h_m(x)$ at each point (x,k_m) of C_m at which $h_m(x)$ is continuous.

Once again, we seek to determine the values $f_m(x)$ of $u(x,y)$ along D_m for each m (though we will at times find it convenient to let $f_m(x)$ denote the values of $u(x,y)$ along <u>all</u> of ℓ_m) and note that we thus obtain the values of $u(x,y)$ along $C \bigcup D$ which divides the plane into n+2 regions in each of which $u(x,y)$ may be determined from its boundary values by the appropriate Poisson integral. Let us denote these regions by

[5] Thus C is the boundary of Ω_n.

(5.2) (a) $S_o = \{(x,y) | y < 0\}$

(b) $S_m = \{(x,y) | k_{m-1} < y < k_m\}$, $(m = 1,2,\ldots,n)$

(c) $S_{n+1} = \{(x,y) | y > k_n\}$

(Of course, if $D_{\binom{o}{n}} = \emptyset$, then we do not consider $S_{\binom{o}{n+1}}$.)

$$(5.3) \quad \therefore \quad u(x,y) = \begin{cases} \sum_{m=o}^{n+1} u_m(x,y) , & y \neq k_m \\ f_m(x) , & y = k_m \end{cases}$$

where

$$(5.4) \quad (a) \quad u_o(x,y) = \frac{-y}{\pi} \int_{-\infty}^{\infty} \frac{f_o(\xi)d\xi}{(\xi-x)^2 + y^2} , \quad y < 0$$

$$u_o(x,y) = 0 , \quad y > 0$$

$$(b) \quad u_{n+1}(x,y) = \frac{y-k_n}{\pi} \int_{-\infty}^{\infty} \frac{f_n(\xi)d\xi}{(\xi-x)^2 + (y-k_n)^2} , \quad y > k_n$$

$$u_{n+1}(x,y) = 0 , \quad y < k_n$$

(c) $u_m(x,y) =$
$(m = 1,2,\ldots,n)$

$$\frac{1}{d_m} e^{\frac{\pi x}{d_m}} \text{Sin} \frac{\pi}{d_m}(y-k_{m-1}) \int_{-\infty}^{\infty} \frac{f_{m-1}(\xi) e^{\frac{\pi\xi}{d_m}} d\xi}{\left(e^{\frac{\pi\xi}{d_m}} - e^{\frac{\pi x}{d_m}} \text{Cos} \frac{\pi}{d_m}(y-k_{m-1})\right)^2 + \left(e^{\frac{\pi x}{d_m}} \text{Sin} \frac{\pi}{d_m}(y-k_{m-1})\right)^2} ,$$

$$+ \frac{1}{d_m} e^{\frac{\pi x}{d_m}} \operatorname{Sin}\frac{\pi}{d_m}(y-k_{m-1}) \int_{-\infty}^{\infty} \frac{f_m(\xi) e^{\frac{\pi\xi}{d_m}} d\xi}{\left(e^{\frac{\pi\xi}{d_m}} + e^{\frac{\pi x}{d_m}} \operatorname{Cos} \frac{\pi}{d_m}(y-k_{m-1})\right)^2 + \left(e^{\frac{\pi x}{d_m}} \operatorname{Sin} \frac{\pi}{d_m}(y-k_{m-1})\right)^2},$$

$$k_{m-1} < y < k_m$$

$$u_m(x,y) = 0 \qquad , y \in [k_{m-1}, k_m]$$

(Of course if $D_{\binom{o}{n}} = \emptyset$, then we do not consider $u_{\binom{o}{n+1}}(x,y)$.)

Equivalent to formula (5.4)(c) is the formula

$$(5.5)\ u_m(x,y) = \frac{1}{d_m} \operatorname{Sin} \frac{\pi}{d_m}(y-k_{m-1}) \int_{-\infty}^{\infty} \frac{f_{m-1}(\xi) d\xi}{e^{\frac{\pi}{d_m}(\xi-x)} - 2\operatorname{Cos}\frac{\pi}{d_m}(y-k_{m-1}) + e^{-\frac{\pi}{d_m}(\xi-x)}}$$

$$+ \frac{1}{d_m} \operatorname{Sin} \frac{\pi}{d_m}(y-k_{m-1}) \int_{-\infty}^{\infty} \frac{f_m(\xi) d\xi}{e^{\frac{\pi}{d_m}(\xi-x)} + 2\operatorname{Cos} \frac{\pi}{d_m}(y-k_{m-1}) + e^{-\frac{\pi}{d_m}(\xi-x)}},\ k_{m-1} < y < k_m$$

$$u(x,y) = 0 \qquad , y \notin [k_{m-1}, k_m]$$

If $P = (x,y)$ denotes an arbitrary point of S_m, Q_m an arbitrary point of ℓ_m, and $\wp_m$ the Poisson kernel of S_m, then we have

(5.6) (a) $\wp_o = \wp_o(P, Q_o)$

(b) $\wp_{n+1} = \wp_{n+1}(P, Q_n)$

$$\text{(c)}\quad \wp_m = \begin{cases} \wp_m(P,Q_{m-1},d_m) \\ \wp_m(P,Q_m,d_m) \end{cases}$$

depending upon whether the boundary point Q lies in ℓ_{m-1} or in ℓ_m. (We include d_m above since the Poisson kernel of each strip, S_m, depends on the strip's thickness, d_m. At times, we will simply write $\wp_m = \wp_m(P,Q)$.)

Thus (5.4) could be rewritten as

$$\text{(5.7)}\quad \text{(a)}\quad u_o(P) = \begin{cases} \displaystyle\int_{Q_o \in \ell_o} f_o(\xi)\wp_o(P,Q_o)d\xi \quad , & P \in S_o \\ 0 \quad , & P \notin S_o \end{cases}$$

$$\text{(b)}\quad u_{n+1}(P) = \begin{cases} \displaystyle\int_{Q_n \in \ell_n} f_n(\xi)\wp_{n+1}(P,Q_n)d\xi \quad , & P \in S_{n+1} \\ 0 \quad , & P \notin S_{n+1} \end{cases}$$

$$\text{(c)}\quad u_m(P) = \quad (m = 1,2,\ldots,n) \quad \begin{cases} \displaystyle\int_{Q_{m-1} \in \ell_{m-1}} f_{m-1}(\xi)\wp_m(P,Q_{m-1},d_m)d\xi \\ \displaystyle + \int_{Q_m \in \ell_m} f_m(\xi)\wp_m(P,Q_m,d_m)d\xi \quad , & P \in S_m \\ 0 \quad , & P \notin S_m \end{cases}$$

We now apply the mean value property to u(P) at the point

$P = P_m = (x,k_m)$ in one of the intervals of D_m and average over the circle $C_R(P_m)$ where

$$(5.8)\quad R = R(x) = \begin{cases} \min\,(x - a_q^{(o)},\ b_q^{(o)} - x,\ d_1)\ , & m = 0 \\ \min\,(x - a_q^{(n)},\ b_q^{(n)} - x,\ d_n), & m = n \\ \min\,(x - a_q^{(m)},\ b_q^{(m)} - x,\ d_m,\ d_{m+1}), & m = 1,2,\ldots,n-1. \end{cases}$$

Along $C_R(P_m)$, $P = P_m + Re^{i\theta}$ lies either in S_{m+1} for θ in $[0,\pi]$ or in S_m for θ in $[\pi,2\pi]$ and R is the largest possible radius under which this condition holds. (At most four points of $C_R(P_m)$ lie on a "boundary" line.) See Figures (5.2) - (5.4).

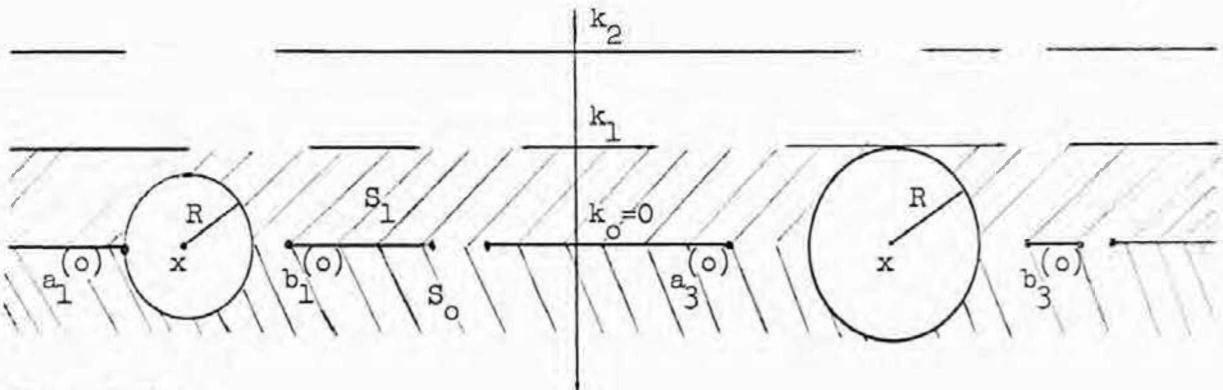

Fig. (5.2):

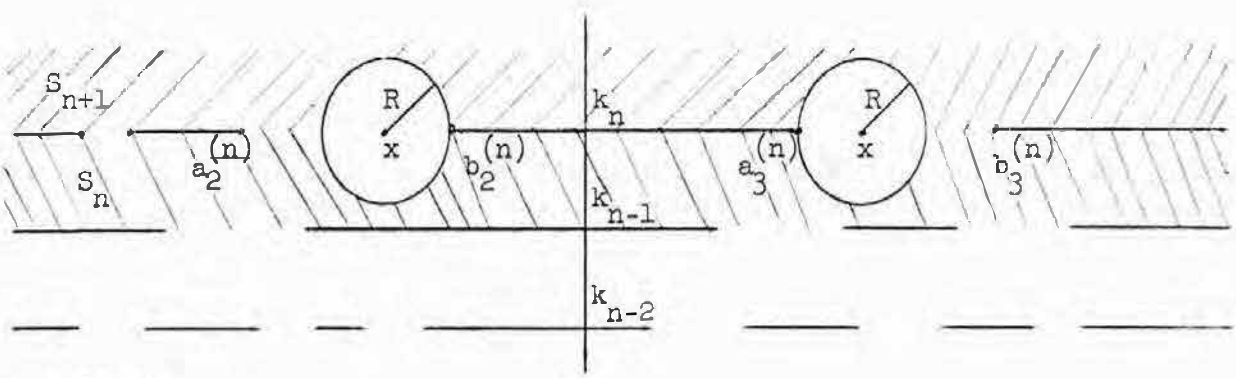

Fig. (5.3):

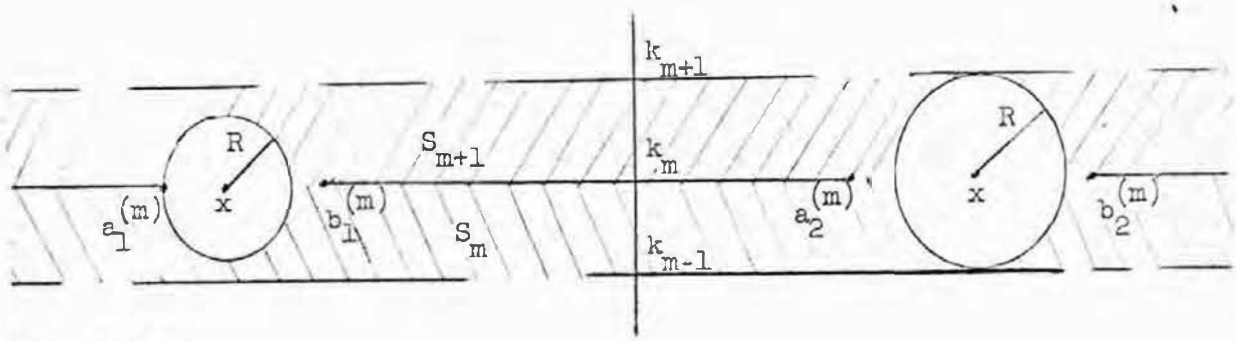

Fig. (5.4):

Therefore the mean value property of $u(P)$ at $P = P_m$ may be expressed as

$$(5.9)\quad f_m(x) = \frac{1}{2\pi}\int_0^{2\pi} u(P_m + Re^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{\pi} u_{m+1}(P_m + Re^{i\theta})d\theta$$

$$+ \frac{1}{2\pi}\int_{\pi}^{2\pi} u_m(P + Re^{i\theta})d\theta .$$

Now by substituting the equations (5.7) in (5.9) and then interchanging the order of integration - as in Section II - we obtain

$$(5.10)\quad f_m(x) = \int_{\Gamma_m} H_m(\xi)K_R^{(m)}(x,\xi)d\xi + \int_{\Delta_m} F_M(\xi)K_R^{(m)}(x,\xi)d\xi$$

$$= G_R^{(m)}(x) + \int_{\Delta_m} F_m(\xi)K_R^{(m)}(x,\xi)d\xi$$

where

$$(5.11)\quad K_R^{(m)}(x,\xi) = \begin{cases} \frac{1}{2\pi}\int_0^{\pi} \wp_{m+1}(P_m + Re^{i\theta}, Q_m, d_{m+1})d\theta \\ + \frac{1}{2\pi}\int_{\pi}^{2\pi} \wp_m(P_m + Re^{i\theta}, Q_m, d_m)d\theta, \quad Q_m = (\xi, k_m) \in \ell_m \end{cases}$$

$$\begin{cases} \frac{1}{2\pi}\int_{0}^{\pi}\varphi_{m+1}(P_m+Re^{i\theta},Q_{m+1},d_{m+1})d\theta, & Q_{m+1}=(\xi,k_{m+1})\epsilon\,\ell_{m+1}, \\ \frac{1}{2\pi}\int_{\pi}^{2\pi}\varphi_m(P_m+Re^{i\theta},Q_{m-1},d_m)d\theta, & Q_{m-1}=(\xi,k_{m-1})\in\ell_{m-1}, \end{cases}$$

and

(5.12) (a) $$\Gamma_m = \begin{cases} C_o \bigcup C_1 & , m = 0 \\ C_{n-1} \bigcup C_n & , m = n \\ C_{m-1} \bigcup C_m \bigcup C_{m+1} & , m = 1,2,\ldots,n-1 \end{cases}$$

(b) $$\Delta_m = \begin{cases} D_o \bigcup D_1 & , m = 0 \\ D_{n-1} \bigcup D_n & , m = n \\ D_{m-1} \bigcup D_m \bigcup D_{m+1} & , m = 1,2,\ldots,n-1 \end{cases}$$

(c) $H_m(\xi)$ = The known values of $u(x,y)$ in Γ_m

(d) $F_m(\xi)$ = The unknown values of $u(x,y)$ in Δ_m

(e) $$G_R^{(m)}(x) = \int_{\Gamma_m} H_m(\xi)K_R^{(m)}(x,\xi)d\xi$$

If we let $K_R^{(m)}(x,\xi) = K_R(x,\xi)$ for ξ the abscissa of a point $Q = (\xi,y)$ in $\Gamma_m \bigcup \Delta_m$ and let $\begin{pmatrix} H_m(\xi) = h(\xi) \\ F_m(\xi) = f(\xi) \end{pmatrix}$ under similar conditions, we may then rewrite (5.10) as

$$(5.13) \quad f(x) = g_R(x) + \int_{\Delta_m} f(\xi)K_R(x,\xi)d\xi, \; x \in D_m, (m = 0,1,2,\ldots,n)$$

where $x \in D_m$ means x is the abcissa of $P_m = (x,k_m)$ in D_m and $G_R^{(m)}(x) = g_R(x)$ for $x \in D_m$.

VI. Ω_n: Solvability of the Integral Equation by Iteration

Equation (5.13) is solvable by iteration if there exists $0 < p < 1$ such that

$$(6.1) \qquad 0 < \int_{\Delta_m} K_R(x,\xi)d\xi \leq p < 1, \quad x \in D_m, (m = 0,1,2,\ldots,n)$$

(The left hand inequality is trivial since $K_R(x,\xi)$ is positive for all x,ξ.)

We begin our proof of inequality (6.1) by taking note of the fact that for P in any of the domains S_m having Poisson kernel $\wp_m = \wp_m(P,Q)$ where $Q = \left(\xi, k_{\binom{m-1}{m}}\right)$ is in the boundary, B_m, of S_m

$$(6.2) \qquad \int_{B_m} \wp_m(P,Q)d\xi = \int_{B_m} 1\cdot\wp_m(P,Q)d\xi = 1$$

Therefore, by (5.11) and by Fubini's theorem, (6.2) implies

$$(6.3) \qquad \int_{\Delta_0} K_R(x,\xi)d\xi = \frac{1}{2\pi}\int_0^{\pi}\left[\int_{D_0}\wp_1(P_0+Re^{i\theta},Q_1,d_1)d\xi + \int_{D_1}\wp_1(P_0+Re^{i\theta},Q_1,d_1)d\xi\right]d\theta$$

$$+ \frac{1}{2\pi}\int_{\pi}^{2\pi}\left[\int_{D_0}\wp_0(P_0+Re^{i\theta},Q_0)d\xi\right]d\theta$$

$$\leq \frac{1}{2\pi}\int_0^{\pi}\left[\int_{B_1}\wp_1(P_0+Re^{i\theta},Q)d\xi\right]d\theta + \frac{1}{2\pi}\int_{\pi}^{2\pi}\left[\int_{B_0}\wp_0(P_0+Re^{i\theta},Q)d\xi\right]d\theta$$

$$= 1/2 + 1/2 = 1$$

A similar statement holds for $\int_{\Delta_m} K_R(x,\xi)d\xi$ for all other values of m.

We now prove inequality (6.1) for x in D_o (assuming D_o is not empty). Since a comparison of (5.7)(a) and (5.4)(a) shows that

$$(6.4) \qquad \varphi_o(P,Q_o) = \frac{-y}{\pi} \frac{1}{(\xi-x)^2 + y^2}, \quad P = (x,y) \in S_o$$

we conclude that

$$(6.5) \qquad \frac{1}{2\pi}\int_{\pi}^{2\pi} \varphi_o(P_o + Re^{i\theta}, Q_o)d\theta = \frac{-R}{2\pi^2}\int_{\pi}^{2\pi} \frac{\sin\theta \, d\theta}{(\xi-x-R\cos\theta)^2 + (R\sin\theta)^2}$$

$$= \left[2\pi^2(\xi-x)\right]^{-1} \log\left|\frac{\xi-x+R}{\xi-x-R}\right|$$

$$(6.6) \qquad \therefore \int_{\Delta_o} K_R(x,\xi)d\xi \le 1/2 + \frac{1}{2\pi^2}\int_{D_o} (\xi-x)^{-1} \log\left|\frac{\xi-x+R}{\xi-x-R}\right| d\xi$$

But it has been shown [1, p. 314] that there exists $0 < \alpha < 1$ such that

$$(6.7) \qquad 0 < \frac{1}{\pi^2}\int_{D_o} (\xi-x)^{-1} \log\left|\frac{\xi-x+R}{\xi-x-R}\right| d\xi \le \alpha < 1 \quad \text{for all } x \in D_o \ni$$

$$R = R(x) = \min(x - a_q^{(o)}, b_q^{(o)} - x).$$

If $R = R(x) = d_1$, i.e., $d_1 < \min(x - a_q^{(o)}, b_q^{(o)} - x)$ for $P_1 = (x,0) \in (a_q^{(o)}, b_q^{(o)})$, we then have $x \in (a_q^{(o)} + d_1, b_q^{(o)} - d_1)$ and

$$(6.8) \qquad \frac{1}{\pi^2}\int_{D_o} (\xi-x)^{-1}\log\left|\frac{\xi-x+d_1}{\xi-x-d_1}\right| d\xi = 1 - \frac{1}{\pi^2}\int_{C_o} (\xi-x)^{-1}\log\left|\frac{\xi-x+d_1}{\xi-x-d_1}\right| d\xi$$

$$\leq 1 - \frac{1}{\pi^2} \int_{b_q^{(o)}}^{a_{q+1}^{(o)}} (\xi - x)^{-1} \log\left|\frac{\xi - x + d_1}{\xi - x - d_1}\right| d\xi$$

$$= 1 - \frac{1}{\pi^2} \int_{(b_q^{(o)} - x)/d_1}^{(a_{q+1}^{(o)} - x)/d_1} u^{-1} \log\left|\frac{1+u}{1-u}\right| du$$

$$\leq 1 - \frac{1}{\pi^2} \int_{(b_q^{(o)} - a_q^{(o)})/d_1}^{(a_{q+1}^{(o)} - a_q^{(o)})/d_1} u^{-1} \log\left|\frac{1+u}{1-u}\right| du = \alpha_q{}^{7} < 1$$

The last integral inequality in (6.8) follows from the fact that the interval $\left[(b_q^{(o)} - x)/d_1, (a_{q+1}^{(o)} - x)/d_1\right]$ lies to the right of the point $u = 1$ (since $b_q^{(o)} - x > d_1$) and has constant length for all x. Furthermore the function $u^{-1} \log\left|\frac{1+u}{1-u}\right|$ is monotonically decreasing for $u > 1$ and consequently its integral over intervals of constant length lying to the right of $u = 1$ decreases as the intervals move further to the right.

$\therefore x \in (a_q^{(o)} + d_1, b_q^{(o)} - d_1) \Longrightarrow$

(6.9) $$\frac{1}{\pi^2} \int_{(b_q^{(o)}-x)/d_1}^{(a_{q+1}^{(o)}-x)/d_1} u^{-1}\log\left|\frac{1+u}{1-u}\right| du \geq \frac{1}{\pi^2} \int_{(b_q^{(o)}-a_q^{(o)})/d_1}^{(a_{q+1}^{(o)}-a_q^{(o)})/d_1} u^{-1}\log\left|\frac{1+u}{1-u}\right| du$$

---

[7] Let $\alpha_q = 0$ if $R = R(x) < d_1$ for $x \quad (a_q^{(o)}, b_q^{(o)})$

and thus (6.8) follows. (If $q = N(1)$, then $a_{q+1}^{(o)} = \infty$.)

Letting $\alpha' = \max_{q=1,2,\ldots,N(1)} (\alpha,\alpha_q)$, we have $\alpha' < 1$ and therefore

$$(6.10) \quad 0 < \frac{1}{\pi^2} \int_{D_o} (\xi-x)^{-1} \log\left|\frac{\xi-x+R}{\xi-x-R}\right| d\xi \leq \alpha' < 1 \text{ for all } x \in D_o$$

$$(6.11) \quad \therefore \int_{\Delta_o} K_R(x,\xi)d\xi \leq 1/2 + \alpha'/2 = p < 1. \qquad \text{Q.E.D.}$$

If x is in D_n, we prove the inequality (6.1) by a method similar to that used for x in D_o. Therefore we now focus our attention on x in D_m; $m = 1,2,\ldots,n-1$. Without loss of generality, we may pick a specific such m and assume $d_{m+1} = \pi$.

By the comment following (6.3), we may write

$$(6.12) \quad \int_{\Delta_m} K_R(x,\xi)d\xi \leq \frac{1}{2\pi} \int_o^\pi \left[\int_{D_m} \wp_{m+1}(P_m+Re^{i\theta},Q_m,\pi)d\xi\right] d\theta$$

$$+ \frac{1}{2\pi} \int_o^\pi \left[\int_{\ell_{m+1}} \wp_{m+1}(P_m+Re^{i\theta}, Q_{m+1},\pi)d\xi\right] d\theta + 1/2$$

$$= 1/2 - \frac{1}{2\pi} \int_o^\pi \left[\int_{C_m} \wp_{m+1}(P_m+Re^{i\theta},Q_{m+1},\pi)d\xi\right] d\theta + 1/2$$

$$\leq 1 - \frac{1}{2\pi} \int_o^\pi \left[\int_{b_{q-1}^{(m)}}^{a_q^{(m)}} \wp_{m+1}(P_m+Re^{i\theta},Q_{m+1},\pi)d\xi\right] d\theta$$

(where $b_{q-1}^{(m)} = -\infty$ if $q = 1$).

(6.13) Let (a) $a_q^{(m)} = a$

(b) $b_{q-1}^{(m)} = b$

(c) $b_q^{(m)} = b_1$.

By (6.12), we need only show that

(6.14) $$\frac{1}{2\pi} \int_0^{\pi} \left[\int_b^a \varphi_{m+1}(P_m + Re^{i\theta}, Q_m, \pi)\,d\xi \right] d\theta > 0, \; x \in D_m$$

in order to prove (6.1).

(6.15) Letting (a) $(a + b_1)/2 = a_1$

(b) $\min(a + \pi, a_1) = a_2$

(c) $\max(b_1 - \pi, a_1) = b_2$

we have the following four possibilities for $R = R(x)$ when $P_m = (x, k_m) \in (a_q^{(m)}, b_q^{(m)})$.

(6.16) (a) $a < x \leq a_2 \Longrightarrow R = x - a$ (c) $a_1 \leq x \leq b_1 - \pi \Longrightarrow R = \pi$

(b) $a + \pi \leq x \leq a_1 \Longrightarrow R = \pi$ (d) $b_2 \leq x \leq b_1 \Longrightarrow R = b_1 - x$.

We need not consider the possibility $d_m < \pi$ and $R = d_m$ since the proof of (6.1) for this case is similar to that of the case $R = \pi$. Furthermore, we may assume without loss of generality that P_m lies in the left half of (a, b_1) and therefore it suffices to consider only the possibilities (6.16)(a),(b).

A comparison of (5.7)(c) and (5.4)(c) shows that

(6.17) $\varphi_{m+1}(P_m+Re^{i\theta},Q_m,\pi) =$

$$\frac{e^{\xi}}{\pi}\ \frac{\exp(x+R\cos\theta)\sin(R\sin\theta)}{(e^{\xi}-\exp(x+R\cos\theta)\cos(R\sin\theta))^2+(\exp(x+R\cos\theta)\sin(R\sin\theta))^2}$$

(6.18) $\therefore\ \frac{1}{2\pi}\int_0^{\pi}\left[\int_b^a \varphi_{m+1}(P_m + Re^{i\theta},Q_m,\pi)d\xi\right]d\theta$

$$= \frac{1}{2\pi^2}\int_0^{\pi} \operatorname{Tan}^{-1}\left[\frac{(e^a-e^b)\exp(x+R\cos\theta)\sin(R\sin\theta)}{\exp[2(x+R\cos\theta)]-(e^a+e^b)\exp(x+R\cos\theta)\cos(R\sin\theta)+e^{a+b}}\right]d\theta$$

Let the argument of the arctangent in (6.18) be $J(x,R,\theta)$. Of course $J(x,R,\theta)$ is non-negative since the Poisson kernel is non-negative. Therefore, in order to prove that $\frac{1}{2\pi^2}\int_0^{\pi}\operatorname{Tan}^{-1}[J(x,r,\theta)]d\theta$ is positive, it suffices to prove the same of $J(x,R,\theta)$ for θ in some subset of $[0,\pi]$ having positive measure.

Under the conditions of (6.16)(b), we have

(6.19) $$J(x,R,\theta)=J(x,\pi,\theta)=\frac{(e^a-e^b)\exp(x+\pi\cos\theta)\sin(\pi\sin\theta)}{\exp[2(x+\pi\cos\theta)]-(e^a+e^b)\exp(x+\pi\cos\theta)\cos(\pi\sin\theta)+e^{a+b}}$$

(6.20) $\therefore\ \theta\in[\pi/6,\ 5\pi/6] \Longrightarrow J(x,\pi,\theta)\geq$

$$\frac{(e^a-e^b)\exp[a+\pi(1+\cos\theta)]\sin(\pi\sin\theta)}{\exp[2(a_1+\pi\cos\theta)]-(e^a+e^b)\exp(a_1+\pi\cos\theta)\cos(\pi\sin\theta)+e^{a+b}} = L(\pi,\theta) > 0$$

For the case (6.16)(a), we note that $J(x,R,\theta) = J(x,x-a,\theta)$ which is indeterminate at $x = a$. However, using L'Hospital's rule, we find that

(6.21) $$J(x,x-a,\theta) \longrightarrow \sin\theta/(1+\cos\theta) = \tan\theta/2 \quad \text{as } x \to a^+$$

Therefore, if we define

(6.22) $$\tan^{-1}[J(x,x-a,\theta)] = \theta/2$$

then given $\epsilon > 0$, $\exists\ \delta(\epsilon) > 0 \ni a < x < a+\delta \Longrightarrow$

(6.23) $$\frac{1}{2\pi^2}\int_0^\pi \tan^{-1}[J(x,x-a,\theta)]\,d\theta > \frac{1}{2\pi^2}\int_0^\pi (\theta/2-\epsilon)\,d\theta = 1/8 - \epsilon/2\pi$$

$$> 0 \text{ if } \epsilon < \pi/4$$

Finally, if $a + \delta \le x \le a_2$, then

(6.24) $$J(x,x-a,\theta) =$$

$$\frac{(e^a-e^b)\exp[x(1+\cos\theta)-a\cos\theta]\sin[(x-a)\sin\theta]}{\exp[2x(1+\cos\theta)-2a\cos\theta] - (e^a+e^b)\exp[x(1+\cos\theta)-a\cos\theta]\cos[(x-a)\sin\theta] + e^{a+b}}$$

Using the facts that for θ in $[0,\pi/2]$, $\cos\theta$ is non-negative and $\sin\theta$ and $\cos\theta$ satisfy the respective inequalities

(6.25) (a) $\sin\theta \ge 2\theta/\pi$

(b) $\cos\theta \ge 1 - 2\theta/\pi$

we find that

(6.26) $$\theta \in (0,\pi/6),\ a + \delta \le x \le a_2 \Longrightarrow J(x,x-a,\theta) \ge$$

$$\frac{2\delta}{\pi} \cdot \frac{(e^{a}-e^{b})\exp[a+\delta(1+\cos\theta)]\sin\theta}{\exp[2a_2+2(a_2-a)\cos\theta]-(e^{a}+e^{b})\exp[a+\delta(1+\cos\theta)]\left[1-\frac{2(a_2-a)}{\pi}\sin\theta\right]+e^{a+b}}$$

$$= L_1(\theta) > 0.$$

Obviously the numerator of $L_1(\theta)$ is positive. To see that its denominator, $L_2(\theta)$, is also positive, we must consider the separate cases

(6.27) (a) $a_2 = a + \pi$

(b) $a_2 = a_1 = (a + b_1)/2$

(6.28) $\therefore\ a_2 = a + \pi \Longrightarrow L_2(\theta) > \exp[2a+2\pi(1+\cos\theta)]-2\exp[2a+\delta(1+\cos\theta)]$

$> \exp[2a+\delta(1+\cos\theta)]\cdot(\exp[\pi(1+\cos\theta)]-2) > 0$

(since $\theta \in (0,\pi/6)$).

(6.29) $a_2 = (a + b_1)/2 \longrightarrow L_2(\theta) >$

$\exp[b_1(1+\cos\theta)+a(1-\cos\theta)] - (e^{a}+e^{b})\exp[a+\delta(1+\cos\theta)] + e^{a+b} =$

$e^{a}\Big(\exp[b_1(1+\cos\theta)-a\cos\theta] - \exp[a+\delta(1+\cos\theta)]\Big) - e^{a+b}\Big(\exp[\delta(1+\cos\theta)] - 1\Big) =$

$\exp[2a+\delta(1+\cos\theta)]\Big(\exp[(b_1-(a+\delta))(1+\cos\theta)]-1\Big) - e^{a+b}\Big(\exp[\delta(1+\cos\theta)]-1\Big) >$

$e^{a+b}\Big(\exp[\delta(1+\cos\theta)]-1\Big) - e^{a+b}\Big(\exp[\delta(1+\cos\theta)]-1\Big) = 0$

since

(6.30) (a) $a > b$

(b) $b_1 - (a+\delta) > b_1 - (a+b_1)/2 = (b_1-a)/2 > \delta$

since

(c) $a < a + \delta < (a+b_1)/2 \Longrightarrow 0 < \delta < (b_1 - a)/2$ Q.E.D.

We now formulate our result in a theorem.

Theorem (6.1): Let Ω_n be a "bounded" parallel slit domain as described in the beginning of section V. Let the boundary values $h_m(s)$ be prescribed along the boundary slits C_m $(m = 0,1,2,\ldots,n)$ of Ω_n where s is the parameter of linear arc length along C_m. Let $h_m(s)$ be bounded on each component of C_m (for each m) and let it possess at most a finite number of discontinuities.

Then the Dirichlet problem for Ω_n with the stated boundary values can be solved in a constructive way by determining the values $f_m(s)$ of its solution $u(x,y)$ along the complementary intervals, D_m, of the boundary slits, C_m. The values $f_m(s)$ are given as the unique bounded solution of the integral equation (5.13) and are obtainable by iteration.

Thus with the values of u known all along of each of the lines ℓ_m $(\ell_m = C_m \bigcup D_m)$, $u(x,y)$ may be determined in the remainder of Ω_n by appropriate Poisson integral formulas.

Corollary (6.1): Let

$$(6.31) \quad (a) \quad u_t(x,y) = \begin{cases} \sum_{m=0}^{n} u_{tm}(x,y) & , y \neq k_m \\ f_{tm}(x) & , y = k_m \end{cases}$$

(b) $f_{tm}(\xi) = f_t(\xi)$, ξ the abcissa of a point $Q = (\xi,y)$ in D_m.

If $(x,y) = P$ in S_m, then let us define

$$u_t(P) = u_{tm}(P) = \int_{Q_m \in B_m} f_t(\xi)\varphi_m(P,Q)d\xi \tag{6.32}$$

(For $P \notin S_m$, $u_{tm}(P) = 0$)

Then: If $\{f_t(x)\}$ are the iterative approximations of $f(x)$, $u_t(P)$ converges uniformly to $u(P)$.

Proof: Since $f_t(\xi)$ converges uniformly to $f(\xi)$ - see the remarks following (1.10) - given any $\epsilon > 0$ we may choose $t(\epsilon)$ such that

$$t > t(\epsilon) \Longrightarrow |f(\xi) - f_t(\xi)| < \epsilon, \text{ for all } m \tag{6.33}$$

$$\therefore\ t > t(\epsilon) \Longrightarrow |u(P)-u_t(P)| \leq \int_{Q_m \in B_m} |f(\xi)-f_t(\xi)|\varphi_m(P,Q)d\xi \tag{6.34}$$

$$< \epsilon \int_{Q_m \in B_m} \varphi_m(P,Q)d\xi = \epsilon, \text{ for all } m.$$

VII. Ω_n-Extensions

Theorem (7.1): If Ω_n satisfies $D_{\binom{o}{n}} = \emptyset$, then we may allow $D_{\binom{1}{n-1}}$ to be unbounded and the Dirichlet problem for Ω_n will still be solvable by iteration.

Proof: As in the case of "bounded" Ω_n, we derive the integral equation (5.13) and seek to prove that the inequality (6.1) holds for all x in D_m ($m = 1,2,\ldots,n-1$). It is immediately evident that (6.1) holds for $m = 2,3,\ldots,n-2$ since the corresponding D_m are all bounded. Therefore, we need only prove (6.1) for x in $D_{\binom{1}{n-1}}$. Without loss of generality, we may confine our attention to one of these - say D_1 - and assume it is unbounded on the left. We may also assume that $d_1 = \pi$, $k_o = 0$. Let $P_1 = (x,\pi)$ in D_1 be in the leftmost interval of D_1. There $x < b_o^{(1)}$.

If $|x-b_o^{(1)}| \leq \pi$, then $R(x) = |x-b_o^{(1)}|$ and $\int_{\Delta_1} K_R(x,\xi)d\xi$ is bounded below unity just as in section VI. (See (6.24),(6.26).) Thus we need only consider the case $|x-b_o^{(1)}| > \pi$ which implies $R = \pi$. (See Fig. (7.1).

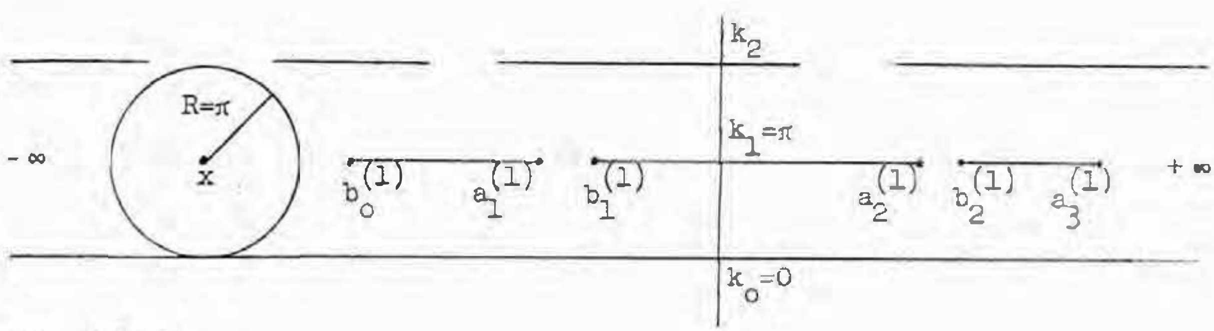

Fig. (7.1):

(Once again we need not consider the possibility that $d_2 < \pi$ - and therefore $R = d_2$ for x far out enough - since the proof of (6.1) for x in D_1 is similar in both cases.)

Since $D_o = \emptyset$, we have, for $R = \pi$

$$(7.1) \quad \int_{\Delta_1} K_\pi(x,\xi) = \frac{1}{2\pi} \int_o^\pi \left[\int_{D_1} \wp_2(P_1+\pi e^{i\theta}, Q_1, d_2)d\xi + \int_{D_2} \wp_2(P_1+\pi e^{i\theta}, Q_2, d_2)d\xi \right] d\theta$$

$$+ \frac{1}{2\pi} \int_\pi^{2\pi} \left[\int_{D_1} \wp_1(P_1+\pi e^{i\theta}, Q_1, \pi)d\xi \right] d\theta \le 1/2 + \frac{1}{2\pi} \int_\pi^{2\pi} \left[\int_{\ell_1} \wp_1(P_1+\pi e^{i\theta}, Q_1, \pi)d\xi \right] d\theta$$

By substitution in (5.4)(c), we find

$$(7.2) \quad \frac{1}{2\pi} \int_\pi^{2\pi} \left[\int_{\ell_1} \wp_1(P_1+ \pi e^{i\theta}, Q_1, \pi)d\xi \right] d\theta =$$

$$\frac{1}{2\pi^2} \int_\pi^{2\pi} \left[\int_{-\infty}^{\infty} \frac{\exp(\xi+x+\pi \mathrm{Cos}\theta)\mathrm{Sin}(\pi+\pi \mathrm{Sin}\theta)d\xi}{[e^\xi+\exp(x+\pi \mathrm{Cos}\theta)\mathrm{Cos}(\pi+\pi \mathrm{Sin}\theta)]^2+[\exp(x+\pi \mathrm{Cos}\theta)\mathrm{Sin}(\pi+\pi \mathrm{Sin}\theta)]^2} \right] d\theta$$

$$= 1/2 - 1/\pi$$

$$\therefore \quad x \in D_1, x < b_o^{(1)}, |x - b_o^{(1)}| > \pi \Longrightarrow$$

$$(7.3) \quad 0 < \int_{\Delta_1} K_\pi(x,\xi)d\xi \le 1/2 + (1/2 - 1/\pi) < 1, \qquad \text{Q.E.D.}$$

Corollary (7.1): The methods of theorem (7.1) can be extended to unbounded D_m if either D_{m-1} or D_{m+1} is bounded but not if they are both unbounded.

Proof: According to the methods of theorem (7.1), we let $d_{m-1} = \pi$ and integrate $K_\pi(x,\xi)$ over all of B_{m+1}, obtaining the bound $1 - 1/\pi$. Therefore, a necessary and sufficient condition for any extension of these methods to the case of nonempty D_{m-1} is that for all "large" x in D_m,

$$(7.4) \qquad \int_{D_{m-1}} K_\pi(x,\xi)d\xi < 1/\pi$$

Now if D_{m-1} is included in $[\alpha,\beta]$, it can be shown that

$$(7.5) \qquad \int_{D_{m-1}} K_\pi(x,\xi)d\xi \leq \frac{1}{2\pi^2}\int_0^\pi M(\beta,\alpha,x,\theta)d\theta \qquad \text{where}$$

$$(7.6) \quad (a) \quad M(\beta,\alpha,x,\theta) = I(\beta,x,\theta) - I(\alpha,x,\theta) \quad \text{and}$$

$$(b) \quad I(t,x,\theta) = \mathrm{Tan}^{-1}\left[\frac{e^t + \exp(x-\pi\mathrm{Cos}\theta)\mathrm{Cos}(\pi\mathrm{Sin}\theta)}{\exp(x - \pi\mathrm{Cos}\theta)\mathrm{Sin}(\pi\mathrm{Sin}\theta)}\right]$$

Since $M(\beta,\alpha,x,\theta)$ is nonnegative and approaches zero uniformly as $|x|$ approaches infinity, there is a positive number α' such that for all θ,

$$(7.7) \quad |x| > \alpha' \Longrightarrow M(\beta,\alpha,x,\theta) < 1$$

$$(7.8) \quad \therefore \quad |x| > \alpha' \Longrightarrow \int_{D_{m-1}} K_\pi(x,\xi)d\xi < \frac{1}{2\pi} < \frac{1}{\pi}$$

Now it is easily verified that

$$(7.9) \qquad M(\beta,\alpha,x,\theta) \leq N(\alpha,x,\theta) = \pi/2 - I(\alpha,x,\theta)$$

and that $N(\alpha,x,\theta)$ increases monotonically from zero to $\pi \mathrm{Sin}\theta$ as x increases from minus infinity to infinity.

Therefore, if $|x| \leq \alpha'$, then

$$(7.10) \qquad \int_{D_{m-1}} K_\pi(x,\xi)d\xi \leq \frac{1}{2\pi^2} \int_0^\pi N(\alpha,\alpha',\theta)d\theta < \frac{1}{2\pi^2} \int_0^\pi \pi \mathrm{Sin}\theta d\theta = 1/\pi$$

If D_{m-1} is unbounded, say on the right, and therefore contains some interval $[\beta,\infty)$, then it can be shown that

$$(7.11) \qquad \int_{D_{m-1}} K_\pi(x,\xi)d\xi \geq \frac{1}{2\pi^2} \int_0^\pi N(\beta,x,\theta)d\theta \longrightarrow \frac{1}{2\pi^2} \int_0^\pi \pi \mathrm{Sin}\theta d\theta = 1/\pi$$

as x approaches infinity.

(A similar method yields the same results for D_{m+1}(un)bounded.) Q.E.D.

We are now prepared to consider the solution of the Dirichlet problem for arbitrary Ω_n - i.e., to remove the boundedness restriction on D_m for any or all $m = 0,1,2,\ldots,n$. We have thus far delayed consideration of the "unbounded" Ω_n since the methods developed for proving (6.1) will not work for x "too large" in one of the unbounded intervals of D.

To be more precise, let us recall that we proved (6.1) by proving (6.14) for each of the possible values of R stated in (6.16). Now the unbounded intervals of D will be either the leftmost interval of D_m, the rightmost interval of D_m, or both for one or more m. Suppose, for some m, the rightmost interval of D_m is unbounded. (Therefore, $b^{(m)}_{N(m)} = \infty$.) As in section V, we may assume without loss of generality that $d_{m+1} = \pi$ (unless $m = n$, then $d_{n+1} = 0$). Then, if $m \neq n$, and x in D_m satisfies $x - a^{(m)}_{N(m)} > \pi > d_m$ -- and is thus "too large"-- $R = R(x) = \pi$, and the left side of (6.14) assumes the form

$$(7.12) \qquad \frac{1}{2\pi} \int_0^{\pi} \left[\int_b^a \wp_{m+1}(P_m + \pi e^{i\Theta}, Q_m, \pi) d\xi \right] d\Theta =$$

$$\frac{1}{2\pi^2} \int_0^{\pi} \mathrm{Tan}^{-1} \left[\frac{(e^a - e^b)\exp(x + \pi \mathrm{Cos}\Theta)\mathrm{Sin}(\pi \mathrm{Sin}\Theta)}{\exp[2(x+\pi\mathrm{Cos}\Theta)] - (e^a + e^b)\exp(x+\pi\mathrm{Cos}\Theta)\mathrm{Cos}(\pi\mathrm{Sin}\Theta) + e^{a+b}} \right] d\Theta$$

(where $P_m = (x, k_m)$, $a = a^{(m)}_{N(m)}$, $b = b^{(m)}_{(N-1)(m)}$)

as in (6.18).

However, the right side of (7.12) approaches zero as x approaches infinity, and therefore (6.14) is not satisfied.

If $d_m < \pi$, then $x-a > d_m$ implies $R = R(x) = d_m$ and thus d_m replaces π in the above integrands. Nevertheless the right side of (7.12) still approaches zero as x approaches infinity and therefore (6.14) is still not satisfied.

If $m = n$, we assume without loss of generality that $d_n = \pi$ and (7.12) becomes

$$(7.13) \qquad \frac{1}{2\pi} \int_0^{\pi} \left[\int_b^a \wp_{n+1}(P_n + \pi e^{i\Theta}, Q_n) d\xi \right] d\Theta = \frac{1}{2\pi^2} \int_b^a (\xi - x)^{-1} \log \left| \frac{\xi - x + \pi}{\xi - x - \pi} \right| d\xi$$

$$= \frac{1}{2\pi^2} \int_{(b-x)/\pi}^{(a-x)/\pi} u^{-1} \log \left| \frac{1+u}{1-u} \right| du$$

Once again, the integrals in (7.13) approach zero as x approaches infinity and therefore (6.14) is satisfied for no unbounded D_m.[8] This is why our methods have thus far been unable to yield a better extension of "bounded" Ω_n than that in corollary (7.1).

It will be noticed that as long as x is close enough to a - i.e., less than π away, or boundedly far away - (6.14) is true. Thus a new method of defining R may be deemed advisable - since until now we defined R as a bounded function of the distance from x to a. However, even allowing R = x - a for all x > a --and hence allowing R to approach infinity along with the distance-- fails to prove (6.14).

Thus we use a "limiting" iteration method for the constructive solution of the Dirichlet problem for "unbounded" Ω_n. Admittedly, this method falls short of the elegance of the "ordinary" iteration method since it requires the solution of an infinite sequence of Dirichlet problems for "bounded" Ω_n. It is the writer's hope that a satisfactory extension of the

[8] Note: If (6.12) had been written instead as

$$\int_{\Delta_m} K_R(x,\xi)d\xi \leq 1 - \frac{1}{2\pi}\int_{\pi}^{2\pi}\left[\int_b^a \wp_m(P_m+Re^{i\theta},Q_m,d_m)d\xi\right]d\theta$$

(i.e., if the bounding were to be performed over the lower semicircle) we would still find that

$$\frac{1}{2\pi}\int_{\pi}^{2\pi}\left[\int_b^a \wp_m(P_m+Re^{i\theta},Q_m,d_m)d\xi\right]d\theta \longrightarrow 0 \text{ as } x \longrightarrow \infty$$

exactly as in (7.12) and (7.13).

"ordinary" iteration method to "unbounded" Ω_n will subsequently be found. For the moment, however, we must be content with the "limiting" iteration method which we now describe.

Without loss of generality, we may assume that all the D_m are unbounded - in fact on both sides - thus making all the C_m bounded. Since there are only finitely many, $(n+1)$, C_m, there exists a positive number α_o such that $|x| < \alpha_o$ for all x in C.

(7.14) (a) Let $\Omega_{n\alpha} = \Omega_n - C'_\alpha$ where

(b) $C'_\alpha = \bigcup_{m=o}^{n} C_{m\alpha}$ where

(c) $C_{m\alpha} = \left\{ (x,k_m) \Big| |x| \geq \alpha > \alpha_o \right\}$

(d) Let $C_\alpha = C \bigcup C'_\alpha$

(e) Let $D_\alpha = \bigcup_{m=o}^{n} \ell_m - C_\alpha$

(Thus D_α is the union of the complementary intervals of C_α.)

Therefore we have

Theorem (7.2): Given the Dirichlet problem for Ω_n, which we may assume, without loss of generality, has non-negative boundary values, let there be formulated corresponding Dirichlet problems for all the domains $\Omega_{n\alpha}$ specifying the same boundary values on C as those given in the problem for Ω_n and the boundary values zero along all of C'_α. If the solutions of these Dirichlet problems --solvable by iteration-- are denoted $u_\alpha(x,y)$ and if the solution of the Dirichlet problem for Ω_n is $u(x,y)$, then

(7.15) $$u(x,y) = \lim_{\alpha \to \infty} u_\alpha(x,y)$$

Proof:

(7.16) Let (a) $f_\alpha(x)$ be the values of $u_\alpha(x,y)$ along D_α

(b) $h_\alpha(x)$ be the prescribed values of $u_\alpha(x,y)$ along C_α

Therefore,

(7.17) $$h(x) = h_\alpha(x)\Big|_{x\epsilon C}$$

Therefore, given any $\beta \geq \alpha_o$, $\alpha \geq \beta$ implies that the functions $u_\alpha(x,y)$ are all harmonic and uniformly bounded in $\Omega_{n\beta}$. Furthermore

(7.18) $$\alpha_2 > \alpha_1 \geq \beta \Longrightarrow u_{\alpha_2}(x,y) \geq u_{\alpha_1}(x,y) \text{ in } \Omega_{n\beta}.$$

These facts follow from the maximum and minimum principles for functions harmonic and bounded in $\Omega_{n\beta}$ by the following reasoning.

(7.19) Let (a) $I = \operatorname{Inf}\left(0, \operatorname{Inf}_{x\epsilon C} h(x)\right)$

(b) $S = \operatorname{Sup}\left(0, \operatorname{Sup}_{x\epsilon C} h(x)\right)$

(7.20) $$\therefore \quad I \leq \operatorname{Inf}_{x\epsilon C_\alpha} h_\alpha(x) \leq u_\alpha(x,y) \leq \operatorname{Sup}_{x\epsilon C_\alpha} h_\alpha(x) \leq S$$

by the maximum and minimum principles, hence the first assertion.

Along any line ℓ_m,

(7.21) (a) $\alpha_1 \leq |x| \leq \alpha_2 \Longrightarrow u_{\alpha_1}(x,k_m) = 0$ and

(b) $u_{\alpha_2}(x,k_m) \geq 0$ (by the minimum principle applied to $\Omega_{n\alpha_2}$) and

(c) $|x| > \alpha_2 \Longrightarrow u_{\alpha_1}(x,k_m) = u_{\alpha_2}(x,k_m) = 0$ (See Fig. (7.2).)

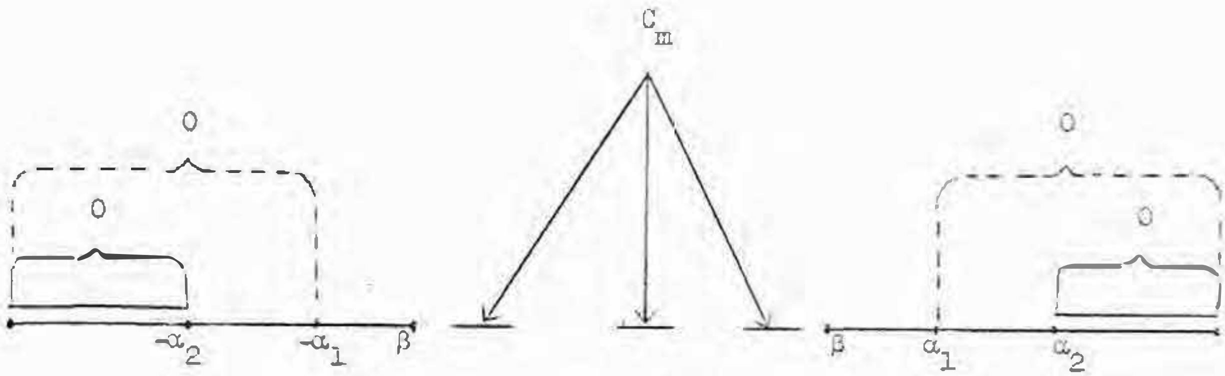

Fig. (7.2): Picture of C_α (for several α) along ℓ_m

Therefore, since $u_{\alpha_2}(x,y) = u_{\alpha_1}(x,y)$ along C, but $u_{\alpha_2}(x,y) \geq u_{\alpha_1}(x,y)$ along C_{α_1}, the minimum principle tells us that $u_{\alpha_1}(x,y) \leq u_{\alpha_2}(x,y)$ in $\Omega_{n\alpha_1}$. Finally, the fact that $\Omega_{n\beta}$ is included in $\Omega_{n\alpha}$, implies that $u_{\alpha_1}(x,y) \leq u_{\alpha_2}(x,y)$ in $\Omega_{n\beta}$ as asserted in (7.9).

Thus for each $\beta \geq \alpha_0$ and any sequence $\{\alpha_j\}$ such that $\alpha_j \geq \beta$ for all j, and α_j approaches infinity, the corresponding sequence $\{u_{\alpha_j}(x,y)\}$ is a monotone nondecreasing sequence of functions harmonic and uniformly bounded in $\Omega_{n\beta}$. Therefore, by Harnack's theorem of monotone convergence, they converge uniformly on compact subsets to a function V(x,y) harmonic and bounded in $\Omega_{n\beta}$.

The limit function V(x,y) is independent of the sequence chosen; for if $\{\alpha_j\}$ is the sequence yielding V(x,y) and $\{\gamma_j\}$ is any other sequence (of the "right" type), then their common refinement $\{\sigma_j\}$ yields the corresponding sequence $\{u_{\sigma_j}(x,y)\}$ which converges to W(x,y) harmonic and bounded in

$\Omega_{n\beta}$. But since the subsequence $\{u_{\alpha_\ell}(x,y)\}$ of $\{u_{\sigma_j}(x,y)\}$ converges to $V(x,y)$ in $\Omega_{n\beta}$, we conclude that $W(x,y) \equiv V(x,y)$ in $\Omega_{n\beta}$.

However, now the function $V(x,y)$ is harmonic and bounded in $\Omega_{n\beta}$ for all $\beta > \alpha_0$ since we may order the sequence $\{\alpha_j\}$ by "size places", and by casting off the "right" finite number of terms, begin it above any β thus making $V(x,y)$ harmonic and bounded in $\Omega_{n\beta}$ for any, and hence all, $\beta > \alpha_0$.

Therefore, $V(x,y)$ is harmonic and bounded throughout Ω_n; for if not, then there exists some $\beta > \alpha_0$ such that $V(x,y)$ is not harmonic at (β, k_m) for some m. But, given any positive ϵ, we can show that $V(x,y)$ is harmonic and bounded in $\Omega_{n(\beta+\epsilon)}$ and therefore at the point (β, k_m).

Finally, $V(x,y)$ has the same boundary values as $u(x,y)$ and hence, by the uniqueness principle for functions harmonic and bounded in Ω_n, we have

$$V(x,y) \equiv u(x,y) \text{ in } \Omega_n. \qquad (7.22) \qquad \text{Q.E.D.}$$

Thus the (limiting) iteration method may be used to solve the Dirichlet problem for any domain whose conformal map onto "unbounded" Ω_n is known.

Example: Let X_n be a half-plane with a finite number of linear slits removed along (n-1) rays emanating from a fixed point on the bounding straight line. Without loss of generality, we may take this domain to be an upper half-plane, the bounding line to be the x-axis and the fixed point to be the origin. Let the rays be labeled according to increasing argument as ℓ_m $(m = 1,2,\ldots,n-1)$ and let C_m and D_m be taken as before. We denote the non-negative and non-positive halves of the x-axis by ℓ_0 and ℓ_n respectively.

Thus

$$(7.23)\quad \ell_m = \{(r,\theta)\,|\,\theta = \theta_m,\ r \geq 0\}$$

$$(m = 0,1,2,\ldots,n;\ 0 = \theta_o < \theta_1 < \ldots < \theta_{n-1} < \theta_n = \pi)$$

See Fig. (7.3)

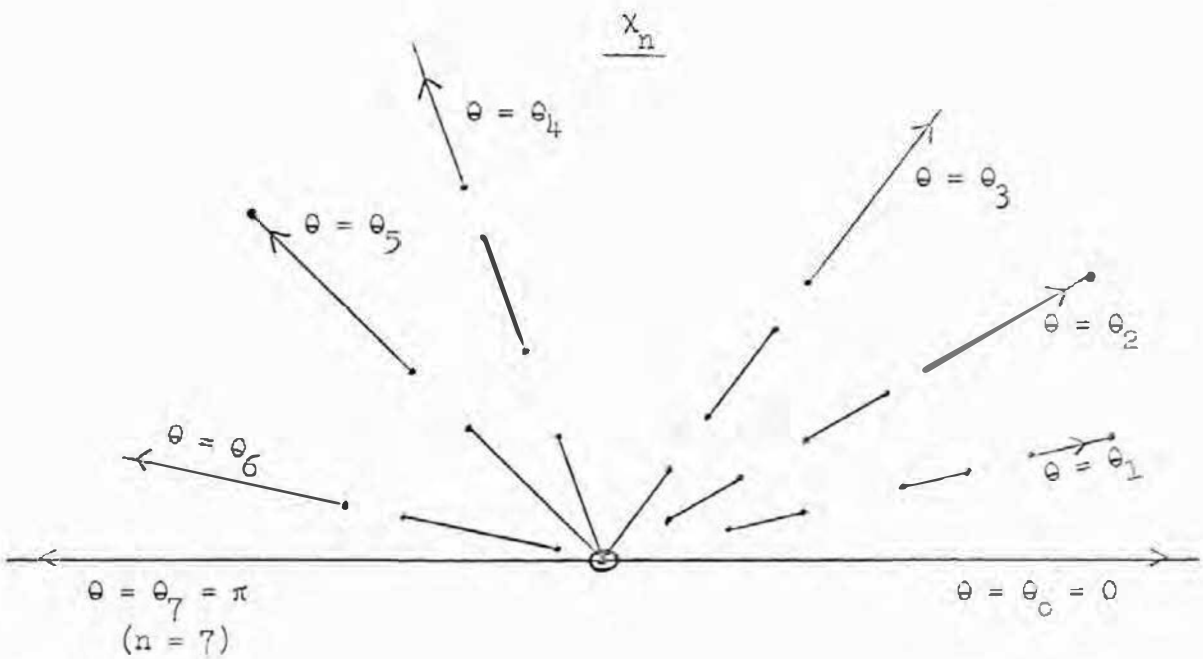

Fig. (7.3):

The function $\omega = (k/\pi)\log z$ (principle value) maps X_n onto the domain Ω_n consisting of a strip of width k (based on the x-axis) from which a finite number of linear slits lying on (n-1) lines parallel to the strip's bases have been removed. Thus the Dirichlet problem for X_n may be solved in a constructive way by solving the corresponding Dirichlet problem for Ω_n. If for some ℓ_m $(m = 1,2,\ldots,n-1)$, D_m is unbounded (i.e., contains points arbitrarily close to or far from the origin), then (by corollary (7.1)) the "ordinary" iteration method is applicable as long as either D_{m-1} or D_{m+1}

is bounded.[9] Otherwise, the Dirichlet problem for X_n may be solved by the "limiting" iteration method.

[9] In Fig. (7.3) the $\begin{pmatrix}\text{boundedness}\\ \text{unboundedness}\end{pmatrix}$ of a particular D_m is indicated by the $\begin{pmatrix}\text{absence}\\ \text{presence}\end{pmatrix}$ of a dot at the extreme end(s) of the corresponding C_m.

VIII. Applications I

The harmonic measure $u_j(x,y)$, associated with the boundary slit A_j of the domain Ω_n, is defined as the function harmonic in Ω_n, continuous and bounded in $\bar{\Omega}_n$, whose boundary values are unity on A_j and zero on the remainder of C. If $v_j(x,y)$ is the harmonic conjugate of $u_j(x,y)$ and A_r is any boundary slit of Ω_n, then the period p_{jr} of $v_j(x,y)$ as the point (x,y) performs a circuit about A_r is given by

$$(8.1) \qquad p_{jr} = \oint_{E_r} [\partial v_j(x,y)/\partial s]\, ds = \oint_{E_r} [\partial u_j(x,y)/\partial n]\, ds$$

where E_r is any (sufficiently smooth) curve described in the positive sense surrounding only the boundary component A_r of C and $\partial/\partial n$ indicates differentiation with respect to the outward pointing normal.

The periods p_{jr} of the functions $v_j(x,y)$ have several important uses in the theory of conformal mapping, one of which is in the construction of a conformal map of given domain onto one of its canonical domains. Our application will be the determination of p_{jr} purely in terms of the values $f_{mj}(x)$ of the function $u_j(x,y)$ on the lines $y = k_m$. In this section we determine p_{jr} for a particular class of the domains Ω_n; in the next section we determine p_{jr} for arbitrary Ω_n.

Let Ω_n be a parallel slit domain having at least one finite boundary slit and having the property that the projections of any two of its finite boundary slits (on the x-axis) do not overlap either each other or the infinite boundary component. Furthermore, we assume that there is an infinite boundary component which extends along each line to $-a$ from the left and to a from the right. ($a > 0$. See Fig. (8.1).)

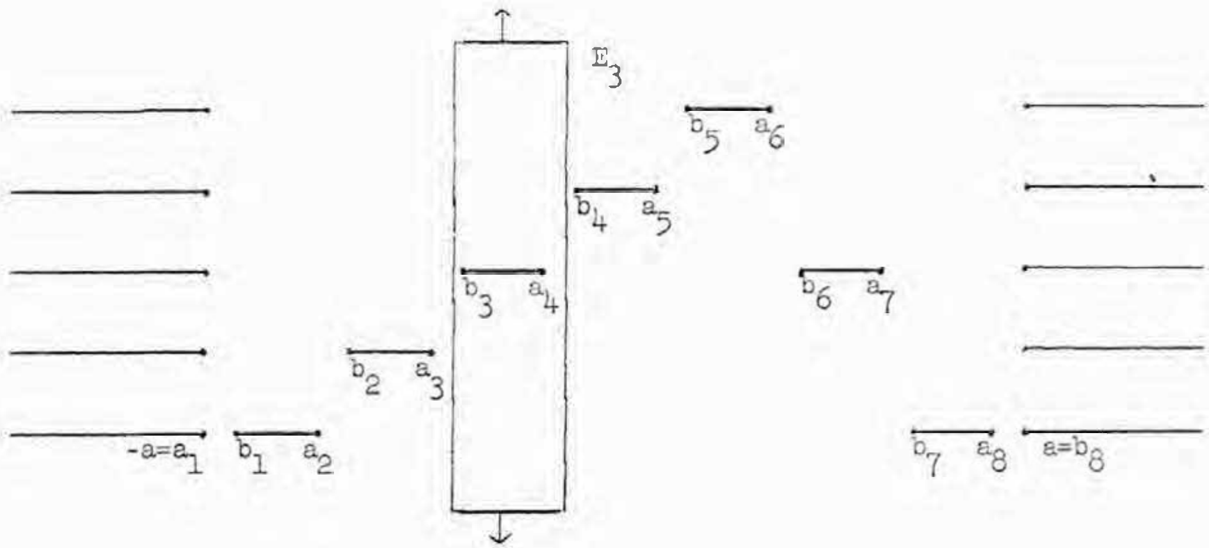

Fig. (8.1):

We label the finite boundary slits in order of the appearance from left to right of their projections on the x-axis as

$$A_j = [b_j, a_{j+1}], \quad \left(j = 1,2,3,\ldots, \sum_{m=o}^{n} [N(m)-1] \right), \tag{8.2}$$

where $N(m)$ is the number of complementary intervals along ℓ_m and hence $N(m)-1$ is the number of boundary slits along ℓ_m. Let A_J,

$$J = 1 + \sum_{m=o}^{n} [N(m)-1] \tag{8.3}$$

represent the infinite boundary component.

Since the projections of the A_r do not overlap, it is possible --for each fixed r-- to take as E_r a rectangle whose vertical sides are parallel to the y-axis and pass through the gaps (a_r, b_r), (a_{r+1}, b_{r+1}). (For $r = J$, we replace r+1 by 1.) We will subsequently see that it is permissible to allow the horizontal sides of E_r to recede to infinity. (See Fig. (8.1).) Therefore, we may rewrite (8.1) as

(8.4) $$p_{jr} = I_{jr} - I_{j(r+1)}$$

where for all x in (a_r, b_r)

(8.5) $$I_{jr}(x) = \int_{-\infty}^{\infty} \left[\partial u_j(x,y)/\partial x\right] dy = I_{jr}$$

[We will subsequently show that I_{jr} is constant for x in (a_r, b_r).]

Before proceeding, we recall several properties of the periods p_{jr} which we shall have occasion to use:

(8.6) (a) $p_{jr} = p_{rj}$

(b) $$\sum_{r=1}^{J} p_{jr} = 0 \ , \quad (j = 1,2,\ldots,J)$$

Thus it suffices to determine the p_{jr} for $j < J$, $r < J$. Therefore, the horizontal sides of all E_r under consideration are each of finite length --in fact they are at most of length 2a. Thus, in letting these sides recede to infinity, we need only show that $\partial u_j(x,y)/\partial y$ approaches zero uniformly as y approaches infinity whenever x is in $[-a, a]$. (This will also prove the constancy of $I_{jr}(x)$ for x in (a_r, b_r).) Also, since we need only consider $j < J$, we have for all m and for all x such that $|x| \geq a$,

$$f_{mj}(x) \equiv 0 \tag{8.7}$$

Now as the horizontal sides of E_r recede to infinity, they eventually enter and remain in $S\binom{o}{n+1}$ respectively where the values of $u_j(x,y)$ are given by the formulas $(5.4)\binom{a}{b}$ respectively. Since (for all $m = 0,1,2,\ldots,n$) the functions $f_{mj}(\xi)$ are uniformly bounded --in fact $0 \leq f_{mj}(\xi) \leq 1$ by the maximum and minimum principles --upon differentiation of both sides of $(5.4)\binom{a}{b}$, we find that as y approaches infinity,

$$\text{(8.8)}\quad \text{(a)}\quad \partial u_j(x,y)/\partial x = O\left(1/|y|^3\right)$$

$$\text{(b)}\quad \partial u_j(x,y)/\partial y = O\left(1/y^2\right)$$

By (8.8), we see that (8.4) holds. We prove the constancy of $I_{jr}(x)$ for x in (a_r,b_r) by considering $\oint_{E'_r} [\partial u_j(x,y)/\partial n]ds$ where E'_r is a rectangle both of whose vertical sides pass through the gap (a_r,b_r) and are parallel to the y-axis. Since E'_r encloses a region whose closure is interior to Ω_n, we conclude that

$$\oint_{E'_r} [\partial u_j(x,y)/\partial n]ds = 0 \tag{8.9}$$

Thus as the horizontal sides of E'_r recede to infinity, the assertion (8.9) remains true. But by (8.8)(b), the contribution of the horizontal sides to the integral in (8.9) approaches zero. Thus, in the limit, for all α, β in (a_r, b_r)

(8.10) $$\int_{-\infty}^{\infty} [\partial u_j(\beta,y)/\partial x - \partial u_j(\alpha,y)/\partial x]\, dy = 0$$

Therefore, (8.3) holds. Thus, if we integrate both sides of (8.3) over (a_r, b_r) --as done in [2, p. 127]-- we obtain

(8.11) $$\int_{a_r}^{b_r} I_{jr}\, dx = (b_r - a_r) I_{jr} = \int_{a_r}^{b_r} \int_{-\infty}^{\infty} [\partial u_j(x,y)/\partial x]\, dy dx$$

$$= \int_{-\infty}^{\infty} \int_{a_r}^{b_r} [\partial u_j(x,y)/\partial x]\, dx dy = \int_{-\infty}^{\infty} [u_j(b_r,y) - u_j(a_r,y)]\, dy$$

where the inversion of order of integration is justified by (8.8)(a).

(8.12) $$(b_r - a_r) I_{jr} = \int_{-\infty}^{0} [u_j(b_r,y) - u_j(a_r,y)]\, dy$$

$$+ \sum_{m=1}^{n} \int_{k_{m-1}}^{k_m} [u_j(b_r,y) - u_j(a_r,y)]\, dy + \int_{k_n}^{\infty} [u_j(b_r,y) - u_j(a_r,y)]\, dy$$

By (5.4)(a) and (8.7) we obtain --after twice interchanging the order of integration--

(8.13) $$\int_{-\infty}^{0} [u_j(b_r,y) - u_j(a_r,y)]\, dy = \frac{1}{\pi} \int_{-a}^{a} f_{oj}(\xi) \log\left|\frac{\xi - a_r}{\xi - b_r}\right| d\xi$$

Similarly,

$$(8.14)\quad \int_{k_n}^{\infty} \left[u_j(b_r,y)-u_j(a_r,y)\right]dy - \frac{1}{\pi}\int_{-a}^{a} f_{nj}(\xi)\log\left|\frac{\xi-a_r}{\xi-b_r}\right|d\xi$$

By (5.5) and (8.5) we obtain --after twice interchanging the order of integration--

$$(8.15)\quad \int_{k_{m-1}}^{k_m} \left[u_j(b_r,y)-u_j(a_r,y)\right]dy =$$

$$\frac{1}{\pi}\int_{-a}^{a}\left[f_{(m-1)j}(\xi)+f_{mj}(\xi)\right]\log\left|\mathrm{Tanh}\left[(\pi/2d_m)(\xi-a_r)\right]\mathrm{Ctnh}\left[(\pi/2d_m)(\xi-b_r)\right]\right|d\xi$$

$$(8.16)\quad \therefore\ I_{jr} = \frac{1}{\pi(b_r-a_r)}\int_{-a}^{a}\left[f_{oj}(\xi)+f_{nj}(\xi)\right]\log\left|\frac{\xi-a_r}{\xi-b_r}\right|d\xi +$$

$$\frac{1}{\pi(b_r-a_r)}\sum_{m=1}^{n}\int_{-a}^{a}\left[f_{(m-1)j}(\xi)+f_{mj}(\xi)\right]\log\left|\mathrm{Tanh}\left[(\pi/2d_m)(\xi-a_r)\right]\mathrm{Ctnh}\left[(\pi/2d_m)(\xi-b_r)\right]\right|d\xi$$

IX. Applications II

We now extend the results of the previous section to arbitrary Ω_n. This time, however, it will be more convenient for us to label the finite boundary slits --of which we assume Ω_n has at least one-- in order of their appearance from left to right along their respective lines ℓ_m from m = 0 to m = n as

(9.1) $$A_j + [a_j, b_j], \quad \left[j = 1,2,3,\ldots, \sum_{m=o}^{n} (N(M)-1) \right]$$

Again A_J, where

(9.2) $$J = 1 + \sum_{m=o}^{n} N(m)-1$$

will represent the infinite boundary component (if there is one) and we need only consider p_{jr} for $j < 1$, $r < J$ (by 8.6). As before we set $f_{mj}(x) = u_j(x,k_m)$ and seek to determine p_{jr} in terms of the values $f_{mj}(x)$. We do so by considering first those A_r that lie on an "interior" line (i.e., a line ℓ_m such that $m \neq 0,n$) and then those A_r, if any, that lie on an "end" line.

If A_r lies on an "interior" line, we choose E_r as the boundary of $\{(x,y) | a_r \leq x \leq b_r, \bar{k}_m \leq y \leq \bar{k}_{m+1}\}$ where $\bar{k}_m = (k_{m-1} + k_m)/2$. (See Fig. (9.1).)

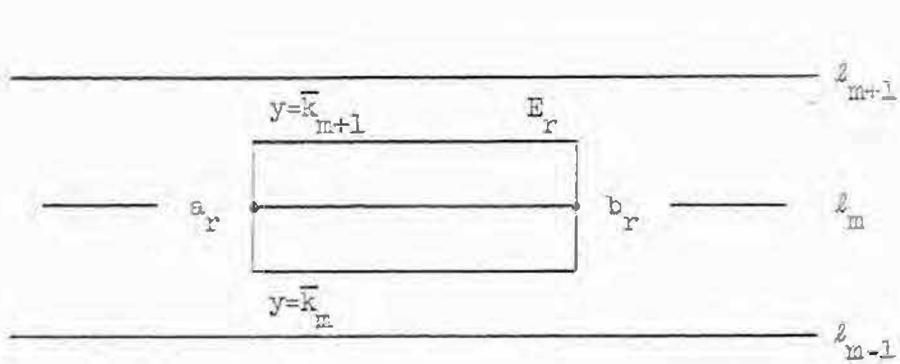

Fig.(9.1):

$$(9.3)\qquad \therefore\ p_{jr} = \int_{a_r}^{b_r} \left[\partial u_j(x,\bar{k}_m)/\partial y - \partial u_j(x,\bar{k}_{m+1})/\partial y\right] dx$$

$$+ \int_{\bar{k}_m}^{\bar{k}_{m+1}} \left[\partial u_j(a_r,y)/\partial x - \partial u_j(b_r,y)/\partial x\right] dy$$

By differentiating both sides of (5.5) and by interchanging the order of integration, we find that

$$(9.4)\qquad \int_{a_r}^{b_r} \left[\partial u_j(x,k_m)/\partial y - \partial u_j(x,k_{m+1})/\partial y\right] dx =$$

$$\frac{1}{d_{m+1}} \int_{-\infty}^{\infty} \frac{\left[f_{mj}(\xi)-f_{(m+1)j}(\xi)\right]\left(\exp\left[(2\pi/d_{m+1})(\xi-a_r)\right]-\exp\left[(2\pi/d_{m+1})(\xi-b_r)\right]\right)d\xi}{\left(1+\exp\left[(2\pi/d_{m+1})(\xi-a_r)\right]\right)\left(1+\exp\left[(2\pi/d_{m+1})(\xi-b_r)\right]\right)}$$

$$+ \frac{1}{d_m} \int_{-\infty}^{\infty} \frac{\left[f_{mj}(\xi)-f_{(m-1)j}(\xi)\right]\left(\exp\left[2\pi/d_m)(\xi-a_r)\right] - \exp\left[(2\pi/d_m)(\xi-b_r)\right]\right)d\xi}{\left(1+\exp\left[(2\pi/d_m)(\xi-a_r)\right]\right)\left(1+\exp\left[(2\pi/d_m)(\xi-b_r)\right]\right)}$$

and that

$$(9.5)\qquad \int_{\bar{k}_m}^{\bar{k}_{m+1}} [\partial u_j(a_r,y)/\partial x - \partial u_j(b_r,y)/\partial x]dy =$$

$$\int_{\bar{k}_m}^{k_m} [\partial u_j(a_r,y)/\partial x-\partial u_j(b_r,y)/\partial x]dy + \int_{k_m}^{\bar{k}_{m+1}} [\partial u_j(a_r,y)/\partial x-\partial u_j(b_r,y)/\partial x]dy \quad =$$

$$\frac{1}{2d_m}\int_{-\infty}^{\infty} f_{(m-1)j}(\xi)\left[\mathrm{Tanh}\frac{\pi}{2d_m}(\xi-a_r)\mathrm{Sech}\frac{\pi}{d_m}(\xi-a_r)-\mathrm{Tanh}\frac{\pi}{2d_m}(\xi-b_r)\mathrm{Sech}\frac{\pi}{d_m}(\xi-b_r)\right]d\xi \quad +$$

$$\frac{1}{2d_m}\, P\int_{-\infty}^{\infty} f_{mj}(\xi)\left[\mathrm{Ctnh}\frac{\pi}{2d_m}(\xi-a_r)\mathrm{Sech}\frac{\pi}{d_m}(\xi-a_r)-\mathrm{Ctnh}\frac{\pi}{2d_m}(\xi-b_r)\mathrm{Sech}\frac{\pi}{d_m}(\xi-b_r)\right]d\xi \quad +$$

$$\frac{1}{2d_{m+1}}\, P\int_{-\infty}^{\infty} f_{mj}(\xi)\left[\mathrm{Ctnh}\frac{\pi}{2d_{m+1}}(\xi-a_r)\mathrm{Sech}\frac{\pi}{d_{m+1}}(\xi-a_r)-\mathrm{Ctnh}\frac{\pi}{2d_{m+1}}(\xi-b_r)\mathrm{Sech}\frac{\pi}{d_{m+1}}(\xi-b_r)\right]d\xi$$

$$+\ \frac{1}{2d_{m+1}}\int_{-\infty}^{\infty} f_{(m+1)j}(\xi)\left[\mathrm{Tanh}\frac{\pi}{2d_{m+1}}(\xi-a_r)\mathrm{Sech}\frac{\pi}{d_{m+1}}(\xi-a_r)\right]d\xi$$

$$-\ \frac{1}{2d_{m+1}}\int_{-\infty}^{\infty} f_{(m+1)j}(\xi)\left[\mathrm{Tanh}\frac{\pi}{2d_{m+1}}(\xi-b_r)\mathrm{Sech}\frac{\pi}{d_{m+1}}(\xi-b_r)\right]d\xi$$

where "P" denotes the Cauchy principal value of the integral.

It should be noted, however, that for $r \neq j$, $f_{mj}(\xi) \equiv 0$ on A_r (which is $[a_r,b_r]$). Furthermore, since (a_r,k_m) and (b_r,k_m) are points of continuity of the boundary values of $u_j(x,y)$, $f_{mj}(x) = u_j(x,k_m)$ approaches zero as x approaches $\begin{pmatrix} b_r^+ \\ a_r^- \end{pmatrix}$. In fact, in a $\begin{pmatrix} \text{right} \\ \text{left} \end{pmatrix}$ neighborhood of $\begin{pmatrix} b_r \\ a_r \end{pmatrix}$, $f_{mj}(x) = u_j(x,k_m)$ can be expanded in a power series of which $\begin{pmatrix} b_r \\ a_r \end{pmatrix}$ is a zero of at least the first order. Since the hyperbolic co-tangent has a pole of only the first order at the simple zeroes of its argument, the behavior of $f_{mj}(\xi)$ near $\begin{pmatrix} b_r \\ a_r \end{pmatrix}$ nullifies the effect of this pole. Thus, for $r \neq j$, we may remove the symbol "P" in (9.5). We may then justify interchanging the order of integration by the absolute integrability of the respective integrands. For $r = j$, the justification is the same except in the case of those integrals whose Cauchy principal value is taken. Thus (9.4) and (9.5) give the value of p_{jr} for A_r on an interior line.[10]

If A_r lies on an "end" line --say ℓ_n-- we determine p_{jr} by letting E_r be the boundary of $\{(x,y) \mid a_r \leq x \leq b_r,\ y \geq \bar{k}_n\}$. (See Fig. (9.2).) The justification

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If A_J has $\begin{pmatrix} \text{one} \\ \text{two} \end{pmatrix}$ finite end points $\begin{pmatrix} \alpha_m \\ \beta_m \end{pmatrix}$ along ℓ_m, $(-\infty < \alpha_m < \beta_m < \infty)$, then the intervals of integration of the right sides of (9.4) and (9.5) will be finite.

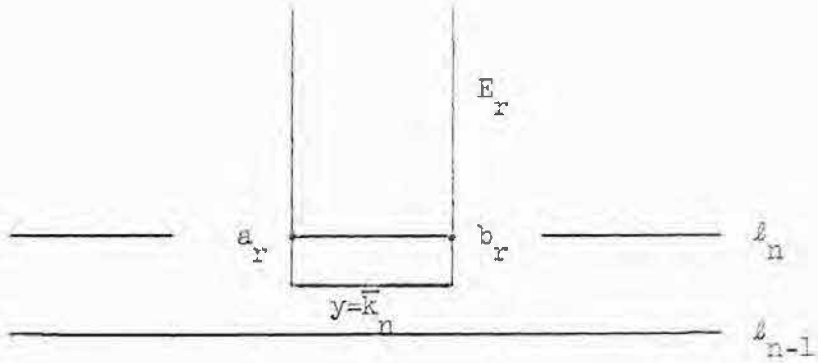

Fig. (9.2):

of this choice follows from the inequality (8.8)(b).

$$(9.6) \qquad \therefore\ p_{jr} = \int_{\bar{k}_n}^{\infty} [\partial u_j(a_r,y)/\partial x - \partial u_j(b_r,y)/\partial x]\,dy + \int_{a_r}^{b_r} [\partial u_j(x,\bar{k}_n)/\partial y]\,dx$$

By differentiating both sides of (5.5) and (5.4)(b), and by interchanging the order of integration, we find that

$$(9.7) \qquad \int_{a_r}^{b_r} [\partial u_j(x,\bar{k}_n)/\partial y]\,dx =$$

$$\frac{1}{d_n}\int_{-\infty}^{\infty} \frac{[f_{nj}(\xi)-f_{(n-1)j}(\xi)](\exp[(2\pi/d_n)(\xi-a_r)] - \exp[(2\pi/d_n)(\xi-b_r)])\,d\xi}{(1+\exp[(2\pi/d_n)(\xi-a_r)])(1+\exp[(2\pi/d_n)(\xi-b_r)])}.$$

$$(9.8) \qquad \int_{\bar{k}_n}^{\infty} [\partial u_j(a_r y)/\partial x - \partial u_j(b_r,y)/\partial x]\,dy =$$

$$\int_{\bar{k}_n}^{k_n} \left[\partial u_j(a_r,y)/\partial x - \partial u_j(b_r,y)/\partial x\right] dy + \int_{k_n}^{\infty} \left[\partial u_j(a_r,y)/\partial x - \partial u_j(b_r,y)/\partial x\right] dy =$$

$$\frac{1}{2d_n} \int_{-\infty}^{\infty} f_{(n-1)j}(\xi) \left[\mathrm{Tanh}\frac{\pi}{2d_n}(\xi - a_r)\mathrm{Sech}\frac{\pi}{d_n}(\xi - a_r) - \mathrm{Tanh}\frac{\pi}{2d_n}(\xi - b_r)\mathrm{Sech}\frac{\pi}{d_n}(\xi - b_r)\right] d\xi$$

$$+ \frac{1}{2d_n} P \int_{-\infty}^{\infty} f_{nj}(\xi) \left[\mathrm{Ctnh}\frac{\pi}{2d_n}(\xi - a_r)\mathrm{Sech}\frac{\pi}{d_n}(\xi - a_r) - \mathrm{Ctnh}\frac{\pi}{2d_n}(\xi - b_r)\mathrm{Sech}\frac{\pi}{d_n}(\xi - b_r)\right] d\xi$$

$$+ \frac{(a_r - b_r)}{\pi} P \int_{-\infty}^{\infty} \frac{f_{nj}(\xi) d\xi}{(\xi - a_r)(\xi - b_r)}$$

Of course, the remarks following (9.5) apply equally well here. Thus (9.7) and (9.8) give the value of p_{jr} for A_r on ℓ_n. For A_r on ℓ_o we obtain formulas completely analogous to those in (9.6)-(9.8).

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