AN N-DIMENSIONAL MUNTZ-JACKSON THEOREM

by

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INTRODUCTION

The main theorem of approximation theory is that of Weierstrass, [7], which states that the closure of the monomials $\{1,x,x^2,\ldots\}$, on a compact set, X , is the set of all continuous function on X, C (X). There are many generalizations of this theorem, and two, in particular, are of concern in this paper.

The first is Muntz' theorem, [3], which states that the closure of the monomials, $\{1,x^{\lambda_1},x^{\lambda_2},\ldots\}$, is C(X) if and and only if $\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty$. Here $\{\lambda_i\}$ is an arbitrary sequence of positive numbers.

The theorems of Weierstrass and Muntz are quantitative in nature, in that they deal only with the possibility of approximation; i.e., given $f(x)\epsilon$ C [0,1], and given $\epsilon > 0$, there exists a finite linear combination of $\{1,x,x^2,\ldots\}$ or $\{1,x^{\lambda_1},x^{\lambda_2},\ldots\}$, say p(x), such that $| f(x)-p(x) | < \epsilon$ for all $x \epsilon [0,1]$. No information is furnished as to the degree of p(x), considered as a function of ϵ. The answer to this problem is given in Jackson's theorem, [2], which forms the basis of quantitative approximation theory. To be explicit, let $f(x)\epsilon C[0,1]$, let P_n be the space of polynomials of degree less than or equal to n, and let

$$W_f(\delta) = \sup_{x\epsilon[0,1]} \sup_{|t|\leq\delta} |f(x+t) - f(x)|$$

be the modulus of continuity of f(x). Jackson's theorem states that there exists $p(x) \epsilon P_n$ such that

$$|f(x) - p(x)| \leq c_1 W_f(\tfrac{1}{n}) ,$$

and there exists $f(x) \epsilon C[0,1]$ such that

$$|f(x) - p(x)| \geq c_2 W_f(\tfrac{1}{n}) .$$

for all $p(x) \epsilon P_n$. Here c_1 and c_2 are constants independent of f(x) and n.

The above three theorems hold in the space $L^2[0,1]$, with the appropriate modifications.

D.J. Newman, [4], found a quantitative version of Muntz' theorem in $L^2[0,1]$. Specifically, he proved the following:

Let $1 = \lambda_0 < \lambda_1 < \ldots < \lambda_n$ be a finite set of integers satisfying $\lambda_{i+1} - \lambda_i \geq 2$. Let $\epsilon_\Lambda = \prod_{i=1}^{n} \frac{\lambda_i - 1/2}{\lambda_i + 3/2}$.

Then for $f(x) \epsilon L^2[0,1]$, there exist constants c_i , i=0,1,...,n, such that

$$\| f(x) - \sum_{i=0}^{n} c_i x^{\lambda_i} \|_{L^2} \leq 3 W_f^{L^2} (\epsilon_\Lambda),$$

and there exists $f(x) \epsilon L^2[0,1]$ such that

$$\| f(x) - \sum_{i=0}^{n} c_i x^{\lambda_i} \|_{L^2} \geq 1/4 W_f^{L^2} (\epsilon_\Lambda)$$ for any choice of c_i.

The results of this paper stem from attempts to generalize Newman's theorem. The first is a "Muntz- Jackson" theorem in n-dimensions. Here, however, a qualification is in order, since the corresponding Muntz theorem is not known in n-dimensions; i.e., there are no known necessary and sufficient conditions for a set of functions $\{x_1^{\lambda_{1i}} x_2^{\lambda_{2i}} \ldots x_n^{\lambda_{ni}}\}$, $i = 0,1,2,\ldots$, to be dense in either the continuous functions or the L^2 functions on the n-dimensional unit cube. Sufficient conditions are known, and this paper treats one such case, namely that of a product set.

The second result came from an attempt to prove the Muntz-Jackson theorem in the uniform norm. The method used was unsuccessful, but yielded an elementary proof of Jackson's theorem.

I. The Muntz-Jackson Theorem

In this section we prove the main theorem, preceded by the Muntz theorem in n-dimensions.

Let $\{ \lambda_{ik} \}$ $i=1,2,\ldots,n$; $k=0,1,2,\ldots$ be sequences of real numbers, satisfying $\lambda_{ik} > -1/2$.

Let $T_i = \{x_i^{\lambda_{ik}}\}$ $k=0,1,2,\ldots$

and $T = T_1 \times T_2 \times \ldots \times T_n$.

Denote by U.C. the n-dimensional unit cube, $0 \leq x_i \leq 1$, $i=1,2,\ldots,n$

For a function of n variables, $f(x_1,\ldots,x_n)$, we write $f(\overline{x})$, whenever no confusion will result. We also write

$\int f(\overline{x})d\overline{x}$ instead of $\iint \cdots \int f(x_1,\ldots,x_n)\, dx_1 \ldots dx_n$.

<u>Theorem</u>: A necessary and sufficient condition that T be dense in L^2 [U.C.] is that

$$\sum_{k=0}^{\infty} \frac{1}{\lambda_{ik}} = \infty \qquad i = 1,2,\ldots,n \; .$$

<u>Proof</u>:

<u>Sufficiency</u>: Suppose $\varphi(\overline{x})$ is orthogonal to T, $\varphi \in L^2$ [U.C.].

Fix $\lambda_{2k}, \lambda_{3k}, \ldots, \lambda_{nk}$ and let

$$g_1(x_1) = \int_0^1 \cdots \int_0^1 \prod_{i=2}^{n} x_i^{\lambda_{ik}} \varphi(x_1,\ldots,x_n)\, dx_2 \ldots dx_n \; .$$

Then $g_1(x_1) \in L^2 [0,1]$, and

$$\int_0^1 x_1^{\lambda_{1k}} g_1(x_1) dx_1 = 0 \qquad k = 0,1,2,\ldots$$

and, hence, $g_1(x_1) \equiv 0$ by the 1-dimensional Muntz theorem.

Let $g_2(x_2) = \int_0^1 \cdots \int_0^1 \prod_{i=3}^{n} x_i^{\lambda_{ik}} \varphi(x_1,\ldots,x_n)\, dx_3 \ldots dx_n$.

Then $g_2(x_2) \in L^2\,[0,1]$, and

$$\int_0^1 x_2^{\lambda_{2k}}\, g_2(x_2) dx_2 = 0 \qquad k = 0,1,2,\ldots$$

and, hence, $g_2(x_2) \equiv 0$.

Repetition of this process yields

$$\int_0^1 x_n^{\lambda_{nk}}\, \varphi\,(x_1,\ldots,x_n) dx_n = 0 \qquad k = 0,1,2,\ldots$$

so that $\varphi(\overline{x}) \equiv 0$.

Therefore T is dense in L^2 [U.C.].

<u>Necessity:</u> Suppose $\sum_{k=0}^{\infty} \frac{1}{\lambda_{1k}} < \infty$.

Then there exists $\varphi(x) \in L^2\,[0,1]$, such that

$$\int_0^1 x^{\lambda_{1k}}\, \varphi(x) dx = 0 \qquad k = 0,1,2,\ldots$$

and $\varphi(x) \not\equiv 0$.

Since $\varphi(x) \in L^2$ [U.C.], $\int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} x_i^{\lambda_{ik}}\, \varphi(x_1) dx_1 \ldots dx_n$

exists, and, by Fubini's theorem, is equal to

$$\int_0^1 \cdots \int_0^1 \prod_{i=2}^{n} x_i^{\lambda_{ik}} \left[\int_0^1 x_1^{\lambda_{1k}}\, \varphi(x_1) dx_1 \right] dx_2 \ldots dx_n$$

$$= 0.$$

Thus φ is orthogonal to T, and $\varphi \not\equiv 0$.

Therefore T is not dense.

Q.E.D.

We turn now to the corresponding Muntz-Jackson theorem.

Let $f(\overline{x}) \in L^2$ [U.C.]. We continue f so that

$$f(x_1,\ldots,x_i + 1,\ldots,x_n) = f(x_1,\ldots,x_i,\ldots,x_n) \qquad i = 1,2,\ldots,n .$$

The L^2 modulus of continuity of f is defined by

$$W_f^{L^2}(\delta) = \sup_{\sum_{i=1}^{n} h_i^2 \le \delta^2} \| f(\overline{x} + \overline{h}) - f(\overline{x}) \|_{L^2} .$$

If $W_f^{L^2}(\delta) \le \delta$, we say that f is an L^2 - shrinker, and denote the class of L^2-shrinkers by S.

Lemma 1: If $f(\overline{x}) \in S$, then $f_{x_i}(\overline{x})$ exists almost everywhere, and satisfies $\| f_{x_i} \|_{L^2} \le 1$, $i = 1,\ldots,n$.

Proof: Since $f(\overline{x}) \in S$,

$$\| f(x_1 + t, x_2,\ldots,x_n) - f(x_1,x_2,\ldots x_n) \|_{L^2} \le t$$

and, hence, the set of functions

$$f_t(\overline{x}) = \frac{f(x_1 + t, x_2,\ldots,x_n) - f(x_1,x_2,\ldots,x_n)}{t}$$

has L^2 norm uniformly bounded by one. We may extract a subsequence $\{f_{t_k}\}$ which converges weakly to $\varphi(\overline{x})$ as $t_k \to 0$.

By weak convergence

$$\lim_{t_k \to 0} \int_0^{x_n} \cdots \int_0^{x_1} f_{t_k}(\bar{u})\, d\bar{u} = \int_0^{x_n} \cdots \int_0^{x_1} \varphi(\bar{u}) d\bar{u} \ .$$

But $\lim_{t_k \to 0} \int_0^{x_n} \cdots \int_0^{x_1} f_{t_k}(\bar{u}) d\bar{u} = \int_0^{x_n} \cdots \int_0^{x_2} [f(x_1,u_2,\ldots,u_n)-$

$f(0,u_2,\ldots,u_n)]d\bar{u}$ for almost all x_1.

Thus $\int_0^{x_n} \cdots \int_0^{x_2} [f(x_1,u_2,\ldots,u_n) - f(0,u_2,\ldots,u_n)]d\bar{u} =$

$$\int_0^{x_n} \cdots \int_0^{x_1} \varphi(\bar{u}) d\bar{u} \ .$$

Differentiating successively with respect to $x_n,\ldots,x_2$, we obtain

$$f(x_1,x_2,\ldots,x_n) = f(0,x_2,\ldots,x_n) + \int_0^{x_1} \varphi(u,x_2,\ldots,x_n) du$$

and hence

$$f_{x_1}(\bar{x}) = \varphi(\bar{x}).$$

Since $\|f_{t_k}(\bar{x})\|_{L^2} \leq 1$, and $\varphi(\bar{x})$ is the weak limit of $f_{t_k}(\bar{x})$, we have $\|\varphi(\bar{x})\|_{L^2} \leq 1$, or $\|f_{x_1}(\bar{x})\|_{L^2} \leq 1$.

Similarly for $f_{x_i}(\bar{x})$.

Q.E.D.

We now prove a result of independent interest.

<u>Lemma 2</u>: Let $\{a_{ij}(t)\}$ be a complete orthonormal set in $L^2[0,1]$,

$i = 1,2,\ldots,n; \quad -\infty < j < \infty$. Let $\varphi(\bar{t}) \in L^2$ [U.C.], and suppose $\int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} a_{ij_i}(t_i)\, \varphi(\bar{t})d\bar{t} = 0$ whenever the j_i satisfy simultaneously, $0 \leq j_i \leq N_i$.

Then there exist $\varphi_1(\bar{t}), \varphi_2(\bar{t}),\ldots, \varphi_n(\bar{t}) \in L^2$ [U.C.], such that

(a) $\varphi = \varphi_1 + \varphi_2 + \ldots + \varphi_n$

(b) $(\varphi_i, \varphi_j) = 0 \quad i \neq j$

(c) $\| \varphi_i \| \leq \| \varphi \| \quad i = 1,2,\ldots,n$

(d) $\int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} a_{ij_i}(t_i)\, \varphi_k(\bar{t})d\bar{t} = 0$ for $0 \leq j_k \leq N_k$,

for all j_i $(i \neq k)$.

Proof: φ has a Fourier series,

$$\varphi(\bar{t}) = \sum_{i_1,\ldots,i_n=-\infty}^{\infty} c_{i_1,\ldots i_n} a_{1i_1}(t_1)\ldots a_{ni_n}(t_n)$$

where $c_{i_1,\ldots,i_n} = 0$ for those indices which satisfy simultaneously $0 \leq i_k \leq N_k \quad k = 1,2,\ldots,n.$

We choose the φ_i in the following way:

$$\text{Let } \varphi_1 = \sum_{i_1,\ldots,i_n=-\infty}^{\infty} d_{i_1,\ldots,i_n} \prod_{k=1}^{n} a_{ki_k}(t_k)$$

where $d_{i_1,\ldots,i_n} = c_{i_1,\ldots,i_n}$ for those indices for which $i_1 < 0$ or $i_1 > N_1$, and $d_{i_1,\ldots,i_n} = 0$ otherwise .

Let $\varphi_2 = \sum_{i_1,\ldots,i_n=-\infty}^{\infty} d_{i_1,\ldots,i_n} \prod_{k=1}^{n} a_{ki_k}(t_k)$

where $d_{i_1,\ldots,i_n} = c_{i_1,\ldots,i_n}$ for the remaining indices for which $i_2 < 0$ or $i_2 > N_2$, and $d_{i_1,\ldots,i_n} = 0$ otherwise.

Choose $\varphi_3, \ldots, \varphi_{n-1}$ in a similar fashion, and

let $\varphi_n = \sum_{i_1,\ldots,i_n=-\infty}^{\infty} d_{i_1,\ldots,i_n} \prod_{k=1}^{n} a_{ki_k}(t_k)$

where $d_{i_1,\ldots,i_n} = c_{i_1,\ldots,i_n}$ for the remaining indices, which consist of the remaining indices for which $i_n < 0$ or $i_n > N_n$, and those indices for which $0 \leq i_j \leq N_j$ for all j.

The latter coefficients are, of course, all zero.

Clearly (a) and (b) are satisfied.

Now $\| \varphi \|^2 = \| \varphi_1 + \ldots + \varphi_n \|^2 = \| \varphi_1 \|^2 + \ldots + \| \varphi_n \|^2$

$$+ 2 \sum_{i \neq j} \operatorname{Re} (\varphi_i, \varphi_j)$$

$$= \| \varphi_1 \|^2 + \ldots + \| \varphi_n \|^2 \text{ , since } (\varphi_i, \varphi_j) = 0 \quad i \neq j .$$

Therefore, $\| \varphi_i \| \leq \| \varphi \|$.

Since $c_{i_1,\ldots,i_n} = \int_0^1 \cdots \int_0^1 \prod_{k=1}^{n} a_{ki_k}(t_k)\, \varphi\, (\bar{t}) d\bar{t}$,

we have $\int_0^1 \cdots \int_0^1 \prod_{k=1}^{n} a_{ki_k}(t_k)\, \varphi_j\, (\bar{t})\, d\bar{t}$

$= d_{i_1,\ldots,i_n}$

$= 0 \quad \text{for} \quad 0 \leq i_j \leq N_j$

Q.E.D.

Let $\Lambda_i = \{ 0 = \lambda_{i0} < \lambda_{i1} < \cdots < \lambda_{iN_i} \}$ be a finite set of integers, $i = 1,2,\ldots,n$,

satisfying,

$$\lambda_{i(k+1)} - \lambda_{ik} \geq 2 .$$

Let $T_i = \{ x_i^{\lambda_{ik}} \}$, $k = 0,1,\ldots,N_i$

and $T = T_1 \times T_2 \times \ldots \times T_n$

Let C_T be the set of all $\varphi \in L^2$ [U.C.], $\| \varphi \|_{L^2} = 1$,

and φ orthogonal to all $t \in T$.

$$\text{Let } \epsilon_{\Lambda_i} = \prod_{k=1}^{N_i} \frac{\lambda_{ik} - 1/2}{\lambda_{ik} + 3/2},$$

and let $P_T = \sup_{f \in S} \inf_{t \in T} \| f(\bar{x}) - t\,(\bar{x}) \|_{L^2}$ be the approximation index.

We now state the main

<u>Theorem:</u> $\quad 1/4 \sum_{i=1}^{n} \epsilon_{\Lambda_i} \leq P_T \leq 2/3 \sum_{i=1}^{n} \epsilon_{\Lambda_i}$

Proof: We assume, without loss of generality, that all the functions in our space have mean-value zero. For suppose $f(\bar{x}) \in S$.

Let $g(\bar{x}) = f(\bar{x}) - \int_0^1 f(\bar{x})d\bar{x}$.

Then $g(\bar{x})$ has mean-value zero. Suppose $t(\bar{x})$ is an approximation to $g(\bar{x})$,

$$\| g(\bar{x}) - t(\bar{x}) \| < \epsilon.$$

Then $\| f(\bar{x}) - [t(\bar{x}) + \int_0^1 f(\bar{x})d\bar{x}] \| < \epsilon$.

Thus, it is sufficient to approximate $g(\bar{x})$.

If f has mean-value zero, then its Fourier series has no constant term. The set of all such functions forms a Hilbert space.

Suppose $\varphi \in C_T$

$$\text{Let } F(\bar{z}) = \int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} t_i^{(z_i - 1/2)} \varphi(\bar{t})d\bar{t} \qquad (1)$$

Then $F(\lambda_{1k_1} + 1/2, \lambda_{2k_2} + 1/2, \ldots, \lambda_{nk_n} + 1/2) = 0$

for $k_i = 0,1,\ldots,N_i$.

Let $z_j = x_j + iy_j$.

Definition: $G(\bar{z})$ is said to be in the Paley-Weiner class, P, for $\bar{x} > 0$ if

(a) $G(\bar{z})$ is analytic for $\bar{x} > 0$, and

$\int_{-\infty}^{\infty} | G(\bar{x} + i\bar{y}) |^2 d\bar{y} < M$

for all $\bar{x} > 0$,

or, equivalently,

$$\text{(b)} \quad G(\bar{z}) = \int_{-\infty}^{0} f(\bar{u}) e^{(u_1 z_1 + \ldots + u_n z_n)} \, d\bar{u}$$

where $\int_{-\infty}^{0} |f(\bar{u})|^2 \, d\bar{u} < \infty$

For a proof of the equivalence of (a) and (b), and for the general Paley-Weiner theory, the reader is referred to [1] and [5].

We now show that $F(\bar{z}) \in P$.

Letting $t_i = e^{u_i}$, (1) is transformed into

$$\int_{-\infty}^{0} \cdots \int_{-\infty}^{0} e^{(u_1 z_1 + \ldots + u_n z_n)} \left[e^{1/2 (u_1 + \ldots + u_n)} \varphi(e^{u_1}, \ldots, e^{u_n}) \right] d\bar{u}$$

Now $\int_{-\infty}^{0} \cdots \int_{-\infty}^{0} e^{(u_1 + \ldots + u_n)} | \varphi(e^{u_1}, \ldots, e^{u_n}) |^2 \, d\bar{u}$

$$= \int_0^1 | \varphi(\bar{t}) |^2 \, d\bar{t} = 1 .$$

Therefore $F(\bar{z}) \in P$.

By Lemma 2, there exist $\varphi_1, \ldots, \varphi_n$ such that

$$F(\bar{z}) = \sum_{j=1}^{n} \int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} t_i^{(z_i - 1/2)} \varphi_j(\bar{t}) \, d\bar{t}$$

where $(\varphi_i, \varphi_j) = 0$, and φ_i is orthogonal to T_i .

Let $B_i(z) = \prod_{k=0}^{N_i} \dfrac{z - (\lambda_{ik} + 1/2)}{z + (\lambda_{ik} + 1/2)}$, and let

$$B_i(z_i) g_i(z_1, \ldots, z_n) = \int_0^1 \cdots \int_0^1 \prod_{j=1}^{n} t_j^{(z_j - 1/2)} \varphi_i(\bar{t}) \, d\bar{t} \quad i = 1, 2, \ldots, n$$

Then $F(\overline{z}) = \sum_{i=1}^{n} B_i(z_i) g_i(z_1, z_2, \ldots z_n)$ where $g_i(\overline{z}) \in P$.

Let $C^*_{T_i} = \{ \varphi : \varphi \in L^2 [U.C.]; \varphi$ is orthogonal to T_i; $\| \varphi \| \leq 1 \}$

Then $\varphi_i \in C^*_{T_i}$.

<u>Lemma 3:</u> $$P_T \leq \sum_{i=1}^{n} \sup_{\varphi_i \in C^*_{T_i}} \left\| \int_0^{x_i} \varphi_i(x_1, \ldots, x_n) dx_i \right\|_{L^2}$$

<u>Proof:</u> $$P_T = \sup_{f \in S} \inf_{t \in T} \| f(\overline{x}) - t(\overline{x}) \|$$

$$= \sup_{f \in S} \sup_{\varphi \in C_T} (f, \varphi) \quad \text{(see [6])}$$

$$= \sup_{f \in S} \sup_{(\sum_{i=1}^{n} \varphi_i) \in C_T} \left[\sum_{i=1}^{n} (f, \varphi_i) \right]$$

$$\leq \sup_{f \in S} \sum_{i=1}^{n} \sup_{\varphi_i \in C^*_{T_i}} (f, \varphi_i)$$

$$\leq \sum_{i=1}^{n} \sup_{f \in S} \sup_{\varphi_i \in C^*_{T_i}} (f, \varphi_i)$$

$$= \sum_{i=1}^{n} \sup_{\varphi_i \in C^*_{T_i}} \sup_{f \in S} \int_0^1 f(\overline{x}) \varphi_i(\overline{x}) d\overline{x} \qquad (2)$$

Integrating by parts, a procedure which is valid by Lemma 1, we obtain

$$(2) = \sum_{i=1}^{n} \sup_{\varphi_i \in C^*_{T_i}} \sup_{f \in S} \int_0^1 \cdots \int_0^1 \Big[f(x_1,\ldots,x_n) \int_0^{x_i} \varphi_i(x_1,\ldots,x_n)dx_i \Big|_0^1$$

$$-\int_0^1 \{ f_{x_i}(x_1,\ldots,x_n) \int_0^{x_i} \varphi_i(x_1,\ldots,x_n)dx_i \} dx_i \Big] dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n$$

Consider $\int_0^{x_i} \varphi(\bar{x})dx_i \Big|_0^1 = \int_0^1 \varphi(\bar{x})dx_i$.

Since φ has mean-value zero, its Fourier series,

$$\sum_{m_1,\ldots,m_n=-\infty}^{\infty} C_{m_1,\ldots,m_n} e^{2\pi i (m_1x_1+\ldots+m_nx_n)}$$

has no constant term.

Therefore, $\int_0^1 \varphi_i(\bar{x})dx_i = \int_0^1 \sum_{m_1,\ldots,m_n=-\infty}^{\infty} C_{m_1,\ldots,m_n} e^{2\pi i (m_1x_1+\ldots+m_nx_n)}$

$= 0$ by term-by-term integration.

$$\text{Thus } P_T \le \sum_{i=1}^{n} \sup_{\varphi_i \in C^*_{T_i}} \sup_{f \in S} - \int_0^1 \cdots \int_0^1 \Big[f_{x_i}(\bar{x}) \int_0^{x_i} \varphi_i(\bar{x})dx_i \Big] d\bar{x}$$

$$\le \sum_{i=1}^{n} \sup_{\varphi_i \in C^*_{T_i}} \left\| \int_0^{x_i} \varphi_i(\bar{x})\, dx_i \right\|_{L^2} .$$

The last inequality is true because $\|f_{x_i}\|_{L^2} \le 1$. The sup over all f with $\| f_{x_i} \| \leq$ would be equal to $\| \int_0^{x_i} \varphi_i(\bar{x})\, dx_i \|_{L^2}$.

Hence the lemma is proved. Returning to the proof of the main theorm, we have

$$B_k(z_k) g_k(\bar{z}) = \int_0^1 \cdots \int_0^1 \prod_{j=1}^{n} t_j^{(z_j - 1/2)} \varphi_k(\bar{t})\, d\bar{t} \qquad (3)$$

$$B_k(iy_k)\, g_k(\overline{iy}) = \int_0^1 \cdots \int_0^1 \prod_{j=1}^{n} t_j^{(iy_j - 1/2)} \varphi_k(\bar{t})\, d\bar{t}$$

Letting $t_j = e^{u_j}$,

$B_k(iy_k) g_k(\overline{iy}) =$

$$\int_{-\infty}^{0} \cdots \int_{-\infty}^{0} e^{i(u_1 y_1 + \cdots + u_n y_n)} \left[e^{1/2(u_1 + \cdots + u_n)} \varphi_k(e^{u_1}, \ldots, e^{u_n}) \right] d\bar{u}$$

Define $\varphi_k(e^{u_1}, \ldots, e^{u_n}) = 0$ for $u_j \epsilon (0, \infty)$ $j = 1, \ldots, n$

Then Parseval's identity yields, since $| B_k(iy_k) | = 1$,

$$\frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} | g_k(\overline{iy}) |^2 \, d\bar{y} =$$

$$\int_{-\infty}^{0} \cdots \int_{-\infty}^{0} e^{(u_1 + \cdots + u_n)} | \varphi_k(e^{u_1}, \ldots, e^{u_n}) |^2 \, d\bar{u}$$

$$= \int_0^1 | \varphi_k(\bar{t}) |^2 \, d\bar{t} = \| \varphi_k(\bar{t}) \|^2$$

Integrating (3) by parts, we obtain

$B_k(z_k) g_k(\bar{z}) =$

$$\int_0^1 \cdots \int_0^1 \Big[t_k^{(z_k - 1/2)} \int_0^{t_k} \varphi_k(\bar{t}) dt_k \Big|_0^1 -$$

$$(z_k - 1/2) \int_0^1 \{ t_k^{(z_k - 3/2)} \int_0^{t_k} \varphi_k(\bar{t}) dt_k \} dt_k \Big] \prod_{\substack{j=1 \\ j \neq k}}^{n} t_j^{(z_j - 1/2)} d\bar{t}'$$

where $d\bar{t}' = dt_1 \cdots dt_{k-1}\, dt_{k+1} \cdots dt_n$.

As shown on page (11),

$$\int_0^1 \varphi_k(\bar{t})\, dt_k = 0$$

Thus

$$\frac{B_k(z_k) g_k(\bar{z})}{(z_k - 1/2)} = - \int_0^1 \Big[\prod_{\substack{j=1 \\ j \neq k}}^{n} t_j^{(z_j - 1/2)}\, t_k^{(z_k - 3/2)} \int_0^{t_k} \varphi_k(\bar{t}) dt_k \Big] d\bar{t}$$

and, hence,

$$\frac{B_k(1+iy_k) g_k(iy_1, \ldots, iy_{k-1}, 1+iy_k, iy_{k+1}, \ldots, iy_n)}{iy_k + 1/2}$$

$$= - \int_0^1 \Big[\prod_{j=1}^{n} t_j^{(iy_j - 1/2)} \int_0^{t_k} \varphi_k(\bar{t}) dt_k \Big] d\bar{t}$$

and, again by Parseval,

$$\int_{-\infty}^{\infty} \frac{|B_k(1+iy_k) g_k(iy_1, \ldots, iy_{k-1}, 1+iy_k, iy_{k+1}, \ldots, iy_n)|^2}{(2\pi)^n \;(y_k^2 + 1/4)}\, d\bar{y} =$$

$$\int_0^1 \Big| \int_0^{t_k} \varphi(\bar{t}) dt_k \Big|^2 d\bar{t} = \Big\| \int_0^{t_k} \varphi_k(\bar{t})\, dt_k \Big\|^2_{L^2} .$$

Therefore $\sup\limits_{\varphi_k \epsilon C^*_{T_k}} \| \int_0^{t_k} \varphi_k(\bar{t}) dt_k \|^2 =$

$$\sup_{g\epsilon P} \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \frac{|B_k(1+iy_k) g_k(iy_1, \ldots, iy_{k-1}, 1+iy_k, iy_{k+1}, \ldots iy_n)|^2}{y_k^2 + 1/4} d\bar{y}$$

$$\leq \sup_{g\epsilon P} \sup_{y_k} \frac{|B_k(1+iy_k)|^2}{y_k^2 + 1/4} \quad x$$

$$\frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} |g_k(iy_1, \ldots, iy_{k-1}, 1+iy_k, iy_{k+1}, \ldots, iy_n)|^2 d\bar{y} \quad .$$

Since $g_k(\bar{z}) \epsilon P$, there exists $\Psi_k(\bar{t}) \epsilon L^2[U.G.]$, such that

$$g_k(\bar{z}) = \int_0^1 \prod_{j=1}^{n} t_j^{(z_j - 1/2)} \Psi_k(\bar{t}) d\bar{t}$$

By Parseval,

$$\frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} |g_k(\overline{iy})|^2 d\bar{y} = \int_{-\infty}^{0} e^{(u_1 + \ldots + u_n)} |\Psi_k(e^{u_1}, \ldots, e^{u_n})|^2 d\bar{u}$$

$$= \int_0^1 | \Psi_k(\bar{t}) |^2 d\bar{t},$$

while, again by Parseval,

$$\frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} |g_k(iy_1, \ldots, iy_{k-1}, 1+iy_k, iy_{k+1}, \ldots, iy_n)|^2 d\bar{y}$$

$$= \int_{-\infty}^{0} e^{(u_1 + \ldots + u_n)} e^{2u_k} |\Psi_k(e^{u_1}, \ldots, e^{u_n})|^2 d\bar{u}$$

$$= \int_0^1 t_k^2 |\Psi_k(\bar{t})|^2 d\bar{t} \leq \int_0^1 |\Psi_k(\bar{t})|^2 d\bar{t}$$

$$\frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} |g_k(i\bar{y})|^2 d\bar{y} \ .$$

Thus, $\| \int_0^{t_k} \varphi_k(\bar{t}) dt_k \|^2$

$$\leq \sup_{y_k} \frac{|B_k(1+iy_k)|^2}{y_k^2 + 1/4} \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} |g_k(\bar{iy})|^2 d\bar{y}$$

$$= \sup_{y_k} \frac{|B_k(1+iy_k)|^2}{y_k^2 + 1/4} \| \varphi_k(\bar{t}) \|^2 \ .$$

$$\leq \sup_{y_k} \frac{|B_k(1+iy_k)|^2}{y_k^2 + 1/4} \quad \text{since } \|\varphi_k(\bar{t})\| \leq 1 \ .$$

Now $B_k(z_k) = \prod_{j=0}^{N_k} \frac{z - (\lambda_{kj} + 1/2)}{z + (\lambda_{kj} + 1/2)}$

and $$\frac{|B_k(1+iy_k)|^2}{y_k^2 + 1/4} = \frac{1}{y_k^2 + 1/4} \prod_{j=0}^{N_k} \frac{(\lambda_{kj} - 1/2)^2 + y_k^2}{(\lambda_{kj} + 3/2)^2 + y_k^2}$$

$$= \frac{1}{y_k^2 + 1/4} \prod_{j=0}^{N_k} \frac{(\lambda_{kj} - 1/2)^2}{(\lambda_{kj} + 3/2)^2} \left[\frac{1 + \frac{y_k^2}{(\lambda_{kj} - 1/2)^2}}{1 + \frac{y_k^2}{(\lambda_{kj} + 3/2)^2}} \right]$$

$$= 4/9 \prod_{j=1}^{N_k} \frac{(\lambda_{kj} - 1/2)^2}{(\lambda_{kj} + 3/2)^2} \prod_{j=1}^{N_k} \left[\frac{1 + \frac{y_k^2}{(\lambda_{kj} - 1/2)^2}}{1 + \frac{y_k^2}{(\lambda_{k(j-1)} + 3/2)^2}} \right] \frac{1}{1 + \frac{y_k^2}{(\lambda_{N_k} + 3/2)^2}} \quad (4)$$

Since $(\lambda_{kj} - 1/2) \geq (\lambda_{k(j-1)} + 3/2)$ for all j, we have,

$$(4) \leq 4/9 \prod_{j=1}^{N_k} \frac{(\lambda_{kj} - 1/2)^2}{(\lambda_{kj} + 3/2)^2}$$

Therefore, $\|\int_0^{t_k} \varphi_k(\bar{t})dt_k\| \leq 2/3 \prod_{j=1}^{N_k} \frac{(\lambda_{kj} - 1/2)}{(\lambda_{kj} + 3/2)} = 2/3\, \epsilon_{\Lambda_k}$

and hence, by Lemma 3,

$$P_T \leq 2/3 \sum_{k=1}^{n} \epsilon_{\Lambda_k} \ .$$

To obtain the lower bound, we orthonormalize the sets T_i and form a complete orthonormal set, $\varphi_{1k}(x_1),\ldots,\varphi_{nk}(x_n)$. If $f(\bar{x})L^2$ [U.C.], then $f(\bar{x})$ has a Fourier series,

$$f(\bar{x}) \sim \sum_{k_1,\ldots,k_n=0}^{\infty} C_{k_1,\ldots,k_n} \varphi_{1k_1}(x_1)\, \varphi_{2k_2}(x_2)\ldots\varphi_{nk_n}(x_n).$$

Now suppose that $f(\bar{x})$ is of the form $f(\bar{x}) = f_1(x_1) + f_2(x_2) + \ldots + f_n(x_n)$, and suppose that $f_i(x_i)$ has the Fourier series $\sum_{k=0}^{\infty} C_k\, \varphi_{ik}(x_i)$. Then the Fourier series of $f(\bar{x})$ is

$\sum_{i=1}^{n} \sum_{k=0}^{\infty} C_k \varphi_{ik}(x_i)$, and the best approximation to f $(\bar{x})$ is

$\sum_{i=1}^{n} \sum_{k=0}^{N_i} C_k \varphi_{ik}(x_i)$; i.e., the sum of the best approximations to $f_1(x_1), f_2(x_2), \ldots, f_n(x_n)$.

Newman, [4], has shown that for any set $\{x^{\lambda_i}\}$, $i=0\ldots,k$, there exists a function which cannot be approximated better than $1/4\ \epsilon_\Lambda$. For each of the sets T_i, let $g_i(x_i)$ be the corresponding "bad" function, and let $f(\bar{x}) = g_1(x_1) + \ldots g_n(x_n)$.

Then $\| f(\bar{x}) - t(\bar{x}) \| \geq 1/4 \sum_{i=1}^{n} \epsilon_{\Lambda_i}$ for all $t \epsilon T$, and hence

$$P_T \geq 1/4 \sum_{i=1}^{n} \epsilon_{\Lambda_i} .$$

This completes the proof.

Q.E.D.

The above theorem is applicable only to shrinkers. To extend our result to functions with arbitrary modulus of continuity, we use the following construction.

Given $f(\bar{x}) \in L^2$ [U.C.],

$$\text{let } g(\bar{x}) = \frac{1}{\epsilon^n} \int_0^\epsilon f(\bar{x} + \bar{t})\, d\bar{t} .$$

Then $g_{x_1}(\bar{x}) = \frac{1}{\epsilon^n} \int_0^\epsilon f_{x_1}(\bar{x} + \bar{t}) d\bar{t}$

$$= \frac{1}{\epsilon^n} \int_0^\epsilon \cdots \int_0^\epsilon [\, f(x_1+\epsilon, x_2 + t_2, \ldots, x_n + t_n) -$$

$$f(x_1, x_2 + t_2, \ldots, x_n + t_n)\,]\, dt_2 \ldots dt_n .$$

Thus $\|g_{x_1}(\bar{x})\|_{L^2} \leq \frac{1}{\epsilon^n} \int_0^{\epsilon} \cdots \int_0^{\epsilon} \| f(x_1 + \epsilon, x_2 + t_2, \ldots, x_n + t_n) -$

$f(x_1, \ x_2 + t_2, \ldots, x_n + t_n) \| dt_2 \ldots dt_n$

by Schwarz' inequality,

and, hence,

$$\|g_{x_1}(\bar{x})\|_{L^2} \leq \frac{W_f(\epsilon)}{\epsilon} \ .$$

Similarly, $\|g_{x_i}(\bar{x})\|_{L^2} \leq \dfrac{W_f(\epsilon)}{\epsilon}$.

Now $W_g(\delta) = \sup\limits_{\sum t_i^2 \leq \delta^2} \|g(x_1 + t_1, \ldots, x_n + t_n) - g(x_1, \ldots, x_n)\|$

$$\leq \sup_{\sum t_i^2 \leq \delta^2} \|g(x_1 + t_1, \ldots, x_n + t_n) - g(x_1, x_2 + t_2, \ldots, x_n + t_n)\|$$

$$+ \sup_{\sum t_i^2 \leq \delta^2} \|g(x_1, x_2 + t_2, \ldots, x_n + t_n) - g(x_1, x_2, x_3 + t_3, \ldots, x_n + t_n)\|$$

$$+ \cdots + \sup_{\sum t_i^2 \leq \delta^2} \|g(x_1, x_2, \ldots, x_{n-1} x_n + t_n) - g(x_1, \ldots, x_n)\|$$

$$\leq n \ \frac{W_f(\epsilon)}{\epsilon}$$

Moreover, $\|g(\bar{x}) - f(\bar{x})\| = \| \frac{1}{\epsilon^n} \int_0^{\epsilon} [f(\bar{x} + \bar{t}) - f(\bar{x})] d\bar{t} \|$

$$\leq \frac{1}{\epsilon^n} \int_o^\epsilon \|f(\bar{x} + \bar{t}) - f(\bar{x})\| d\bar{t}$$

$$\leq W(\sqrt{n}\epsilon) \leq W(n\epsilon) \leq n\, W(\epsilon) \; .$$

Since $\frac{\epsilon}{nW_f(\epsilon)} \; g(\bar{x}) \in S$, there exists $t(\bar{x}) \in T$ such that

$$\left\| \frac{\epsilon}{nW_f(\epsilon)} \; g(\bar{x}) - t(\bar{x}) \right\| \leq \sum_{i=1}^{n} \epsilon_{\Lambda_i} \; .$$

Thus $\| g(\bar{x}) - \frac{nW_f(\epsilon)}{\epsilon} \; t(\bar{x}) \| \leq \frac{nW_f(\epsilon)}{\epsilon} \sum_{i=1}^{n} \epsilon_{\Lambda_i}$

and, hence, $\| f(\bar{x}) - \frac{nW_f(\epsilon)}{\epsilon} \; t(\bar{x}) \|$

$$\leq \|f(\bar{x}) - g(\bar{x})\| + \|g(\bar{x}) - \frac{nW_f(\epsilon)}{\epsilon} \; t(\bar{x}) \|$$

$$\leq nW_f(\epsilon) + \frac{nW_f(\epsilon)}{\epsilon} \sum_{i=1}^{n} \epsilon_{\Lambda_i} \; .$$

Letting $\epsilon = \sum_{i=1}^{n} \epsilon_{\Lambda_i}$, $P(\bar{x}) = \frac{nW_f(\epsilon)}{\epsilon} \; t(\bar{x})$,

we obtain,

$$\|f(\bar{x}) - P(\bar{x})\| \leq 2\, n\, W_f\left(\sum_{i=1}^{n} \epsilon_{\Lambda_i} \right) .$$

II. An Elementary Proof of Jackson's Theorem

In this chapter we give an elementary proof of Jackson's Theorem. For convenience we prove the theorem on the interval [-1,1], although the proof carries over to an arbitrary interval with just a few modifications.

Denote by P_n the space of polynomials of degree less than or equal to n.

For $f \epsilon C[-1,1]$, denote by $W_f(\delta)$ the modulus of continuity of f(x);

$$W_f(\delta) = \sup_{x \epsilon [-1,1]} \sup_{|t| \leq \delta} |f(x+t) - f(x)| .$$

Theorem: Let $f(x) \epsilon C[-1,1]$. Then there exists $p(x) \epsilon P_n$ such that $\sup_{x \epsilon [-1,1]} |f(x) - p(x)| \leq c\, W_f(\frac{1}{n})$ where c is an independent constant.

Proof: Divide the interval [-1,1] into 2n equal subintervals $\left[\frac{k}{n}, \frac{k+1}{n} \right]$ $-n \leq k \leq n-1$.

Define L(x) in the following way.

Let $L\left(\frac{k}{n}\right) = f\left(\frac{k}{n}\right)$, and let L (x) be linear in the subinterval $\left[\frac{k}{n}, \frac{k+1}{n}\right]$. L(x) is thus a piecewise linear function, and $| L(x) - f(x) | \leq W_f\left(\frac{1}{n}\right)$ in [-1,1] .

L(x) can be written as $A + \sum_{k=-n+1}^{n} a_k \left| x - \frac{k}{n} \right|$. To see this,

let $M(x) = \sum_{k=-n+1}^{n} a_k \left| x - \frac{k}{n} \right|$. We will find a_k so that $L(x) = A + M(x)$.

Let S_{k+1} be the slope of L(x) in $\left[\frac{k}{n}, \frac{k+1}{n} \right]$.

In the subinterval $\left[\frac{j}{n}, \frac{j+1}{n} \right]$,

$$M(x) = \sum_{k=-n+1}^{j} a_k \left(x - \frac{k}{n} \right) + \sum_{k=j+1}^{n} a_k \left(\frac{k}{n} - x \right)$$

and

$$M'(x) = \sum_{k=-n+1}^{j} a_k - \sum_{k=j+1}^{n} a_k .$$

Setting $M'(x) = S_{j+1}$ in $\left[\frac{j}{n}, \frac{j+1}{n} \right]$, we obtain a system of equations,

$$\begin{aligned}
-a_{-n+1} - a_{-n+2} - \cdots - a_n &= S_{-n+1} \\
a_{-n+1} - a_{-n+2} - \cdots - a_n &= S_{-n+2} \\
a_{-n+1} + a_{-n+2} - \cdots - a_n &= S_{-n+3} \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
a_{-n+1} + a_{-n+2} + \cdots + a_{n-1} - a_n &= S_n
\end{aligned}$$

This system has the solution

$$\begin{cases} a_k = \dfrac{S_{k+1} - S_k}{2} & -n+1 \le k \le n-1 \\ a_n = \dfrac{-S_n - S_{-n+1}}{2} \end{cases}$$

Thus M(x) and L(x) are piecewise linear functions, having equal slopes in each subinterval. They can thus differ by at most a constant.

Therefore $L(x) = A+M(x)$.

We can write

$$\sum_{k=-n+1}^{n} a_k \left| x - \frac{k}{n} \right| = \int_{-1}^{1} |x-t| dg(t)$$

where g(t) is a step function having a jump at $x = \frac{k}{n}$ equal to a_k, and $g(-1) = 0$.

Thus $\left| f(x) - A - \int_{-1}^{1} |x - t| dg(t) \right| \leq W_f \left(\frac{1}{n} \right)$.

<u>Lemma 1</u>: If $\int_{-2}^{2} \left| d \{ |x| - p(x) \} \right| \leq \frac{c}{n}$

then $\left| f(x) - A - \int_{-1}^{1} p(x-t)\, dg(t) \right| \leq (2c+1)\, W_f \left(\frac{1}{n} \right)$.

<u>Proof</u>: $\left| f(x) - A - \int_{-1}^{1} p\,(x-t)\, dg(t) \right|$

$$\leq \left| f(x) - A - \int_{-1}^{1} |x-t| dg(t) \right| + \left| \int_{-1}^{1} \{ |x-t| - p(x-t) \}\, dg(t) \right|$$

$$\leq W_f\left(\frac{1}{n} \right) + \left| \{ |x-t| - p(x-t) \}\, g(t) \Big|_{-1}^{1} - \int_{-1}^{1} g(t) d \{ |x-t| - p(x-t) \} \right|$$

by integration by parts,

$$\leq W_f \left(\frac{1}{n} \right) + |g(1)| \frac{c}{n} + \max_{-1 \leq t \leq 1} |g(t)| \frac{c}{n}$$

Now $\max_{-1 \leq t \leq 1} |g(t)| = \max_{j} \left| \sum_{k=-n+1}^{j} a_k \right|$

$$= \max_{j} \left| \frac{S_{j+1} - S_{-n+1}}{2} \right| \leq \max_{j} |S_j| \leq \frac{W_f\left(\frac{1}{n} \right)}{\frac{1}{n}} = n\, W_f\left(\frac{1}{n} \right)$$

Thus $\left| f(x) - A - \int_{-1}^{1} p(x-t)\, dg\,(t) \right|$

$$\leq W_f\left(\frac{1}{n}\right) + 2\, nW_f\left(\frac{1}{n}\right)\frac{c}{n}$$

$$= (2c+1)\, W_f\left(\frac{1}{n}\right)$$

Q.E.D.

<u>Lemma 2</u>:

There exists $p(x) \in P_n$ such that

$$\int_{-2}^{2} \left| d\{\, |x| - p(x)\} \right| \leq \frac{6\pi}{n}$$

<u>Proof</u>: Since $|x| - p(x)$ is absolutely continuous in $[-2,2]$,

$$\int_{-2}^{2} \left| d\{\, |x| - p(x)\,\} \right| = \int_{-2}^{2} |\, s(x) - p'(x)\, | dx$$

where $s(x) = \begin{cases} -1 & -2 \leq x < 0 \\ 1 & 0 \leq x \leq 2 \end{cases}$

We seek the best L^1 polynomial approximation to $s(x)$; i.e., we wish to minimize

$$\int_{-2}^{2} \left| s(x) - \sum_{k=0}^{n-1} c_k x^k \right| dx \qquad (1)$$

over all possible c_k.

Considering (1) as a function of c_k, $k = 0, 1, \ldots, n-1$, and differentiating with respect to c_j in order to obtain the best approximation, we find that

$$\int_{-2}^{2} \operatorname{sgn}\left\{ s(x) - \sum_{k=0}^{n-1} c_k x^k \right\} x^j\, dx = 0 \qquad j = 0,1,\ldots,n-1 \quad (2)$$

In order for (2) to hold, $s(x) - \sum_{k=0}^{n-1} c_k x^k$ must change sign at least n times.

Let $b_0 = -2$, $b_{n+1} = 2$, and let $b_1, b_2, \ldots, b_n$ be the points where $s(x) - \sum_{k=0}^{n-1} c_k x^k$ changes sign. Assume, initially, that $n = 4m+1$. The b's are symmetrical about the origin. Thus $b_{\frac{n+1}{2}} = 0$.

(2) yields,

$$\int_{b_0}^{b_1} x^j\,dx - \int_{b_1}^{b_2} x^j dx + \int_{b_2}^{b_3} x^j dx - \ldots + \int_{b_{n-1}}^{b_n} x^j dx - \int_{b_n}^{b_{n+1}} x^j dx = 0$$

or $(-b_0)^{j+1} + 2b_1^{j+1} - 2b_2^{j+1} + \ldots + 2b_n^{j+1} - b_{n+1}^{j+1} = 0$.

We thus have a system of equations,

$$b_1^k - b_2^k + b_3^k - \ldots + b_n^k = 2^{k-1}\,[1 + (-1)^k] \qquad k=1,2,\ldots,n \quad (3)$$

The solution to this system is $b_j = 2\cos\left(\frac{n+1-j}{n+1}\right)\pi$, as will be shown.

The equations with odd exponents are satisfied by cancellation. Let k be even. We must show that

$$\sum_{j=1}^{n} (-1)^{j+1} \cos^k\left(\frac{n+1-j}{n+1}\right)\pi = 1 .$$

Now $\cos^k x = \left[\dfrac{e^{ix} + e^{-ix}}{2}\right]^k$

$$= \frac{1}{2^k}\left[e^{ikx} + \binom{k}{1} e^{i(k-2)x} + \dots + \binom{k}{\frac{k}{2}} + \dots + e^{-ikx}\right]$$

$$= \frac{1}{2^{k-1}}\left[\cos kx + \binom{k}{1}\cos (k-2)x + \dots + \binom{k}{\frac{k}{2}-1}\cos 2x + \frac{1}{2}\binom{k}{\frac{k}{2}}\right]$$

Thus $$\sum_{j=1}^{n} (-1)^{j+1} \cos^k \left(\frac{n+1-j}{n+1}\right)\pi \qquad (4)$$

$$= \sum_{j=1}^{n} (-1)^{j+1} \cos^k \left(\frac{j}{n+1}\right)\pi$$

$$= \frac{1}{2^{k-1}} \sum_{j=1}^{n} (-1)^{j+1} \sum_{p=1}^{\frac{k}{2}-1} \left[\binom{k}{p} \cos\left(\frac{k-2p}{n+1}\right) j\pi + 1/2 \binom{k}{\frac{k}{2}}\right]$$

$$= \frac{1}{2^{k-1}} \sum_{p=1}^{\frac{k}{2}-1} \sum_{j=1}^{n} (-1)^{j+1} \left[\binom{k}{p} \cos\left(\frac{k-2p}{n+1}\right) j\pi + 1/2 \binom{k}{\frac{k}{2}}\right]$$

$$= \frac{1}{2^{k-1}} \sum_{p=1}^{\frac{k}{2}-1} \operatorname{Re} \sum_{j=1}^{n} (-1)^{j+1} \left[\binom{k}{p} e^{i\left(\frac{k-2p}{n+1}\right)j\pi} + 1/2 \binom{k}{\frac{k}{2}}\right]$$

$$= \frac{1}{2^{k-1}} \sum_{p=1}^{\frac{k}{2}-1} \binom{k}{p} \operatorname{Re} \left[\frac{e^{\frac{i(2n+1)(k-2p)\pi}{2(n+1)}} + e^{\frac{i(k-2)\pi}{2(n+1)}}}{2\cos \frac{(k-2p)\pi}{2(n+1)}}\right] + \frac{1}{2^k}\binom{k}{\frac{k}{2}}$$

$$= \frac{1}{2^{k-1}} \sum_{p=1}^{\frac{k}{2}-1} \binom{k}{p} \frac{\cos\left(\frac{2n+1}{2n+2}\right)(k-2p)\pi + \cos\frac{(k-2p)\pi}{2n+2}}{2\cos\frac{(k-2p)\pi}{2n+2}} + \frac{1}{2^k}\binom{k}{\frac{k}{2}}$$

$$= \frac{1}{2^{k-1}} \sum_{p=1}^{\frac{k}{2}-1} \binom{k}{p} \left[\frac{1}{2} + \frac{1}{2} \quad \frac{\cos\left[(k-2p)\pi - \dfrac{(k-2p)\pi}{2n+2}\right]}{\cos \dfrac{(k-2p)\pi}{2n+2}} \right] + \frac{1}{2^k} \binom{k}{\frac{k}{2}}$$

$$= \frac{1}{2^{k-1}} \sum_{p=1}^{\frac{k}{2}-1} \binom{k}{p} \left[\frac{1}{2} + \frac{1}{2} \left\{ \cos (k-2p)\pi + \sin (k-2p)\pi \tan \frac{(k-2p)\pi}{2n+2} \right\} \right] +$$

$$\frac{1}{2^k} \binom{k}{\frac{k}{2}}$$

$$= \frac{1}{2^{k-1}} \sum_{p=1}^{\frac{k}{2}-1} \binom{k}{p} + \frac{1}{2^k} \binom{k}{\frac{k}{2}} . \qquad (5)$$

Now $\displaystyle\sum_{p=1}^{k} \binom{k}{p} = 2^k ,$

and $\displaystyle\sum_{p=1}^{k} \binom{k}{p} = 2 \sum_{p=1}^{\frac{k}{2}-1} \binom{k}{p} + \binom{k}{\frac{k}{2}}$

Thus $\displaystyle\sum_{p=1}^{\frac{k}{2}-1} \binom{k}{p} = 2^{k-1} - \frac{1}{2} \binom{k}{\frac{k}{2}} \qquad (6)$

and, hence, combining (5) and (6), we have

$$(4) = \frac{1}{2^{k-1}} \left[2^{k-1} - \frac{1}{2} \binom{k}{\frac{k}{2}} \right] + \frac{1}{2^k} \binom{k}{\frac{k}{2}}$$

$$= 1 .$$

Thus all the equations are satisfied.

Thus $(1) = \int_{b_1}^{b_0} [-1- \sum_{k=0}^{n-1} c_k x^k]dx + \int_{b_1}^{b_2} [-1- \sum_{k=0}^{n-1} c_k x^k]dx$

$+\ldots+ \int_{b_{\frac{n+1}{2}}}^{b_{\frac{n-1}{2}}} [-1- \sum_{k=0}^{n-1} c_k x^k]dx + \int_{b_{\frac{n+1}{2}}}^{b_{\frac{n+3}{2}}} [1- \sum_{k=0}^{n-1} c_k x^k]dx$

$+\ldots+ \int_{b_n}^{b_{n+1}} [1- \sum_{k=0}^{n-1} c_k x^k]dx$

In $[-2,0)$ the integral is $- x - \sum_{k=0}^{n-1} \frac{c_k}{k+1} x^{k+1}$,

and in $[0,2]$ the integral is $x - \sum_{k=0}^{n-1} \frac{c_k}{k+1} x^{k+1}$.

Thus $(1) = [- x - \sum_{k=0}^{n-1} \frac{c_k}{k+1} x^{k+1}]_{b_1}^{b_0} + [-x- \sum_{k=0}^{n-1} \frac{c_k}{k+1} x^{k+1}]_{b_1}^{b_2}$

$+\ldots+ [-x- \sum_{k=0}^{n-1} \frac{c_k}{k+1} x^{k+1}]_{b_{\frac{n+1}{2}}}^{b_{\frac{n-1}{2}}} + [x- \sum_{k=0}^{n-1} \frac{c_k}{k+1} x^{k+1}]_{b_{\frac{n+1}{2}}}^{b_{\frac{n+3}{2}}}$

$+\ldots+ [x- \sum_{k=0}^{n-1} \frac{c_k}{k+1} x^{k+1}]_{b_n}^{b_{n+1}}$

$$= [-b_0 + 2(b_1 - b_2 + \dots - b_{\frac{n-1}{2}}) + b_{\frac{n+1}{2}} - b_{\frac{n+1}{2}} + 2(b_{\frac{n+3}{2}} - \dots - b_n) + b_{n+1}]$$

$$- \sum_{k=0}^{n-1} \frac{c_k}{k+1} [b_0^{k+1} - 2(b_1^{k+1} - b_2^{k+1} + \dots + b_n^{k+1}) + b_{n+1}^{k+1}] ,$$

and all terms but the first are zero, since the b's are solutions to the system (3).

Thus (1) $= -b_0 + 2(b_1 - b_2 + \dots - b_{\frac{n-1}{2}} + b_{\frac{n+3}{2}} - , \dots - b_n) + b_{n+1}$

$$= 4 + 4(b_1 - b_2 + \dots - b_{\frac{n-1}{2}}) \text{ by symmetry}$$

$$= 4 + 4\left(2\cos\left(\frac{n}{n+1}\right)\pi - 2\cos\left(\frac{n-1}{n+1}\right)\pi + \dots - 2\cos\left(\frac{\frac{n+3}{2}}{n+1}\right)\pi\right)$$

$$= 4 - 8\left(\cos\frac{\pi}{n+1} - \cos\frac{2\pi}{n+1} + \dots - \cos\frac{(n-1)\pi}{2(n+1)}\right)$$

$$= 4 - 8 \operatorname{Re} \sum_{k=1}^{\frac{n-1}{2}} (-1)^{k+1} e^{\frac{ik\pi}{n+1}}$$

$$= 4 - 8\left[\frac{1}{2} - \frac{\cos\frac{n\pi}{2(n+1)}}{2\cos\frac{\pi}{2(n+1)}}\right]$$

$$= 4\,\frac{\cos\frac{n\pi}{2n+2}}{\cos\frac{\pi}{2n+2}} = 4\tan\frac{\pi}{2n+2}$$

$$= \frac{4\pi}{2n+2} + \frac{4\pi^3}{8(n+1)^3}\,\frac{1}{3!}\,[\,2\sec^4\theta + 4\sec^2\theta\tan^2\theta\,]$$

where $0 \leq \theta \leq \frac{\pi}{2n+2}$.

Since n can be as small as 1, θ may be as large as $\pi/4$.

Thus $2 \sec^4\theta + 4 \sec^2\theta \tan^2\theta \leq 16$

Therefore $(1) \leq \frac{2\pi}{n+1}\left[1 + \frac{\pi^2}{6}\right]$

$\leq \frac{6\pi}{n+1}$.

If $n = 4m+3$, the above analysis yields the same result, (1) $\leq \frac{6\pi}{n+1}$.

Now let n be even, $n = 2m$. Then we can certainly approximate to within $\frac{6\pi}{(2m-1)+1} = \frac{6\pi}{2m} = \frac{6\pi}{n}$

Therefore $(1) \leq \frac{6\pi}{n}$, and the proof is complete.

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