ABSTRACT

Completeness Theorems for Special Classes of Trigonometric Polynomials

by

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In this paper, we consider the completeness of some special classes of trigonometric polynomials. These classes are special in that some conditions are placed on the ratio of the coefficient of sin nx to the coefficient of cos nx. To compensate for this restriction, we have to shorten the interval of completeness and restrict the space of functions in which we have completeness.

The main theorems proven in this paper are:

1) Let λ be a non-zero real number and let $p > 1$. Then $\{\cos nx + \lambda \sin nx\}_{n=1}^{\infty}$ is complete in $L^p[0,\pi]$ iff $p \leq \frac{1}{1 - |\beta|}$ where $\beta = \frac{2 \arctan \lambda}{\pi}$.

2) Let λ be a real number. Then $\{\cos nx + \lambda \sin nx\}_{n=0}^{\infty}$ is complete in $C[0,\pi]$.

3) $\{\cos nx + \lambda_n \sin nx\}_{n=1}^{\infty}$ is complete in $L^2[0,\pi]$ if $|\lambda_n| > 1$.

4) There exist $\{\lambda_n\}$ such that $|\lambda_n| < 1$ and $\{\cos nx + \lambda_n \sin nx\}_{n=1}^{\infty}$ is incomplete in $L^2[0,\pi]$.

5) $\{\cos nx + \lambda_n \sin nx\}_{n=0}^{\infty}$ is complete in $C[0,\pi]$ if $|\lambda_n| < 1$.

6) There exist $\{\lambda_n\}$ such that $|\lambda_n| > 1$ and $\{\cos nx + \lambda_n \sin nx\}_{n=0}^{\infty}$ is incomplete in $C[0,\pi]$.

7) Let $P(z)$ and $Q(z)$ be algebraic polynomials and let $P_E(z)$ be the even part of $P(z)$ and $P_o(z)$ the odd part. Then, if $P_E(z)Q_E(z) - P_o(z)Q_o(z) \not\equiv 0$, $\{P(n)\cos nx + Q(n) \sin nx\}_{n=0}^{\infty}$ is complete in $C[-a,a]$ for all $a < \pi$.

8) If $P_E(z)Q_E(z) - P_O(z)Q_O(z) \equiv 0$, then $\{P(n)\cos nx + Q(n)\sin nx\}_{n=0}^{\infty}$ is incomplete in $L^1[-\epsilon,\epsilon]$ for any $\epsilon > 0$.

9) For each $n = 0, 1, 2, \ldots$ let $f_n(x)$ be either e^{inx} or e^{-inx}. Then $\{f_n(x)\}_{n=0}^{\infty}$ is complete in $C[-\frac{\pi}{2}, \frac{\pi}{2}]$.

10) $\{e^{i(-1)^n nx}\}_{n=0}^{\infty}$ is incomplete in $L^2[-\frac{\pi}{2} - \epsilon, \frac{\pi}{2} + \epsilon]$ for any $\epsilon > 0$.

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I - INTRODUCTION AND BACKGROUND

The classic theorem in trigonometric polynomial approximation is that of Weierstrass: Any continuous function with period 2π can be uniformly approximated by trigonometric polynomials. Since both the trigonometric polynomials and the functions have period 2π, all that we really have to consider is approximation on $[-\pi,\pi]$. In this paper, we will restrict the interval to have length less than 2π and we will see whether, by not demanding approximation on as large an interval, we can demand some restriction on the form of the trigonometric polynomials. As an example, in chapter two, we consider the interval $[0,\pi]$ and restrict the polynomials to be linear combinations of $\{\cos n x + \lambda \sin nx\}_{n=0}^{\infty}$ where λ is a fixed real number. As might be expected, Fourier analysis plays a big role in some of the proofs. What is surprising, is the great amount of complex variables used, especially H^P spaces and growth theorems for entire functions. Some of the other subfields of analysis used are functional analysis and differential equations.

<u>DEFINITION</u> A set $\{V_n\}_{n=0}^{\infty}$ of vectors in a normed vector space N is said to be <u>complete</u> in N if any vector $X \in N$ can be approximated to any degree of accuracy by linear combinations of the V_n (where, of course, accuracy of approximation is measured by the norm of N).

One way to prove completeness is to find an approximating linear combination for an arbitrary vector $X \in N$. The next theorem, while not showing how the approximating linear combination vectors can be chosen, proves the existence of such vectors.

<u>THEOREM 1.1 : A set of vectors $\{V_n\}_{n=0}^{\infty}$ in a normed vector space N is complete N iff the only bounded linear functional L(x) on N for which $L(V_n) = 0 \quad n = 0,1,2,\ldots$ is the identically zero linear functional.</u>

Proof : See [1, P.49].

In this paper, the normed vector spaces we deal with will be Banach spaces of functions and because of the Riesz Representation Theorem, the bounded linear functionals have a special form.

NOTATION: 1) By $L^P[a,b]$, where $1 \le P < \infty$, we mean all measurable functions $f(x)$ on $[a,b]$ such that

$\int_a^b |f(x)|^P \, dx < \infty$. If $f(x) \in L^P[a,b]$ then by $\|f\|_p$ we mean $(\int_a^b |f(x)|^P \, dx)^{1/P}$

2) By $L^\infty[a,b]$ we mean all bounded measurable functions on $[a,b]$. We, of course, identify functions which are equal almost everywhere.

3) By $C[a,b]$ we mean all continuous functions on $[a,b]$. If $f(x) \in C[a,b]$ then by $\|f\|$ we mean $\max_{x \in [a,b]} |f(x)|$.

It is well known [5,P.6], that $L^P[a,b]$, $1 \le P < \infty$ and $C[a,b]$ are Banach spaces with the described norms.

The Riesz Representation Theorem provides a convenient means of characterizing the bounded linear functionals on $L^P[a,b]$, $1 \le P < \infty$ and on $C[a,b]$.

RIESZ REPRESENTATION THEOREM:

1) $L(f)$ is a bounded linear functional on $C[a,b]$ iff there exists a finite measure $d\mu(x)$ such that $\int_a^b f(x)d\mu(x) = L(f)$ for all $f \in C[a,b]$. (If we are dealing with the space of functions in $C[-\pi,\pi]$ with period 2π, then $d\mu(x)$ must be periodic also: i.e., if $\mu(x)$ has a point mass at $-\pi$ and π these masses must be the same.)

2) Let $1 < P < \infty$. Then $L(f)$ is a bounded linear functional on $L^P[a,b]$ iff there exists a function $g(x) \in L^q[a,b]$ where $\frac{1}{P} + \frac{1}{q} = 1$,

such that $L(f) = \int_a^b f(x)g(x)\,dx$ for all $f \in L^P[a,b]$

3) $L(f)$ is a bounded linear functional on $L^1[a,b]$ iff there exists a function $g(x) \in L^\infty[a,b]$ such that $L(f)=\int_a^b f(x)g(x)\,dx$ for all $f \in L^1[a,b]$.

Proof: See [5]

NOTATION: If $\int_a^b f_n(x)g(x)dx = 0$ (or $\int_a^b f_n(x)d\mu(x) = 0$) $n = 0,1,2,\ldots$ then we say that $g(x)$ (or $d\mu(x)$) is orthogonal to $\{f_n\}_{n=0}^\infty$ on $[a,b]$. It is written as $g \perp f_n$ $(d\mu \perp f_n)$ $n = 0,1,2,\ldots$

Weierstrass' theorem can now be stated in two equivalent forms.

1) $\{e^{inx}\}_{n=-\infty}^\infty$ or $\{\cos nx, \sin nx\}_{n=0}^\infty$ is complete among the periodic functions in $C[-\pi,\pi]$.

2) If $\int_{-\pi}^{\pi} e^{inx}\,d\mu(x) = 0 \quad n = 0, \pm 1, \pm 2, \ldots$

or $\int_{-\pi}^{\pi} \cos nx\,d\mu(x) = \int_{-\pi}^{\pi} \sin nx\,d\mu(x) = 0$ for $n = 0,1,2,\ldots$ and $d\mu(x)$ is a periodic measure then $d\mu(x) \equiv 0$.

One of the most useful tools in analysis is the following [5,P.3].

Holder's Inequality: If $f \in L^P[a,b]$ & $g \in L^q[a,b]$ where $\frac{1}{P} + \frac{1}{q} = 1$ then $f\,g \in L^1[a,b]$ and

$$\int_a^b |f(x)g(x)|\,dx \le \left(\int_a^b |f(x)|^P\,dx\right)^{\frac{1}{P}} \left(\int_a^b |g(x)|^q\,dx\right)^{\frac{1}{q}}$$

Letting $g(x) \equiv 1$ we get the well-known result that if $f \in L^{P_1}[a,b]$ $P_1 > 1$ then $f \in L^{P_2}[a,b]$, $1 \le P_2 < P_1$.

In the completeness theory, there is a certain "hierarchy" among the various $L^P[a,b]$ spaces and $C[a,b]$.

THEOREM 1.2: Let $\{f_n(x)\}_{n=0}^\infty$ be a sequence of continuous functions on $[a,b]$. Then

1) If $\{f_n(x)\}_{n=0}^\infty$ is complete in $C[a,b]$, it is complete in $L^P[a,b]$ $\quad 1 \le P < \infty$

2) <u>If $\{f_n(x)\}_{n=0}^{\infty}$ is complete in $L^{P_1}[a,b]$, $1 < P_1 \leq \infty$, it is complete in $L^{P_2}[a,b]$ $1 \leq P_2 \leq P_1$</u>.

3) <u>If $\{f_n(x)\}_{n=0}^{\infty}$ is incomplete in $L^{P_1}[a,b]$ $1 \leq P_1 \leq \infty$ it is incomplete in $L^{P_2}[a,b]$ for $P_1 \leq P_2 < \infty$ and $C[a,b]$</u>.

<u>Proof</u>: 1) Assume $\{f_n(x)\}_{n=0}^{\infty}$ is incomplete in $L^P[a,b]$ for some P such that $1 \leq P < \infty$. Then there exists a $g(x)$ such that

$$\int_a^b f_n(x)g(x)dx = 0 \quad n = 0,1,2,\ldots$$

where $g(x) \in L^{\infty}[a,b]$ if $P = 1$ and $g(x) \in L^q[a,b]$ where $\frac{1}{P} + \frac{1}{q} = 1$ if $1 < P < \infty$.

In either case let $d\mu(x) = g(x)\,dx$.

Then $d\mu(x)$ is a finite measure on $[a,b]$ and

$$\int_a^b f_n(x)\,d\mu(x) = 0 \quad n = 0,1,2,\ldots$$

Since $\{f_n(x)\}_{n=0}^{\infty}$ is complete in $C[a,b]$, $d\mu(x) \equiv 0$.

Hence $g(x) = 0$ a.e. and we have completeness in $L^P[a,b]$ $1 \leq P < \infty$.

2) Assume $\{f_n(x)\}_{n=0}^{\infty}$ is incomplete in $L^{P_2}[a,b]$ for some P_2 such that $1 \leq P_2 < P_1$. Then there exists a $g(x)$ such that

$$\int_a^b f_n(x)g(x)dx = 0 \quad n = 0,1,2,\ldots$$

and $g(x) \in L^{\infty}[a,b]$ if $P_2 = 1$ or $g(x) \in L^{q_2}[a,b]$ where $\frac{1}{P_2} + \frac{1}{q_2} = 1$ if $P_2 > 1$

In either case $g(x) \in L^{q_1}[a,b]$ where $\frac{1}{P_1} + \frac{1}{q_1} = 1$

Since $\{f_n(x)\}_{n=0}^{\infty}$ is complete in $L^{P_1}[a,b]$ and since $\int_a^b f_n(x)g(x)\,dx = 0$ $n = 0,1,2,\ldots$

$g(x) = 0$ a.e. Hence we have completeness in $L^{P_2}[a,b]$ for $1 \leq P_2 < P_1$

3) This is just the converse of parts 1) and 2).

We are now ready to prove some of the completeness theorems which are similar to or based on the Weierstrass theorem.

THEOREM 1.3: $\{e^{inx}\}_{n=-\infty}^{\infty}$ (or $\{\cos n x, \sin n x\}_{n=0}^{\infty}$) is complete in $L^P[-\pi,\pi]$ $1 \le P < \infty$.

Proof: This is just a simple corollary to the theorem that the Cesaro average of the Fourier series of an $L^P[-\pi,\pi]$ function, $1 \le P < \infty$ converges to the function in the $L^P[-\pi,\pi]$ norm, (see [5, P.17]).

THEOREM 1.4: $\{e^{i n x}\}_{n=-\infty}^{\infty}$ is complete in $C[-a,a]$ if $0 \le a < \pi$.

Proof: Take any $f(x) \in C[-a,a]$. Extend $f(x)$ to $[-\pi,\pi]$ such that $f(x)$ is continuous on $[-\pi,\pi]$ and $f(-\pi) = f(\pi)$. This can obviously be done. Then, by the Weierstrass theorem, $f(x)$ can be uniformly approximated by trigonometric polynomials on $[-\pi,\pi]$ and hence surely on $[-a,a]$.

THEOREM 1.5: $\{\cos n x\}_{n=0}^{\infty}$ is complete in $C[0,\pi]$(and hence in $L^P[0,\pi]$ $1 \le P < \infty$).

Proof: Let $f(x)$ be continuous on $[0,\pi]$. Extend $f(x)$ evenly to $[-\pi,0]$. i.e., $f(-x) = f(x)$. Then $f(x)$ is periodic (since $f(-\pi)=f(\pi)$) and continuous. By the Weierstrass theorem, given $\epsilon > 0$, there exists a linear combination of cosines and sines within ϵ of $f(x)$ on $[-\pi,\pi]$. i.e., there exists $C(x)$ and $S(x)$ such that $|F(x) - C(x) - S(x)| < \epsilon$ for $x \in [-\pi,\pi]$ where $C(x) = \sum_{n=0}^{N} a_n \cos n x$ and $S(x) = \sum_{n=1}^{N} b_n \sin n x$

For $x \in [-\pi,0]$ we can write $-x$ where $x \in [0,\pi]$. Therefore $|f(-x) - C(-x) - S(-x)| < \epsilon$ for $x \in [0,\pi]$. However, since $f(x)$ and $C(x)$ are even while $S(x)$ is odd we have that $|f(x) - C(x) + S(x)| < \epsilon$ for $x \in [0,\pi]$. Combining this with $|f(x) - C(x) - S(x)| < \epsilon$ for $x \in [0,\pi]$ gives us that $|S(x)| < \epsilon$ for $x \in [0,\pi]$. Hence, when approximating $f(x)$, we can forget about $S(x)$ and get an arbitrarily small error.

i.e., $|f(x) - C(x)| < 2\,\epsilon$. $x \in [0,\pi]$.

Since $f(x)$ and ϵ were arbitrary we have completeness.

THEOREM 1.6: $\{\cos n x\}_{n=1}^{\infty}$ is incomplete in $L^1[0,\pi]$ (and hence in $L^P[0,\pi]$, $1 \le P < \infty$ and in $C[0,\pi]$)

Proof: To prove incompleteness in $L^1[0,\pi]$ it suffices to find a non-trivial function $f(x) \in L^{\infty}[0,\pi]$ such that $\int_0^{\pi} \cos n x\, f(x)\, dx = 0$ $n = 1,2,\ldots$ Let $f(x) \equiv 1$ and we are done.

THEOREM 1.7: $\{\sin n x\}_{n=1}^{\infty}$ is 1) complete in $L^P[0,\pi]$ and 2) incomplete in $C[0,\pi]$.

Proof: 2) The incompleteness in $C[0,\pi]$ comes from the fact that, $\sin nx$ vanishes at $x = 0$ and $x = \pi$ and any linear combination of them does also. Therefore we cannot approximate any function which does not vanish at 0 and π.

1) In approximating a function in $L^P[0,\pi]$ the value of the function at any point does not matter and the above proof does not go through. By the hierarchy theorem it will suffice to prove completeness in $L^P[0,\pi]$ $1 < P < \infty$. Assume $\{\sin n x\}_{n=1}^{\infty}$ is incomplete in $L^P[0,\pi]$ for some P. Then, there exists a function $f(x) \in L^q[0,\pi]$ where $\frac{1}{P} + \frac{1}{q} = 1$ such that $\int_0^{\pi} f(x) \sin n x\, dx = 0$ $n = 1,2,3,\ldots$ Extend $f(x)$ to $(-\pi,0)$ by defining it to be odd. i.e., $f(-x) = -f(x)$. Then $f(x) \in L^q[-\pi,\pi]$ and

$$\int_{-\pi}^{\pi} f(x) \sin n x\, dx = 2\int_0^{\pi} f(x) \sin n x\, dx = 0 \quad n = 1,2,\ldots$$

As $f(x)$ is an odd function, $\int_{-\pi}^{\pi} f(x) \cos n x\, dx = 0$ $n = 0,1,2,\ldots$ Hence all the Fourier coefficients of $f(x)$ as well as its Cesaro average are identically zero. As the Cesaro average of $f(x)$ converges to $f(x)$ in the $L^q[-\pi,\pi]$ norm, $\int_{-\pi}^{\pi} |f(x)|^q\, dx = 0$. Therefore $f(x) = 0$ a.e.

By comparing Theorem 1.5 with Theorem 1.6, we see that the addition of the constant can change an $L^1[0,\pi]$ incomplete sequence into a $C[0,\pi]$ complete sequence. One might be tempted to say that since $\{\sin n x\}_{n=1}^{\infty}$ is $L^P[0,\pi]$ complete for $1 \le P < \infty$, then $\{1, \sin n x\}_{n=1}^{\infty}$ is $C[0,\pi]$ complete. However, this is not the case.

THEOREM 1.8: $\{1, \sin n x\}_{n=1}^{\infty}$ is incomplete in $C[0,\pi]$.

Proof: The constant function and all the sines, individually, have the same value at $x = 0$ and $x = \pi$. Hence any linear combination of them would have the same property and we could not approximate any function $f(x)$ such that $f(\pi) \neq f(0)$.

In Theorem 1.5 we saw that $\{\cos n x\}_{n=0}^{\infty}$ is complete in $C[0,\pi]$. The next theorem will prove that if the interval is shifted ever so slightly, we end up with an $L^1[-\epsilon,\epsilon]$ incomplete (and certainly $C[-\epsilon,\epsilon]$ incomplete) sequence.

THEOREM 1.9: Let $\epsilon > 0$. Then $\{\cos n x\}_{n=0}^{\infty}$ is incomplete in $L^1[-\epsilon,\epsilon]$.

Proof: Let $f(x)$ be any odd function in $C[-\epsilon,\epsilon]$. Then $f(x)$ is certainly in $L^{\infty}[-\epsilon,\epsilon]$. As $f(x)$ is odd and $\cos n x$ is even,

$\int_{-\epsilon}^{\epsilon} \cos n x \, f(x) \, dx = 0 \quad n = 0,1,2,\ldots$

NOTE: To prove $\{\sin n x\}_{n=1}^{\infty}$ is incomplete in $L^1[-\epsilon,\epsilon]$ we could take $f(x)$ to be even. If we wanted to prove that $\{1, \sin n x\}_{n=1}^{\infty}$ is $L'[-\epsilon,\epsilon]$ incomplete we would take any even continuous $f(x)$ with the additional property that $\int_0^{\epsilon} f(x) \, dx = 0$.

Thus, we see that the completeness of $\{\cos n x\}_{n=0}^{\infty}$ is not merely dependent upon the length of the interval, but also on its location. However, with "pure" exponentials, this is not the case, as seen in the following theorem.

THEOREM 1.10: If $\{e^{i\lambda_n x}\}_{n=0}^{\infty}$ is complete in $C[a,b]$ (or $L^P[a,b]$) it's complete in $C[a-L, b-L]$ (or $L^P[a-L, b-L]$) for any real L.

Proof: Assume $\int_{a-L}^{b-L} e^{i\lambda_n x}\, d\mu(x) = 0 \quad n = 0,1,2,\ldots$

Make a change variable $t = x + L$ and let $d\mu_1(x) = d\mu(x - L)$.

Then, $0 = \int_{a-L}^{b-L} e^{i\lambda_n(t-L)}\, d\mu(t - L) = e^{-i\lambda_n L} \int_a^b e^{i\lambda_n t}\, d\mu_1(t)$

As $e^{-i\lambda_n L} \neq 0$, we have that

$$\int_a^b e^{i\lambda_n t}\, d\mu_1(t) = 0 \qquad n = 0,1,2,\ldots$$

By the completeness of $\{e^{i\lambda_n x}\}_{n=0}^{\infty}$ in $C[a,b]$ $d\mu_1(x) \equiv 0$ and hence $d\mu(x) \equiv 0$.

(For $L^P[a,b]$ we have instead of $d\mu, f(x)dx$).

The next theorem, while proving an important result, also demonstrates a frequently used and valuable technique.

THEOREM 1.11: $\{e^{inx}\}_{n=0}^{\infty}$ is complete in $C[a,b]$ if $b-a < 2\pi$

Proof: By the previous theorem, it suffices to prove completeness on $[-L,L]$ where $2L = b-a$ (and hence $L < \pi$).

Assume there exists $d\mu$ such that $\int_{-L}^{L} e^{inx}\, d\mu(x) = 0, \quad n = 0,1,2,\ldots$

Let $F(z) = \int_{-L}^{L} e^{izx}\, d\mu(x)$

$F(z)$ is an entire function and vanishes at all the non-negative integers.

$|F(z)| \leq \int_{-L}^{L} |e^{izx}|\, |d\mu(x)| \leq V e^{L|z|}$ where V is the total variation of $d\mu$ on $[-L,L]$.

We will now use the following theorem. [9, P.186]

THEOREM: If $F(z)$ is entire, vanishes at all the non-negative integers and satisfies $F(z) = O\, e^{(\pi - \epsilon)|z|}$ for some $\epsilon > 0$, then $F(z) \equiv 0$.

Therefore $\int_{-L}^{L} e^{izx}\, d\mu(x) \equiv 0$.

By setting $z = n$ we get that $d\mu \perp e^{inx} \quad n = 0, \pm 1, \pm 2, \ldots$

Since $\{e^{inx}\}_{n=-\infty}^{\infty}$ is complete in $C[-L,L]$, $d\mu(x) \equiv 0$.

II. Completeness of $\{\cos n x + \lambda \sin n x\}$ on $[0,\pi]$.

In this chapter we discuss the completeness of the sequence of functions $f_n(x) = \cos nx + \lambda \sin nx$ where λ is a real number $\neq 0$. We consider the two sequences $\{f_n(x)\}$ $n = 1,2,\ldots$ and $\{f_n(x)\}$ $n=0,1,\ldots$ and prove results concerning their completeness in various $L^P[0,\pi]$ spaces and in $C[0,\pi]$.

In our discussion we can always exclude the case of $\lambda=0$ since we've already proven that $\{\cos nx\}$ $n=1,2,\ldots$ is incomplete in $L^1[0,\pi]$ (and hence in $L^P[0,\pi]$, $P > 1$) and we've proven that $\{\cos nx\}$ $n=0,1,2\ldots$ is complete in $C[0,\pi]$.

In the discussion we use some theorems about H^P spaces. For completeness we define H^P spaces and state the theorems used.

NOTATION: 1) Let $F(z)$ be defined in the open unit disc. Then by $F_r(\theta)$ we mean $F(re^{i\theta})$ $0 \leq r < 1$

2) By $P_r(\theta)$ we mean the Poisson kernel $\sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} =$

$$= \frac{1-r^2}{1-2\,r\cos\theta + r^2}$$

3) Let $f(\theta)$ and $g(\theta)$ be in $L^1[-\pi,\pi]$. Then by $(f*g)(\theta)$ we mean the convolution of f and g; i.e., $(f*g)(\theta)= \frac{1}{2\pi}\int^{\pi} f(\theta-t)g(t)dt$. By use of Fubini's theorem, we get the well-known result that $f*g \in L^1[-\pi,\pi]$ if f and g are.

DEFINITION: A function $F(z)$, defined for z in the open unit disc, is said to be in the class H^P, $P > 0$, if $F(z)$ is analytic and $M_r = \int_{-\pi}^{\pi} |F_r(\theta)|^P d\theta$ is bounded for all r such that $0 \leq r < 1$.

THEOREM 2.1 Let $F(z)$ be in H^P where $P \geq 1$. Then, for almost all θ, the radial limit, $\lim_{r\to 1} F_r(\theta)$, exists. Moreover, if $\lim_{r\to 1} F_r(\theta)$ is called $\widetilde{F}(\theta)$, $\widetilde{F}(\theta) \in L^P[-\pi,\pi]$, and $F_r(\theta) = (\widetilde{F} * P_r)(\theta)$

Proof: See [5, PP. 38 and 51]

THEOREM 2.2 Let $F(z)$ be in H^P where $P \geq 1$. Then

$$\int_{-\pi}^{\pi} \tilde{F}(\theta)e^{i n \theta}\, d\theta = 0 \quad n = 1,2,3,$$

Proof: See [5, PP. 38 and 51]

THEOREM 2.3 Let $F(z)$ be in H^P where $P \geq 1$. Then $\tilde{F}(\theta)$ is neither real nor pure imaginary a.e. unless $F(z)$ is constant.

Proof: By Theorem 2.1 $F(re^{i\theta}) = (\tilde{F}*P_r)(\theta)$. Since $P_r(\theta)$ is real, $F(re^{i\theta})$ is real or pure imaginary as $\tilde{F}(\theta)$ is. As $F(re^{i\theta})$ is analytic, either case is clearly impossible unless $F(re^{i\theta})$ is constant.

THEOREM 2.4 Let $f(\theta)$ and $g(\theta)$ be in $L'[-\pi,\pi]$. Then the n-th Fourier coefficient of $(f*g)(\theta)$ is equal to the product of the n-th Fourier coefficient of $f(\theta)$ and the n-th Fourier coefficient of $g(\theta)$.

Proof: See [5, P.21].

THEOREM 2.5 Let $f(\theta)$ be in $L^P[-\pi,\pi]$ where $P \geq 1$. Then $\lim_{r \to 1} (f*P_r)(\theta) = f(\theta)$ a.e. in $[-\pi,\pi]$.

Proof: See [5, P.38]

THEOREM 2.6 Let $f(\theta)$ be in $L^P[-\pi,\pi]$ where $P \geq 1$. Then there exists a number M such that $\|f*P_r\| \leq M$ for r in $[0,1)$ where $\| \;\|$ is the $L^P[-\pi,\pi]$ norm.

Proof: See [5, P.32]

THEOREM 2.7 Let $F(z)$ be in H^P where $P > 1$. Then $F(z)$ is in H^1.

Proof: Since $F(z)$ is in H^P $F(z)$ is analytic. To prove $F(z)$ is in H^1 we must show that $\int_{-\pi}^{\pi} |F_r(\theta)|d\theta$ is bounded for all r in $[0,1)$.

By Holder's inequality

$$\int_{-\pi}^{\pi} |F_r(\theta)|d\theta \leq \left(\int_{-\pi}^{\pi}|F_r(\theta)|^P\, d\theta\right)^{\frac{1}{P}} (2\pi)^{\frac{P-1}{P}}$$ which is bounded for all r in $[0,1]$ since $F(z) \in H^P$.

LEMMA 1 - Let P be > 1 and let λ be real and non-zero. Then $\{\cos n x + \lambda \sin n x\}_{n=1}^{\infty}$ is incomplete in $L^{\frac{P}{P-1}}[0,\pi]$ if and only if there exists a non-trivial function $F(z)$ in H^P such that the Taylor series of $F(z)$ about 0 has real coefficients and $\operatorname{Re} \tilde{F}(\theta) = -\frac{1}{\lambda} \operatorname{Im} \tilde{F}(\theta)$ a.e. on $(0,\pi)$.

Proof: a) Assume there exists such an $F(z)$ in H^P, $P > 1$.

Let $F(z) = U(z) + i\, V(z)$ and let $\tilde{F}(\theta) = \tilde{U}(\theta) + i\, \tilde{V}(\theta)$ where, of course, $\tilde{U}(\theta) = \lim_{r \to 1} U_r(\theta)$ a.e. and $\tilde{V}(\theta) = \lim_{r \to 1} V_r(\theta)$ a.e.

Using Theorem 2.2 we get that

2.1) $\int_{-\pi}^{\pi} \tilde{F}(\theta)(\cos n\theta + i \sin n\theta)\, d\theta = 0 \qquad n = 1,2,3,\ldots$

Since $F(z)$ has real coefficients, $F(\overline{z}) = \overline{F}(z)$, or

$$U_r(-\theta) + i\, V_r(-\theta) = U_r(\theta) - i\, V_r(\theta)$$

Hence U_r is an even function of θ and V_r is odd. Taking limits as $r \to 1$ gives us that $\tilde{U}(\theta) = \tilde{U}(-\theta)$ and $\tilde{V}(\theta) = -\tilde{V}(-\theta)$. a.e. in $(-\pi,\pi)$.

Expressing 2.1) in terms of $\tilde{U}$ and $\tilde{V}$, we get

2.2) $$\int_{-\pi}^{\pi} [\tilde{U}(\theta) \cos n\theta - \tilde{V}(\theta) \sin n\theta]\, d\theta + i \int_{-\pi}^{\pi} [\tilde{U}(\theta) \sin n\theta + \tilde{V}(\theta) \cos n\theta]\, d\theta = 0 \quad n = 1,2,3,\ldots$$

By the evenness of $\tilde{U}(\theta)$ and the oddness of $\tilde{V}(\theta)$, 2.2) becomes

2.3) $\int_{0}^{\pi} [\tilde{U}(\theta) \cos n\theta - \tilde{V}(\theta) \sin n\theta]\, d\theta = 0 \quad n = 1,2,\ldots.$

But now, since we are given that $\tilde{V}(\theta) = -\lambda\, \tilde{U}(\theta)$ a.e. on $(0,\pi)$, 2.3) becomes

2.4) $\int_{0}^{\pi} \tilde{U}(\theta)(\cos n\theta + \lambda \sin n\theta)\, d\theta = 0 \quad n = 1,2,3,\ldots$

Hence, once we've proven that $\tilde{U}(\theta)$ is a non-trivial function in $L^P[0,\pi]$, we will have the result that $\{\cos n\theta + \lambda \sin n\theta\}_{n=1}^{\infty}$ is incomplete in $L^q[0,\pi]$, $\frac{1}{p} + \frac{1}{q} = 1$ or $q = \frac{p}{p-1}$. As $F(z)$ is in H^P, $\tilde{F}(\theta)$ is in $L^P[-\pi,\pi]$ by theorem 2.1.

Since $\int_{-\pi}^{\pi} |\tilde{U}(\theta)|^P d\theta \le \int_{-\pi}^{\pi} |\tilde{U}(\theta) + i\, \tilde{V}(\theta)|^P d\theta = \int_{-\pi}^{\pi} |\tilde{F}(\theta)|^P\, d\theta$, we get that $\tilde{U}(\theta)$ is in $L^P[-\pi,\pi]$ and certainly in $L^P[0,\pi]$. Thus all we have to prove is that $\tilde{U}(\theta) \not\equiv 0$ on $(0,\pi)$. If it were, then since $\tilde{U}(\theta) = \tilde{U}(-\theta)$ a.e. we would have that $\tilde{F}(\theta)$ is pure imaginary a.e. on $(-\pi,\pi)$ which by Theorem 2.3 implies that $F(z) \equiv C$ where C is pure imaginary or zero. As $F(z)$ has real coefficients we get that $F(z) \equiv 0$ which contradicts the hypothesis that $F(z)$ is non-trivial.

b) Assume $\{\cos n\,\theta + \lambda \sin n\,\theta\}_{n=1}^{\infty}$ is incomplete in $L^{\frac{P}{P-1}}[0,\pi]$ where $P > 1$: i.e., there exists a non-trivial function $f(\theta)$ such that $f(\theta)$ is in $L^P[0,\pi]$ and $\int_0^{\pi} f(\theta)(\cos n\theta + \lambda \sin n\theta)d\theta = 0$ $n = 1,2,3,\ldots$ Since Re $f(\theta)$ has these two properties, we can assume that $f(\theta)$ is real. Now define $f(\theta)$ to be 0 on $[-\pi,0)$. Then $f(\theta)$ is in $L^P[-\pi,\pi]$ and

2.5) $$\int_{-\pi}^{\pi} f(\theta)(\cos n\theta + \lambda \sin n\theta)d\theta = 0 \qquad n = 1,2,3,\ldots$$

As $P > 1$, $f(\theta)$ is integrable and has a corresponding Fourier series : i.e., $f(\theta) \sim \sum_{n=0}^{\infty} a_n \cos n\theta + b_n \sin n\theta$. From equation 2.5) $a_n + \lambda b_n = 0$. $n = 1,2,\ldots$ Therefore

2.6) $$f(\theta) \sim \sum_{n=0}^{\infty} a_n \left(\cos n\theta - \frac{1}{\lambda} \sin n\theta\right)$$

We will now define a function $F(z) = \sum_{n=0}^{\infty} a_n z^n$ and prove that this function has all the properties of our required function: i.e., $F(z)$ is in H^P, has real coefficients, Re $\tilde{F}(\theta) = -\frac{1}{\lambda}$ Im $\tilde{F}(\theta)$ a.e. on $(0,\pi)$ and is non-trivial. Since $f(\theta)$ is non-trivial, at least one of the a_n is not zero, and therefore $F(z)$ is non-trivial. As $f(\theta)$ is real all the a_n are real and hence $F(z)$ has real coefficients. The a_n, as Fourier coefficients, are bounded. Therefore $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for $|z| < 1$ and hence $F(z)$ is defined for all z in the open unit disc. We now consider the function $(f * P_r)(\theta)$ We can write the Fourier series of $f(\theta)$ as $\sum_{n=\infty}^{\infty} c_n e^{i\,n\theta}$ where, by equation 2.6)

$$C_n = \begin{cases} \frac{a_n}{2}\left(1 - \frac{1}{i\lambda}\right) & n > 0 \\ a_o & n = 0 \\ \frac{a_{-n}}{2}\left(1 + \frac{1}{i\lambda}\right) & n < 0 \end{cases}$$

As $P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{i n \theta}$, by Theorem 2.4

$(f*P_r)(\theta) \sim \sum_{n=-\infty}^{\infty} C_n\, r^{|n|} e^{i n \theta}$

Writing the Fourier series of $(f*P_r)(\theta)$ in terms of sines and cosines gives us that

$$(f*P_r)(\theta) \sim \sum_{n=0}^{\infty} a_n\, r^n \left(\cos n\,\theta - \frac{1}{\lambda} \sin n\,\theta\right) \text{ which is } = \operatorname{Re}\left(1 + \frac{i}{\lambda}\right) F_r(\theta)$$

By the uniqueness of Fourier series and continuity in θ of both functions for all r, $0 \le r < 1$, we get

$$(f*P_r)(\theta) = \operatorname{Re}\left(1 + \frac{i}{\lambda}\right) F_r(\theta) \text{ for all } \theta \text{ and all } r,\ 0 \le r < 1.$$

By Theorem 2.5, $\lim_{r\to 1} (f*P_r)(\theta) = f(\theta)$ a.e.

Therefore $\lim_{r\to 1} \operatorname{Re}\left(1 + \frac{i}{\lambda}\right) F_r(\theta) = f(\theta)$ a.e. or, if $F_r(\theta) = U_r(\theta) + iV_r(\theta)$ we have that

2.7) $$\lim_{r\to 1} \left(U_r(\theta) - \frac{1}{\lambda} V_r(\theta)\right) = f(\theta) \text{ a.e.}$$

Since $F(z)$ has real coefficients, $\overline{F(re^{i\theta})} = F(re^{-i\theta})$ or

$U_r(\theta) - iV_r(\theta) = U_r(-\theta) + iV_r(-\theta)$. Therefore, $U_r(\theta) = U_r(-\theta)$ and $V_r(\theta) = -V_r(-\theta)$.

Equation 2.7) can therefore be written as

2.8) $$\lim_{r\to 1} \left(U_r(\theta) + \frac{1}{\lambda} V_r(\theta)\right) = f(-\theta) \text{a.e.}$$

Adding equations 2.7) and 2.8), we get

2.9) $$2 \lim_{r\to 1} U_r(\theta) = f(\theta) + f(-\theta) \text{ a.e.}$$

Subtracting equation 2.7) from equation 2.8), we get

2.10) $$\frac{2}{\lambda} \lim_{r\to 1} V_r(\theta) = f(-\theta) - f(\theta)$$

Since $f(\theta) = 0$ for $-\pi < \theta < 0$, we have

$\lim_{r\to 1} U_r(\theta) = \frac{f(\theta)}{2}$ a.e. on $(0,\pi)$ and

$\lim_{r\to 1} V_r(\theta) = -\frac{\lambda}{2} f(\theta)$ a.e. on $(0,\pi)$.

Thus $F(z)$ has the right boundary values.

We just have to check that $F(z)$ is in H^P.

By Theorem 2.6 $\|f*P_r\| \le M$ for $0 \le r$ 1 where the norm is the $L^P[-\pi,\pi]$ norm.

Therefore, since $(f*P_r)(\theta) = U_r(\theta) - \frac{1}{\lambda} V_r(\theta) = U_r(-\theta) + \frac{1}{\lambda} V_r(-\theta)$

we have that $\left(\int_{-\pi}^{\pi} |U_r(\theta) - \frac{1}{\lambda} V_r(\theta)|^P d\theta\right)^{1/P} = \left(\int_{-\pi}^{\pi} |U_r(-\theta) + \frac{1}{\lambda} V_r(-\theta)|^P d\theta\right)^{\frac{1}{P}} \le M$

i.e. $\|U_r - \frac{1}{\lambda} V_r\| \le M$ and $\|U_r + \frac{1}{\lambda} V_r\| \le M$

By the triangle inequality, $\|U_r\| \le M$ and $\|V_r\| \le |\lambda|\, M$.

Hence $\|F_r\| = \|U_r + i\, V_r\| \le \|U_r\| + \|V_r\| \le M(1 + |\lambda|)$

Therefore $F(z)$ is in H^P.

<u>LEMMA 2: Let P be a real number and let $F(z) = (\frac{1+z}{1-z})^P$ where $F(0)$ is defined as 1. Then $F(z) \in H^q$, $q > 0$ iff $q\,|P| < 1$</u>

<u>Proof</u>: Without loss of generality we can assume $P > 0$ (otherwise we are considering $(\frac{1-z}{1+z})^{|P|}$ and the proofs are similar).

Assume $q\,P < 1$. $F(z)$ is obviously analytic for $|z| < 1$.

To show that $F(z) \in H^q$ we have to show that $M_r = \int_{-\pi}^{\pi} \left|\frac{1 + r\, e^{i\theta}}{1 - r\, e^{i\theta}}\right|^{Pq} d\,\theta$

is bounded for all $r \in [0,1)$. As $F(z)$ is continuous on the compact set $S = \{z : |z| \le \frac{1}{2}\}$, M_r is obviously bounded for $0 \le r \le \frac{1}{2}$. We, therefore, just have to consider M_r for $r \in (\frac{1}{2},1)$

$$M_r \le 2^{Pq} \int_{-\pi}^{\pi} \frac{1}{|1 - r\, e^{i\theta}|^{Pq}}\, d\theta$$

$$= 2^{Pq} \int_{-\pi}^{\pi} \frac{1}{(1 + r^2 - 2\, r \cos\theta)^{Pq/2}}\, d\,\theta$$

$$= 2^{Pq+1} \int_{0}^{\pi/2} \frac{d\,\theta}{(1 + r^2 - 2\, r \cos\theta)^{Pq/2}} + 2^{Pq+1} \int_{\pi/2}^{\pi} \frac{d\,\theta}{(1 + r^2 - 2r\cos\theta)^{pq/2}}$$

2.11) $\le 2^{Pq+1} \int_{0}^{\pi/2} \frac{d\,\theta}{(^2\,\theta^2\, r)^{Pq/2}} + 2^{Pq} \frac{\pi}{(\ \ \ ^2)^{Pq/2}}$

$$= 2^{Pq+1}\,\pi^{Pq/2}\,\frac{1}{2^{Pq/2}}\cdot\frac{1}{r^{Pq/2}}\quad\frac{(\pi/2)^{1-Pq}}{1-Pq}\quad+\frac{2^{Pq}\,\pi}{(1+r^2)^{Pq/2}}$$

$$\leq 2^{2Pq}\quad\frac{\pi^{1-Pq/2}}{1-Pq}+2^{Pq}\,\pi$$

In the inequality of line 2.11) we used $(1 + r^2 - 2r\cos\theta) \geq \frac{2\theta^2 r}{\pi}$

for $|\theta| \leq \frac{\pi}{2}$ and $r < 1$ which is proven in the appendix. Hence, $'M_r$ is bounded for $r \in (\frac{1}{2}, 1)$ and therefore for $r \in [0,1)$.

Now assume $F(z) \in H^q$ where $q\,P \geq 1$, i.e., there exists M such that

$$\int_{-\pi}^{\pi}\left|\frac{1 + r\,e^{i\,\theta}}{1 - r\,e^{i\,\theta}}\right|^{Pq} d\theta \leq M \text{ for all } r \in [0,1).$$

Let $G(z) = (\frac{1+z}{1-z})$. Since $F(z) \in H^q$, $G(z) \in H^{Pq}$. As $Pq \geq 1$, by Theorem 2.1, $\tilde{G}(\theta) \in L^{Pq}\,[-\pi,\pi]$ i.e., $\int_{-\pi}^{\pi}\left|\frac{1+e^{i\,\theta}}{1-e^{i\,\theta}}\right|^{Pq} d\,\theta < \infty$

$$\text{But } \int_{-\pi}^{\pi}\left|\frac{1+e^{i\,\theta}}{1-e^{i\,\theta}}\right|^{Pq} d\theta = \int_{-\pi}^{\pi}\left|\frac{\cos\,\theta/2}{\sin\,\theta/2}\right|^{Pq} d\theta \;\geq \int_{-\pi/2}^{\pi/2}\left|\frac{\cos\,\theta/2}{\sin\,\theta/2}\right|^{Pq} d\theta$$

$$\geq \int_{-\pi/2}^{\pi/2}\left|\frac{1}{\theta}\right|^{Pq} d\theta = \infty \quad \text{since } Pq \geq 1$$

In the last inequality we use $\left|\frac{\cos\,\theta/2}{\sin\,\theta/2}\right| \geq \frac{1}{|\theta|}$ for $|\theta| \leq \pi/2$

which is proven in the appendix. Thus $F(z) \notin H^q$ for $q\,P \geq 1$.

In the next lemma, we will use the following theorems.

Study's Theorem: If the function $F(z)$ transforms conformally the open unit disc $|z| < 1$ into a convex region, then every circle $|z|=r$ $r \in (0,1)$ is transformed into a convex region by $F(z)$.

Proof: See [7, P.224].

Carlson's Theorem: If C is a closed convex curve, lying in $|z| \leq 1$ and $F(z)$ is analytic in $|z| \leq 1$, then

$$\int_C |F(z)|\;|dz| \leq 2\int_{|z|=1} |F(z)|\;|dz|$$

Proof: See [3]

Let $\omega(z) = \dfrac{(1+z)^2 - i(1-z)^2}{(1+z)^2 + i(1-z)^2}$

$\omega(z)$ takes the upper half of the open unit disc schlichtly onto the open unit disc and takes the boundary into the boundary. As a schlicht function, it has an inverse function, $z(\omega)$.

$$z(\omega) = \frac{\sqrt{i\left(\frac{1+\omega}{1-\omega}\right)} - 1}{\sqrt{i\left(\frac{1+\omega}{1-\omega}\right)} + 1} \qquad \text{where } \sqrt{i\left(\tfrac{1+\omega}{1-\omega}\right)}\,\Big|_{\omega=0} = \frac{1+i}{\sqrt{2}}$$

$z(\omega)$ maps the open unit disc conformally onto the upper half of the open unit disc.

<u>LEMMA 3) Let $F(z)$ be in H^1. Then $\dfrac{F(z(\omega))}{\sqrt{1-\omega^2}} \in H^1$</u>

<u>Proof</u>: The function is obviously analytic (since $\sqrt{1-\omega^2}$ can be defined to be analytic in open unit disc). We must prove that

$M_r = \int_{|\omega|=r} \left|\dfrac{F(z(\omega))}{\sqrt{1-\omega^2}}\right| |d\,\omega|$ is bounded for all $r \in [0,1)$.

Let G_r be the image of $|\omega| = r$ under $z(\omega)$, $r \in [0,1)$

Since $z(\omega)$ takes the open unit disc onto a convex domain, G_r is convex by Study's theorem. By transforming the domain of integration from $|\omega| = r$ to z on G_r we get that

$$M_r = \int_{|\omega|=r} \left|\frac{F(z(\omega))}{\sqrt{1-\omega^2}}\right| |d\,\omega| = 4 \int_{G_r} \left|\frac{F(z)}{(1+z)^2 + i(1-z)^2}\right| |dz|$$

By the continuity of $z(\omega)$ and the compactness of $\{\omega : |\omega| = r\}$ there exists a disc of radius ρ which completely contains G_r and such that $\frac{1}{2} < \rho < 1$. Letting $g(z) = \dfrac{F(z\rho)}{(1+z\rho)^2 + i(1-z\rho)^2}$, we get that

$M_r = 4\,\rho \int_{\frac{G_r}{\rho}} |g(\xi)d\xi|$ where $\frac{G_r}{\rho}$ is G_r multiplied by $\frac{1}{\rho}$

$\frac{G_r}{\rho}$ is still convex and z on $\frac{G_r}{\rho}$ implies $|z| < 1$

$g(z)$ is analytic in $|z| \le 1$ and, by Carlson's theorem,

$$M_r = 4\,\rho \int_{\frac{G_r}{\rho}} |g(\xi)d\xi| \le 4\,\rho \int_{|\xi| = 1} |g(\xi)|\,|d\xi|$$

$$= 4\,\rho \int_{|\xi| = 1} \left| \frac{F(\xi\,\rho)}{(1 + \xi\,\rho)^2 + i\,(1 - \xi\,\rho)^2} \right| |d\xi|$$

$$= 4 \int_{|z| = \rho} \left| \frac{F(z)}{(1 + z)^2 + i\,(1 - z)^2} \right| |dz|$$

$$\le 16 \int_{|z| = \rho} |F(z)|\,|dz| \quad (\text{since } |(1 + z)^2 + i\,(1 - z)^2| \ge \tfrac{1}{4} \text{ for } \tfrac{1}{2} \le |z| \le 1$$

which is proven in the appendix).

Thus $M_r \le 16 \int_{|z| = \rho} |F(z)|\,|dz|$ which is bounded since $F(z) \in H^1$

LEMMA 4) Let $F(z)$ be in H^1 and have real coefficients. Let γ be such that $|\gamma| < \pi/2$ and $e^{-i\gamma}\tilde{F}(\theta)$ is real a.e. for $0 \le \theta \le \pi$. Then $F(z) = C\,(\frac{1+z}{1-z})^{2\gamma/\pi}$ where C is real.

Proof: Let $z(\omega)$ be as before, let $S_1 = \{z|\ |z| = 1,\ \text{im } z > 0\}$ and let $S_2 = \{z|\ |z| = 1\ \ \text{im } z < 0\}$. $z(\omega)$ maps S_1 onto itself and S_2 onto the real axis. Therefore $e^{-i\gamma} F(z(\omega))$ is real a.e. on S_1 and $F(z(\omega))$ is real a.e. on S_2, since $F(z)$ has real coefficients

Let $g(\omega) = (\frac{1-\omega}{1+\omega})^{\gamma/\pi}$. Then

$$\arg\ g(\omega) = \left\{ \begin{matrix} -\gamma/2 & \omega \in S_1 \\ \gamma/2 & \omega \in S_2 \end{matrix} \right\}$$

Therefore $e^{-i\gamma/2} g(\omega) F(z(\omega))$ is real a.e. on $|\omega| = 1$

However $e^{-i\gamma/2} g(\omega) F(z(\omega)) = e^{-i\gamma/2} (1-\omega)^{\gamma/\pi + 1/2} (1+\omega)^{\frac{1}{2} - \gamma/\pi} \dfrac{F(z(\omega))}{\sqrt{1 - \omega^2}}$

Since $|\gamma| < \pi/2$ $(1 - \omega)^{\gamma/\pi + \frac{1}{2}} (1+\omega)^{\frac{1}{2} - \gamma/\pi}$ is continuous for $|\omega| \le 1$ and hence its absolute value is bounded there. Combining this with $\dfrac{F(z(\omega))}{\sqrt{1-\omega^2}}$ being in H^1 gives us that $e^{-i\gamma/2} g(\omega) F(z(\omega))$ is in H^1. Since we have shown it to be real a.e. on the boundary, by Theorem 2.3

$$e^{-i\,\gamma/2}\, g(\omega)\, F(z(\omega)) = C \text{ where } C \text{ is real.}$$

i.e. $$F(z(\omega) = C\, e^{i\,\gamma/2}\, \left(\frac{1+\omega}{1-\omega}\right)^{\gamma/\pi}$$

Setting $\omega = \omega(z)$ which was defined before, gives us

$$F(z) = C\, \left(\frac{1+z}{1-z}\right)^{2\gamma/\pi}$$

<u>THEOREM 2.8 Let λ be a non-zero real number and let $P > 1$. Then $\{\cos n\,x + \lambda \sin n\,x\}_{n=1}^{\infty}$ is complete in $L^P[0,\pi]$, iff $P \le \frac{1}{1-|\beta|}$ where $\beta = \frac{2 \text{ arc tan } \lambda}{\pi}$</u>

<u>Proof</u>: a) Assume $\{\cos n\,x + \lambda \sin n\,x\}_{n=1}^{\infty}$ is incomplete in $L^P[0,\pi]$. By lemma 1, there exists a non-trivial $F(z) \in H^{\frac{P}{P-1}}$ with real coefficients and such that $\text{Re}\,\tilde{F}(\theta) = -\frac{1}{\lambda}\,\text{Im}\,\tilde{F}(\theta)$ a.e. for $0 \le \theta \le \pi$. Therefore $e^{-i \text{ arc tan } \lambda}\,\tilde{F}(\theta)$ is real a.e. for $0 \le \theta \le \pi$. Since $\frac{P}{P-1} > 1$, $F(z) \in H^1$ by Theorem 2.7. Observing that $F(z)$ now satisfies the hypothesis of lemma 4 with $\gamma = \text{arc tan } \lambda$, gives us that $F(z) = C\,\left(\frac{1+z}{1-z}\right)^{\frac{2 \text{ arc tan } \lambda}{\pi}} = C\,\left(\frac{1+z}{1-z}\right)^{\beta}$. Since $F(z) \in H^{\frac{P}{P-1}}$, by lemma 2 we get that

$$\frac{P}{P-1}\,|\beta| < 1 \text{ or that } P > \frac{1}{1-|\beta|}\,. \text{ Therefore } P \le \frac{1}{1-|\beta|} \text{ implies}$$

completeness.

b) Now take $P > \frac{1}{1-|\beta|}$, and we'll prove incompleteness.

By lemma 1, it will suffice to show that there exists a non-trivial $F(z) \in H^{\frac{P}{P-1}}$ with real coefficients and such that $\text{Re}\,\tilde{F}(\theta) = -\frac{1}{\lambda}\,\text{Im}\,\tilde{F}(\theta)$ a.e. on $(0,\pi)$.

Let $F(z) = \left(\frac{1-z}{1+z}\right)^{\beta}$ and such that $F(0) = 1$.

$F(z)$ is obviously analytic for $|z| < 1$ and $F(z)$ has real coefficients since, by continuity, it's real on the real axis. Take any θ, $0 < \theta < \pi$.

By continuity, $\lim_{r\to 1} F_r(\theta) = (\frac{1-e^{i\theta}}{1+e^{i\theta}})^{\beta} = (-i \tan \frac{\theta}{2})^{\beta} = e^{\frac{-\pi i \beta}{2}} (\tan \frac{\theta}{2})^{\beta}$

$$= [\cos (\arctan \lambda) - i \sin (\arctan \lambda)] (\tan \frac{\theta}{2})^{\beta}$$

$$= (\frac{1-\lambda i}{\sqrt{1+\lambda^2}}) (\tan \frac{\theta}{2})^{\beta}$$

Therefore Re $\tilde{F}(\theta) = -\frac{1}{\lambda}$ Im $\tilde{F}(\theta)$ on $(0,\pi)$

By lemma 2, to prove that $F(z) \in H^{\frac{P}{P-1}}$, it is only necessary to prove $\frac{P|\beta|}{P-1} < 1$. However, this is equivalent to $P > \frac{1}{1-|\beta|}$ which is given.

Hence, we have incompleteness in $L^P [0,\pi]$ for $P > \frac{1}{1-|\beta|}$.

<u>LEMMA 5) Let $f(x)$ be continuous on $[0,\pi]$ and let $\epsilon > 0$ be given. Then there exists a polygonal function $L(x)$, such that $|f(x)-L(x)| < \epsilon$ for all $x \in [0,\pi]$</u>

<u>Proof</u>: Since $f(x)$ is continuous on a compact set, it is uniformly continuous there. Let δ correspond to $\frac{\epsilon}{2}$ i.e. $|f(x)-f(y)| < \frac{\epsilon}{2}$ for $x, y \in [0,\pi]$ and $|x - y| < \delta$. As any smaller δ would also work we may assume $\delta = \frac{\pi}{n}$ for some integer n.

Let $x_K = K\delta \quad K = 0,1,2,\ldots,n$.

For $X \in [x_K, x_{K+1}]$ define $L(x)$ as

$$L(x) = \frac{f(x_{K+1})[x-x_K] + f(x_K)[x_{K+1}-x]}{x_{K+1} - x_K}$$

i.e. $(x,L(x))$ is on the line joining $(x_K,f(x_K))$ and $(x_{K+1},f(x_{K+1}))$

Since $L(x_K) = f(x_K)$ and $L(x_{K+1}) = f(x_{K+1})$, $L(x)$ is continuous and since it is piecewise linear, it is a polygonal function. Let x be any point in $[x_K, x_{K+1}]$.

$$|f(x) - L(x)| \le |f(x) - f(x_K)| + |f(x_K) - L(x_K)| + |L(x_K) - L(x)|$$

$$< \frac{\epsilon}{2} \qquad + 0 \qquad + \epsilon/2 \quad = \quad \epsilon$$

where $|L(x_K) - L(x)| < \frac{\epsilon}{2}$ since by the way $L(x)$ was defined

$$|L(x_K) - L(x)| \leq |L(x_K) - L(x_{K+1})|$$

$$= |f(x_K) - f(x_{K+1})| < \frac{\varepsilon}{2}$$

Hence $\sup_{x \in [0,\pi]} |f(x) - L(x)| < \varepsilon$

<u>THEOREM 2.9 $\{\cos n x + \lambda \sin n x\}_{n=0}^{\infty}$ where λ is real, is complete in $C[0,\pi]$.</u>

<u>Proof</u>: If $\lambda = 0$ it has been proven in the introduction [Theorem 1.5].

If $\lambda \neq 0$ set $\alpha = \frac{-1}{\lambda}$.

By Theorem 2.8 there exists a $P > 1$ such that $\{\cos n x + \alpha \sin n x\}_{n=1}^{\infty}$ is complete in $L^P[0,\pi]$. Let $f(x)$ be any continuous function and let $\varepsilon > 0$ be given. By lemma 5, there exists a polygonal function $L(x)$ such that $|f(x) - L(x)| < \varepsilon$ for all $x \in [0,\pi]$. $L'(x)$ exists except at a finite number of points and as a bounded step function, $L'(x) \in L^P[0,\pi]$. By the completeness of $\{\cos n x + \alpha \sin n x\}_{n=1}^{\infty}$ in $L^P[0,\pi]$ we have

$$\int_0^{\pi} |L'(x) - \sum_{n=1}^{N} a_n (\cos n x + \alpha \sin n x)|^P \, dx < \varepsilon^P$$

for some $\{a_n\}$ $n = 1,2,\ldots,N$.

Therefore,

$$2.12) \quad |\int_0^x [L'(t) - \sum_{n=1}^{N} a_n (\cos n t + \alpha \sin n t)] \, dt|$$

$$\leq \int_0^{\pi} |L'(t) - \sum_{n=1}^{N} a_n (\cos n t + \alpha \sin n t| \, dt$$

$$\leq (\int_0^{\pi} |L'(t) - \sum_{n=1}^{N} a_n (\cos n t + \alpha \sin n t|^P)^{\frac{1}{P}} (\int_0^{\pi} dx)^{\frac{P-1}{P}}$$

$$< \varepsilon \, \pi.$$

Evaluating the integral in line 2.12) we get

$$2.13) \quad |L(x) - L(0) - \sum_{n=1}^{N} \frac{a_n}{n} (\sin n x - \alpha \cos n x) - \sum_{n=1}^{N} \frac{\alpha \, a_n}{n}| < \varepsilon \, \pi$$

Let $b_o = L(0) - \frac{1}{\lambda} \sum_{n=1}^{N} \frac{a_n}{n}$ and $b_n = \frac{a_n}{n\lambda}$.

Therefore, line 2.13 becomes

2.14) $$|L(x) - \sum_{n=0}^{N} b_n (\cos n x + \lambda \sin n x| < \epsilon \pi$$

where the inequality holds for all $x \in [0,\pi]$.

Therefore,

$$|f(x) - \sum_{n=0}^{N} b_n (\cos n x + \lambda \sin n x)|$$

$$\leq |f(x) - L(x)| + |L(x) - \sum_{n=0}^{N} b_n (\cos n x + \lambda \sin n x)|$$

$$< \epsilon + \epsilon\pi = \epsilon (1 + \pi)$$

Hence, $\{\cos n x + \lambda \sin n x\}_{n=0}^{\infty}$ is complete in $C[0,\pi]$.

<u>COROLLARY: Let λ be a non-zero real number and let $P > 1$. Then there does not exist a non-trivial function $F(z) \in H^P$ such that $F(z)$ has real coefficients, $F(0) = 0$ and $\text{Re } \tilde{F}(\theta) = -\frac{1}{\lambda} \text{Im } \tilde{F}(\theta)$ a.e. on $(0,\pi)$.</u>

<u>Proof</u>: Assume for some non-zero real λ and some $P > 1$ there exists such a function $F(z) \in H^P$.

Let $\tilde{U}(\theta) = \text{Re } \tilde{F}(\theta)$.

Then, as proven in lemma 1, $\tilde{U}(\theta)$ is a non-trivial even function in $L^P[-\pi,\pi]$ such that

$$\int_0^{\pi} \tilde{U}(\theta)(\cos n \theta + \lambda \sin n \theta)d\theta = 0 \qquad n = 1,2,\ldots$$

Since $F(z) \in H^P$, $P > 1$, by Theorem 2.1

2.15) $$F(re^{i\theta}) = (F*P_r)(\theta)$$

Setting $r = 0$ in equation 2.15, gives us that

$$0 = F(0) = (F*P_o)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{F}(\theta)d\theta$$

2.16) Therefore $$\int_{-\pi}^{\pi} \tilde{U}(\theta)d\theta = 0$$

However, since $\tilde{U}(\theta) = \tilde{U}(-\theta)$, equation 2.16 becomes

$$\int_0^{\pi} \tilde{U}(\theta)d\theta = 0$$

Hence, $\widetilde{U}(\theta)$ is a non-trivial function in $L^P[0,\pi]$ such that

$$\int_0^\pi \widetilde{U}(\theta)(\cos n\,\theta + \lambda \sin n\,\theta)\,d\theta = 0 \qquad n = 0,1,2,\ldots$$

i.e. $\{\cos n\,\theta + \lambda \sin n\,\theta\}_{n=0}^\infty$ is incomplete in $L^{\frac{P}{P-1}}[0,\pi]$.

But in Theorem 2.9 we proved that

$\{\cos n\,\theta + \lambda \sin n\,\theta\}_{n=0}^\infty$ is complete in $C[0,\pi]$ and, hence certainly in $L^{\frac{P}{P-1}}[0,\pi]$.

Therefore, there does not exist such a function $F(z) \in H^P$.

Of course, we could have proven this corollary by observing that by Lemma 4, $F(z)$ would have to be of the form $c(\frac{1+z}{1-z})^\beta$ which is not zero at $z = 0$ (without letting c be zero).

III - Completeness of $\{\cos n x + \lambda_n \sin n x\}$ on $[0,\pi]$.

In this chapter we consider the completeness of $\{\cos n x + \lambda_n \sin n x\}$ and $\{\lambda_n \cos n x + \sin n x\}$ where the λ_n are constants. If $|\lambda_n| < 1$ for all n, then the dominant term in $\cos n x + \lambda_n \sin n x$ is $\cos n x$ and we might expect the results to be similar to the completeness of $\{\cos n x\}$. If $|\lambda_n| > 1$ for all n, then the dominant term is $\sin n x$ and we would expect the results to be similar to the completeness of $\{\sin n x\}$. In general, this turns our to be true. The corollary to Theorem 3.1 proves that $\{\cos nx + \lambda_n \sin nx\}_{n=1}^{\infty}$ is complete in $L^2[0,\pi]$ if $|\lambda_n| > 1$. Theorem 3.2 proves the existence of λ_n such that $|\lambda_n| < 1$ and $\{\cos nx + \lambda_n \sin nx\}_{n=1}^{\infty}$ is incomplete in $L^1[0,\pi]$. Theorem 3.3 proves that $\{\cos nx + \lambda_n \sin nx\}_{n=0}^{\infty}$ is complete in $C[0,\pi]$ if $|\lambda_n| < 1$. Finally, Theorems 3.4 and 3.5 will prove the existence of λ_n such that $|\lambda_n| > 1$ and $\{\cos nx + \lambda_n \sin nx\}_{n=0}^{\infty}$ is incomplete in $C[0,\pi]$.

LEMMA: Let $f(x)$ be in $L^2[0,\pi]$ and let $a_n = \frac{1}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$ and $b_n = \frac{1}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$. Then

$$\frac{1}{2\pi} \int_0^{\pi} |f(x)|^2 \, dx = \frac{1}{2} |a_0|^2 + \sum_{n=1}^{\infty} |a_n|^2 = \sum_{n=1}^{\infty} |b_n|^2$$

Proof: Let $g(x)$ be defined on $[-\pi,\pi]$ by extending $f(x)$ evenly. Let $h(x)$ be defined on $[-\pi,\pi]$ by extending $f(x)$ oddly. Then $g(x)$ and $h(x)$ are both in $L^2[-\pi,\pi]$.

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos n x\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos n x\,dx = 2a_n \qquad n = 0,1,2,\ldots$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin n x\,dx = 0 \qquad n = 0,1,2,\ldots$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\cos n x\,dx = 0 \qquad n = 0,1,2,\ldots$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\sin n x\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin n x\,dx = 2b_n \qquad n = 1,2,\ldots$$

Then, by using Parseval's identity [9, P.422] we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |g(x)|^2\,dx = 4\left[\tfrac{1}{2}|a_0|^2 + \sum_{n=1}^{\infty}|a_n|^2\right]$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |h(x)|^2\,dx = 4\sum_{n=1}^{\infty}|b_n|^2$$

However, since $\frac{1}{\pi}\int_{-\pi}^{\pi}|g(x)|^2\,dx = \frac{2}{\pi}\int_{0}^{\pi}|f(x)|^2\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}|h(x)|^2\,dx$

we have the desired result.

THEOREM 3.1 Let $|\lambda_n| < 1$ for $n = 1,2,\ldots$. Then $\{\lambda_n \cos n x + \sin n x\}_{n=1}^{\infty}$ is complete in $L^2[0,\pi]$.

Proof: Assume there exists $f(x) \in L^2[0,\pi]$ such that

3.1) $$\int_0^{\pi} f(x)\,(\lambda_n \cos n x + \sin n x)\,dx = 0 \qquad n = 1,2,\ldots$$

Let a_n and b_n be as in the lemma. Then, equation 3.1 becomes

$$\lambda_n a_n + b_n = 0 \qquad n = 1,2,\ldots$$

$$\tfrac{1}{2}|a_0|^2 + \sum_{n=1}^{\infty}|a_n|^2 = \sum_{n=1}^{\infty}|b_n|^2 = \sum_{n=1}^{\infty}|\lambda_n a_n|^2 < \sum_{n=1}^{\infty}|a_n|^2$$ which

is impossible unless

$$a_n = 0 \qquad n = 0,1,2,\ldots .$$

Since $\frac{1}{2\pi}\int_0^{\pi}|f(x)|^2\,dx = \frac{1}{2}|a_0|^2 + \sum_{n=1}^{\infty}|a_n|^2 = 0$,

we have that $f(x) = 0$ a.e.

Hence $\{\lambda_n \cos n x + \sin n x\}_{n=1}^{\infty}$ is complete in $L^2[0,\pi]$.

COROLLARY - Let $|\lambda_n| > 1 \quad n = 1,2,\ldots$. Then $\{\cos n x + \lambda_n \sin n x\}_{n=1}^{\infty}$ is complete in $L^2[0,\pi]$.

To compare with this corollary, we have the following.

THEOREM 3.2 - There exist λ_n such that $|\lambda_n| < 1$ for $n = 1,2,\ldots$ and $\{\cos n x + \lambda_n \sin n x\}_{n=1}^{\infty}$ is incomplete in $L^1[0,\pi]$.

Proof: Let λ_n be defined as

$$\lambda_n = - \frac{\int_0^{\pi} x \cos n x \, dx}{\int_0^{\pi} x \sin n x \, dx} \qquad n = 1,2,\ldots$$

Integrating by parts, we get that

$$\lambda_n = - \frac{\frac{(-1)^n - 1}{n^2}}{\frac{-(-1)^n \pi}{n}} = \frac{1 - (-1)^n}{n\pi}$$

Since $\frac{(-1)^n \pi}{n}$ is never zero, our definition of λ_n makes sense. Also, we have

$$|\lambda_n| = \left|\frac{1 - (-1)^n}{n\pi}\right| \le \frac{2}{n\pi} < 1$$

By the way λ_n was defined,

3.2) $$\int_0^{\pi} x \,(\cos n x + \lambda_n \sin n x)\, dx = 0 \qquad n = 1,2,\ldots$$

Since $f(x) = x$ is in $L^{\infty}[0,\pi]$, equation 3.2 tells us that $\{\cos n x + \lambda_n \sin n x\}_{n=1}^{\infty}$ is incomplete in $L^1[0,\pi]$.

LEMMA: If $\{- \lambda_n \cos n x + \sin n x\}_{n=1}^{\infty}$ is complete in $L^2[0,\pi]$, then $\{\cos n x + \lambda_n \sin n x\}_{n=0}^{\infty}$ is complete in $C[0,\pi]$.

Proof: Let $f(x)$ be in $C[0,\pi]$. As in lemma 5 of chapter 2, we can approximate $f(x)$ uniformly by a polygonal function $g(x)$

3.3) i.e. $|g(x) - f(x)| < \epsilon$ for all $x \in [0,\pi]$.

$g'(x)$, as a step function, is in $L^2\,[0,\pi]$.

Since $\{-\lambda_n \cos n x + \sin n x\}_{n=1}^{\infty}$ is $L^2[0,\pi]$ complete, there exist a_n such that

$$\int_0^{\pi} |g'(x) - \sum_{n=1}^{N} a_n (-\lambda_n \cos n x + \sin n x)|^2 \, dx < \epsilon^2$$

$$3.4) \qquad |g(x) - g(o) - [\sum_{n=1}^{N} a_n (-\frac{\lambda_n \sin n x}{n} - \frac{\cos n x}{n}) + \sum_{n=1}^{N} \frac{a_n}{n}]|$$

$$= |\int_0^x (g'(t) - \sum_{n=1}^{N} a_n (-\lambda_n \cos n t + \sin n t))\, dt|$$

$$\leq \int_0^x |g'(t) - \sum_{n=1}^{N} a_n (-\lambda_n \cos n t + \sin n t)|\, dt$$

$$\leq \int_0^{\pi} |g'(t) - \sum_{n=1}^{N} a_n (-\lambda_n \cos n t + \sin n t)|\, dt$$

$$\leq \int_0^{\pi} |g'(t) - \sum_{n=1}^{N} a_n (-\lambda_n \cos n t + \sin n t)|^2 dt]^{\frac{1}{2}} \sqrt{\pi}$$

$$\leq \epsilon\, \pi$$

Let $b_o = g(o) + \sum_{n=1}^{N} \frac{a_n}{n}$ and let $b_n = \frac{-a_n}{n}$ $n = 1,\ldots,N$

Hence, equation 3.4) becomes

$$|g(x) - \sum_{n=0}^{N} b_n (\cos n x + \lambda_n \sin n x)| \leq \epsilon\, \pi.$$

Combining this with equation 3.3), we get

$$|f(x) - \sum_{n=0}^{N} b_n (\cos n x + \lambda_n \sin n x)| \leq \epsilon\, (\pi + 1)$$

Since $f(x)$ was arbitrary, we have that

$\{\cos n x + \lambda_n \sin n x\}_{n=0}^{\infty}$ is complete in $C[0,\pi]$.

<u>THEOREM 3.3 Let $|\lambda_n| < 1$ for $n = 1,2,\ldots$. Then $\{\cos n x + \lambda_n \sin n x\}_{n=0}^{\infty}$ is complete in $C[0,\pi]$.</u>

<u>Proof</u>: By Theorem 3.1, $\{-\lambda_n \cos n x + \sin n x\}_{n=1}^{\infty}$ is complete in $L^2[0,\pi]$. By the lemma,

$\{\cos n x + \lambda_n \sin n x\}_{n=0}^{\infty}$ is complete in $C[0,\pi]$.

Comparing Theorem 3.2 with Theorem 3.3, we see that just the addition of the constant term, can change an $L^2[0,\pi]$ incomplete sequence into a $C[0,\pi]$ complete sequence. One might be tempted to conjecture that since $\{\cos n x + \lambda_n \sin n x\}_{n=1}^{\infty}$ is $L^2[0,\pi]$ complete if $|\lambda_n| > 1$, then $\{\cos n x + \lambda_n \sin n x\}_{n=0}^{\infty}$ is $C[0,\pi]$ complete if $|\lambda_n| > 1$. This is not the case, as the following theorems demonstrate

THEOREM 3.4 - Let λ_n be periodic with period 6, let $\lambda_5 = -\lambda_1$, $\lambda_2 = -\lambda_4$ and $\lambda_1\lambda_2 = -3$. Then $\{\cos n x + \lambda_n \sin n x\}_{n=0}^{\infty}$ is incomplete in $C[0,\pi]$.

NOTE: In this theorem, as well as in Theorem 3.5, we can choose λ_n to satisfy $|\lambda_n| > 1$ $n = 1,2,\ldots$.

Proof: Let $g_n(x) = \cos n x + \lambda_n \sin n x$.

We will prove that we can find non-trivial C_1, C_2, C_3 and C_4 such that

3.5) $$C_1 g_n(0) + C_2 g_n(\tfrac{\pi}{3}) + C_3 g_n(\tfrac{2\pi}{3}) + C_4 g_n(\pi) = 0 \quad n = 0,1,\ldots .$$

Once this is done we will have proven incompleteness since any linear combination of $\{g_n(x)\}_{n=0}^{\infty}$ will also have this property and hence we would be unable to approximate a function $f(x)$ such that

$$C_1 f(0) + C_2 f(\tfrac{\pi}{3}) + C_3 f(\tfrac{2\pi}{3}) + C_4 f(\pi) \neq 0$$

Since, obviously there are functions $f(x) \in C[0,\pi]$ with this property we will have proven incompleteness. In finding C_1, C_2, C_3 and C_4 we notice that since λ_n has period 6, $g_n(0)$, $g_n(\frac{\pi}{3})$, $g_n(\frac{2\pi}{3})$ and $g_n(\pi)$ all have period 6. Therefore, equation 3.5) has to be satisfied only for $n = 0,1,\ldots 5$ and all other values of n follow by periodicity. We therefore have the following 6 simultaneous equations to be satisfied non-trivially.

3.6) $n = 0$ $$C_1 + C_2 + C_3 + C_4 = 0$$

3.7) $n = 1$ $$C_1 + C_2(\tfrac{1}{2} + \lambda_1 \tfrac{\sqrt{3}}{2}) + C_3(-\tfrac{1}{2} + \lambda_1 \tfrac{\sqrt{3}}{2}) - C_4 = 0$$

3.8) $n = 2$ $$C_1 + C_2(-\tfrac{1}{2} + \lambda_n \tfrac{\sqrt{3}}{2}) + C_3(-\tfrac{1}{2} - \lambda_2 \tfrac{\sqrt{3}}{2}) + C_4 = 0$$

3.9) $n = 3$ $$C_1 + C_2(-1) + C_3 + C_4(-1) = 0$$

3.10) $\quad n = 4 \qquad C_1 + C_2\left(-\frac{1}{2} - \lambda_4\frac{\sqrt{3}}{2}\right) + C_3\left(-\frac{1}{2} + \lambda_4\frac{\sqrt{3}}{2}\right) + C_4 = 0$

3.11) $\quad n = 5 \qquad C_1 + C_2\left(\frac{1}{2} - \lambda_5\frac{\sqrt{3}}{2}\right) + C_3\left(-\frac{1}{2} - \lambda_5\frac{\sqrt{3}}{2}\right) - C_4 = 0$

By the conditions $\lambda_1 = -\lambda_5$ and $\lambda_2 = -\lambda_4$ equations 3.10) and 3.11) are the same as equations 3.8) and 3.7) respectively. If we set $C_1 = -C_3$ and $C_2 = -C_4$ then equations 3.6) and 3.9) are satisfied. Hence we are just left with equations 3.7) and 3.8). After we substitute for C_3 and C_4, these two equations become

3.12) $\qquad C_1\left(\frac{3}{2} - \lambda_1\frac{\sqrt{3}}{2}\right) + C_2\left(\frac{3}{2} + \lambda_1\frac{\sqrt{3}}{2}\right) = 0$

3.13) $\qquad C_1\left(\frac{3}{2} + \lambda_2\frac{\sqrt{3}}{2}\right) + C_2\left(-\frac{3}{2} + \lambda_2\frac{\sqrt{3}}{2}\right) = 0$

Equations 3.12) and 3.13) can be solved non-trivially iff

$$\begin{vmatrix} \frac{3}{2} - \lambda_1\frac{\sqrt{3}}{2} & \frac{3}{2} + \lambda_1\frac{\sqrt{3}}{2} \\ \frac{3}{2} + \lambda_2\frac{\sqrt{3}}{2} & -\frac{3}{2} + \lambda_2\frac{\sqrt{3}}{2} \end{vmatrix} = 0$$

This reduces to $\lambda_1\lambda_2 + 3 = 0$ which is given. Thus we can solve non-trivially for C_1, C_2, C_3 and C_4.

<u>THEOREM 3.5 - Let λ_n have period 8, let $\lambda_7 = -\lambda_1$, $\lambda_6 = -\lambda_2$, $\lambda_5 = -\lambda_3$ and let $\lambda_2(\lambda_1 + \lambda_3) = 2\sqrt{2}$. Then $\{\cos n x + \lambda_n \sin n x\}_{n=0}^{\infty}$ is incomplete in $C[0,\pi]$.</u>

<u>Proof</u>: As in Theorem 3.4 let $g_n(x) = \cos n x + \lambda_n \sin n x$. This time we wish to find non-trivial C_1, C_2, C_3, C_4 and C_5 such that

3.14) $\qquad C_1\, g_n(0) + C_2\, g_n(\frac{\pi}{4}) + C_3\, g_n(\frac{\pi}{2}) + C_4\, g_n(\frac{3\pi}{4}) + C_5\, g_n(\pi) = 0$

for $n = 0,1,\ldots$

Once this is done we will have proven incompleteness since we couldn't approximate an $f(x)$ such that

$$C_1\, f(0) + C_2\, f(\tfrac{\pi}{4}) + C_3\, f(\tfrac{\pi}{2}) + C_4\, f(\tfrac{3\pi}{4}) + C_5\, f(\pi) \neq 0.$$

Since λ_n has period 8, $g_n(0)$, $g_n(\frac{\pi}{4})$, $g_n(\frac{\pi}{2})$, $g_n(\frac{3\pi}{4})$ and $g_n(\pi)$ all have period 8 and we only have to check equation 3.14) for $n = 0,1,\ldots,7$. Since $\lambda_7 = -\lambda_1$, $\lambda_6 = -\lambda_2$ and $\lambda_5 = -\lambda_3$ the equations for $n = 7$ and $n = 1$ coincide as well as the equations for $n = 6$ and $n = 2$ and those of $n = 5$ and $n = 3$. We thus have to satisfy equation 3.14 for $n = 0,1,2,3,4$. We have the following five simultaneous equations.

3.15) $n = 0$ $$C_1 + C_2 + C_3 + C_4 + C_5 = 0$$

3.16) $n = 1$ $$C_1 + C_2\left(\frac{1}{\sqrt{2}} + \lambda_1 \frac{1}{\sqrt{2}}\right) + C_3(\lambda_1) + C_4\left(-\frac{1}{\sqrt{2}} + \frac{\lambda_1}{\sqrt{2}}\right) + C_5(-1) = 0$$

3.17) $n = 2$ $$C_1 + C_2(\lambda_2) + C_3(-1) + C_4(-\lambda_2) + C_5 = 0$$

3.18) $n = 3$ $$C_1 + C_2\left(-\frac{1}{\sqrt{2}} + \frac{\lambda_3}{\sqrt{2}}\right) + C_3(-\lambda_3) + C_4\left(\frac{1}{\sqrt{2}} + \frac{\lambda_3}{\sqrt{2}}\right) + C_5(-1) = 0$$

3.19) $n = 4$ $$C_1 - C_2 + C_3 - C_4 + C_5 = 0$$

If we set $C_1 + C_3 + C_5 = 0$ and $C_2 + C_4 = 0$ then equations 3.15) and 3.19) are satisfied. After we substitute for C_4 and C_5, equations 3.16), 3.17) and 3.18 become

3.20) $$C_1(2) + C_2(\sqrt{2}) + C_3(1 + \lambda_1) = 0$$

3.21) $$C_1(0) + C_2(2\lambda_2) + C_3(-2) = 0$$

3.22 $$C_1(2) + C_2(-\sqrt{2}) + C_3(1 - \lambda_3) = 0$$

These equations can be solved non-trivially for C_1, C_2 and C_3 iff

$$\begin{vmatrix} 2 & \sqrt{2} & 1 + \lambda_1 \\ 0 & 2\lambda_2 & -2 \\ 2 & -\sqrt{2} & 1 - \lambda_3 \end{vmatrix} = 0$$

This reduces to $\lambda_2(\lambda_1 + \lambda_3) + 2\sqrt{2} = 0$ which is given. Hence we can solve non-trivially for C_1, C_2, C_3, C_4 and C_5 and have established incompleteness.

In Theorem 3.3 we saw that if $|\lambda_n| < 1$ for $n = 1,2,\ldots$ then $\{\cos n x + \lambda_n \sin n x\}_{n=0}^{\infty}$ is $C[0,\pi]$ complete. As a proof of the delicateness of that theorem we have the following theorem.

THEOREM 3.6 - There exists $\{\lambda_n\}_{n=1}^{\infty}$ such that $|\lambda_n| < 1$ for $n = 2,3,\ldots$ and $\{\cos n x + \lambda_n \sin n x\}_{n=0}^{\infty}$ is incomplete in $L^1[0,\pi]$.

Proof: Define λ_n as

$$\lambda_n = - \frac{\int_0^{\pi} (3 x^2 - \pi^2) \cos n x \, dx}{\int_0^{\pi} (3 x^2 - \pi^2) \sin n x \, dx} \qquad n = 1,2,\ldots$$

Once we have shown that our definition of λ_n makes sense, then we will have that

$$\int_0^{\pi} (3 x^2 - \pi^2) (\cos n x + \lambda_n \sin n x) \, dx = 0 \qquad n = 0,1,2,\ldots$$

because of the way λ_n was defined. Since $f(x) = 3 x^2 - \pi^2 \in L^{\infty}[0,\pi]$, we will have proven that

$$\{\cos n x + \lambda_n \sin n x\}_{n=0}^{\infty} \text{ is incomplete in } L^1[0,\pi].$$

Integration by parts gives us that

$$\lambda_n = \begin{cases} \dfrac{2}{\pi n} & n \text{ even} \\[2ex] \dfrac{6\pi n}{\pi^2 n^2 - 12} & n \text{ odd} \end{cases}$$

Thus $\quad |\lambda_n| \le \frac{1}{\pi} < 1 \quad$ for n even.

For n an odd integer ≥ 3 consider the function

$$P(x) = \frac{6 \pi x}{\pi^2 x^2 - 12} \qquad \text{where} \ge 3$$

$0 < P(3) < 1, \lim_{x \to \infty} P(x) = 0 \quad$ and $\quad P'(x) < 0$ for $x \ge 3$.

Hence $\quad 0 < P(x) \le P(3) < 1 \quad$ for $\quad x \ge 3$.

Therefore $0 < \lambda_n \le P(3) < 1$ for n an odd integer ≥ 3.

Thus $|\lambda_n| < 1$ for $n = 2,3,\ldots$

THEOREM 3.7 - For each n, let $\lambda_n = 1$ or -1. Then, $\{\cos n x + \lambda_n \sin n x\}_{n=0}^{\infty}$ is complete in $C[0,\pi]$.

Proof: Assume there exists a finite measure $d\mu_1(x)$ on $[0,\pi]$ such that

$$\int_0^{\pi} (\cos n x + \lambda_n \sin n x)\, d\mu_1(x) = 0 \qquad n = 0,1,2,\ldots$$

3.23) Then $\left(\int_0^{\pi} \cos n x\, d\mu_1(x)\right)^2 = \left(\int_0^{\pi} \sin n x\, d\mu_1(x)\right)^2 \quad n = 0,1,2,\ldots$

Let $U = \frac{\pi}{2} - x$. Then

$$\cos n x + \lambda_n \sin n x = \begin{cases} \cos n U - \lambda_n \sin n U & n = 4K \\ \sin n U + \lambda_n \cos n U & n = 4K+1 \\ -\cos n U + \lambda_n \sin n U & n = 4K+2 \\ -\sin n U - \lambda_n \cos n U & n = 4K+3 \end{cases}$$

Let $d\mu(U) = d\mu_1\left(\frac{\pi}{2} - U\right)$

Then, by letting $U = \frac{\pi}{2} - x$, equation 3.23) becomes

3.24) $\left(\int_{-\pi/2}^{\pi/2} \cos n U\, d\mu(U)\right)^2 = \left(\int_{-\pi/2}^{\pi/2} \sin n U\, d\mu(U)\right)^2 \quad n = 0,1,2,\ldots$

Actually equation 3.24) holds for $-n$ also.

Now let $F(z) = \left(\int_{-\pi/2}^{\pi/2} \cos z U\, d\mu(U)\right)^2 - \left(\int_{-\pi/2}^{\pi/2} \sin z U\, d\mu(U)\right)^2$

$F(z)$ is an entire function, $F(\pm n) = 0$ and $|F(z)| \le M e^{\pi|z|}$ for some M.

We will use the following theorem [2, P.156].

THEOREM: Let $F(z)$ be entire, vanish at all the integers and satisfy $|F(z)| \le M e^{\pi|z|}$ for some M. Then $F(z) \equiv C \sin \pi z$ for some constant C.

Thus, in our case $F(z) \equiv C \sin \pi z$.

However, our $F(z)$ is an even function of z while $C \sin \pi z$ is odd.

Hence C must be 0.

Therefore

$$\left(\int_{-\pi/2}^{\pi/2} \cos z U \, d\mu(U)\right)^2 \equiv \left(\int_{-\pi/2}^{\pi/2} \sin z U \, d\mu(U)\right)^2$$

Since both $\int_{-\pi/2}^{\pi/2} \cos z U \, d\mu(U)$ and $\int_{-\pi/2}^{\pi/2} \sin z U \, d\mu(U)$ are entire functions, we must have that

$$\int_{-\pi/2}^{\pi/2} \cos z U \, d\mu(U) = \pm \int_{-\pi/2}^{\pi/2} \sin z U \, d\mu(U)$$

where we have either + for all z or - for all z. In either case we have even function equal an odd function which is impossible unless both are identically zero.

Therefore

$$\int_{-\pi/2}^{\pi/2} \cos n U \, d\mu(U) = \int_{-\pi/2}^{\pi/2} \sin n U \, d\mu(U) = 0 \qquad n = 0,1,\ldots$$

Since $\{\cos n x, \sin n x\}_{n=0}^{\infty}$ is complete in $C[-\pi/2, \pi/2]$ $d\mu(U) \equiv 0$

But $d\mu(U) = d\mu_1(x)$

Hence $d\mu_1(x) \equiv 0$ and we have completeness.

<u>THEOREM 3.8 - Let λ_n be real and such that</u>

<u>$|\lambda_n| \le 1 \qquad n = 1,2,\ldots$. Then $\{\lambda_n \cos n x + \sin n x\}_{n=1}^{\infty}$</u>

<u>is complete in $L^2[0,\pi]$.</u>

<u>Proof</u>: Assume there exists an $f(x) \in L^2[0,\pi]$ such that

$$\int_0^{\pi} f(x) (\lambda_n \cos n x + \sin n x) \, dx = 0 \quad n = 1,2,\ldots$$

Let a_n and b_n be as in the lemma to Theorem 3.1.

Then $\lambda_n a_n + b_n = 0 \qquad n = 1,2,\ldots$

$$\tfrac{1}{2}|a_0|^2 + \sum_{n=1}^{\infty} |a_n|^2 = \sum_{n=1}^{\infty} |b_n|^2 = \sum_{n=1}^{\infty} |\lambda_n a_n|^2$$

Let $S = \{n : |\lambda_n| < 1\}$ and let $\overline{S} = \{n : |\lambda_n| = 1\}$.

Then $\tfrac{1}{2}|a_0|^2 + \sum_{n\in S} |a_n|^2 + \sum_{n\in\overline{S}} |a_n|^2 = \sum_{n\in S} |\lambda_n a_n|^2 + \sum_{n\in\overline{S}} |\lambda_n a_n|^2$

$$= \sum_{n\in S} |\lambda_n a_n|^2 + \sum_{n\in\overline{S}} |a_n|^2$$

Therefore $\tfrac{1}{2}|a_0|^2 + \sum_{n\in S} |a_n|^2 = \sum_{n\in S} |a_n \lambda_n|^2 < \sum_{n\in S} |a_n|^2$

This is impossible unless $a_o = 0$ and $n \in S$ implies $a_n = 0$.

Since $\lambda_n a_n + b_n = 0$ we have $b_n = 0$ for $n \in S$.

Thus, $\int_o^\pi f(x)\ (\cos n x + \gamma_n \sin n x)\ dx = \pi\ (a_n + \gamma_n b_n)$

$= 0$ for $n \in S$, $n = 0$ and for any γ_n .

In particular let

$$\gamma_n = \left\{ \begin{matrix} 1 & n \in S \\ \frac{1}{\lambda_n} & n \in \overline{S} \end{matrix} \right\}$$

Since λ_n is real and $|\lambda_n| = 1$ for $n \in \overline{S}$ we have that $\gamma_n = +1$ or -1 for any n.

By the way γ_n was defined

$$\int_o^\pi f(x)\ (\cos n x + \gamma_n \sin n x)\ dx = 0 \quad n = 0,1,2,\ldots$$

Since in Theorem 3.7 we proved that

$\{\cos n x + \gamma_n \sin n x\}_{n=0}^\infty$ is complete in $C[0,\pi]$, it is certainly complete in $L^2[0,\pi]$.

Hence $\quad f(x) = 0 \quad$ a.e.

COROLLARY: Let λ_n be real and such that $|\lambda_n| \geq 1$.
Then $\{\cos n x + \lambda_n \sin n x\}_{n=1}^\infty$ is complete in $L^2[0,\pi]$.

NOTE: In this theorem and in the following one, we don't need all the λ_n to be real. All that is needed is that all those λ_n such that $|\lambda_n| = 1$ should be real.

THEOREM 3.9 - Let λ_n be real and such that $|\lambda_n| \leq 1$. Then $\{\cos n x + \lambda_n \sin n x\}_{n=0}^\infty$ is complete in $C[0,\pi]$.

Proof: By Theorem 3.8 $\{-\lambda_n \cos n x + \sin n x\}_{n=1}^\infty$ is complete in $L^2[0,\pi]$. By the lemma to Theorem 3.3

$\{\cos n x + \lambda_n \sin n x\}_{n=0}^\infty$ is complete in $C[0,\pi]$.

In Chapter II we proved that $\{\cos n x + \lambda \sin n x\}_{n=0}^{\infty}$ is complete in $C[0,\pi]$ if λ is real. For λ complex, if $|\lambda| < 1$ we get completeness by Theorem 3.3. The following theorems take care of the case when $|\lambda| = 1$.

<u>THEOREM 3.10 - Let $|\lambda| = 1$. Then $\{\cos n x + \lambda \sin n x\}_{n=1}^{\infty}$ is complete in $L^2[0,\pi]$.</u>

<u>Proof</u>: Assume there exists $f(x) \in L^2[0,\pi]$ such that

3.25) $\quad \int_0^{\pi} f(x) (\cos n x + \lambda \sin n x)\, dx = 0 \qquad n = 1,2,\ldots$

Let a_n and b_n be as in the lemma to Theorem 3.1. Then

$$a_n + \lambda b_n = 0 \qquad n = 1,2,\ldots$$

$$\tfrac{1}{2} |a_o|^2 + \sum_{n=1}^{\infty} |a_n|^2 = \sum_{n=1}^{\infty} |b_n|^2 = \sum_{n=1}^{\infty} \left|\frac{a_n}{\lambda}\right|^2 = \sum_{n=1}^{\infty} |a_n|^2$$

Hence $\quad a_o = 0$

Therefore equation 3.25) holds for $n = 0$ also. Let L be the $L^2[0,\pi]$ closure of $\{\cos n x + \lambda \sin n x\}_{n=1}^{\infty}$ and let L_1 be the $L^2[0,\pi]$ closure of $\{\cos n x + \lambda \sin n x\}_{n=0}^{\infty}$. We have proven that $L^{\perp} = L_1{}^{\perp}$ where, as usual $L^{\perp}$ is the closed subspace of annihilators of L. Since $L^2 = L \oplus L^{\perp} = L_1 \oplus L_1^{\perp}$ we must have that $L = L_1$

Hence $g(x) \equiv 1 \in L$

i.e. given $\epsilon > 0$ there exist $a_1 \ldots a_N$ such that

3.26) $\quad \int_0^{\pi} \left| 1 - \sum_{n=1}^{N} a_n (\cos n x + \lambda \sin n x)\right|^2 dx < \epsilon^2$

Letting $\|\quad\|$ be the $L^2[0,\pi]$ norm, inequality 3.26) can be written as

$$\left\| 1 - \sum_{n=1}^{N} a_n (\cos n x + \lambda \sin n x) \right\| < \epsilon$$

Since $|\cos x| \leq 1$, $\qquad \left\| \left[1 - \sum_{n=1}^{N} a_n (\cos n x + \lambda \sin n x)\right] \cos x \right\| < \epsilon$

However, $(\cos n x + \lambda \sin n x) \cos x$

$$= \tfrac{1}{2} [\cos (n+1) x + \lambda \sin (n+1) x] + \tfrac{1}{2} [\cos (n-1) x + \lambda \sin (n-1) x]$$

Therefore, letting $b_n = \dfrac{a_{n+1} + a_{n-1}}{2}$ (where $a_o = a_{N+1} = a_{-1} = a_{N+2} = 0$)

we have

$$\left\|\cos x - \sum_{n=0}^{N+1} b_n (\cos n x + \lambda \sin n x)\right\| < \epsilon$$

i.e. $\cos x \in L_1 = L$

Since $\cos x + \lambda \sin x \in L$ we have that

$$\sin x \in L.$$

Now, inductively, assume that 1, $\cos x$, $\sin x$, ..., $\cos M x$, $\sin M x$ are all in L and we'll prove that $\cos(M+1)x$ and $\sin(M+1)x$ are in L. Again we have

$$\left\| 1 - \sum_{n=1}^{N} a_n(\cos n x + \lambda \sin n x) \right\| < \epsilon$$

3.27) Therefore $\left\| [1 - \sum_{n=1}^{N} a_n(\cos n x + \lambda \sin n x)]\cos (M+1)x\right\| < \epsilon$

We will use the relationship

$$(\cos n x + \lambda \sin n x) \cos (M+1)x =$$

$$\tfrac{1}{2} [\cos(n+M+1)x + \lambda \sin(n+M+1)x]+\tfrac{1}{2}[\cos(n-M-1)x+\lambda\sin(n-M-1)x]$$

Let $b_n = \dfrac{a_{n+M+1} + a_{n-M-1}}{2}$ where $a_n = 0 \quad n > N$ or $n \leq 0$

Then inequality 3.27 becomes

$$\left\| \cos(M+1) x - \sum_{n=-M}^{N+M+1} b_n(\cos n x + \lambda \sin n x) \right\| < \epsilon$$

Thus $\cos(M+1)x$ can be approximated by linear combinations of $\{\cos n x + \lambda \sin n x\}_{n=1}^{\infty}$ and $\{\cos n x, \sin n x\}_{n=0}^{M}$

Since, by induction, all of $\{\cos n x, \sin n x\}_{n=0}^{M}$ is contained in L we have that $\cos(M+1)x \in L$. Since $\cos(M+1)x + \lambda \sin(M+1)x$ is also in L, we have that $\sin(M+1)x$ is in L.

Hence, by induction, all of $\{\cos n x, \sin n x\}_{n=0}^{\infty}$ is contained in L.

As L is closed it contains the $L^2[0,\pi]$ closure of $\{\cos n x, \sin n x\}_{n=0}^{\infty}$ which is $L^2[0,\pi]$.

Since, obviously, $L \subseteq L^2[0,\pi]$ we have that $L = L^2[0,\pi]$. Therefore

$\{\cos n x + \lambda \sin n x\}_{n=1}^{\infty}$ is complete in $L^2[0,\pi]$.

THEOREM 3.11 - Let $|\lambda| = 1$. Then $\{\cos n x + \lambda \sin n x\}_{n=0}^{\infty}$ is complete in $C[0,\pi]$.

Proof: By Theorem 3.10 $\{\cos n x - \frac{1}{\lambda} \sin n x\}_{n=1}^{\infty}$ is complete in $L^2[0,\pi]$. Hence, $\{-\lambda \cos nx + \sin nx\}_{n=1}^{\infty}$ is also complete in $L^2[0,\pi]$. Therefore, by the lemma to Theorem 3.3 $\{\cos nx + \lambda \sin nx\}_{n=0}^{\infty}$ is complete in $C[0,\pi]$.

THEOREM 3.12 - $\{1, n \cos nx + \lambda \sin nx\}_{n=1}^{\infty}$ is complete in $C[0,\pi]$, if $\lambda \neq 2ki$, k a non-zero integer

Proof: If $\lambda = 0$ it is trivially true. Therefore, we can assume $\lambda \neq 0$. Take any function $f(x) \epsilon\ C[0,\pi]$. We want to approximate by something of the form $\sum_{n=1}^{N} a_n(n \cos n x + \lambda \sin n x) + a_o\lambda$

Let $P(x) = \sum_{n=1}^{N} a_n \sin n x + a_o$

Then we want to approximate by $P'(x) + \lambda P(x)$. The idea of the proof is to solve the differential equation $Y' + \lambda Y = f$ and approximate the solution by polynomials. The general solution of the differential equation is [4, P.75] $Y(x) = C e^{-\lambda x} + e^{-\lambda x} \int_o^x e^{\lambda t} f(t)dt$ where C is some arbitrary constant. If $\int_o^{\pi} Y'(x)dx = 0$ then $Y'(x)$ can be uniformly approximated by linear combinations of $\{\cos n x\}_{n=1}^{\infty}$ i.e. given $\epsilon > 0$ there exist $a_1, \ldots, a_N$ such that

$$|Y'(x) - \sum_{n=1}^{N} a_n n \cos n x| < \epsilon \text{ for all } x\ \epsilon[0,\pi].$$

$$|Y(x) - Y(0) - \sum_{n=1}^{N} a_n \sin n x| = |\int_o^x (Y'(t) - \sum_{n=1}^{N} a_n n\cos nt)dt|$$

$$\leq \int_o^x |Y'(t) - \sum_{n=1}^{N} a_n n \cos n t|dt < \epsilon\ x \leq \epsilon\ \pi$$

Let $a_o = Y(0)$. Then, since $Y'(x) + \lambda\ Y(x) = f(x)$

$$|f(x) - \sum_{n=1}^{N} a_n(n \cos n x + \lambda \sin n x) - \lambda\ a_o| < \epsilon\ (1 + |\lambda|\ \pi)$$

Since $f(x)$ was arbitrary, we would have that

$\{1, n \cos n x + \lambda \sin n x\}_{n=1}^{\infty}$ is complete in $C[0,\pi]$.

All we have to show is that $0 = \int_0^{\pi} Y'(x)dx = Y(\pi) - Y(0)$ or that

3.28) $$C e^{-\lambda\pi} + e^{-\lambda\pi} \int_0^{\pi} e^{\lambda t} f(t)dt - C = 0$$

Since $\lambda \neq 2ki$, $e^{-\lambda\pi} \neq 1$. Therefore, we can find C to satisfy equation 3.28.

<u>THEOREM 3.13 - $\{n \cos nx + \lambda \sin nx\}_{n=1}^{\infty}$, is incomplete in $L'[0,\pi]$. Moreover, if $\lambda = 2ki$, k a non-zero integer, then $\{1, n \cos nx + \lambda \sin nx\}_{n=1}^{\infty}$ is incomplete in $L^1[0,\pi]$.</u>

<u>Proof</u>: $\int_0^{\pi} e^{\lambda x}(n \cos nx + \lambda \sin nx)\, dx = 0$ $n = 1,2,\ldots$ (and if $\lambda = 2ki$, it is true for $n = 0$ also.) Since $e^{\lambda x} \in L^{\infty}[0,\pi]$, we have proven incompleteness in $L^1[0,\pi]$.

The motivation for the linear functional is the following. In Theorem 3.12, the constant term is $Y(0)$ which is C. If we don't have a constant term to approximate with, we would want $C = Y(0)$ to $= 0$. In equation 3.28 this leads to $\int_0^{\pi} e^{\lambda x} f(x)\, dx = 0$. If we have a constant term, but $\lambda = 2ki$, then, in equation 3.28, we again have $\int_0^{\pi} e^{\lambda x} f(x)dx = 0$ which is not always true for $f \in C[0,\pi]$. However, it does suggest to orthogonal function $e^{\lambda x}$.

<u>THEOREM 3.14 - $\{\lambda \cos n x + n \sin n x\}_{n=0}^{\infty}$ is incomplete in $C[0,\pi]$.</u>

<u>Proof</u>: As in Theorem 3.12 we consider a differential equation $Y' - \lambda Y = f$ and want to approximate $Y'(x)$ uniformly by $\sum_{n=1}^{N} a_n \sin nx$. However, this can only be done if $Y'(0) = Y'(\pi) = 0$. The general solution of the differential equation is [4,p.75]

$$Y(x) = C e^{\lambda x} + e^{\lambda x} \int_0^{x} e^{-\lambda t} f(t)dt$$

Hence,

$$0 = Y'(0) = \lambda C + f(0) \qquad \text{and}$$

$$0 = Y'(\pi) = \lambda C e^{\lambda\pi} \int_0^{\pi} e^{-\lambda t} f(t)dt + f(\pi)$$

Combining the two equations to eliminate C, we get

$$-f(0)\ e^{\lambda\pi} + \lambda e^{\lambda\pi} \int_0^\pi e^{-\lambda t} f(t)\ dt + f(\pi) = 0$$

This suggests an orthogonal measure. Let

$$d\mu(x) = -\ \delta(x)\ e^{\lambda\pi} + \lambda e^{\lambda\pi}\ e^{-\lambda x} + \delta(x-\pi)$$

where $\delta(x)$ is the usual delta measure. An easy calculation shows this $d\mu(x)$ works:

i.e. $$\int_0^\pi (\lambda \cos n\,x + n \sin n\,x)\ d\mu(x) = 0 \quad n = 0,1,2,\ldots$$

<u>THEOREM 3.15 - $\{\lambda \cos n\,x + n \sin n\,x\}_{n=0}^\infty$ is complete in $L^P[0,\pi]$ for any $P \geq 1$.</u>

<u>Proof</u>: If $\lambda = 0$ it is trivial as we have $\{\sin n\,x\}_{n=1}^\infty$. We can assume $\lambda \neq 0$. Take any $f(x) \in L^P[0,\pi]$. Let $g(x) = C\,e^{\lambda x} + e^{\lambda x} \int_0^x e^{-\lambda t} f(t)\ dt$ where C is a constant. Then, $g(x)$ is absolutely continuous on $[0,\pi]$, differentiable a.e. on $[0,\pi]$ and $g'(x) = \lambda g(x) + f(x)$ a.e. Also, $g'(x) \in L^P[0,\pi]$.

As $\{\sin n\,x\}_{n=1}^\infty$ is complete in $L^P[0,\pi]$, there exist $a_1,\ldots,a_N$ such that

3.29) $$\|g'(x) - \sum_{n=1}^N a_n\, n \sin n\,x\| < \epsilon \text{ where } \|\quad\| \text{ is the } L^P[0,\pi] \text{ norm.}$$

Since $g(x)$ is absolutely continuous $\int_0^x g'(t)dt = g(x) - g(0)$

Therefore,

$$\begin{aligned}
&\left|g(x) - g(0) + \sum_{n=1}^N a_n \cos n\,x - \sum_{n=1}^N a_n\right| \\
&= \left|\int_0^x \left(g'(t) - \sum_{n=1}^N a_n\, n \sin n\,t\right) dt\right| \\
&\leq \int_0^x \left|g'(t) - \sum_{n=1}^N a_n\, n \sin n\,t\right| dt \\
&\leq \int_0^\pi \left|g'(t) - \sum_{n=1}^N a_n\, n \sin n\,t\right| dt \\
&\leq \left\|g'(t) - \sum_{n=1}^N a_n\, n \sin n\,t\right\| \left(\int_0^\pi dt\right)^{\frac{P-1}{P}} \quad \text{(by Holder's inequality)} \\
&< \epsilon\,\pi
\end{aligned}$$

Let $a_0 = -\ g(0) - \sum_{n=1}^N a_n$. Then

3.30) $\quad |g(x) + \sum_{n=0}^{N} a_n \cos n x| < \epsilon \pi$ for all $x \epsilon [0,\pi]$.

Therefore $\|g(x) + \sum_{n=0}^{N} a_n \cos n x\| < \epsilon \pi^{1+\frac{1}{P}} \leq \epsilon \pi^2$

Combining this with inequality 3.29) we get

$$\|g'(x) - \lambda g(x) - \sum_{n=0}^{N} a_n (\lambda \cos n x + n \sin n x)\| < \epsilon (1+|\lambda| \pi^2)$$

Since $g'(x) - \lambda g(x) = f(x)$ a.e. we have

$$\|f(x) - \sum_{n=0}^{N} a_n (\lambda \cos n x + n \sin n x)\| < \epsilon (1 + |\lambda| \pi^2).$$

Since $f(x)$ is an arbitrary $L^P[0,\pi]$ function, we have completeness in $L^P[0,\pi]$.

<u>THEOREM 3.16 - $\{\lambda \cos n x + n \sin n x\}_{n=1}^{\infty}$ is complete in $L^P[0,\pi]$ for all $P \geq 1$ iff $\lambda \neq 2Ki$ where K is a non-zero integer.</u>

<u>Proof</u>: Assume $\lambda \neq 2Ki$. Let $f(x)$ be in $L^P[0,\pi]$.

In Theorem 3.15 we proved completeness if we are allowed to use the constant function. In inequality 3.30 we had

$$|g(x) + \sum_{n=1}^{N} a_n \cos n x + a_0| < \epsilon \pi \text{ for all } x \epsilon [0,\pi].$$

$$|\int_0^\pi g(x)dx + a_0\pi| = |\int_0^\pi (g(x) + \sum_{n=1}^{N} a_n \cos n x + a_0)\, dx|$$

$$\leq \int_0^\pi |g(x) + \sum_{n=1}^{N} a_n \cos n x + a_0| dx$$

$$< \epsilon \pi^2$$

Hence if $\int_0^\pi g(x)dx = 0$ then $|a_0| < \epsilon \pi$

i.e. we get an arbitrarily small error by forgetting about the constant term. $g(x) = C e^{\lambda x} + e^{\lambda x} \int_0^x e^{-\lambda t} f(t)dt$, and we want to choose C so that $\int_0^\pi g(x)dx = 0$. If $\lambda = 0$, $\int_0^\pi g(x) = C \pi + \int_0^\pi \int_0^x f(t)dt\, dx$ and C can obviously be chosen so that $\int_0^\pi g(x)dx = 0$. If $\lambda \neq 0$ we have

$$\int_0^\pi g(x)dx = \frac{C[e^{\lambda\pi}-1]}{\lambda} + \int_0^\pi e^{\lambda x} \int_0^x e^{-\lambda t} f(t)dt\, dx$$

Since $\lambda \neq 2Ki$, $e^{\lambda\pi} - 1 \neq 0$ and we can choose C so that $\int_0^\pi g(x)dx = 0$.

On the other hand, assume $\lambda = 2Ki$ where K is a non-zero integer.

Since we want $\int_0^\pi g(x)dx = 0$ we get

$$\int_0^\pi e^{\lambda x} \int_0^x e^{-\lambda t} f(t)\, dt\, dx = 0.$$

This suggests the bounded linear functional

$$L(f) = \int_0^\pi e^{\lambda x} \int_0^x e^{-\lambda t} f(t)\, dt\, dx$$

$$L(\lambda \cos n x + n \sin n x) = \int_0^\pi e^{\lambda x} \int_0^x e^{-\lambda t} (\lambda \cos n t + n \sin n t) dt\, dx$$

$$= \int_0^\pi e^{\lambda x} [-e^{-\lambda x} \cos n x + 1]dx$$

$$= \int_0^\pi (- \cos n x + e^{\lambda x})\, dx$$

$$= \frac{e^{\lambda\pi} - 1}{\lambda} = 0 \text{ since } \lambda = 2Ki.$$

Hence $\{\lambda \cos n x + n \sin n x\}_{n=1}^{\infty}$ is incomplete in $L^P[0,\pi]$.

IV: Completeness of $\{P(n)\cos nx + Q(n)\sin nx\}$ on $[-a,a]$, $o < a < \pi$

We now consider the completeness of

$\{P(n)\ \cos nx + Q(n)\ \sin nx\}_{n=0}^{\infty}$ where $P(x)$ and $Q(x)$ are algebraic

polynomials. Whereas, in chapters two and three, we considered completeness on $[o,\pi]$, here we deal with $[-a,a]$ where $o < a < \pi$. The results are very surprising when compared with the same sequences on $[o,\pi]$.

NOTATION: For an algebraic polynomial $P(z)$, we let $P_e(z)$ be the even part of $P(z)$ and $P_o(z)$ be the odd part of $P(z)$.

We will use the following theorem [9, P. 186].

THEOREM A: Let $F(z)$ be analytic and of the form $O(e^{k|z|})$ where $K < \pi$, for $\mathrm{Re}\ z \geq 0$ and let $F(z) = 0$ for $z = 0,1,2, \ldots$ Then $F(z) = 0$.

THEOREM 4.1 - Let $D(z) = P_e(z)\ Q_e(z) - P_o(z)\ Q_o(z)$ and let a be such that $0 < a < \pi$. Then, if $D(z) \not\equiv 0$, $\{P(n)\cos nx + Q(n)\ \sin\ nx\}_{n=0}^{\infty}$ is complete in $C[-a,a]$.

Proof: Assume there exists a measure $d\mu(x)$ such that

$$P(n) \int_{-a}^{a} \cos nx\ d\mu(x) + Q(n) \int_{-a}^{a} \sin nx\ d\mu(x) = 0$$

$$n = 0,1,2\ldots$$

Let $F(z) = P(z) \int_{-a}^{a} \cos zx\ d\mu(x) + Q(z) \int_{-a}^{a} \sin zx\ d\mu(x)$

Then $F(z)$ is an entire function and $F(z) = 0$ for $z = 0,1,2...$

Let $K = a + \frac{\pi - a}{2}$. Then $a < K < \pi$.

Also, $|P(z)| \leq M e^{\frac{(\pi-a)}{2}|z|}$ and $|Q(z)| \leq M e^{\frac{(\pi-a)}{2}|z|}$ for some M.

Thus $|F(z)| \leq M_1 e^{K|z|}$ for some M_1.

By theorem A, $F(z) \equiv 0$.

4.1) $P_e(z) \int_{-a}^{a} \cos zx \, d\mu(x) + Q_o(z) \int_{-a}^{a} \sin zx \, d\mu(x) = \frac{F(z) + F(-z)}{2} \equiv 0$

4.2) $P_o(z) \int_{-a}^{a} \cos zx \, d\mu(x) + Q_e(z) \int_{-a}^{a} \sin zx \, d\mu(x) = \frac{F(z) - F(-z)}{2} \equiv 0$

Multiplying equation 4.1) by $Q_e(z)$ and equation 4.2) by $-Q_o(z)$ and adding, we get

$[P_e(z)\ Q_e(z) - P_o(z)\ Q_o(z)] \int_{-a}^{a} \cos zx \, d\mu(x) \equiv 0$

or

$$D(z) \int_{a}^{a} \cos zx \, d\mu(x) \equiv 0$$

Since $D(z) \not\equiv 0$, $\int_{-a}^{a} \cos zx \, d\mu(x) \equiv 0$

Similarly, multiplying equation 4.1) by $P_o(z)$ and equation 4.2) by $-P_e(z)$ and adding, we get

$$D(z) \int_{-a}^{a} \sin zx \, d\mu(x) \equiv 0$$

Thus $\int_{-a}^{a} \sin zx \, d\mu(x) \equiv 0$

Since $\int_{-a}^{a} \cos nx \, d\mu(x) = \int_{-a}^{a} \sin nx \, d\mu(x) = 0$ for $n = 0,1,2,\ldots$ we have that all the Fourier coefficients of $d\mu(x)$ are zero. By the completeness of

$\{\cos nx, \sin nx\}_{n=0}^{\infty}$ in $C[-a,a]$, we must have that $d\mu(x) \equiv 0$.

Theorem 4.1 proves that if $D(z) \not\equiv 0$, then $\{P(n) \cos nx + Q(n) \sin nx\}_{n=0}^{\infty}$ is complete in $C[-a,a]$ for all a such that $0 < a < \pi$. In a general sense, we have completeness in the "largest" interval under the "strongest" norm. The next theorem will prove that if $D(z) \equiv 0$, then we get incompleteness in $L^1[-\epsilon,\epsilon]$ for any $\epsilon > 0$, which is sort of the "smallest" interval under the "weakest" norm. Not only is the sequence incomplete, but even the addition of any finite number of integrable functions, still leaves the sequence incomplete.

First we will prove the following lemma.

LEMMA 1). Let $P(z)$ and $Q(z)$ be algebraic polynomials where $P(z)$ is even and $Q(z)$ is odd. Let $g_1(x),\ldots,g_N(t)$ be any integrable functions on $[-a,a]$ where $0 < a < \pi$. Then, there exists a continuous non-trivial function $f(x)$ on $[-a,a]$ such that

1) $P(z) \int_{-a}^{a} f(x) \cos zx \, dx + Q(z) \int_{-a}^{a} f(x) \sin zx \, dx \equiv 0$

2) $\int_{-a}^{a} f(x) g_n(x) \, dx = 0 \qquad n = 1,\ldots, N.$

Proof: Assume $P(z) = a_o + a_2 z^2 + \ldots + a_{2k} z^{2k}$ and $Q(z) = a_1 z + \ldots + a_{2k-1} z^{2k-1}$ where any of the a_i (including a_{2k})

may be zero.

Let $M = N + K$.

Let $g(x)$ be in $C^{2M}[-\pi,\pi]$, non trivial, odd and zero on $[-\pi,-a]$ and $[a,\pi]$.

Then $g(\pm a) = g'(\pm a) = \ldots = g^{(2M)}(\pm a) = 0$

Integration by parts, combined with the vanishing of $g(x)$ and its first 2M derivatives at $\pm a$, gives us that

4.3) $\int_{-a}^{a} g^{(2n-1)}(x) \cos zx\, dx = (-1)^{n-1} z^{2n-1} \int_{-a}^{a} g(x) \sin zx\, dx$ $\qquad n \leq M$

and

4.4) $\int_{-a}^{a} g^{(2n)}(x) \sin zx\, dx = (-1)^{n} z^{2n} \int_{-a}^{a} g(x) \sin zx\, dx \qquad n \leq M.$

By the fact that $g(x)$ and all its even numbered derivatives are odd and all its odd numbered derivatives are even, we have

4.5) $\int_{-a}^{a} g^{(2n-1)}(x) \sin zx\, dx = 0 \qquad n \leq M$

and

4.6) $\int_{-a}^{a} g^{(2n)}(x) \cos zx\, dx = 0 \qquad n \leq M.$

Let $f(x) = c_0 g(x) + c_1 g'(x) + \ldots + c_{2M} g^{(2M)}(x)$ where $\{c_i\}_{i=0}^{2M}$ will be determined later.

Let $F(z) = P(z) \int_{-a}^{a} f(x) \cos zx\, dx + Q(z) \int_{-a}^{a} f(x) \sin zx\, dx$

By equations 4.3), 4.4), 4.5) and 4.6),

$$F(z) = P(z)\ (c_1 z - c_3 z^2 + \ldots + (-1)^{M-1}\ c_{2M-1}\ z^{2M-1}) \int_{-a}^{a} g(x) \sin zx\, dx$$

$$+ Q(z)\ (c_0 - c_2\ z^2 + \ldots + (-1)^M\ c_{2M} z^{2M}) \int_{-a}^{a} g(x) \sin zx\, dx.$$

Let $R(z) = P(z)\ (c_1 z - c_3 z^3 + \ldots + (-1)^{M-1}\ c_{2M-1}\ z^{2M-1})$

$$+ Q(z)\ (c_0 - c_2 z^2 + \ldots + (-1)^M\ c_{2M} z^{2M})$$

Then $F(z) = R(z) \int_{-a}^{a} g(x) \sin zx\, dx$

$R(z)$ is an odd polynomial of degree at most $2M + 2K - 1$. Its coefficients are linear combinations of $\{c_i\}_{i=0}^{2M}$. For $R(z)$ to be identically zero, all its coefficients would have to be zero. That gives us $M + K$ linear homogeneous equations in $\{c_i\}_{i=0}^{2M}$ since there are $M + K$ odd integers between 1 and $2M + 2K - 1$. If, in addition, we require that

$$\int_{-a}^{a} f(x)\ g_n(x)\, dx = 0 \qquad n = 1, \ldots, N,$$

we get N more linear homogeneous equations in $\{c_i\}_{i=0}^{2M}$. Altogether we get $M + K + N$ equations in $2M + 1$ unknowns. However, since $K + N = M$, we have more unknowns than equations and can solve non-trivially. Thus we have produced a continuous function $f(x)$ on $[-a,a]$ such that

1) $P(z) \int_{-a}^{a} f(x) \cos zx\, dx + Q(z) \int_{-a}^{a} f(x) \sin zx\, dx \equiv 0$

2) $\int_{-a}^{a} f(x)\ g_n(x)\, dx = 0 \qquad n = 1, \ldots, N$

All that is left to prove is that $f(x) \not\equiv 0$.

If it were, then $g(x)$ would be a non-trivial solution to the differential equation

$$c_{2M}\, Y^{(2M)}(x) + c_{2M-1}\, Y^{(2M-1)}(x) + \ldots + c_1\, Y'(x) + c_o\, Y(x) \equiv 0$$

satisfying the boundary conditions

$$Y(a) = Y'(a) = \ldots = Y^{(2M)}(a) = 0$$

However, by [4, P.67], this can only be solved by the trivial function. Hence, $f(x)$ is non-trivial.

THEOREM 4.2 - Let $P(z)$ and $Q(z)$ be algebraic polynomials satisfying $D(z) = P_e(z)\, Q_e(z) - P_o(z)\, Q_o(z) \equiv 0$ and let a be such that $0 < a < \pi$. Then $\{P(n) \cos nx + Q(n) \sin nx\}_{n=0}^{\infty}$ is incomplete in $L^1[-a,a]$. Moreover, if $g_1(x), g_2(x), \ldots, g_N(x)$ are any functions in $L^1[-a,a]$, then $\{g_1(x), \ldots, g_N(x), P(n) \cos nx + Q(n) \sin nx\}_{n=0}^{\infty}$ is incomplete in $L^1[-a,a]$

Proof: Case I: $Q_e(z) \equiv P_o(z) \equiv 0$.

Since $P(z)$ is even and $Q(z)$ is odd, lemma 1) applies. Therefore, there exists a continous, non-trivial function $f(x)$ such that

1) $P(z) \int_{-a}^{a} f(x) \cos zx\, dx + Q(z) \int_{-a}^{a} f(x) \sin zx\, dx \equiv 0$ and

2) $\int_{-a}^{a} f(x)\, g_n(x)\, dg = 0 \qquad n = 1, \ldots, N$

1)certainly implies that

$$P(n) \int_{-a}^{a} f(x) \cos nx\, dx + Q(n) \int_{-a}^{a} f(x) \sin nx\, dx = 0 \qquad n = 0,1,2, \ldots$$

Since $f(x)$ is continuous on $[-a,a]$, $f(x) \in L^{\infty}[-a,a]$

Hence $\{g_1(x), \ldots, g_N(x), P(n) \cos nx + Q(n) \sin nx\}_{n=0}^{\infty}$

is incomplete in $L^1[-a,a]$.

Case II: $P_o(z) \not\equiv 0$

By lemma 1), since $z P_o(z)$ is even and $zQ_e(z)$ is odd, there exists a continuous non-trivial function $f(x)$ such that

4.7) 1) $z P_o(z) \int_{-a}^{a} f(x) \cos zx \, dx + zQ_e(z) \int_{-a}^{a} f(x) \sin zx \, dx \equiv 0$

2) $\int_{-a}^{a} f(x) g_n(x) \, dx = 0 \qquad n = 1, \ldots, N$

However, since $h(z) = z$ is not identically zero, equation 4.7) gives us that

4.8) $P_o(z) \int_{-a}^{a} f(x) \cos zx \, dx + Q_e(z) \int_{-a}^{a} f(x) \sin zx \, dx \equiv 0$

Multiplying equation 4.8) by $P_e(z)$ gives us

4.9) $P_e(z) P_o(z) \int_{-a}^{a} f(x) \cos zx \, dx + P_e(z) Q_e(z) \int_{-a}^{a} f(x) \sin zx \, dx \equiv 0$

However, since $P_e(z) Q_e(z) - P_o(z) Q_o(z) \equiv 0$, equation 4.9) becomes

4.10) $P_e(z) P_o(z) \int_{-a}^{a} f(x) \cos zx \, dx + P_o(z) Q_o(z) \int_{-a}^{a} f(x) \sin zx \, dx \equiv 0$

As we are assuming $P_o(z) \not\equiv 0$, we can divide equation 4.10) by $P_o(z)$ and get

4.11) $P_e(z) \int_{-a}^{a} f(x) \cos zx \, dx + Q_o(z) \int_{-a}^{a} f(x) \sin zx \, dx \equiv 0$

Adding equations 4.8) and 4.11) we have

$$P(z) \int_{-a}^{a} f(x) \cos zx\, dx + Q(z) \int_{-a}^{a} f(x) \sin zx\, dx \equiv 0$$

Hence, $P(n) \int_{-a}^{a} f(x) \cos nx\, dx + Q(n) \int_{-a}^{a} f(x) \sin nx\, dx = 0$

$n = 0,1,2,\ldots$

Since $f(x) \in L^{\infty}[-a,a]$ and since we already have proven

$\int_{-a}^{a} f(x)\, g_n(x)\, dx = 0 \qquad n = 1,\ldots, N$

we have the incompleteness of

$\{g_1(x), \ldots, g_N(x), P(n) \cos nx + Q(n) \sin nx\}_{n=0}^{\infty}$ in $L^1[-a,a]$.

Case III: $Q_e(z) \not\equiv 0$

As in case II, we can find a function $f(x) \in L^{\infty}[-a,a]$ such that

4.12) 1) $P_o(z) \int_{-a}^{a} f(x) \cos zx\, dx + Q_e(z) \int_{-a}^{a} f(x) \sin zx\, dx \equiv 0$

2) $\int_{-a}^{a} f(x)\, g_n(x)\, dx = 0 \qquad n = 1, \ldots, N$

Multiplying equation 4.12) by $Q_o(z)$, we get

4.13) $P_o(z)\, Q_o(z) \int_{-a}^{a} f(x) \cos zx\, dx + Q_e(z)\, Q_o(z) \int_{-a}^{a} f(x) \sin zx\, dx \equiv 0$

Since $P_e(z)\, Q_e(z) - P_o(z)\, Q_o(z) \equiv 0$, equation 4.13) becomes

4.14) $P_e(z)\, Q_e(z) \int_{-a}^{a} f(x) \cos zx\, dx + Q_e(z)\, Q_o(z) \int_{-a}^{a} f(X) \sin zx\, dx \equiv 0$

As we are assuming $Q_e(z) \not\equiv 0$, we can divide equation 4.14) by $Q_e(z)$ and get

4.15) $P_e(z) \int_{-a}^{a} f(x) \cos zx\, dx + Q_o(z) \int_{-a}^{a} f(x) \sin zx\, dx \equiv 0$

Adding equations 4.12) and 4.15) we have

$$P(z)\int_{-a}^{a} f(x)\cos zx\,dx + Q(z)\int_{-a}^{a} f(x)\sin zx\,dx \equiv 0$$

and have established incompleteness.

In theorem 4.1 we proved that if $D(z) \not\equiv 0$, then we have completeness in $C[-a,a]$ for any a such that $0 < a < \pi$. A question which arises is "Do we need all the terms from $n = 0,1\ldots$ or can some be eliminated without affecting completeness?" In the following theorem we will prove that actually an infinite number of terms can be omitted, proved the omitted set of integers is "sparse" among the set of positive integers.

THEOREM 4.3. Let S be a set of non negative integers such that there exists an $\alpha < 1$ such that

$$\sum_{\substack{n\in S\\ n\neq 0}} \frac{1}{n^{\alpha}} < \infty.$$

Let $\overline{S}$ be the complement of S in the set of non-negative integers. Let $P(z)$ and $Q(z)$ be algebraic polynomials and let a be such that $0 < a < \pi$. Then if $P_e(z)\,Q_e(z) - P_o(z)\,Q_o(z) \not\equiv 0$, $\{P(n)\cos nx + Q(n)\sin nx\}\ n \in \overline{S}$ is complete in $C[-a,a]$.

Proof: Assume there exists a measure $d\mu(x)$ such that

$$P(n)\int_{-a}^{a} \cos nx\,d\mu(x) + Q(n)\int_{-a}^{a} \sin nx\,d\mu(x) = 0 \quad N \in \overline{S}$$

Let $F(z) = P(z)\int_{-a}^{a} \cos zx\,d\mu(x) + Q(z)\int_{-a}^{a} \sin zx\,d\mu(x)$

Let $\epsilon = \dfrac{\pi - a}{3}$. Then

$$|F(z)| \le M\,e^{(a+\epsilon)|z|} \quad \text{for some } M.$$

Let $W(z) = z\prod_{\substack{n\in S\\ n\neq 0}} \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}}$

By a theorem [2, P.19], the order of $W(z)$ is α. Therefore

$$|W(z)| \leq M_1 e^{\epsilon |z|} \text{ for some } M_1.$$

Let $G(z) = F(z)\, W(z)$. $G(z)$ is an entire function, vanishes at all the non-negative integers and satisfies $|G(z)| \leq MM_1\, e^{(a+2\epsilon)|z|}$. Since $0 < a + 2\epsilon < \pi$, by theorem A, $G(z) \equiv 0$. As $W(z)$ is obviously not identically zero, we must have that $F(z) \equiv 0$. Therefore

$$P(n) \int_{-a}^{a} \cos nx \, d\mu(x) + Q(n) \int_{-a}^{a} \sin nx \, d\mu(x) = 0 \quad n = 0,1,2\ldots$$

By theorem 4.1,

$$\{P(n) \cos nx + Q(n) \sin nx\}_{n=0}^{\infty} \text{ is complete in } C[-a,a]$$

Hence $d\mu(x) \equiv 0$ and $\{ P(n) \cos nx + Q(n) \sin nx\}\ n \in \overline{S}$ is complete in $C[-a,a]$.

It is interesting to take some particular $P(z)$ and $Q(z)$ and see how the change of interval from $[0,\pi]$ to $[-a,a]$ affects completeness.

I - Let $P(z) \equiv 1$ and $Q(z) \equiv 0$.

We then have $\{\cos nx\}_{n=0}^{\infty}$ which is complete in $C[0,\pi]$ and incomplete in $L^1[-\epsilon,\epsilon]$.

II - Let $P(z) \equiv 1$ and $Q(z) \equiv \lambda$ where λ is real. We then have $\{\cos nx + \lambda \sin nx\}_{n=0}^{\infty}$ which is complete in $C[0,\pi]$ and in $C[-a,a]$ for any $a < \pi$. The difference is, that when we discard the constant function, we have incompleteness in some $L^P[0,\pi]$ spaces as well as in $C[0,\pi]$, whereas we still have completeness in $C[-a,a]$.

III - Let $P(z) \equiv z$ and $Q(z) \equiv \lambda$, $\lambda \neq 2ki$, k a non-zero integer. Then we have $\{n \cos nx + \lambda \sin nx\}_{n=1}^{\infty}$ which is incomplete in both $L^1[0,\pi]$ and $L^1[-\epsilon,\epsilon]$. The difference is that if we add in the constant function we get completeness in $C[0,\pi]$ and still have incompleteness in $L^1[-\epsilon,\epsilon]$ for any $\epsilon > 0$.

IV - Let $P(z) \equiv \lambda$ and $Q(z) \equiv z$.

Then we have $\{\lambda \cos nx + n \sin nx\}_{n=0}^{\infty}$ which is complete in $L^P[0,\pi]$ for any $P \geq 1$ and is incomplete in $L^1[-\epsilon,\epsilon]$ for any $\epsilon > 0$.

V. COMPLETENESS OF $\{e^{\pm\, i\, nx}\}_{n=0}^{\infty}$

In this chapter we give some answers to the following question: "If, for each non-negative integer n, we take either e^{inx} or e^{-inx}, what is the length of the greatest interval of completeness?". Since we are dealing with exponentials, we can consider the length of the interval of completeness, since, as proven in the introduction, it makes no difference where the interval is situated. Of course, the length of an interval of completeness has to be strictly less than 2π since each of those exponentials not chosen is orthogonal to the entire selected set over an interval of length 2π. However, if we take $\{e^{inx}\}_{n=0}^{\infty}$, then we get completeness in $C[2\pi-\epsilon]$ for any $\epsilon > 0$. Thus the least upper bound on the possible lengths of completeness is 2π (and it can never be achieved). Theorem 5.1 will prove that π is a lower bound and theorem 5.2 will prove that it is actually achieved (as a length of a greatest interval of completeness).

<u>THEOREM 5.1 - For each $n = 0,1,2...$ let $f_n(x)$ be either e^{inx} or e^{-inx}. Then $\{f_n(x)\}_{n=0}^{\infty}$ is complete in $C[-\pi/2,\pi/2]$.</u>

<u>Proof</u>: Assume there exist $d\mu(x)$ such that

$$\int_{-\pi/2}^{\pi/2} f_n(x)\, d\mu(x) = 0 \qquad n = 0,1,2....$$

Let $$F(z) = \int_{-\pi/2}^{\pi/2} e^{izx}\, d\mu(x)$$

Then for each $n = 0,1,2...$ either $F(n) = 0$ or $F(-n) = 0$. Also $|F(z)| \leq M\, e^{\pi/2\, |z|}$ for some M. Let $G(z) = F(z)\, F(-z)$. Since $G(z)$ is an even function of z and since for each $n = 0,1,2...$

either $F(n)$ or $F(-n)$ is zero, we get that $G(n) = 0$ for n an integer (positive, negative or zero). Also $|G(z)| = |F(z)||F(-z)| \leq M^2 e^{\pi|z|}$. By the theorem quoted in Theorem 3.8, $G(z) \equiv c \sin \pi z$ for some constant c. However, $G(z)$ is even and $c \sin \pi z$ is odd. Therefore $c = 0$ and $G(z) \equiv 0$. As $F(z)$ is entire and $F(z)\, F(-z) \equiv 0$, we must have that $F(z) \equiv 0$.

i.e. $$\int_{-\pi/2}^{\pi/2} e^{izx}\, d\mu(x) \equiv 0.$$

Since $\int_{-\pi/2}^{\pi/2} e^{inx}\, d\mu(x) = 0$ for all integers n and since $\{e^{inx}\}_{n=-\infty}^{\infty}$ is complete in $C[-\pi/2, \pi/2]$, we have that $d\mu(x) \equiv 0$.

LEMMA: Let $G(z) = \dfrac{1}{\Gamma(\frac{z}{2})\Gamma(\frac{1}{2} - \frac{z}{2})}$ where $\Gamma(z)$ is the gamma function.

Then $G(z)$ is entire and satisfies (for some K)

1) $|G(z)| \leq K \left|\cos \frac{\pi z}{2}\right| \sqrt{|z|}$ $\qquad$ Re $z \geq 0$

2) $|G(z)| \leq K \left|\sin \frac{\pi z}{2}\right| \sqrt{|z|}$ $\qquad$ Re $z \leq 0$, $|z| \geq 1$

Proof: $G(z)$ is obviously entire as $\dfrac{1}{\Gamma(z)}$ is entire. For the estimates on $G(z)$ we will use the following form of Stirling's formula for $\Gamma(z)$ [10, P.239]: For $0 < a \leq 1$ and $|\arg z + a| \leq \pi - \delta$ and $|\arg z| \leq \pi - \delta$,

$$\log \Gamma(z + a) = (z + a - \tfrac{1}{2}) \log z - z + \tfrac{1}{2} \log 2\pi + o(1)$$ where $o(1)$ refers to $|z| \to \infty$. We will also use the following well-known relationships for $\Gamma(z)$ [10, P.239]:

1) $\Gamma(z)\,\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$

2) $\Gamma\,(1+ z) = z\,\Gamma(z)$

Consequently, $\Gamma(\frac{1}{2} - \frac{z}{2}) = \Gamma\,(1-(\frac{z+1}{2})) = \dfrac{\pi}{\Gamma(\frac{z+1}{2})\,\sin \pi(\frac{z+1}{2})}$

$$= \frac{\pi}{\Gamma(\frac{z+1}{2}) \cos \frac{\pi z}{2}}$$

Also $\Gamma\,(\frac{z}{2}) = \frac{2}{z}\,\Gamma(1 + \frac{z}{2})$

Therefore, $G(z) = \dfrac{z\ \Gamma(\frac{1}{2} + \frac{z}{2}) \cos \frac{\pi z}{2}}{2\Gamma\,(1 + \frac{z}{2}) \cdot \pi}$

For Re $z \geq 0$ we can apply Stirlings formula (with $a = \frac{1}{2}$ in the numerator and $a = 1$ in the denominator), and get

$$G(z) = \frac{z \cos \frac{\pi z}{2}}{2\pi} \cdot \frac{(z/2)^{z/2}\ e^{-z/2}\sqrt{2\pi}\ e^{o(1)}}{(z/2)^{z/2+\frac{1}{2}}\ e^{-z/2}\sqrt{2\pi}\ e^{o(1)}}$$

$$= \frac{z \cos \frac{\pi z}{2}}{2\pi} \cdot \frac{\sqrt{2}}{\sqrt{z}} \cdot \frac{e^{o(1)}}{e^{o(1)}} \qquad \text{(we can assume } z \neq 0)$$

Therefore $|G(z)| \leq K \,|\cos \frac{\pi z}{2}|\,\sqrt{|z|}$ for Re $z \geq 0$

For Re $z \leq 0$ let $w = -z$. Then Re $w \geq 0$.

$G(z) = G(-w) = \dfrac{1}{\Gamma\,(\frac{-w}{2})\,\Gamma(\frac{1}{2} + \frac{w}{2})}$ which can be rewritten as

$$G(-w) = \frac{\sin(\pi\, w/2)}{-\pi}\ \frac{\Gamma(1 + \frac{w}{2})}{\Gamma(\frac{1}{2} + \frac{w}{2})}\ .$$

Since Re $w \geq 0$ we can use Stirling's formula and get

$$G(-w) = \frac{\sin \frac{\pi w}{2}}{\pi}\ \frac{(\frac{w}{2})^{w/2+\frac{1}{2}}\ e^{-w/2}\sqrt{2\pi}\ e^{o(1)}}{(\frac{w}{2})^{w/2}\ e^{-w/2}\ \sqrt{2\pi}\ e^{o(1)}}$$

Therefore $|G(-w)| \leq K \left|\sin \frac{\pi w}{2}\right| \sqrt{|w|}$

Since $w = -z$

$$|G(z)| \leq K\left|\sin \frac{\pi z}{2}\right|\sqrt{|z|} \quad \text{for Re } z \leq 0, \ |z| \geq 1$$

<u>THEOREM 5.2 $\{e^{-i(-1)^n nx}\}_{n=0}^{\infty}$ is incomplete in $L^2\,[-\frac{\pi}{2} - \epsilon, \frac{\pi}{2} + \epsilon]$ for any $\epsilon > 0$.</u>

<u>Proof</u>: For a given $\epsilon > 0$, take a positive integer n such that $\frac{\pi}{2n} < \epsilon$.

$$\text{Let } G(z) = \frac{1}{\Gamma(\frac{z}{2})\ \Gamma(\frac{1}{2} - \frac{z}{2})}.$$

Since $\frac{1}{\Gamma(z)}$ vanishes at $z = 0,-1,-2,\ldots$, $G(z)$ vanishes at $z = 0,1,-2,3,-4\ldots$

By the lemma, $G(z) = 0\,(\sqrt{|z|}\,)$ on the real axis and elsewhere

$G(z) = 0\,(\sqrt{|z|}\ e^{\frac{\pi\,|z|}{2}})$.

Let $F(z) = \dfrac{G(z)\ G(\frac{z}{2n})}{z(z + 2n)}$

Then $F(z)$ is an entire function in $L^2(-\infty,\infty)$ and satisfies

$F(z) = 0\,(e^{\pi\,(\frac{1}{2} + \frac{1}{2n})|z|})$

We will now use the Paley-Wiener theorem on Fourier transforms which is [6, P.13],

<u>THEOREM</u>: Let $F(z)$ be entire. Then $F(z) = 0(e^{A|z|})$ and $F(z) \in L^2(-\infty,\infty)$ iff $F(z)=\int_{-A}^{A} e^{izx}\, f(x)\, dx \quad f \in L^2\,[-A,A]$.

Hence our

$$F(z) = \int_{-\pi(\frac{1}{2}+\frac{1}{2}n)}^{\pi(\frac{1}{2}+\frac{1}{2}n)} e^{izx} f(x)dx \qquad f \in L^2 \left[-\frac{\pi}{2} - \frac{\pi}{2n}, \frac{\pi}{2} + \frac{\pi}{2n} \right]$$

Since $F(z) \not\equiv 0$ $f(x)$ is non-trivial

As $F(z) = 0$ for $z = 0, (-1)^{n+1}n$ we have that

$$\{e^{-i(-1)^n nx}\}_{n=0}^{\infty} \text{ is incomplete in } L^2 \left[-\frac{\pi}{2} - \frac{\pi}{2n}, \frac{\pi}{2} + \frac{\pi}{2n}\right]$$

and certainly in $L^2 \left[-\frac{\pi}{2} - \epsilon, \frac{\pi}{2} + \epsilon\right]$.

Now that we know that $\{e^{-i(-1)^n nx}\}_{n=0}^{\infty}$ is complete in an interval of length π (and no greater), a natural question which arises is "When can the constant term be dropped without affecting completeness?"

THEOREM 5.3. $\{e^{-i(-1)^n nx}\}_{n=1}^{\infty}$ is complete in $L^P \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ if $1 \leq P \leq 2$

Proof: Since we are dealing with pure exponentials, it will suffice to prove completeness on $[0,\pi]$. Also, by the "hierarchy" theorems, it will suffice to prove completeness in $L^2[0,\pi]$. Therefore, assume there exists $f(x) \in L^2[0,\pi]$ such that

$$\int_0^{\pi} e^{-i(-1)^n nx} f(x)\, dx = 0 \quad n = 1,2, \ldots$$

Then $\int_0^{\pi} \cos nx\, f(x)\, dx - i(-1)^n \int_0^{\pi} \sin nx\, f(x)\, dx = 0 \quad n = 1,2, \ldots$

Let a_n and b_n be as in the lemma to theorem 3.1.

Then $a_n - i(-1)^n b_n = 0 \qquad n = 1,2,\ldots$

Therefore $|a_n| = |b_n| \qquad n = 1,2,\ldots$

By the lemma to theorem 3.1,

$$\tfrac{1}{2}\,|a_o|^2 + \sum_{n=1}^{\infty} |a_n|^2. = \sum_{n=1}^{\infty} |b_n|^2$$

Therefore $a_o = 0$

i.e., $\int_o^{\pi} f(x)\, dx = 0.$

Thus $\int_o^{\pi} e^{-i(-1)^n nx} f(x)\, dx = 0 \quad n = 0,1,2,\ldots$

Since $\{e^{-i(-1)^n nx}\}_{n=0}^{\infty}$ is complete in $L^2[0,\pi]$, $f(x) = 0$ a.e.

<u>THEOREM 5.4. $\{e^{-i(-1)^n nx}\}_{n=1}^{\infty}$ is incomplete in $L^P[-\frac{\pi}{2}, \frac{\pi}{2}]$</u>
<u>if $2 < P < \infty$.</u>

<u>Proof:</u> We will use the following Fourier transform formula [8, P.186].

$$\int_{-\pi/2}^{\pi/2} (\cos t)^{a-2}\, e^{ixt}\, dt = \frac{\pi\,\Gamma\,(a-1)}{2^{a-2}\;\Gamma(\frac{a}{2}+\frac{x}{2})\;\Gamma(\frac{a}{2}-\frac{x}{2})} \qquad a > 1$$

Let $a = \frac{3}{2}$ and let $x = z + \frac{1}{2}$. Then

$$\int_{-\pi/2}^{\pi/2} \frac{e^{it/2}\, e^{izt}}{\sqrt{\cos t}}\, dt = \frac{\pi\,\Gamma\,(\frac{1}{2})\,\sqrt{2}}{\Gamma\,(1+\frac{z}{2})\;\Gamma(\frac{1}{2}-\frac{z}{2})}$$

which vanishes for $z = 1,-2,3,-4,\ldots$

Since $\dfrac{e^{it/2}}{\sqrt{\cos t}} \in L^P\,[-\frac{\pi}{2}\ \frac{\pi}{2}]$ if $P < 2$

$\{e^{ix}, e^{-2ix}, e^{3ix}, \ldots\}$ is incomplete in $L^P\,[-\pi/2, \pi/2]$ if $P > 2$.

THEOREM 5.5. $\{1,e^{ix},e^{-2ix},e^{3ix},e^{4ix},e^{-5ix},\ldots\}$ is complete in $C[-\frac{2\pi}{3}+\epsilon, \frac{2\pi}{3}-\epsilon]$ for any $\epsilon > 0$.

Proof: Assume there exists $d\mu(x)$ such that

$$\int_{-L}^{L} e^{izx}\, d\mu(x) = 0 \quad \text{for } z = 0,1,-2,3,4,5,\ldots \text{ where } 0 < L < \frac{2\pi}{3}.$$

Let $F(z) = \sin \pi(\frac{z+1}{3}) \int_{-L}^{L} e^{izx}\, d\mu(x)$.

Then $F(z)$ is an entire function, vanishes at all the non-negative integers and satisfies

$$F(z) = O(e^{(\frac{\pi}{3}+L)|z|})$$

Since $\frac{\pi}{3} + L < \pi$, by theorem A of chapter 4, $F(z) \equiv 0$.

Since $\sin \pi(\frac{z+1}{3})$ is obviously not identically zero,

$$\int_{-L}^{L} e^{izx}\, d\mu(x) \equiv 0$$

Obviously $\int_{-L}^{L} e^{inx}\, d\mu(x) = 0 \qquad n = 0,\pm 1, \pm 2,\ldots$

Since $L < \pi$ and $\{e^{inx}\}_{n=-\infty}^{\infty}$ is complete on $C[-L,L]$ we get that $d\mu(x) \equiv 0$.

THEOREM 5.6. $\{1, e^{ix}, e^{-i2x}, e^{3ix}, e^{4ix}, e^{-5ix},\ldots\}$ is incomplete in $L^1[-\frac{2\pi}{3}, \frac{2\pi}{3}]$.

Proof: Let $F(z) = \sin \frac{\pi z}{3} \sin\pi(\frac{z+2}{3})$.

Then F(z) is entire, vanishes at $z = 0,1,-2,\pm 3,4,-5,\pm 6, \ldots$

and satisfies $F(z) = O(e^{\frac{2\pi}{3}|z|})$

Let $G(z) = \dfrac{F(z)}{(z+3)(z+6)}$

G(z) is an entire function which vanishes at $z = 0,1,-2,3,4,-5,\ldots$, is in $L^1(-\infty,\infty)$ and satisfies

$$G(z) = O(e^{\frac{2\pi}{3}|z|}).$$

We will now use the following theorem on representation of entire functions as Fourier transforms [2, P.107].

THEOREM: If G(z) is an entire function, belonging to $L^1(-\infty,\infty)$ and satisfying $G(z) = O(e^{A|z|})$, then

$G(z) = \int_{-A}^{A} e^{izx}\,\phi(x)dx$ where ϕ is continuous on $[-A,A]$

(and hence bounded there).

Therefore, our $G(z) = \int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} e^{izx}\,\phi(x)dx$ where $\phi(x) \in L^\infty[-\frac{2\pi}{3},\frac{2\pi}{3}]$.

Since $G(z) = 0$ $z = 0,1,-2,3,4,-5,\ldots$, we have proven

$\{1,e^{ix}, e^{-2ix}, e^{3ix}, e^{4ix}, e^{-5ix},\ldots\}$ is incomplete in $L^1[-\frac{2\pi}{3},\frac{2\pi}{3}]$.

APPENDIX - 3 INEQUALITIES

1) $|(1+z)^2 + i(1-z)^2| \geq \frac{1}{4}$ for $\frac{1}{2} \leq |z| \leq 1$

Proof: Let $z = \rho e^{i\theta}$

$$|(1+z)^2 + i(1-z)^2| = |(1+\rho e^{i\theta})^2 + i(1-\rho e^{i\theta})^2|$$

$$= |[(1+\rho^2)\cos\theta + 2\rho + (1-\rho^2)\sin\theta] +$$

$$i[(1+\rho^2)\cos\theta - 2\rho - (1-\rho^2)\sin\theta]|$$

$$= \sqrt{2(1+\rho^2)^2\cos^2\theta + 2(2\rho + (1-\rho^2)\sin\theta)^2}$$

$$\geq \sqrt{2}\ |2\rho + (1-\rho^2)\sin\theta|$$

$$\geq \sqrt{2}\ (2\rho - (1-\rho^2))$$

$$= \sqrt{2}\ ((\rho+1)^2 - 2)$$

$$\geq \sqrt{2}\ ((\tfrac{1}{2}+1)^2 - 2) = \frac{\sqrt{2}}{4} > \frac{1}{4}$$

2) $\left|\dfrac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right| \geq \dfrac{1}{|\theta|}$ for $0 < |\theta| \leq \frac{\pi}{2}$

Proof. By a change of variable, it will suffice to show

$$\left|\frac{\cos\theta}{\sin\theta}\right| \geq \frac{1}{2|\theta|} \text{ for } 0 < |\theta| \leq \frac{\pi}{4}$$

Since both functions are odd, it will suffice to prove for $0 < |\theta| \leq \frac{\pi}{4}$ and since then everything is positive it will suffice to prove

$$\frac{\cos\theta}{\sin\theta} \geq \frac{1}{2\theta} \text{ for } 0 < \theta \leq \frac{\pi}{4}$$

We will use the following well-known inequalities:

$$\cos\theta \geq 1 - \frac{\theta^2}{2} \text{ and } \sin\theta < \theta \text{ for } \theta > 0$$

Therefore $\dfrac{\cos\theta}{\sin\theta} > \dfrac{1 - \frac{\theta^2}{2}}{\theta}$

Since $\theta \leq \frac{\pi}{4} \leq 1$ $\quad \dfrac{1 - \frac{\theta^2}{2}}{\theta} \geq \dfrac{1}{2\theta}$

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3) $1 + r^2 - 2r\cos\theta \geq \dfrac{2\theta^2 r}{\pi}$ for $|\theta| \leq \dfrac{\pi}{2}$ and $r \epsilon [0,1]$.

Proof: This is equivalent to proving that

$$\frac{1 + r^2}{2r} \geq \frac{\theta^2}{\pi} + \cos\theta \text{ for } |\theta| \leq \frac{\pi}{2} \text{ and } r \epsilon (0,1]$$

since the first inequality is obvious for $r = 0$, Let $F(r) = \dfrac{1 + r^2}{2r}$

and $g(\theta) = \dfrac{\theta^2}{\pi} + \cos\theta$

$F'(r) = \dfrac{-(r^2 - 1)^2}{4r^2} < 0$

$\therefore F(r) \geq F(1) = 1$ for $r \epsilon (0,1]$.

$g'(0) = \dfrac{2\theta}{\pi} - \sin\theta$ which is negative in $[0, \frac{\pi}{2}]$ by a well-known inequality

Therefore $g(\theta) \leq g(0) = 1$ for $\theta \epsilon [0, \pi/2]$

Since $g(\theta) = g(-\theta)$, $g(\theta) \leq 1$ for $\theta \epsilon [-\pi/2, \pi/2]$.

Therefore $\dfrac{1 + r^2}{2r} \geq \dfrac{\theta^2}{\pi} + \cos\theta$ for $|\theta| \leq \pi/2$ and $r \epsilon (0,1]$.

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