ABSTRACT

Intuitionistic Logic Model Theory and Forcing

Melvin Chris Fitting

The independence proofs of Cohen for the axiom of choice, the continuum hypothesis, and the axiom of constructability are re-formulated using S. Kripke's intuitionistic logic model theory. We define transfinite sequences of intuitionistic models with a 'class' model limit in a manner exactly analogous to the definition of Godel in the classical case of a transfinite sequence of (domains of) classical models, M_α, with a 'class' model limit, L. Classical independence results are established by working with the intuitionistic models themselves; no classical models are constructed, no countable classical models are required (though the definition of intuitionistic model is essentially the same as that of forcing.)

An intuitionistic (or forcing) generalization of the R_α sequence (sets with rank) is defined and some connections between it and Scott and Solovay's boolean valued models for set theory are established.

For completeness sake, the first six chapters provide a complete treatment of S. Kripke's intuitionistic logic model theory. Completeness proofs are given for tableau and axiomatic systems, compactness and Skolem-Lowenheim theorems are established, and relations with classical logic are shown. The connection between Kripke model theory and algebraic model theory is shown in the propositional case.

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INTUITIONISTIC LOGIC MODEL THEORY AND FORCING

by

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This work is dedicated to my parents.

Introduction

In 1963 P. Cohen established various fundamental independence results in set theory using a new technique which he called forcing. Since then there has been a deluge of new results of various kinds in set theory, proved using forcing techniques. It is a powerful method. It is, however, a method which is not as easy to interpret intuitively as the corresponding method of Gödel which establishes consistency results.

Gödel defines an intuitively meaningful transfinite sequence of (domains of) classical models, M_α, defines the class L to be the union of the M_α over all ordinals α, and shows L is a classical model for set theory [3; see also 2]. He then shows the axiom of constructability, the generalized continuum hypothesis, and the axiom of choice are true over L, establishing consistency.

In this dissertation we define transfinite sequences of S. Kripke's intuitionistic models [12] in a manner exactly analogous to that of Gödel in the classical case (in fact, the M_α sequence is a particular example). In a reasonable way we define a "class" model for each sequence, which is to be a limit model over all ordinals.

We show all the axioms of set theory are intuitionistically valid in the class models. Finally we show there are particular such sequences which provide: a class model in which the negation of the axiom of choice is intuitionistically valid; a class model in which the axiom of choice and the negation of the continuum hypothesis are intuitionistically valid; a class model in which the axiom of choice, the generalized continuum hypothesis, and the negation of the axiom of constructability are intuitionistically valid. From this, the <u>classical</u> independence results are shown to follow.

The definition of the sequences of intuitionistic models will be seen to be essentially the same as the definition of forcing in [2]. The difference is in the point of view. In Cohen's method one begins with a set M which is a <u>countable</u> model for set theory and, using forcing, one constructs a second countable model N "on top of" M. Forcing enables one to "discuss" N in M even though N is not a sub-model of M. Various such N are constructed for the different independence results. In this dissertation no countable models are required and no classical models are constructed. It is the forcing relation itself that is the center of attention [see 2, page 147], though now it has an intuitive interpretation.

A similar program has been carried out by Vopěnka and others. [See the series of papers: 20, 21, 22, 25, 5, 23, 6, 7, 24, 26]. The primary difference is that these use topological intuitionistic model theory while we use Kripke's, which is much closer in form to forcing. Also, the Vopěnka series uses Gödel-Bernays set theory and generalizes the F_α sequence, while we use Zermelo-Fraenkel set theory and generalize the M_α sequence. The Vopěnka treatment involves substantial topological considerations which we replace by more "logical" ones.

The dissertation is divided into two parts. In Part I we present a thorough treatment of the Kripke intuitionistic model theory. Part II consists of the set theory work discussed above.

Most of the material in Part I is not original but it is collected together and unified for the first time. The treatment is self-contained. Kripke models are defined (in notation different from that of Kripke). Tableau proof systems are defined using signed formulas (due to R. Smullyan), a device which simplifies the treatment. Three completeness proofs are presented (one for an axiom system, two for tableau systems), one due to Kripke [12], one due independently to R. Thomason [19] and the author, and one due to the author. We present proofs of compactness and Löwenheim-Skolem theorems.

Adapting a method of Cohen, we establish a few connections between classical and intuitionistic logic. In the propositional case we give the relationship between Kripke models and algebraic ones [15] (which provides a fourth completeness proof in the propositional case). Finally we present Schutte's proof of the intuitionistic Craig interpolation lemma [16], adapted to Kleene's tableau system G3 as modified by the use of signed formulas. No attempt is made to use methods of proof acceptable to intuitionists.

Chapter 7 begins Part II. In it we define the notion of an <u>intuitionistic</u> Zermelo-Fraenkel (Z-F) model, and the intuitionistic generalization of the Gödel M_α sequence. Most of the chapter is devoted to showing the class models resulting from the sequences of intuitionistic models are intuitionistic <u>Z-F</u> models. This result is demonstrated in rather complete detail, especially section 8 through 13, not because the work is particularly difficult, but because such models are comparatively unfamiliar.

In Chapter 8 the independence of the axiom of choice is shown.

In Chapter 9 we show how ordinals and cardinals may be represented in the intuitionistic models, and establish when such representatives exist.

Chapter 10 establishes the independence of the continuum hypothesis.

In Chapter 11 we give a way to represent constuctable sets in the intuitionistic models, and establish when such representatives exist.

Chapter 12 establishes the independence of the axiom of constructability.

Chapter 13 is a collection of various results. We establish a connection between the sequences of intuitionistic models and the classical M_α sequence. We give some conditions under which the axiom of choice and the generalized continuum hypothesis will be valid in the intuitionistic class models (thus completing chapters 10 and 12). Finally we present Vopěnka's method for producing classical non-standard set theory models from the intuitionistic class models without countability requirements [24].

The set theory work to this point is self-contained, given a knowledge of the Gödel consistency proof [3; in more detail, 2].

In Chapter 14 we present Scott and Solovay's notion ofboolean valued models for set theory [17]. We define an intuitionistic (or forcing) generalization of the R_α sequence (sets with rank) analogous to the Cohen generalization of the M_α sequence, and we establish some connections between intuitionistic and boolean valued models for set theory.

PART I

LOGIC

Chapter 1

Propositional Intuitionistic Logic - Semantics

Section 1

Formulas

We begin with a denumberable set of propositional variables A, B, C, $\cdots$, three binary connectives, $\wedge$, $\vee$, $\supset$, and one unary connective, $\sim$, together with left and right parantheses, (,). We shall informally use square and curly brackets, [,], {,}, for parentheses to make reading simpler.

The notion of well formed formula, or simply formula, is given recursively by the following rules:

F0: If A is a propositional variable, A is a formula.

F1: If X is a formula, so is $\sim X$.

F2,3,4: If X and Y are formulas, so are $(X \wedge Y)$, $(X \vee Y)$, $(X \supset Y)$

Remark: a propositional variable will sometimes be called an atomic formula.

It can be shown that the formation of a formula is unique. That is, for any given formula X, one and only one of the following can hold:

1) X is A for some propositional variable A.
2) There is a unique formula Y such that X is $\sim Y$.
3) There is a unique pair of formulas Y and Z and a unique binary connective b [$\wedge$, $\vee$, or $\supset$] such that X is (YbZ).

We make use of this uniqueness of decomposition but do not prove it here.

We shall omit writing outer parentheses in a formula when no confusion can result.

Until otherwise stated, we shall use A, B, and C for propositional variables, and X, Y, and Z to represent any formula.

The notion of immediate subformula is given by the following rules:

I0: A has no immediate subformula.

I1: $\sim X$ has exactly one immediate subformula, X.

I2,3,4: $(X \wedge Y)$, $(X \vee Y)$, $(X \supset Y)$, each has exactly two immediate subformulas, X and Y.

The notion of subformula is defined as follows:

S0: X is a subformula of X.

S1: If X is an immediate subformula of Y, then X is a subformula of Y.

S2: If X is a subformula of Y, and Y is a subformula of Z, then X is a subformula of Z.

By the degree of a formula is meant the number of occurences of logical connectives [$\sim, \wedge, \vee, \supset$] in the formula.

Section 2

Models and Validity

By a (propositional intuitionistic) model we mean an ordered triple $\langle G, R, \models \rangle$, where G is a non-empty set, R is a transitive, reflexive relation on G, and $\models$ (conveniently read "forces") is a relation between elements of G and formulas, satisfying the following conditions:

For any $\Gamma \varepsilon G$,

P0: if any $\Gamma \models A$ and $\Gamma R \Delta$ then $\Delta \models A$ [recall A is atomic]

P1: $\Gamma \models (X \wedge Y)$ iff $\Gamma \models X$ and $\Gamma \models Y$

P2: $\Gamma \models (X \vee Y)$ iff $\Gamma \models X$ or $\Gamma \models Y$

P3: $\Gamma \models \sim X$ iff for all $\Delta \varepsilon G$ such that $\Gamma R \Delta$, $\Delta \not\models X$.

P4: $\Gamma \models (X \supset Y)$ iff for all $\Delta \varepsilon G$ such that $\Gamma R \Delta$, if $\Delta \models X$, $\Delta \models Y$.

Remark: For $\Gamma \varepsilon G$, by Γ^* we shall mean any $\Delta \varepsilon G$ such that $\Gamma R \Delta$. Thus "for all Γ^*, $\varphi(\Gamma^*)$" shall mean "for all $\Delta \varepsilon G$ such that $\Gamma R \Delta$, $\varphi(\Delta)$" and "there is a Γ^* such that $\varphi(\Gamma^*)$" shall mean "there is a $\Delta \varepsilon G$ such that $\Gamma R \Delta$ and $\varphi(\Delta)$". Thus P3 and P4 can be written more simply as

P3: $\Gamma \models \sim X$ iff for all Γ^*, $\Gamma^* \not\models X$

P4: $\Gamma \models (X \supset Y)$ iff for all Γ^*, if $\Gamma^* \models X$ then $\Gamma^* \models Y$.

A particular formula X is called valid in the model $\langle G, R, \models \rangle$ if for all $\Gamma \varepsilon G$, $\Gamma \models X$.

X is called valid if X is valid in all models.

We will show later that the collection of all valid formulas coincides with the usual collection of propositional intuitionistic logic theorems.

When it is necessary to distinguish between validity in this sense and the more usual notion, we shall refer to the validity defined above as intuitionistic validity, and the usual notion as classical validity.

This notion of an intuitionistic model is due to Saul Kripke, and is presented, in different notation, in [12].

Examples of models will be found in section 5, chapter 2.

Section 3

Motivation

Let $\langle G, R, \models \rangle$ be a model. G is intended to be a collection of possible universes, or more properly, states of knowledge. Thus a particular Γ in G may be considered as a collection of (physical) facts known at a particular time. The relation R represents (possible) time succession. That is, given two states of knowledge, Γ and Δ of G, to say $\Gamma R \Delta$ is to say, if we now know Γ, it is possible that later we will know Δ. Finally, to say $\Gamma \models X$ is to say, knowing Γ, we know X, or, from the collection of facts Γ, we may deduce the truth of X.

Under this interpretation condition P3 of the last section, for example, may be interpreted as follows: from the facts Γ we may conclude $\sim X$ if and only if from no possible additional facts can we conclude X.

We might remark that under this interpretation it would seem reasonable that if $\Gamma \models X$ and $\Gamma R \Delta$ then $\Delta \models X$, that is, if from a certain amount of information we can deduce X, given addition information, we still can deduce X, or if at some time we know X is true, at any later time we still know X is true. We have required that this hold only for the case that X is atomic, but the other cases follow.

For other interpretations of this modeling, see the original paper [12].

For a different but closely related model theory in terms of forcing, see [4].

Section 4

Some properties of models

Lemma 1: Let $\langle G, R, \models \rangle$ and $\langle G, R, \models' \rangle$ be two models such that for any atomic formula A, and any $\Gamma \varepsilon G$, $\Gamma \models A$ iff $\Gamma \models' A$. Then $\models$ and $\models'$ are identical.

Proof: We must show that for any formula X, $\Gamma \models X \iff \Gamma \models' X$. This is done by induction on the degree of X and is straightforward. We present one case as an example.

Suppose X is ~Y and the result is known for all formulas of degree less than that of X [in particular, for Y] We show it for X.

$\Gamma \models X \iff \Gamma \models \sim Y$

(by definition)

$\iff (\forall \Gamma^*) \quad (\Gamma^* \not\models Y)$

(by hypothesis)

$\iff (\forall \Gamma^*) \quad (\Gamma^* \not\models' Y)$

(by definition)

$\iff \Gamma \models' \sim Y$

$\iff \Gamma \models' X$

Q.E.D.

Lemma 2: Let G be a non-empty set and R be a transitive, reflexive relation on G. Suppose $\models$ is a relation between elements of G and atomic formulas. Then $\models$ can be extended to a relation $\models'$ between elements of G and all formulas in such a way that $\langle G, R, \models' \rangle$ is a model.

Proof: We define $\models'$ as follows:

0) if $\Gamma \models A$ then $\Gamma^* \models' A$

1) $\Gamma \models' (X \wedge Y)$ if $\Gamma \models' X$ and $\Gamma \models' Y$

2) $\Gamma \models' (X \vee Y)$ if $\Gamma \models' X$ or $\Gamma \models' Y$

3) $\Gamma \models' \sim X$ if for all Γ^*, $\Gamma^* \not\models' X$

4) $\Gamma \models' (X \supset Y)$ if for all Γ^*, if $\Gamma^* \models' X$, $\Gamma^* \models' Y$.

This is an inductive definition, the induction being on the degree of the formula.

It is straightforward to show that $\langle G, R, \models' \rangle$ is a model.

Q.E.D.

From lemmas 1 and 2 we immediately have

Theorem: Let G be a non-empty set and R be a transitive, reflexive relation on G. Suppose $\models$ is a relation between elements of G and atomic formulas. Then $\models$ can be extended in one and only one way to a relation, also denoted by $\models$, between elements of G and formulas, such that $\langle G, R, \models \rangle$

is a model.

Theorem: Let $\langle G, R, \models\rangle$ be a model, X a formula, and $\Gamma, \Delta \in G$. If $\Gamma \models X$ and $\Gamma R \Delta$, then $\Delta \models X$.

Proof: A straightforward induction on the degree of X (it is known already for X atomic). For example, suppose the result is known for X, and $\Gamma \models \sim X$. By definition, for all Γ^*, $\Gamma^* \not\models X$. But $\Gamma R \Delta$ so any R-successor of Δ is an R-successor of Γ. Hence for all Δ^*, $\Delta^* \not\models X$, so $\Delta \models \sim X$. The other cases are similar.

Q.E.D.

Section 5

Algebraic models

In addition to the Kripke intuitionistic semantics presented above, there is an older algebraic semantics, that of pseudo-boolean algebras. In this section we state the algebraic semantics, and in the next we prove its equivalence with Kripke's semantics. A thorough treatment of pseudo-boolean algebras may be found in [15].

Def: A psuedo-boolean algebra (PBA) is a pair $\langle B, \leq\rangle$ where B is a non-empty set and $\leq$ is a partial ordering relation on B such that for any two elements a and b of B,

1) the least upper bound $(a \cup b)$ exists.

2) the greatest lower bound $(a \cap b)$ exists.

3) the pseudo compliment of a relative to b $(a \Rightarrow b)$, defined to be the largest $x \in B$ such that $a \cap x \leq b$, exists.

4) a least element Λ exists.

Remark: In the context $\Rightarrow$ is a mathematical symbol, not a metamathematical one.

Let $-a$ be $a \Rightarrow \Lambda$
and V be $-\Lambda$

Def: h is called a homomorphism (from the set W of formulas to the PBA $\langle B, \leq \rangle$) if $h: W \rightarrow B$ and

1) $h(X \wedge Y) = h(X) \cap h(Y)$

2) $h(X \vee Y) = h(X) \cup h(Y)$

3) $h(\sim X) = -h(X)$

4) $h(X \supset Y) = h(X) \Rightarrow h(Y)$

If $\langle B, \leq \rangle$ is a PBA and h is a homomorphism, the triple $\langle B, \leq, h \rangle$ is called a (algebraic) model for W, the set of formulas.

If X is a formula, X is called (algebraically) valid in the model $\langle B, \leq, h \rangle$ if $h(X) = V$.

X is called (algebraically) valid if X is valid in every model.

A proof may be found in [15] that the collection of all algebraically valid formulas coincides with the usual collection of intuitionistic theorems.

Section 6

Equivalence of algebraic and Kripke validity

First, let us suppose we have a Kripke model $\langle G, R, \models \rangle$ [we will not use the name "Kripke model" beyond this section.] We will define an algebraic model $\langle B, \leq, h \rangle$ such that for any formula X,

$$h(X) = \bigvee \quad \text{iff for all} \quad \Gamma \varepsilon G, \Gamma \models X.$$

Remark: This proof is based on exercise LXXXVI of [1].

If $b \subseteq G$, we call b R-closed if whenever $\Gamma \varepsilon b$ and $\Gamma R \Delta$, $\Delta \varepsilon b$.

We take for B the collection of all R-closed subsets of G. For the ordering relation $\leq$, we take $\subseteq$, set inclusion. Finally, we define h by

$$h(X) = \{\Gamma \varepsilon G \mid \Gamma \models X\}$$

It is fairly straightforward to show that $\langle B, \leq \rangle$ is a PBA. Of the four required properties, the first two are left to the reader. We now show:

if $a, b \varepsilon B$, there is a largest $x \varepsilon B$ such that $a \cap x \leq b$. We first note that the operations $\cup$ and $\cap$ are just the ordinary union and intersection.

Now, let p be the largest R-closed subset of $(G \dot{-} a) \cup b$ [where by $\dot{-}$ we mean ordinary set complimentation]. We will show that for all $x \varepsilon B$,

$$x \leq p \quad \text{iff} \quad a \cap x \leq b,$$

which suffices.

Suppose $x \leq p$ Then

$x \subseteq (G \dot{-} a) \cup b$

$a \cap x \subseteq a \cap [(G \dot{-} a) \cup b]$

$a \cap x \subseteq a \cap b$

$a \cap x \subseteq b$

$a \cap x \leq b$

Converseley, suppose $a \cap x \leq b$. Then

$(a \cap x) \cup (x \dot{-} a) \subseteq b \cup (x \dot{-} a)$

$x \subseteq b \cup (x \dot{-} a)$

$x \subseteq b \cup (G \dot{-} a)$

but $x \varepsilon B$, so x is R-closed. Hence

$x \subseteq p$

$x \leq p$

The reader may verify that $\phi \varepsilon B$ and is a least element.

Next we remark that h is a homomorphism. We demonstrate only one of the four cases, case 4. Thus we must show that $h(X \supset Y)$ is the largest $x \varepsilon B$ such that

$$h(X) \cap x \leq h(Y)$$

First we show

$$h(X) \cap h(X \supset Y) \leq h(Y)$$

That is,

$$\{\Gamma \mid \Gamma \models X\} \cap \{\Gamma \mid \Gamma \models X \supset Y\} \subseteq \{\Gamma \mid \Gamma \models Y\}$$

But it is clear from the definition that if $\Gamma \models X$ and $\Gamma \models X \supset Y$, then $\Gamma \models Y$.

Next, suppose there is some $b \varepsilon B$ such that $h(X) \cap b \leq h(Y)$ but $h(X \supset Y) < b$. Then there must be some $\Gamma \varepsilon G$ such that $\Gamma \varepsilon b$ but $\Gamma \not\varepsilon h(X \supset Y)$, i.e. $\Gamma \not\models X \supset Y$. Since $\Gamma \not\models X \supset Y$, there must be some Γ^* such that $\Gamma^* \models X$ but $\Gamma^* \not\models Y$. Since b is R-closed, $\Gamma^* \varepsilon b$. But also, $\Gamma^* \varepsilon h(X)$, so $\Gamma^* \varepsilon h(X) \cap b$, and so by assumption, $\Gamma^* \varepsilon h(Y)$, that is, $\Gamma^* \models Y$, a contradiciton. Thus $h(X \supset Y)$ is largest.

Thus $\langle B, \leq, h \rangle$ is an algebraic model. We leave it to the reader to verify that the unit element $\vee$ of B is G itself.

Hence $h(X) = \bigvee$ iff for all $\Gamma \varepsilon G$, $\Gamma \models X$.

Conversely, suppose we have an algebraic model $\langle B, \leq, h \rangle$. We will define a Kripke model $\langle G, R, \models \rangle$ so that for any formula X,

$$h(X) = \bigvee \quad \text{iff} \quad \text{for all } \Gamma \varepsilon G, \ \Gamma \models X.$$

<u>Lemma 1:</u> Let F be a filter in B and suppose $(a \Rightarrow b) \not\varepsilon F$. Then the filter generated by F and a does not contain b.

<u>Proof</u>: If the filter generated by F and a contained b, then [15, pg. 46-8.2] for some $c \varepsilon F$, $c \cap a \leq b$. So $c \leq (a \Rightarrow b)$ and hence $(a \Rightarrow b) \varepsilon F$ by [15, pg. 46, 8.2] again.

Q.E.D.

<u>Lemma 2:</u> Let F be a proper filter in B and suppose $-a \not\varepsilon F$. Then the filter generated by F and a is also proper.

<u>Proof:</u> By lemma 1, since $-a = (a \Rightarrow \bigwedge)$.

Q.E.D.

<u>Lemma 3:</u> Let F be a filter in B and suppose $a \not\varepsilon F$. Then F can be extended to a prime filter P such that $a \not\varepsilon P$.

Proof: (This is a slight modification of [15, pg. 49, 9.2], included for completeness).

Let O be the collection of all filters in B not containing a. O is partially ordered by $\subseteq$.

O is non-empty since $F \in O$.

Any chain in O has an upper bound since the union of any chain of filters is a filter.

By Zorn's lemma, O contains a maximal element P. Of course, $a \notin P$. We need only show P is prime. Suppose P is not prime. Then for some $a_1, a_2 \in B$,

$$a_1 \cup a_2 \in P, \quad a_1 \notin P, \quad a_2 \notin P.$$

Let S_1 be the filter generated by P and a_1, and S_2 be the filter generated by P and a_2.

Suppose $a \in S_1$ and $a \in S_2$. Then [15, pg. 46, 8.2] for some $c_1, c_2 \in P$, $a_1 \cap c_1 \leq a$ and $a_2 \cap c_2 \leq a$.

So, for $c = c_1 \cap c_2$,

$a_1 \cap c \leq a$ and $a_2 \cap c \leq a$.

hence $(a_1 \cup a_2) \cap c \leq a$.

But $c \in P$ and $(a_1 \cup a_2) \in P$

so $a \in P$. But $a \notin P$, so

either $a \notin S_1$ or $a \notin S_2$.

Suppose $a \notin S_1$. By definition, $S_1 \in O$. But S_1 is the filter generated by P and a_1, hence $P \subsetneq S_1$, so P is not maximal, a contradiction. Similarly if $a \notin S_2$.

Thus P is prime

Q.E.D.

Now we proceed with the main result. Recall, we have $\langle B, \leq, h \rangle$.

Let G be the collection of all proper prime filters in B.

Let R be $\subseteq$, set inclusion.

For any $\Gamma \varepsilon G$ and any formula X, let $\Gamma \models X$ if $h(X) \varepsilon \Gamma$.

To show the resulting structure $\langle G, R, \models \rangle$ is a model, we note property P0 is immediate. To show P1:

$$\begin{aligned} \Gamma \models (X \wedge Y) \quad & \text{iff} \quad h(X \wedge Y) \varepsilon \Gamma \\ & \text{iff} \quad h(X) \cap h(Y) \varepsilon \Gamma \\ & \text{iff} \quad h(X) \varepsilon \Gamma \text{ and } h(Y) \varepsilon \Gamma \\ & \text{iff} \quad \Gamma \models X \text{ and } \Gamma \models Y \end{aligned}$$

[using the facts that h is a homomorphism and Γ is a filter].

Similarly we show P2 using the fact that Γ is prime.

To show P3 :

Suppose $\Gamma \models \sim X$. Then $h(\sim X) \varepsilon \Gamma$,

$$\begin{aligned} \text{so} \quad & (\forall \Delta \varepsilon G)\ (\Gamma \subseteq \Delta \text{ implies } h(\sim X)\ \varepsilon \Delta) \\ & (\forall \Delta \varepsilon G)\ (\Gamma \subseteq \Delta \text{ implies } h(X) \not\varepsilon \Delta) \\ & (\forall \Delta \varepsilon G)\ (\Gamma R \Delta \text{ implies } \Delta \not\models X) \end{aligned}$$

i.e. for all Γ^*, $\Gamma^* \not\models X$.

[using the fact that $h(\sim X) \varepsilon \Delta$ and $h(X) \varepsilon \Delta$ imply $-h(X) \cap h(X) \varepsilon \Delta$, so $\Lambda \varepsilon \Delta$ and Δ is not proper].

Suppose $\Gamma \not\models \sim X$. Then $h(\sim X) \not\in \Gamma$, or $-h(X) \not\in \Gamma$. By lemma 2, the filter generated by Γ and $h(X)$ is proper. By lemma 3, this filter can be extended to a proper prime filter Δ. Then $\Gamma \subseteq \Delta$ and $h(X) \in \Delta$. So $(\exists \Delta \in G)$ $(\Gamma R \Delta$ and $\Delta \models X)$

i.e. for some Γ^*, $\Gamma^* \models X$.

P4 is shown in the same way, but using lemma 1 instead of lemma 2.

Thus $\langle G, R, \models \rangle$ is a model.

Finally, to establish the desired equivalence, suppose first, $h(X) = \bigvee$. Since $\bigvee$ is an element of every filter, for all $\Gamma \in G$, $\Gamma \models X$. Converseley, suppose $h(X) \neq \bigvee$. But $\{\bigvee\}$ is a filter and $h(X) \not\in \{\bigvee\}$. By lemma 3, we can extend $\{\bigvee\}$ to a proper prime filter Γ such that $h(X) \not\in \Gamma$. Thus $\Gamma \in G$ and $\Gamma \not\models X$.

Thus we have shown

<u>Theorem:</u> X is Kripke valid if and only if X is algebraically valid.

CHAPTER 2

Propositional Intuitionistic Logic - Proof Theory

Section 1

Beth tableaus

In this section we present a modified version of a proof system due originally to Beth. It is based on [1, section 145], but at the suggestion of R. Smullyan, we have introduced signed formulas and single trees in place of the unsigned formulas and dual trees of Beth.

By a signed formula we mean TX or FX where X is a formula.

If S is a set of signed formulas and H is a single signed formula, we will write $S \cup \{H\}$ simply as {S, H} or sometimes, S, H.

First we state the reduction rules, then we describe their use. S is any set (possibly empty) of signed formulas, and X and Y are any formulas.

$$T\wedge \quad \frac{S,\ TX\wedge Y}{S,\ TX,\ TY} \qquad\qquad F\wedge \quad \frac{S,\ FX\wedge Y}{S,\ FX \mid S,\ FY}$$

T∨ $$\frac{S,\ TX\vee Y}{S,\ TX \mid S,\ TY}$$

F∨ $$\frac{S,\ FX\vee Y}{S,\ FX,\ FY}$$

T~ $$\frac{S,\ T{\sim}X}{S,\ FX}$$

F~ $$\frac{S,\ F{\sim}X}{S_T,\ TX}$$

T⊃ $$\frac{S,\ TX\supset Y}{S,\ FX \mid S,\ TY}$$

F⊃ $$\frac{S,\ FX\supset Y}{S_T,\ TX,\ FY}$$

In rules F~ and F⊃ above, S_T means $\{TX \mid TX \varepsilon S\}$.

Remark: S is a set, and hence {S, TX} is the same as {S, TX, TX}. Thus duplication and elimination rules are not necessary.

If $\mathcal{U}$ is a set of signed formulas, we say one of the above rules, call it rule R, applies to $\mathcal{U}$ if by appropriate choice of S, X, and Y, the collection of signed formulas above the line in rule R becomes $\mathcal{U}$.

By an application of rule R to the set $\mathcal{U}$ we mean the replacement of $\mathcal{U}$ by $\mathcal{U}_1$ (or by $\mathcal{U}_1$ and $\mathcal{U}_2$ if R is F∧, T∨, or T⊃) where $\mathcal{U}$ is the set of formulas above the line in rule R (after suitable substition for S, X, and Y) and $\mathcal{U}_1$ (or $\mathcal{U}_1$, $\mathcal{U}_2$) is the set of formulas below. This assumes R applies to $\mathcal{U}$. Otherwise, the result is again $\mathcal{U}$. For example, by applying rule F⊃ to the set {TX, FY, FZ⊃W} we may get the set {TX, TZ, FW}. By applying rule T∨ to the set {TX, FY, TZ∨W} we may get the two sets {TX, FY, TZ} and {TX, FY, TW}.

By a configuration we mean a finite collection $\{S_1, S_2, \ldots, S_n\}$ of sets of signed formulas.

By an application of the rule R to the configuration $\{S_1, S_2, \ldots, S_n\}$ we mean the replacement of this configuration with a new one which is like the first except for containing, instead of some S_i, the result (or results) of applying rule R to S_i.

By a tableau we mean a finite sequence of configurations $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ in which each configuration except the first is the result of applying one of the above rules to the preceeding configuration.

A set S of signed formulas is closed if it contains both TX and FX for some formula X.

A configuration $\{S_1, S_2, \ldots, S_n\}$ is closed if each S_i in it is closed.

A tableau $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ is closed if some $\mathcal{C}_i$ in it is closed.

By a tableau for a set S of signed formulas, we mean a tableau $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ in which $\mathcal{C}_1$ is $\{S\}$.

A finite set of signed formulas S, is inconsistent if some tableau for S is closed. Otherwise S is consistent.

X is a theorem if {FX} is inconsistent, and a closed tableau for {FX} is called a proof of X. If X is a theorem, we write $\vdash_I X$.

We will show in the next few sections the correctness and completeness of the above system relative to the semantics of Chapter 1.

Examples of proofs in this system may be found in Section 5.

We have presented this system in a very formal fashion because it makes talking about it easier. In practice there are many simplifications which will become obvious in any attempt to use the method. Also, proofs may be written in a tree form. We find the resulting simplified system the easiest to use of all the intuitionistic proof systems, except in some cases, the system resulting by the same simplifications from the closely related one presented in Section 4 of Chapter 6. A full treatment of the corresponding classical tableau system, with practical simplifications, may be found in [18].

Section 2

Correctness of Beth Tableaus

Def: We call a set of signed formulas,

$$\{TX_1, \ldots, TX_n, FY_1, \ldots, FY_m\},$$

realizable if there is some model $\langle G, R, \models \rangle$ and some $\Gamma \in G$ such that $\Gamma \models X_1, \ldots, \Gamma \models X_n, \Gamma \not\models Y_1, \ldots, \Gamma \not\models Y_m$. We say that Γ realizes the set.

If $\{S_1, S_2, \ldots, S_n\}$ is a configuration, we call it realizable if some S_i in it is realizable.

<u>Theorem:</u> Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ be a tableau. If $\mathcal{C}_i$ is realizable, so is $\mathcal{C}_{i+1}$.

<u>Proof</u>: We have eight cases, depending on the rule whose application produced $\mathcal{C}_{i+1}$ from $\mathcal{C}_i$.

<u>Case 1:</u> $\mathcal{C}_i$ is $\{\ldots, \{S, TX \vee Y\}, \ldots\}$ and $\mathcal{C}_{i+1}$ is $\{\ldots, \{S,TX\}, \{S,TY\}, \ldots\}$. Since $\mathcal{C}_i$ is realizable, some element of it is realizable. If that element is not $\{S,TX \vee Y\}$, the same element of $\mathcal{C}_{i+1}$ is realizable. If that element is $\{S,TX \vee Y\}$, then for some model $\langle G, R, \models\rangle$ and some $\Gamma \varepsilon G$, Γ realizes $\{S,TX \vee Y\}$. That is, Γ realizes S and $\Gamma \models (X \vee Y)$. Then $\Gamma \models X$ or $\Gamma \models Y$, so either Γ realizes $\{S,TX\}$ or $\{S,TY\}$. In either case, $\mathcal{C}_{i+1}$ is realizable.

<u>Case 2:</u> $\mathcal{C}_i$ is $\{\ldots, \{S, F{\sim}X\}, \ldots\}$ and $\mathcal{C}_{i+1}$ is $\{\ldots, \{S_T, TX\}, \ldots\}$. $\mathcal{C}_i$ is realizable, and it suffices to consider the case that $\{S, F{\sim}X\}$ is the realizable element. Then, there is a model $\langle G, R, \models\rangle$ and a $\Gamma \varepsilon G$ such that Γ realizes S and $\Gamma \not\models {\sim}X$. Since $\Gamma \not\models {\sim}X$, for some $\Gamma^* \varepsilon G$, $\Gamma^* \models X$. But clearly, if Γ realizes S, Γ^* realizes S_T [by the second theorem of Chapter 1, section 4], hence Γ^* realizes $\{S_T,TX\}$ and $\mathcal{C}_{i+1}$ is realizable.

The other six cases are similar.

Q.E.D.

Corollary: The system of Beth tableaus is correct; that is, if $\vdash_I X$, X is valid.

Proof: We show the contrapositive. Suppose X is not valid. Then there is a model $\langle G, R, \models \rangle$ and a $\Gamma \varepsilon G$ such that $\Gamma \not\models X$. In other words, $\{F X\}$ is realizable. But a proof of X is a closed tableau $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ in which $\mathcal{C}_1$ is $\{\{F X\}\}$. But $\mathcal{C}_1$ is realizable, hence each $\mathcal{C}_i$ is realizable. But obviously a realizable configuration cannot be closed. Hence $\not\vdash_I X$.

Q.E.D.

Section 3

Hintikka collections

In classical logic, a set S of signed formulas is sometimes called downward saturated, or a Hintikka set, if

$$TX \wedge Y \ \varepsilon\ S \Rightarrow TX\ \varepsilon\ S \quad \text{and} \quad TY\ \varepsilon\ S$$

$$FX \vee Y \ \varepsilon\ S \Rightarrow FX\ \varepsilon\ S \quad \text{and} \quad FY\ \varepsilon\ S$$

$$TX \vee T \ \varepsilon\ S \Rightarrow TX\ \varepsilon S \quad \text{or} \quad TY\ \varepsilon\ S$$

$$FX \wedge Y \ \varepsilon\ S \Rightarrow FX\ \varepsilon\ S \quad \text{or} \quad FY\ \varepsilon\ S$$

$T{\sim}X \varepsilon S \Rightarrow FX \varepsilon S$

$TX \supset Y \varepsilon S \Rightarrow FX \varepsilon S$ or $TY \varepsilon S$

$F{\sim}X \varepsilon S \Rightarrow TX \varepsilon S$

$FX \supset Y \varepsilon S \Rightarrow TX \varepsilon S$ and $FY \varepsilon S$

Remark: The names Hintikka set and downward saturated set were given by Smullyan [18]. Hintikka, their originator, called them model sets.

Hintikka showed that any consistent downward saturated set could be included in a set for which the above properties hold with => replaced by <=>. From this follows the completeness of certain classical tableau systems. This approach is thoroughly developed by Smullyan in [18].

We now introduce a corresponding notion in intuitionistic logic, which we call a Hintikka collection. While its intuitive appeal may not be as immediate as in the classical case, its usefulness is as great.

Def: Let G be a collection of consistent sets of signed formulas. We call G a Hintikka collection if, for any $\Gamma \varepsilon G$,

$TX \wedge Y \varepsilon \Gamma \Rightarrow TX \varepsilon \Gamma$ and $TY \varepsilon \Gamma$

$FX \vee Y \varepsilon \Gamma \Rightarrow FX \varepsilon \Gamma$ and $FY \varepsilon \Gamma$

$T X \vee Y \in \Gamma \quad \Rightarrow \quad TX \in \Gamma \text{ or } TY \in \Gamma$

$F X \wedge Y \in \Gamma \quad \Rightarrow \quad FX \in \Gamma \text{ or } FY \in \Gamma$

$T{\sim}X \in \Gamma \quad \Rightarrow \quad FX \in \Gamma$

$T X \supset Y \in \Gamma \quad \Rightarrow \quad FX \in \Gamma \text{ or } TY \in \Gamma$

$F{\sim}X \in \Gamma \quad \Rightarrow \quad$ for some $\Delta \in G$, $\Gamma_T \subseteq \Delta$ and $TX \in \Delta$

$F X \supset Y \in \Gamma \quad \Rightarrow \quad$ for some $\Delta \in G$, $\Gamma_T \subseteq \Delta$, $TX \in \Delta$, $FY \in \Delta$

Def: Let G be a Hintikka collection. We call $\langle G, R, \models \rangle$ a model for G if

1) $\langle G, R, \models \rangle$ is a model
2) $\Gamma_T \subseteq \Delta \quad \Rightarrow \quad \Gamma R \Delta$
3) $TX \in \Gamma \quad \Rightarrow \quad \Gamma \models X$

   $FX \in \Gamma \quad \Rightarrow \quad \Gamma \not\models X$

Theorem: There is a model for any Hintikka collection.

Proof: Let G be a Hintikka collection. Define R by: $\Gamma R \Delta$ if $\Gamma_T \subseteq \Delta$. If A is atomic, let $\Gamma \models A$ if $TA \in \Gamma$, and extend $\models$ to produce a model $\langle G, R, \models \rangle$. Showing property 3) is a straightforward induction on the degree of X. We give one case as illustration. Suppose X is ${\sim}Y$ and the result is known for Y.

Then $T{\sim}X\varepsilon\Gamma \Rightarrow (\forall \Delta\varepsilon G)(\Gamma_T \subseteq \Delta \Rightarrow T{\sim}X\varepsilon\Delta)$

$\Rightarrow (\forall \Delta\varepsilon G)(\Gamma_T \subseteq \Delta \Rightarrow FX\varepsilon\Delta)$

$\Rightarrow (\forall \Delta\varepsilon G)(\Gamma R \Delta \Rightarrow \Delta \not\models X)$

$\Rightarrow \Gamma \models {\sim}X$

$F{\sim}X\varepsilon\Gamma \Rightarrow (\exists \Delta\varepsilon G)(\Gamma_T \subseteq \Delta \text{ and } TX\varepsilon\Gamma)$

$\Rightarrow (\exists \Delta\varepsilon G)(\Gamma R \Delta \text{ and } \Delta \models X)$

$\Gamma \not\models {\sim}X$

Q.E.D.

It follows from this theorem that to show the completeness of Beth tableaus we need only show the following: If $\not\vdash_I X$, then there is a Hintikka collection G such that for some $\Gamma\varepsilon G$, $FX\varepsilon\Gamma$.

Section 4

Completeness of Beth tableaus

Let S be a set of signed formulas. By $\mathcal{F}(S)$ we mean the collection of all signed subformulas of formulas in S. If S is finite, $\mathcal{F}(S)$ is finite.

Let S be a finite, consistent set of signed formulas. We define a reduction sequence for S (there may be many) as follows:

Let S_o be S.

Having defined S_n, a finite consistent set of signed formulas, suppose one of the following Beth reduction rules applies to S_n: $T\wedge$, $F\wedge$, $T\vee$, $F\vee$, $T\sim$, or $T\supset$. Choose one which applies, say $F\wedge$. Then S_n is $\{U, FX\wedge Y\}$. This is consistent, so clearly, either $\{U, FX\wedge Y, FX\}$ or $\{U, FX\wedge Y, FY\}$ is consistent. Let S_{n+1} be $\{U, FX\wedge Y, FX\}$ if consistent, otherwise, let S_{n+1} be $\{U, FX\wedge Y, FY\}$. Similarly, if $T\wedge$ applies and was chosen, then S_n is $\{U, TX\wedge Y\}$. Since this is consistent, $\{U, TX\wedge Y, TX, TY\}$ is consistent. Let this be S_{n+1}. In this way we define a sequence $S_0, S_1, S_2, \cdots$. This sequence has the property $S_n \subseteq S_{n+1}$. Further, each S_n is finite, and consistent. Since each $S_n \subseteq \mathcal{F}(S)$, there are only a finite number of different possible S_n. Consequently, there must be a member of the sequence, say S_n, such that the application of any one of the rules (except $F\sim$ or $F\supset$) produces S_n again. Call such an S_n a <u>reduced</u> set of S, and denote it by S'. Clearly any finite, consistent set of signed formulas has a finite, consistent reduced set. Moreover, if S' is a reduced set, it has the following suggestive properties:

$$TX\wedge Y\varepsilon S' \Rightarrow TX\varepsilon S' \text{ and } TY\varepsilon S'$$
$$FX\vee Y\varepsilon S' \Rightarrow FX\varepsilon S' \text{ and } FY\varepsilon S'.$$
$$TX\vee Y\varepsilon S' \Rightarrow TX\varepsilon S' \text{ or } TY\varepsilon S'$$
$$FX\wedge Y\varepsilon S' \Rightarrow FX\varepsilon S' \text{ or } FY\varepsilon S'$$
$$T\sim X\varepsilon S' \Rightarrow FX\varepsilon S'$$
$$TX\supset Y\varepsilon S' \Rightarrow FX\varepsilon S' \text{ or } TY\varepsilon S'$$

S' is consistent.

Now, given any finite, consistent set of signed formulas, S, we form the collection of associated sets as follows:

If $F{\sim}X \varepsilon S$, $\{S_T, TX\}$ is an associated set.

If $FX \supset Y \varepsilon S$, $\{S_T, TX, FY\}$ is an associated set.

Let $\mathcal{A}(S)$ be the collection of all associated sets of S. $\mathcal{A}(S)$ is finite, since $U \varepsilon \mathcal{A}(S)$ implies $U \subseteq \mathcal{F}(S)$ and $\mathcal{F}(S)$ is finite.

$\mathcal{A}(S)$ has the following properties: if S is consistent, any associated set is consistent, and

$F{\sim}X \varepsilon S \Rightarrow$ for some $U \varepsilon \mathcal{A}(S)$
$S_T \subseteq U$, $TX \varepsilon U$

$FX \supset Y \varepsilon S \Rightarrow$ for some $U \varepsilon \mathcal{A}(S)$
$S_T \subseteq U$, $TX \varepsilon U$, $FY \varepsilon U$

Now we proceed with the proof of completeness.

Suppose $\nvdash_I X$. Then $\{FX\}$ is consistent. Extend it to its reduced set, S_0.

Form $\mathcal{A}(S_0)$. Let the elements of $\mathcal{A}(S_0)$ be $U_1, U_2, \ldots, U_n$. Let S_1 be the reduced set of $U_1, \ldots,$ S_n be the reduced set of U_n. Thus, we have the sequence $S_0, S_1, S_2, \ldots, S_n$.

Next form $\mathcal{A}(S_1)$. Call its elements U_{n+1}, $U_{n+2}, \ldots, U_m$. Let S_{n+1} be the reduced set of U_{n+1} and so on. Thus, we have the sequence $S_0, S_1, \ldots, S_n, S_{n+1}, \ldots, S_m$. Now we repeat the process with S_2, and so on.

In this way we form a sequence $S_0, S_1, S_2, \ldots$. Since each $S_i \subseteq \mathcal{F}(S)$, there are only finitely many possible different S_i. Thus we must reach a point S_k of the sequence such that any continuation repeats an earlier member.

Let G be the collection $\{S_0, S_1, \ldots, S_k\}$ It is easy to see that G is a Hintikka collection. But $FX \varepsilon S_0 \varepsilon G$. Thus we have shown:

<u>Theorem:</u> Beth tableaus are complete.

<u>Remark:</u> This proof also establishes that propositional intuitionistic logic is decidable. For, if we follow the above procedure beginning with FX, after a finite number of steps we will have either a closed tableau for $\{FX\}$, or a counter-model for X. Moreover, the number of steps may be bounded in terms of the degree of X.

The completeness proof presented here is, in essence, the original proof of Kripke [12]. For a different tableau completeness proof, see section 6, chapter 5, where it is given for first order logic. For a completeness proof of an axiom system, see section 10, chapter 5, where it also is given for a first order system. The work in section 6, chapter 1 provides an algebraic completeness proof, since the Lindenbaum algebra of intuitionistic logic is easily shown to be a pseudo-boolean algebra. See [15].

Section 5

Examples

In this section, so that the reader may gain familiarity with the foregoing, we present a few theorems and non-theorems of intuitionistic propositional logic, together with their proofs or counter-models.

We show

1) $\nvdash_I A \vee \sim A$

2) $\vdash_I \sim\sim(A \vee \sim A)$

3) $\nvdash_I \sim\sim A \supset A$

4) $\vdash_I (A \vee B) \supset \sim(\sim A \wedge \sim B)$

5) $\nvdash_I \sim\sim(A \vee B) \supset (\sim\sim A \vee \sim\sim B)$

For the general principle connecting 1) and 2) see section 8, chapter 4.

1) $\nvdash_I A \vee \sim A$

A counter example for this is the following:

$G = \{\Gamma, \Delta\}$

$\Gamma R \Gamma$, $\Gamma R \Delta$, $\Delta R \Delta$

$\Delta \models A$ is the $\models$ relation for atomic formulas, and $\models$ is extended to all formulas as usual.

We may schematically represent this model by

$$\Gamma$$

$$|$$

$$\Delta \models A$$

We claim $\Gamma \not\models A \vee {\sim} A$. Suppose not. If $\Gamma \models A \vee {\sim} A$, either $\Gamma \models A$ or $\Gamma \models {\sim} A$. But $\Gamma \not\models A$. If $\Gamma \models {\sim} A$ then since $\Gamma R \Delta$, $\Delta \not\models A$, but $\Delta \models A$. Hence $\Gamma \not\models A \vee {\sim} A$.

2) $\vdash_I {\sim}{\sim}(A \vee {\sim} A)$

A tableau proof for this is the following, where the reasons for the steps are obvious.

$\{\{F{\sim}{\sim}(A \vee {\sim}A)\}\}$

$\{\{T{\sim}\ (A \vee {\sim}A)\}\}$

$\{\{T{\sim}\ (A \vee {\sim}A),\ \ F\ (A \vee {\sim}A)\}\}$

$\{\{T{\sim}\ (A \vee {\sim}A),\ \ FA,\ F{\sim}A\}\}$

$\{\{T{\sim}\ (A \vee {\sim}A),\ TA\}\}$

$\{\{F\ \ (A \vee {\sim}A),\ TA\}\}$

$\{\{FA,\ F{\sim}A,\ TA\}\}$

3) $\not\vdash_I {\sim}{\sim}A \supset A$.

The model of example 1) has the property that $\Gamma \models {\sim}{\sim}A$ but $\Gamma \not\models A$.

4) $\vdash_I (A \vee B) \supset {\sim}\ ({\sim}A \wedge {\sim}B)$

The following is a proof:

$$\{\{\ F((A \vee B) \supset \sim (\sim A \wedge \sim B))\}\}$$
$$\{\{\ T(A \vee B),\ F\sim(\sim A \wedge \sim B)\}\}$$
$$\{\{\ T(A \vee B),\ T(\sim A \wedge \sim B)\}\}$$
$$\{\{\ T(A \vee B),\ T\sim A,\ T\sim B\}\}$$
$$\{\{\ T(A \vee B),\ FA,\ T\sim B\}\}$$
$$\{\{\ T(A \vee B),\ FA,\ FB\}\}$$
$$\{\{TA,\ FA,\ FB\},\ \{TB,\ FA,\ FB\}\}$$

5) $\nvdash_I\ \sim\sim(A \vee B) \supset (\sim\sim A \vee \sim\sim B)$

A counter example is the following:

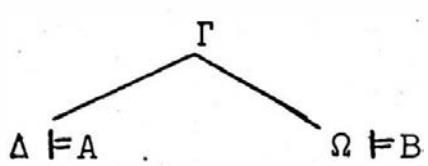
$G = \{\Gamma, \Delta, \Omega\}$

$\Gamma R \Gamma$, $\Delta R \Delta$, $\Omega R \Omega$

$\Gamma R \Delta$, $\Gamma R \Omega$

$\Delta \models A$, $\Omega \models B$ is the $\models$ relation for atomic formulas, and $\models$ is extended as usual.

We may schematically represent this model by



Now, $\Delta \models A$, so $\Delta \models A \vee B$.

Likewise, $\Omega \models A \vee B$. It follows that $\Gamma \models \sim\sim(A \vee B)$

But if $\Gamma \models \sim\sim A \vee \sim\sim B$, either $\Gamma \models \sim\sim A$ or $\Gamma \models \sim\sim B$

If $\Gamma \models \sim\sim A$, it would follow that $\Omega \models A$. If $\Gamma \models \sim\sim B$, it would follow that $\Delta \models B$. Thus $\Gamma \not\models \sim\sim A \vee \sim\sim B$.

CHAPTER 3

Related Systems of Logic

Section 1

f - primitive intuitionistic logic - semantics

This is an alternative formulation of intuitionistic logic in which a symbol f is taken as primitive, instead of $\sim$, which is then re-introduced as a formal abbreviation, $\sim X$ for $X \supset f$. For presentations of this type, see [14] or [16].

Specifically, we change the definition of formula by adding f to our list of propositional variables and removing $\sim$ from the set of connectives. $\sim$ is re-introduced as a metamathematical symbol as above. Our definition of subformula is also changed accordingly.

The definition of model is changed as follows: replace P3 [section 2, chapter 1] by P3': $\Gamma \not\models f$.

This leads to a new definition of validity, which we may call f-validity.

Theorem: Let X be a formula (in the usual sense) and let X' be the corresponding formula with $\sim$ written in terms of f. Then X is valid if and only if X' if f-valid.

Proof: We show that in any model $\langle G, R, \models \rangle$, $\Gamma \models X$ iff $\Gamma \models X'$ (where we use two different senses of $\models$). The proof is by induction on the degree of X (which is the same as the degree of X'). Actually, all cases are easy except that of $\sim$ itself. So, suppose the result is known for all formulas of degree less than that of X, and X is ~Y. Then

$$\begin{aligned} \Gamma \models X &\iff \Gamma \models {\sim}Y \\ &\iff \forall \Gamma^* \quad \Gamma^* \not\models Y \\ &\iff \forall \Gamma^* \quad \Gamma^* \not\models Y' \end{aligned}$$

but clearly this is equivalent to $\Gamma \models Y' \supset f$ since $\Gamma^* \not\models f$. Hence equivalently, $\Gamma \models X'$.

Q.E.D.

Section 2

f-primitive intuitionistic logic-proof theory

In this section we still retain the altered definition of formula in the last section, with f primitive. We give a tableau system for this. The new system is the same as that of section 1, chapter 2 in all but two respects. First, the rules T~ and F~ are removed. Second, a set S of signed formulas is called closed if it contains TX and FX for some formula X, or if it contains Tf.

This leads to a new definition of theorem, which we may call f-theorem.

Theorem: Let X be a formula (in the usual sense) and let X' be the corresponding formula with ~ written in terms of f. Then X is a theorem if and only if X' is an f-theorem.

This follows immediately from the following.

Lemma: Let S be a set of signed formulas (in the usual sense) and let S' be the corresponding set of signed formulas with ~ replaced in terms of f. Then S is inconsistent if and only if S' is f-inconsistent.

Proof: We show this in two halves. First, suppose S is inconsistent. We show the result by induction on the length of the closed tableau for S. There are only two significant cases. Suppose first that the tableau for S is $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$, $\mathcal{C}_1$ is $\{\{U, F{\sim}X\}\}$ and $\mathcal{C}_2$ is $\{\{U_T, TX\}\}$. Then by induction hypothesis, $\{U_T', TX'\}$ is f-inconsistent. Hence, so is $\{U', FX' \supset f\}$, i.e. S'. The other case is if $\mathcal{C}_1$ is $\{\{U, T{\sim}X\}\}$ and $\mathcal{C}_2$ is $\{\{U, FX\}\}$. Then by the induction hypothesis, $\{U', FX'\}$ is f-inconsistent hence so is $\{U', TX' \supset f\}$, i.e. S'.

The converse is shown by induction on the length of the closed f-tableau for S'. If this f-tableau is of length 1, either S' contains TX and FX for some formula X, and we are done, or S' contains Tf, which is not possible since we supposed S' arose from standard set S.

The induction steps are similar to those above.

Q.E.D.

The results of this and the last sections, together with our earlier results give: X' is f-valid if and only if X' is an f-theorem. This is not the complete generality one would like since it holds only for those formulas X' which correspond to standard formulas X. The more complete result is, however, true, as the reader may show by methods similar to those of the last chapter.

Section 3

Minimal logic

Minimal logic is a sublogic of intuitionistic logic in which a false statement need not imply everything. The original paper on minimal logic is Johannson's [8]. Prawitz establishes several results concerning it in [14], and it is treated algebraically by Rasiowa and Sikorski [15].

Semantically, we use the f-models defined in section 1, with the change that we no longer require P3', that is, that $\Gamma \not\models f$.

Proof theoretically, we use the f-tableaus defined in section 2, with the change that we no longer have closure of a set because it contains Tf.

We leave it to the reader to show that $\overset{X}{\mathcal{S}}$ is provable in this tableau system if and only if X is valid in this model sense, using the methods of chapter 2.

Certainly every minimal logic theorem is an intuitionistic logic theorem, but the converse is not true. For example, $(A \wedge {\sim}A) \supset B$ is a theorem of intuitionistic logic, but the following is a minimal counter-model for it, or rather, for $(A \wedge (A \supset f)) \supset B$:

$$G = \{\Gamma\}$$

$$\Gamma R \Gamma$$

$$\Gamma \models A, \quad \Gamma \models f$$

and $\models$ is extended as usual. It is easily seen that $\Gamma \models A \wedge (A \supset f)$, but $\Gamma \not\models B$.

Section 4

Classical logic

Beginning with this section, we return to the usual notions of formula, tableau, and model, that is, with $\sim$ and not f as primitive.

Some authors call a set $\mathcal{F}$ of unsigned formulas a (classical) truth set if

$$X \wedge Y \in \mathcal{F} \iff X \in \mathcal{F} \text{ and } Y \in \mathcal{F}$$
$$X \vee Y \in \mathcal{F} \iff X \in \mathcal{F} \text{ or } Y \in \mathcal{F}$$
$$\sim X \in \mathcal{F} \iff X \notin \mathcal{F}$$
$$X \supset Y \in \mathcal{F} \iff X \notin \mathcal{F} \text{ or } Y \in \mathcal{F}$$

It is a standard result of classical logic that X is a classical theorem if and only if X is in every truth set. There is a proof of this in [15].

Theorem: Any intuitionistic theorem is a classical theorem.

Proof: Suppose X is not a classical theorem. Then there is a truth set $\mathcal{F}$ such that $X \notin \mathcal{F}$. We define a very simple intuitionistic counter-model for X, $\langle G, R, \models \rangle$, as follows:

$$G = \{\mathcal{F}\}$$
$$\mathcal{F} R \mathcal{F}$$

$\mathcal{F} \models A \iff A \varepsilon \mathcal{F}$, for A atomic, and $\models$ is extended as usual. It is easily shown by induction on the degree of Y that

$\mathcal{F} \models Y \iff Y \varepsilon \mathcal{F}$

Hence, $\mathcal{F} \not\models X$ and X is not an intuitionistic theorem.

Q.E.D

That the converse is not true follows since we showed in section 5, chapter 2 that $\not\vdash_I A \vee \sim A$. Thus we have minimal logic is a proper sub-logic of intuitionistic logic which is a proper sub-logic of classical logic.

Section 5

Modal logic, S4 - semantics

In this section we define the set of (propositional) S4 theorems semantically using a model due to Kripke [11]. S4 was originated by Lewis [13], and an algebraic treatment may be found in [15]. A natural deduction treatment is in [14].

The definition of formula is changed by adding $\Box$ to the set of unary connectives. Thus, for example $\sim \Box \sim (A \vee \Box \sim A)$ is a formula. $\Box$ is read "necessarily".

$\Diamond$ is sometimes taken as an abbreviation for $\sim\Box\sim$ and is read "possibly". [In [13], $\Diamond$ was primitive].

The S4 model is defined as follows: It is an ordered triple $\langle G, R, \models\rangle$ where G is a non-empty set, R is a transitive, reflexive relation on G, and $\models$ is a relation between elements of G and formulas, satisfying the following conditions.

M1:	$\Gamma \models X \wedge Y$	iff	$\Gamma \models X$	and	$\Gamma \models Y$
M2:	$\Gamma \models X \vee Y$	iff	$\Gamma \models X$	or	$\Gamma \models Y$
M3:	$\Gamma \models \sim X$	iff	$\Gamma \not\models X$		
M4:	$\Gamma \models X \supset Y$	iff	$\Gamma \not\models X$	or	$\Gamma \models Y$
M5:	$\Gamma \models \Box X$	iff	for all Γ^*,	$\Gamma^* \models X$.	

X is S4 valid in $\langle G, R, \models\rangle$ if for all $\Gamma \varepsilon G$, $\Gamma \models X$. X is S4 valid if X is S4 valid in all S4 models.

The intuitive idea behind this modeling is the following: G is the collection of all possible worlds. $\Gamma R \Delta$ means Δ is a world possible relative to Γ. $\Gamma \models X$ means X is true in the world Γ. Thus M5 may be interpreted: X is necessarily true in Γ if and only if X is true in any world possible relative to Γ. This interpretation is given in [11].

Section 6

Modal logic, S4 - proof theory

We define a tableau system for S4 as follows. Everything in the definition of Beth tableaus in section 1, chapter 2 remains the same except the reduction rules themselves. These are replaced by

$MT\wedge$	$\dfrac{S,\ TX\wedge Y}{S,\ TX,\ TY}$	$MF\wedge$	$\dfrac{S,\ FX\wedge Y}{S, FX \mid S, FY}$
$MT\vee$	$\dfrac{S,\ TX\vee Y}{S,\ TX \mid S,\ TY}$	$MF\vee$	$\dfrac{S,\ FX\vee Y}{S,\ FX,\ FY}$
$MT\sim$	$\dfrac{S,\ T{\sim}X}{S,\ FX}$	$MF\sim$	$\dfrac{S,\ F{\sim}X}{S,\ TX}$
$MT\supset$	$\dfrac{S,\ TX\supset Y}{S,\ FX \mid S,\ TY}$	$MF\supset$	$\dfrac{S,\ FX\supset Y}{S,\ TX,\ FY}$
$MT\Box$	$\dfrac{S,\ T\Box X}{S,\ TX}$	$MF\Box$	$\dfrac{S,\ F\Box X}{S_{\Box},\ FX}$

where, in rule $MF\Box$, $S_{\Box}$ is

$$\{T\Box X \mid T\Box X \varepsilon S\}$$

Again, the methods of chapter 2 can be adapted to S4 to establish the identity of the set of S4 theorems and the set of S4 valid formulas. This is left to the reader. The original proof is in [11]. We are more interested in the relation between S4 and intuitionistic logic.

Section 7

S4 and intuitionistic logic

A map from the set of intuitionistic formulas to the set of S4 formulas is defined by

$$M(A) = \Box A \quad \text{for } A \text{ atomic}$$

$$M(X \vee Y) = M(X) \vee M(Y)$$

$$M(X \wedge Y) = M(X) \wedge M(Y)$$

$$M(\sim X) = \Box \sim M(X)$$

$$M(X \supset Y) = \Box(M(X) \supset M(Y))$$

We wish to show

Theorem: If X is an intuitionistic formula, X is intuitionistically valid if and only if M(X) is S4-valid.

This follows from the next three lemmas.

Lemma 1: Let $\langle G, R, \models_I \rangle$ be an intuitionistic model, and $\langle G, R, \models_{S4} \rangle$ be an S4 model, such that for any $\Gamma \in G$ and any atomic A,

$$\Gamma \models_I A \iff \Gamma \models_{S4} M(A)$$

Then for any formula X,

$$\Gamma \models_I X \iff \Gamma \models_{S4} M(X)$$

Proof: A straightforward induction on the degree of X.

Q.E.D.

Lemma 2 Given an intuitionistic counter-model for X, there is an $S4$ counter-model for $M(X)$.

Proof: We have $\langle G, R, \models_I \rangle$, an intuitionistic model such that for some $\Gamma \varepsilon G$, $\Gamma \not\models_I X$. We take for our S_4 model $\langle G, R, \models_{S4} \rangle$ where $\models_{S4}$ is defined by

$$\Delta \models_{S4} A \quad \text{if} \quad \Delta \models_I A$$

for A atomic and any Δ in G, and $\models_{S4}$ is extended to all formulas.

If A is atomic,

$$\begin{aligned} \Delta \models_{S4} M(A) \quad &\Longleftrightarrow \quad \Delta \models_{S4} \Box A \\ &\Longleftrightarrow \quad (\forall \Delta^*) \quad \Delta^* \models_{S4} A \\ &\Longleftrightarrow \quad (\forall \Delta^*) \quad \Delta^* \models_I A \\ &\Longleftrightarrow \quad \Delta \models_I A \end{aligned}$$

and the result follows by lemma 1.

Q.E.D.

Lemma 3: Given an $S4$ counter-model for $M(X)$, there is an intuitionistic counter-model for X.

Proof: We have $\langle G, R, \models_{S4} \rangle$, an $S4$ model such that for some $\Gamma \varepsilon G$, $\Gamma \not\models_{S4} M(X)$. We take for our intuitionistic model $\langle G, R, \models_I \rangle$ where $\models_I$ is defined by

$$\Delta \models_I A \quad \text{if} \quad \Delta \models_{S4} M(A)$$

for A atomic and any Δ in G, and $\models_I$ is extended to all formulas. Now the result follows by Lemma 1.

Q.E.D.

CHAPTER 4

First Order Intuitionistic Logic - Semantics

Section 1

Formulas

We begin with the following:

1) denumerably many individual variables

   x, y, z, w, ...

2) denumerably many individual parameters

   a, b, c, d, ...

3) for each positive integer n, a denumerable list of n-ary predicates,

   A^n, B^n, C^n, D^n, ...

4) connectives, quantifiers, parantheses,

   $\wedge, \vee, \supset, \sim, \exists, \forall$, (,).

An atomic formula is an n-ary predicate symbol A^n followed by an n-tuple of individual symbols (variables or parameters) thus, $A^n(\alpha_1, \ldots, \alpha_n)$.

A formula is anything resulting from the following recursive rules:

F0: any atomic formula is a formula

F1: If X is a formula, so is $\sim X$

F2,3.4: If X and Y are formulas, so are $(X \wedge Y)$, $(X \vee Y)$, $(X \supset Y)$

F5,6: If X is a formula and x is a variable, $(\forall x)X$ and $(\exists x)X$ are formulas

Subformulas are defined as usual, and the degree of a formula. The property of uniqueness of composition of a formula still holds. We note the usual properties of substitution, and we use the following notation: If X is a formula and α and β are individual symbols, by $X\binom{\alpha}{\beta}$ we mean the result of substituting β for every occurrence of α in X. [every free occurrence in case α is a variable]. We usually denote this informally as follows: we write X as $X(\alpha)$ and $X\binom{\alpha}{\beta}$ as $X(\beta)$. It will be clear from context what is meant.

We again use parentheses is an informal manner and we omit superscripts on predicates.

Although the definition of formula as stated, allows unbound occurrences of variables in formulas, we shall assume, unless otherwise stated, that all variables in a formula are bound. Notation like $X(x)$ however, indicates x may have free occurrences in X.

Section 2

Models and validity

In this section we define the notion of a first order intuitionistic model, and first order intuitionistic validity, referred to respectively as model and validity. This modeling structure is due to Kripke and may be found, in different notion, in [12]. The notions of chapter one, if needed, will be referred to as propositional notions to distinguish them.

If $\mathcal{P}$ is a map to sets of parameters, by $\hat{\mathcal{P}}(\Gamma)$ we mean the set of all formulas which may be constructed using only parameters of $\mathcal{P}(\Gamma)$.

By a (first order intuitionistic) model we mean an ordered quadruple $\langle G, R, \models, \mathcal{P} \rangle$ where G is a non-empty set, R is a transitive, reflexive relation on G, $\models$ is a relation between elements of G and formulas, and $\mathcal{P}$ is a map from G to non-empty sets of parameters, satisfying the following conditions:

for any $\Gamma \varepsilon G$,

Q0: $\mathcal{P}(\Gamma) \subseteq \mathcal{P}(\Gamma^*)$

Q1: $\Gamma \models A \Rightarrow A \varepsilon \hat{\mathcal{P}}(\Gamma)$ for A atomic

Q2: $\Gamma \models A \Rightarrow \Gamma^* \models A$ for A atomic

Q3: $\Gamma \models (X \wedge Y) \Leftrightarrow \Gamma \models X$ and $\Gamma \models Y$

Q4: $\Gamma \models (X \vee Y) \Leftrightarrow (X \vee Y) \varepsilon \hat{\mathcal{P}}(\Gamma)$ and $\Gamma \models X$ or $\Gamma \models Y$

Q5:	$\Gamma \models \sim X$	<=>	$\sim X \varepsilon \hat{\mathcal{P}}(\Gamma)$ and for all Γ^*, $\Gamma^* \not\models X$
Q6:	$\Gamma \models (X \supset Y)$	<=>	$(X \supset Y) \varepsilon \hat{\mathcal{P}}(\Gamma)$ and for all Γ^*, if $\Gamma^* \models X$, $\Gamma^* \models Y$
Q7:	$\Gamma \models (\exists x)X(x)$	<=>	for some $a \varepsilon \mathcal{P}(\Gamma)$, $\Gamma \models X(a)$
Q8:	$\Gamma \models (\forall x)X(x)$	<=>	for every Γ^* and for every $a \varepsilon \mathcal{P}(\Gamma^*)$, $\Gamma^* \models X(a)$

We call a particular formula X valid in the model $\langle G, R, \models, \mathcal{P} \rangle$ if for all $\Gamma \varepsilon G$ such that $X \varepsilon \hat{\mathcal{P}}(\Gamma)$, $\Gamma \models X$.

X is called valid if X is valid in all models.

Section 3

Motivation

The intuitive interpretation given in section 3, chapter 1 for the propositional case may be extended to this first order situation.

In one's usual mathematical work, parameters may be introduced as one proceeds, but having introduced a parameter, of course, it remains introduced. This is what the map $\mathcal{P}$ is intended to represent. That is, for

$\Gamma \varepsilon G$, Γ is a state of knowledge, and $\mathcal{P}(\Gamma)$ is the set of all parameters introduced to reach Γ. [Or, in a stricter intuitive sense, $\mathcal{P}(\Gamma)$ is the set of all mathematical entities constructed by time Γ].

Since parameters, once introduced, do not disappear, we have Q0. Q2-6 are as in the propositional case. Q7 should be obvious. Q8 may be explained: to know $(\forall x) X(x)$ at Γ, it is not enough merely to know $X(a)$ for every parameter a introduced so far [i.e. for all $a \varepsilon \mathcal{P}(\Gamma)$]. Rather, one must know $X(a)$ for all parameters which can ever be introduced [i.e. for all $a \varepsilon \mathcal{P}(\Gamma^*)$, $\Gamma^* \models X(a)$].

The restrictions Q1, and in Q4, Q5, and Q6 are simply to the effect that it makes no sense to say we know the truth of a formula X if X uses parameters we have not yet introduced. It would, of course, make sense to add corresponding restrictions to Q3, Q7, and Q8, but it is not necessary.

The original explanation of Kripke may be found in [12].

For a different but related model theory in terms of forcing see [4].

Section 4

Some properties of models

Theorem: In any model $\langle G, R, \models, \mathcal{P} \rangle$, for any $\Gamma \varepsilon G$, if $\Gamma \models X$, $X \varepsilon \hat{\mathcal{P}}(\Gamma)$.

Proof: A straightforward induction on the degree of X.

Q.E.D.

Theorem: In any model $\langle G, R, \models, \mathcal{P} \rangle$, for any formula X, if $\Gamma \models X$, $\Gamma^* \models X$.

Proof: Also a straightforward induction on the degree of X.

Q.E.D.

Theorem: Let G be a non-empty set, R be a transitive reflexive relation on G, and $\mathcal{P}$ be a map from G to non-empty sets of parameters such that $\mathcal{P}(\Gamma) \subseteq \mathcal{P}(\Gamma^*)$ for all $\Gamma \varepsilon G$. Suppose $\models$ is a relation between elements of G and atomic formulas such that $\Gamma \models A \Rightarrow A \varepsilon \hat{\mathcal{P}}(\Gamma)$. Then $\models$ can be extended in one and only one way to a relation, also denoted by $\models$, between G and formulas, such that $\langle G, R, \models, \mathcal{P} \rangle$ is a model.

Proof: A straightforward extension of the corresponding propositional proof.

Q.E.D

Def: Let $\langle G, R, \models, \mathcal{P} \rangle$ be a model and suppose a is some parameter such that $a \notin \bigcup_{\Gamma \varepsilon G} \mathcal{P}(\Gamma)$. By $\langle G, R, \models, \mathcal{P} \rangle \binom{b}{a}$ we mean the model $\langle G, R, \models', \mathcal{P}' \rangle$ defined as follows: $\mathcal{P}'(\Gamma)$ is the same as $\mathcal{P}(\Gamma)$

except for containing a in place of b if $\mathcal{P}(\Gamma)$ contains b. For A atomic, $\Gamma \models A \Rightarrow \Gamma \models' A\binom{b}{a}$, and $\models'$ is extended to all formulas.

Lemma: Let $\langle G, R, \models, \mathcal{P}\rangle$ be a model, $a \notin \bigcup_{\Gamma \varepsilon G} \mathcal{P}(\Gamma)$, $\langle G, R, \models', \mathcal{P}'\rangle$ be $\langle G, R, \models, \mathcal{P}\rangle\binom{b}{a}$. Then for any formula X not containing a,

$$\Gamma \models X \iff \Gamma \models' X\binom{b}{a}$$

Proof: By an easy induction on the degree of X.

Q.E.D.

Def: Let $\langle G, R, \models, \mathcal{P}\rangle$ be a model and suppose a is some parameter such that $a \notin \bigcup_{\Gamma \varepsilon G} \mathcal{P}(\Gamma)$. By $\langle G, R, \models, \mathcal{P}\rangle_{b = a}$ we mean the model $\langle G, R, \models', \mathcal{P}'\rangle$ defined as follows: $\mathcal{P}'(\Gamma)$ is the same as $\mathcal{P}(\Gamma)$ except for containing a as well as b whenever $\mathcal{P}(\Gamma)$ contains b. For A atomic, $\Gamma \models A \Rightarrow \Gamma \models' A'$ where A' is like A except for containing a at zero or more places where A contains b, and $\models'$ is extended to all formulas.

Lemma: Let $\langle G, R, \models, \mathcal{P}\rangle$ be a model $a \notin \bigcup_{\Gamma \varepsilon G} \mathcal{P}(\Gamma)$, and let $\langle G, R, \models', \mathcal{P}'\rangle$ be $\langle G, R, \models, \mathcal{P}\rangle_{b = a}$. Then if X is any formula not containing a, and if X' is like X except for containing a at zero or more places where X contains b,

$$\Gamma \models X \iff \Gamma \models 'X'$$

Proof: Again an easy induction on the degree of X.

Q.E.D.

Section 5

Examples

We show that two theorems of classical logic are not intuitionistically valid.

$$\vdash_c \sim\sim (\forall x) \ (A(x) \vee \sim A(x))$$

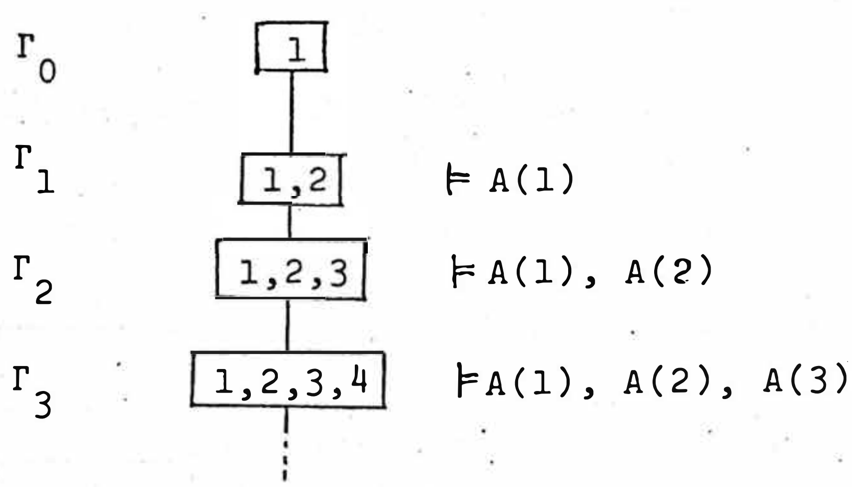
but the following is an intuitionistic counter-model for it. We take the natural numbers as parameters.

Let $G = \{\Gamma_i \mid i = 0, 1, 2, \ldots\}$

$\Gamma_i R \Gamma_j$ iff $i \leq j$

$\mathcal{P}(\Gamma_i) = \{1, 2, \ldots, i, i+1\}$

$\Gamma_n \models A(i)$ iff $i \leq n$ and $\models$ is extended to all formulas. We may give this model schematically.



We claim no $\Gamma_i \models \sim\sim(\forall x)\ (A(x) \vee \sim A(x))$.

Suppose instead that

$$\Gamma_i \models \sim\sim(\forall x)\ (A(x) \vee \sim A(x)).$$

Then for some $j \geq i$,

$$\Gamma_j \models (\forall x)\ (A(x) \vee \sim A(x)).$$

But $j + 1 \in \mathcal{P}(\Gamma_j)$, so

$$\Gamma_j \models A(j + 1) \vee \sim A(j + 1)$$

but $\Gamma_j \not\models A(j + 1)$ since $j + 1 > j$, and if

$\Gamma_j \models \sim A(j + 1)$, then since $\Gamma_j R \Gamma_{j+1}$,

$\Gamma_{j+1} \not\models A(j + 1)$, a contradiction.

$$\vdash_c (\forall x)\ (A \vee B(x)) \supset (A \vee (\forall x)\ B(x))$$

but an intuitionistic counter-model is the following, where again parameters are integers.

$$G = \{\Gamma_1, \Gamma_2\}$$

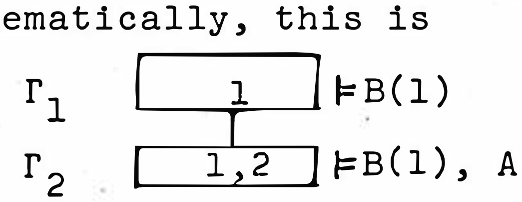
$$\Gamma_1 R \Gamma_2, \quad \Gamma_1 R \Gamma_1, \quad \Gamma_2 R \Gamma_2$$

$$\mathcal{P}(\Gamma_1) = \{1\}, \quad \mathcal{P}(\Gamma_2) = \{1,2\}$$

$$\Gamma_1 \models B(1), \quad \Gamma_2 \models B(1), \quad \Gamma_2 \models A$$

and $\models$ is extended to all formulas.

Schematically, this is



To show this is a counter-model, first we claim;

$$\Gamma_1 \models (\forall x)(A \vee B(x))$$

This follows because $\Gamma_1 \models B(1)$ so

$\Gamma_1 \models A \vee B(1)$, and $\Gamma_2 \models A$ so

$\Gamma_2 \models A \vee B(1)$ and $\Gamma_2 \models A \vee B(2)$

But $\Gamma_1 \not\models A$. Moreover, $\Gamma_1 \not\models (\forall x) B(x)$ since $\Gamma_2 \not\models B(2)$. Thus, $\Gamma_1 \not\models A \vee (\forall x) B(x)$.

Section 6

Truth and almost-truth sets

In classical first order logic, a set $\mathcal{F}$ of formulas is sometimes called a truth set if

1) $X \wedge Y \varepsilon \mathcal{F} \iff X \varepsilon \mathcal{F}$ and $Y \varepsilon \mathcal{F}$
2) $X \vee Y \varepsilon \mathcal{F} \iff X \varepsilon \mathcal{F}$ or $Y \varepsilon \mathcal{F}$
3) $\sim X \varepsilon \mathcal{F} \iff X \not\varepsilon \mathcal{F}$
4) $X \supset Y \varepsilon \mathcal{F} \iff X \not\varepsilon \mathcal{F}$ or $Y \varepsilon \mathcal{F}$
5) $(\exists x) X(x) \varepsilon \mathcal{F} \iff X(a) \varepsilon \mathcal{F}$ for some parameter a
6) $(\forall x) X(x) \varepsilon \mathcal{F} \iff X(a) \varepsilon \mathcal{F}$ for every parameter a

where there is some fixed set of parameters, X and Y are formulas involving only these parameters, and 5) and 6) refer to this set of parameters.

We now call $\mathcal{J}$ an <u>almost-truth</u> set if it satisfies 1) - 5) above and 6a)

$(\forall x)X(x) \varepsilon \mathcal{J} \Rightarrow X(a) \varepsilon \mathcal{J}$ for every parameter a.

It is one form of the classical completeness theorem that for any pure (i.e. with no parameters) formula X, X is a classical theorem if and only if X is in every truth set.

We leave the reader to show:

<u>Theorem:</u> If X is pure and contains no occurrence of the universal quantifier, X is in every truth set if and only if X is in every almost-truth set.

Section 7

Complete sequences

The method used in this section was adapted from forcing techniques, and is due to Cohen [2].

<u>Def:</u> In the model $\langle G, R, \models, \mathcal{P} \rangle$, we call $\mathcal{C}$ an R-chain if

1) $\mathcal{C} \subseteq G$

2) $\Gamma, \Delta \varepsilon \mathcal{C} \Rightarrow \Gamma R \Delta$ or $\Delta R \Gamma$

If $\mathcal{C}$ is an R-chain, by $\bar{\mathcal{C}}$ we mean $\{X \mid$ for some $\Gamma \varepsilon \mathcal{C}, \Gamma \models X\}$

If $\mathcal{C}$ is an R-chain, $\mathcal{C}$ is called complete if, for every formula X with parameters from $\bar{\mathcal{C}}$, $X \vee \sim X \varepsilon \bar{\mathcal{C}}$.

<u>Lemma 1:</u> Let $\mathcal{C}$ be a complete R-chain in the model $\langle G, R, \models, \mathcal{P} \rangle$. Then $\bar{\mathcal{C}}$ is an almost-truth set.

<u>Proof:</u> This is a straightforward verification of the cases. We give case 4) as an illustration.

Suppose $(X \supset Y) \varepsilon \bar{\mathcal{C}}$. Then for some $\Gamma \varepsilon \mathcal{C}$, $\Gamma \models X \supset Y$. Now either $X \not\varepsilon \bar{\mathcal{C}}$ or $X \varepsilon \bar{\mathcal{C}}$. If $X \varepsilon \bar{\mathcal{C}}$, then for some $\Delta \varepsilon \mathcal{C}$, $\Delta \models X$. Let Ω be the R-last of Γ and Δ. Then $\Omega \models X$ and $\Omega \models X \supset Y$, so $\Omega \models Y$ and $Y \varepsilon \bar{\mathcal{C}}$. Thus $X \not\varepsilon \bar{\mathcal{C}}$ or $Y \varepsilon \bar{\mathcal{C}}$.

Conversely, suppose $(X \supset Y) \not\varepsilon \bar{\mathcal{C}}$. Then $\sim X \not\varepsilon \bar{\mathcal{C}}$, since $\bar{\mathcal{C}}$ is closed under modus ponens, and contains $\sim X \supset (X \supset Y)$ as is easily shown. But $X \vee \sim X \varepsilon \bar{\mathcal{C}}$, hence $X \varepsilon \bar{\mathcal{C}}$. Further, $Y \not\varepsilon \bar{\mathcal{C}}$ since again, $Y \supset (X \supset Y) \varepsilon \bar{\mathcal{C}}$.

Q.E.D.

<u>Lemma 2:</u> Let $\langle G, R, \models, \mathcal{P} \rangle$ be a model, $\Gamma \varepsilon G$, and $X \varepsilon \hat{\mathcal{P}}(\Gamma)$. There is some $\Gamma^* \varepsilon G$ such that $\Gamma^* \models X \vee \sim X$.

<u>Proof:</u> Either some $\Gamma^* \models X$ and we are done, or no $\Gamma^* \models X$ in which case $\Gamma \models \sim X$ and we are done.

Q.E.D.

Theorem: Let $\langle G, R, \models, \mathcal{P} \rangle$ be a model and $\Gamma \varepsilon G$. Then Γ can be included in some complete R-chain $\mathcal{C}$ such that $\bar{\mathcal{C}}$ is an almost-truth set.

Proof: There are only countably many formulas, $X_1, X_2, X_3, \cdots$. We define a countable R-chain $\{\Gamma_0, \Gamma_1, \Gamma_2, \ldots\}$ as follows.

Let Γ_0 be Γ.

Having defined Γ_n, if $X_{n+1} \not\varepsilon \hat{\mathcal{P}}(\Gamma_n^*)$ for any $\Gamma_n{}^*$, let Γ_{n+1} be Γ_n. If $X_{n+1} \varepsilon \hat{\mathcal{P}}(\Gamma_n{}^*)$ for some $\Gamma_n{}^*$, then $\Gamma_n{}^*$, by lemma 2, has an R-successor $\Gamma_n{}^{**}$ such that $\Gamma_n{}^{**} \models X_{n+1} \vee {\sim}X_{n+1}$. Let Γ_{n+1} be this $\Gamma_n{}^{**}$.

Let $\mathcal{C}$ be $\{\Gamma_0, \Gamma_1, \Gamma_2, \ldots\}$. Clearly, $\mathcal{C}$ is complete, and by lemma 1, $\bar{\mathcal{C}}$ is an almost-truth set.

Q.E.D.

Section 8

A connection with classical logic

The first theorem of this section is essentially theorem 59(b), pg. 492 [9], but there it is proven proof-theoretically, and here semantically.

Theorem 1: Let X be a pure formula. If X is in every classical almost-truth set, $\sim\sim X$ is intuitionistically valid.

Proof: Suppose $\sim\sim X$ is not valid. Then there is a model $\langle G, R, \models, \mathcal{P} \rangle$ and a $\Gamma \varepsilon G$ such that $\Gamma \not\models \sim\sim X$. Then for some $\Gamma^* \varepsilon G$, $\Gamma^* \models \sim X$. Now Γ^* can, by the theorem of section 7, be included in an R-chain $\mathcal{C}$ such that $\bar{\mathcal{C}}$ is an almost-truth set. But $\sim X \varepsilon \bar{\mathcal{C}}$, so that $X \not\varepsilon \bar{\mathcal{C}}$.

Q.E.D.

Theorem 2: If X is intuitionistically valid, then X is classically valid (for X pure).

Proof: As before, if X is not classically valid, there is a truth set $\mathcal{F}$ not containing X. But it is easily shown that if $G = \{\mathcal{F}\}$, $\mathcal{F} R \mathcal{F}$, $\mathcal{F} \models Y$ iff $Y \varepsilon \mathcal{F}$, and $\mathcal{P}(\mathcal{F})$ is the set of all parameters occurring in $\mathcal{F}$, the resulting $\langle G, R, \models, \mathcal{P} \rangle$ is a model in which X is not valid.

Q.E.D.

Theorem 3: If X is a pure formula with no occurrence of the universal quantifier, then X is classically valid if and only if $\sim\sim X$ is intuitionistically valid.

Proof: ~~X intuitionistically valid =>

~~X classically valid =>

X classically valid.

Conversely, X classically valid =>

X is in every truth set =>

X is in every almost-truth set =>

~~X is intuitionistically valid.

Q.E.D.

Remark: This result will be of fundamental importance in part 2.

Corollary: First order intuitionist logic is undecidable.

Proof: Classical first order logic is undecidable, and every classical formula is classically equivalent to a formula with no universal quantifiers.

Q.E.D.

Remark: That theorem 3 cannot be extended to all formulas is shown by the first example in section 5.

CHAPTER 5

First Order Intuitionistic Logic - Proof Theory

Section 1

Beth tableaus

The following is an extension of the system of section 1, chapter 2, to the first order case. See [1]. Everything is as it was there, except that four reduction rules are added to the list. these are

$T\exists \qquad \dfrac{S,\ T(\exists x)\ X(x)}{S,\ TX\ (a)}$ provided a is new

$F\exists \qquad \dfrac{S,\ F\ (\exists x)\ X(x)}{S,\ FX\ (a)}$

$T\forall \qquad \dfrac{S,\ T(\forall x)\ X(x)}{S,\ TX\ (a)}$

$F\forall \qquad \dfrac{S,\ F(\forall x)\ X(x)}{S_T,\ FX\ (a)}$ provided a is new

[Note the S_T in rule $F\forall$]

In rules $F\exists$ and $T\forall$, a may be any parameter whatsoever. In rules $T\exists$ and $F\forall$, the parameter a introduced must not occur in any formula of S, or in the formula X (x).

As in the propositional case, we proceed to show correctness and completeness (in two ways) of this system.

The following two examples illustrate proofs in the system.

$$\vdash_I (\forall x)\ X\ (x) \supset \sim(\exists x)\ \sim X\ (x)$$

The proof is

$\{\{F\ (\forall x)\ X\ (x) \supset \sim(\exists x)\ \sim X\ (x)\}\}$

$\{\{T\ (\forall x)\ X\ (x),\ F \sim (\exists x)\ \sim X\ (x)\}\}$

$\{\{T\ (\forall x)\ X\ (x),\ T\ (\exists x)\ \sim X(x)\}\}$

$\{\{T\ (\forall x)\ X\ (x),\ T \sim X\ (a)\}\}$

$\{\{T\ X\ (a),\ T \sim X\ (a)\}\}$

$\{\{T\ X\ (a),\ F\ X\ (a)\}\}$

and $\vdash_I \sim(\exists x) \sim [X(x) \supset Y(x)] \supset (\forall x)\ [\sim Y(x) \supset \sim X(x)]$

The proof is

$\{\{F\sim(\exists x)\sim[X(x)\supset Y(x)]\supset(\forall x)\ [\sim Y(x)\supset \sim X(x)]\}\}$

$\{\{T\sim(\exists x)\sim[X(x)\supset Y(x)],\ F(\forall x)[\sim Y(x)\supset \sim X(x)]\}\}$

$\{\{T\sim(\exists x)\sim[X(x)\supset Y(x)],\ F[\sim Y(a)\supset \sim X(a)]\}\}$

$\{\{T\sim(\exists x)\sim[X(x)\supset Y(x)],\ T\sim Y(a),\ F\sim X(a)\}\}$

$\{\{T\sim(\exists x)\sim[X(x)\supset Y(x)],\ T\sim Y(a),\ TX(a)\}\}$

$\{\{F(\exists x)\sim[X(x)\supset Y(x)],\ T\sim Y(a),\ TX(a)\}\}$

$\{\{F\sim[X(a)\supset Y(a)],\ T\sim Y(a),\ TX(a)\}\}$

$\{\{T[X(a)\supset Y(a)],\ T\sim Y(a)\ TX(a)\}\}$

$\{\{FX(a),\ T\sim Y(a),\ TX(a)\},\ \{TY(a),\ T\sim Y(a),\ TX(a)\}\}$

$\{\{FX(a),\ T\sim Y(a),\ TX(a)\},\ \{TY(a),\ FY(a),\ TX(a)\}\}$

Section 2

Correctness of Beth tableaus

Def: Let $\{TX_1, \ldots, TX_n, FY_1 \ldots, FY_m\}$ be a set of signed formulas, $\langle G, R, \models, \mathcal{P}\rangle$ a model, and $\Gamma \varepsilon G$. We say Γ realizes the set if $X_i \varepsilon \hat{\mathcal{P}}(\Gamma)$, $Y_j \varepsilon \hat{\mathcal{P}}(\Gamma)$, and $\Gamma \models X_1, \ldots, \Gamma \models X_n$, $\Gamma \not\models Y_1, \ldots, \Gamma \not\models Y_m$.

A set S is realizable if something realizes it.

A configuration $\mathcal{C}$ is realizable if one of its elements is realizable.

Lemma 1: Let Q stand for either the sign T or the sign F. If S, QX(b) is realizable and if a is a parameter which does not occur in S or in X [so $a \neq b$] then S, QX(a) is realizable.

Proof: Suppose in the model $\langle G, R, \models, \mathcal{P}\rangle$, Γ realizes S, QX(b). Choose a new parameter $c \notin \bigcup_{\Gamma \varepsilon G} \mathcal{P}(\Gamma)$ [we can always construct a new parameter]. Let $\langle G, R, \models', \mathcal{P}'\rangle$ be $\langle G, R, \models, \mathcal{P}\rangle\binom{a}{c}$ [see section 4, chapter 4]. Since a does not occur in S or X, by an earlier lemma, in this new model, Γ realizes S, QX(b). But now, $a \notin \bigcup_{\Gamma \varepsilon G} \mathcal{P}'(\Gamma)$, so we may define a third model $\langle G, R, \models'', \mathcal{P}''\rangle$ as $\langle G, R, \models', \mathcal{P}'\rangle_{b=a}$ By another lemma, in this third model, Γ realizes S,QX(a).

Q.E.D.

Lemma 2: If $S, T(\exists x)X(x)$ is realizable, and if a does not occur in S or $X(x)$, then $S, TX(a)$ is realizable.

Proof: Suppose in the model $\langle G, R, \models, \mathcal{P} \rangle$, Γ realizes $S, T(\exists x)X(x)$. Then $\Gamma \models (\exists x)X(x)$, so for some $b \varepsilon \mathcal{P}(\Gamma)$, $\Gamma \models X(b)$. Thus Γ realizes $S, TX(b)$. If $a=b$ we are done. If not, by lemma 1, we are done.

Q.E.D.

Lemma 3: If $S, F(\exists x)X(x)$ is realizable and if a is any parameter, $S, FX(a)$ is realizable.

Proof: Suppose in the model $\langle G, R, \models, \mathcal{P} \rangle$, Γ realizes $S, F(\exists x)X(x)$. Then, $\Gamma \not\models (\exists x)X(x)$. If $a \varepsilon \mathcal{P}(\Gamma)$, $\Gamma \not\models X(a)$ and we are done. If $a \not\varepsilon \mathcal{P}(\Gamma)$, a cannot occur in S or X by the definition of realizability. But $\mathcal{P}(\Gamma) \neq \phi$ so there is a $b \varepsilon \mathcal{P}(\Gamma)$, $b \neq a$, and $\Gamma \not\models X(b)$. Thus $S, FX(b)$ is realizable. Now use lemma 1.

Q.E.D.

Lemma 4: If $S, T(\forall x)X(x)$ is realizable and if a is any parameter, $S, TX(a)$ is realizable.

Proof: Similar to that of lemma 3.

<u>Lemma 5:</u> If $S, F(\forall x)X(x)$ is realizable and if a is any parameter which does not occur in S or $X(x)$, then $S_T, FX(a)$ is realizable.

<u>Proof:</u> Suppose in the model $\langle G, R, \Vdash, \rho \rangle$, Γ realizes $S, F(\forall x)X(x)$. Then $\Gamma \nVdash (\forall x)X(x)$, but $X(x) \varepsilon \hat{\rho}(\Gamma)$. So there is a Γ^* such that $\Gamma^* \nVdash X(b)$ for some $b \varepsilon \rho(\Gamma^*)$. Of course, Γ^* realizes S_T. If $b=a$ we are done. If not, since $S_T, X(b)$ is realizable, by lemma 1 we are done.

Q.E.D.

<u>Theorem:</u> Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ be a tableau. If $\mathcal{C}_i$ is realizable, so is $\mathcal{C}_{i+1}$.

<u>Proof:</u> We pass from $\mathcal{C}_i$ to $\mathcal{C}_{i+1}$ by the application of some reduction rule. All the propositional rules were dealt with in chapter 2. The four new (first order) rules are handled by lemmas 2-5 above.

Q.E.D.

<u>Corollary:</u> If X is provable, X is valid.

<u>Proof:</u> Exactly as in the propositional situation.

Section 3

Hintikka collections

This generalizes to the first order setting the definition of section 3, chapter 2. Recall, a finite set of signed formulas is consistent if no tableau for it closes. We say an infinite set is consistent if every finite subset is.

Let G be a collection of sets of signed formulas. If $\Gamma \varepsilon G$, by $\mathcal{P}(\Gamma)$ we mean the collection of all parameters occurring in formulas in Γ. If $\Gamma, \Delta \varepsilon G$, by $\Gamma R \Delta$ we mean $\mathcal{P}(\Gamma) \subseteq \mathcal{P}(\Delta)$ and $\Gamma_T \subseteq \Delta$.

We call G a (first order) Hintikka collection if, for any $\Gamma \varepsilon G$, Γ is consistent and

$TX \wedge Y \varepsilon \Gamma \Rightarrow TX \varepsilon \Gamma$ and $TY \varepsilon \Gamma$

$FX \vee Y \varepsilon \Gamma \Rightarrow FX \varepsilon \Gamma$ and $FY \varepsilon \Gamma$

$TX \vee Y \varepsilon \Gamma \Rightarrow TX \varepsilon \Gamma$ or $TY \varepsilon \Gamma$

$FX \wedge Y \varepsilon \Gamma \Rightarrow FX \varepsilon \Gamma$ or $FY \varepsilon \Gamma$

$T{\sim}X \varepsilon \Gamma \Rightarrow FX \varepsilon \Gamma$

$TX \supset Y \varepsilon \Gamma \Rightarrow FX \varepsilon \Gamma$ or $TY \varepsilon \Gamma$

$F{\sim}X \varepsilon \Gamma \Rightarrow$ for some $\Delta \varepsilon G$ $\Gamma R \Delta$ and $TX \varepsilon \Delta$

$FX \supset Y \varepsilon \Gamma \Rightarrow$ for some $\Delta \varepsilon G$, $\Gamma R \Delta$ and $TX \varepsilon \Delta$, $FY \varepsilon \Delta$

$T(\forall x)X(x)\varepsilon\Gamma \Rightarrow TX(a)\varepsilon\Gamma$ for all $a\varepsilon \mathcal{P}(\Gamma)$

$F(\exists x)X(x)\varepsilon\Gamma \Rightarrow FX(a)\varepsilon\Gamma$ for all $a\varepsilon \mathcal{P}(\Gamma)$

$T(\exists x)X(x)\varepsilon\Gamma \Rightarrow TX(a)\varepsilon\Gamma$ for some $a\varepsilon \mathcal{P}(\Gamma)$

$F(\forall x)X(x)\varepsilon\Gamma \Rightarrow$ for some $\Delta\varepsilon G$, $\Gamma R\Delta$, and for some $a\varepsilon \mathcal{P}(\Delta)$, $TX(a)\varepsilon\Delta$.

If G is a Hintikka collection, we call $\langle G, R, \models, \mathcal{P}\rangle$ a model for G if

1) $\langle G, R, \models, \mathcal{P}\rangle$ is a model
2) $\mathcal{P}$ and R are as above
3) $TX\varepsilon\Gamma \Rightarrow \Gamma\models X$
   $FX\varepsilon\Gamma \Rightarrow \Gamma\not\models X$ for all $\Gamma\varepsilon G$

<u>Theorem:</u> There is a model for any Hintikka collection.

<u>Proof:</u> We have a Hintikka collection G. $\mathcal{P}$ and R are as defined. If A is atomic, let $\Gamma\models A$ if $TA\varepsilon\Gamma$, and extend $\models$ to all formulas. The result $\langle G, R, \models, \mathcal{P}\rangle$ is a model. We claim it is a model for G. We show property 3) by induction on the degree of X.

The propositional cases were done in section 3, chapter 2. Of the four new cases, we only do two as illustration.

Suppose the result know for all subformulas of the formula in question.

$T(\forall x)X(x) \varepsilon \Gamma \Rightarrow$

$(\forall \Delta \varepsilon G)(\Gamma R \Delta \Rightarrow T(\forall x)X(x) \varepsilon \Delta)$

[since $\Gamma_T \subseteq \Delta$ if $\Gamma R \Delta$]

$\Rightarrow (\forall \Delta \varepsilon G)(\Gamma R \Delta \Rightarrow ((\forall a \varepsilon \mathcal{P}(\Delta)) TX(a) \varepsilon \Delta))$

$\Rightarrow (\forall \Delta \varepsilon G)(\Gamma R \Delta \Rightarrow ((\forall a \varepsilon \mathcal{P}(\Delta)) \Delta \models X(a)))$

$\Rightarrow \Gamma \models (\forall x)X(x)$

Conversely, $F(\forall x)X(x) \varepsilon \Gamma \Rightarrow$

$(\exists \Delta \varepsilon G)$ $(\Gamma R \Delta$ and $(\exists a \varepsilon \mathcal{P}(\Delta))$ $(FX(a) \varepsilon \Delta))$

$\Rightarrow (\exists \Delta \varepsilon G)$ $(\Gamma R \Delta$ and $(\exists a \varepsilon \mathcal{P}(\Delta))$ $(\Delta \not\models X(a)))$

$\Rightarrow \Gamma \not\models (\forall x)X(x)$.

Q.E.D.

Thus, as in the propositional case, to establish the completeness of Beth tableaus we need only show that if X is not provable, there is a Hintikka collection G and a $\Gamma \varepsilon G$ such that $FX \varepsilon \Gamma$.

Section 4

Hintikka elements

Def: Let Γ be a set of signed formulas and P a set of parameters. We call Γ a Hintikka element with respect to P if Γ is consistent and

$TX \wedge Y \varepsilon \Gamma \Rightarrow TX \varepsilon \Gamma$ and $TY \varepsilon \Gamma$

$FX \vee Y \varepsilon \Gamma \Rightarrow FX \varepsilon \Gamma$ and $FY \varepsilon \Gamma$

$TX \vee Y \varepsilon \Gamma \Rightarrow TX \varepsilon \Gamma$ or $TY \varepsilon \Gamma$

$FX \wedge Y \varepsilon \Gamma \Rightarrow FX \varepsilon \Gamma$ or $FY \varepsilon \Gamma$.

$T{\sim}X \varepsilon \Gamma \Rightarrow FX \varepsilon \Gamma$

$TX \supset Y \varepsilon \Gamma \Rightarrow FX \varepsilon \Gamma$ or $TY \varepsilon \Gamma$

$T(\forall x)X(x) \varepsilon \Gamma \Rightarrow TX(a) \varepsilon \Gamma$ for each $a \varepsilon P$

$F(\exists x)X(x) \varepsilon \Gamma \Rightarrow FX(a) \varepsilon \Gamma$ for each $a \varepsilon P$

$T(\exists x)X(x) \varepsilon \Gamma \Rightarrow TX(a) \varepsilon \Gamma$ for some $a \varepsilon P$

<u>Theorem:</u> Let Γ be an at most countable, consistent set of signed formulas. Let S be the set of all parameters occurring in formulas in Γ. Let $a_1, a_2, a_3, \ldots$ be a countable list of parameters not in S. Let $P = S \cup \{a_1, a_2, a_3, \ldots\}$. Then Γ can be extended to a Hintikka element with respect to P.

<u>Proof:</u> Order the (countable) set of all subformulas of formulas in Γ, using only parameters of P: $X_1, X_2, X_3, \ldots$

We define a (double) sequence of sets of signed formulas.

Let $\Gamma_0 = \Gamma$

Suppose we have defined Γ_n, which is a consistent extension of Γ_0, using only finitely many of $a_1, a_2, a_3, \cdots$. Let $\Delta_n^1 = \Gamma_n$. We define $\Delta_n^2, \ldots, \Delta_n^{n+1}$ and let $\Gamma_{n+1} = \Delta_n^{n+1}$. We do this as follows:

Suppose we have defined Δ_n^k $(1 \le k \le n)$. Consider the formula X_k. At most one of TX_k, FX_k can be in Δ_n^k (since it is consistent). If neither is, let $\Delta_n^{k+1} = \Delta_n^k$. If one is in Δ_n^k, we have several cases.

Case 1a) X_k is $Y \vee Z$ and $TX_k \varepsilon \Delta_n^k$.

Then one of Δ_n^k, TY or Δ_n^k, TZ is consistent. Let Δ_n^{k+1} be Δ_n^k, TY if consistent, otherwise Δ_n^k, TZ.

Case 1b) X_k is $Y \vee Z$ and $FX_k \varepsilon \Delta_n^k$. Then Δ_n^k, FY, FZ is consistent. Let this be Δ_n^{k+1}.

Case 2a) $TX \wedge Y$

Case 2b) $FX \wedge Y$

Case 3) $T{\sim}X$

Case 4) $TX \supset Y$

are all treated in a similar manner.

Case 5a) X_k is $(\exists x)X(x)$ and $TX_k \varepsilon \Delta_n^k$.

Since Δ_n^k uses only finitely many of $a_1, a_2, a_3, \ldots,$ let a_i be the first one unused. Let Δ_n^{k+1} be Δ_n^k, $TX(a_i)$. Since a_i is new, this must also be consistent.

Case 5b) X_k is $(\exists x)X(x)$ and $FX_k \varepsilon \Delta_n^k$. Let Δ_n^{k+1} be Δ_n^k together with $FX(\alpha)$ for each $\alpha \varepsilon S$, and each $\alpha = a_i$ which has been used so far. Then Δ_n^{k+1} is again consistent.

Case 6) $T(\forall x)X(x)$, treated as we did case 5b).

Case 7) If the signed formula does not come under one of the above cases let $\Delta_n^{k+1} = \Delta_n^k$.

Thus we have defined a sequence, $\Gamma_0, \Gamma_1, \Gamma_2, \cdots$. Let $\Pi = \bigcup \Gamma_n$. We claim Π is a Hintikka collection with respect to P. The verification of the properties is straightforward.

Q.E.D.

Section 5

Completeness of Beth tableaus

Supposing X to be not provable, we give a procedure for constructing a sequence of Hintikka elements.

First, we order our countable collection of parameters as follows:

S_1:	a^1_1,	a^1_2,	a^1_3,	...
S_2:	a^2_1,	a^2_2,	a^2_3,	...
S_3:	a^3_1,	a^3_2,	a^3_3,	...
⋮	⋮	⋮	⋮	

where we have placed all the parameters of X in S_1, and let $P_n = S_1 \cup S_2 \cup \ldots \cup S_n$.

For this section only, by an F-formula we mean a signed formula of the form $F{\sim}X$, $FX \supset Y$, or $F(\forall x)X$. We may assume once and for all an ordering of all formulas. Now we proceed.

<u>Step 0)</u> X is not provable, so $\{FX\}$ is consistent. Extend it to a Hintikka element with respect to P_1. Call the result Γ_1.

<u>Step 1)</u> Take the first F-formula of Γ_1. If this is $F{\sim}X$, consider Γ_{1T}, TX. This is consistent. Extend it to a Hintikka element with respect to P_2, call it Γ_2. If the first F-formula is $FX \supset Y$, extend Γ_{1T}, TX, FY to a Hintikka element with respect to P_2, Γ_2. If the first F-formula is $F(\forall x)\, X\,(x)$, extend Γ_{1T}, $FX(a^2_1)$ to a Hintikka element with respect to P_2, Γ_2. In any event, Γ_2 is a consistent Hintikka element with respect to P_2. Now call the first

F-element of Γ_1 "used". The result of step 1 is $\{\Gamma_1, \Gamma_2\}$.

Suppose at the end of step n we have the sequence $\{\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_{2^n}\}$ where each Γ_i is a Hintikka element with respect to P_i.

<u>Step n + 1)</u> Take the first "unused" F-formula of Γ_1, proceed as in step 1 depending on whether the formula is $F{\sim}X$, $FX \supset Y$, $F(\forall x)X$. Produce from Γ_{1T}, TX, or Γ_{1T}, TX, FY, or Γ_{1T}, $FX\,(a_1^{2^n+1})$ a Hintikka element with respect to P_{2^n+1} call it Γ_{2^n+1}. And call the formula in question "used". Repeat the same procedure and the first "unused" F-formula of Γ_2, producing a Hintikka element with respect to P_{2^n+2} call it Γ_{2^n+2}. Continue to Γ_{2^n}, producing a Hintikka element with respect to $P_{2^{n+1}}$, call it $\Gamma_{2^{n+1}}$. The result of the n + 1st step is thus $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_{2^{n+1}}\}$.

Let G be the collection of all Γ_n generated in the above process. We claim G is a Hintikka collection.

Each $\Gamma_n \varepsilon G$ is a Hintikka element with respect to P_n, so $\mathcal{P}(\Gamma_n)$ is P_n. Since Γ_n is a Hintikka element with respect to $\mathcal{P}(\Gamma_n)$, to show G is a Hintikka collection we have only three properties to show.

Suppose for some $\Gamma_n \varepsilon G$, $F(\forall x)X(x)\varepsilon\Gamma_n$. By the above construction there must be some $\Gamma_k \varepsilon G$ such that $\Gamma_{nT} \subseteq \Gamma_k$, $\mathcal{P}(\Gamma_n) \subseteq \mathcal{P}(\Gamma_k)$, and $FX(a)\varepsilon\Gamma_k$ for some parameter a. Thus $(\exists \Gamma_k \varepsilon G)$ $\Gamma_n R \Gamma_k$ and $FX(a)\varepsilon\Gamma_k$ for some $a\varepsilon \mathcal{P}(\Gamma_k)$.

The cases $F\sim$ and $F\supset$ are similar.

Thus G is a Hintikka collection and $FX\varepsilon\Gamma_1\varepsilon G$, so our completeness theorem is established.

We note that in the Hintikka collection G resulting, every formula is a subformula of X.

We remark also that the construction of section 4 and of this section could be combined into a single sequence of steps.

This proof is a modification of the original proof of Kripke [12].

Section 6

Second completeness proof for Beth tableaus

The following is a Henkin type proof and serves as a transition to the completeness of the axiom system presented in the next few sections. A proof along the same lines but using unsigned formulas was discovered independently by Thomason [19]. The similarity to the algebraic work of section 6, chapter 1, is also noted.

Recall that a finite set of signed formulas Γ is consistent if no tableau for it is closed. An infinite set is consistent if every finite subset is.

Def: Let P be a set of parameters and Γ a set of signed formulas. We call Γ maximal consistent with respect to P if

1) every signed formula in Γ uses only parameters of P.
2) Γ is consistent
3) for every formula X with all its parameters from P, either $TX \varepsilon \Gamma$, or $FX \varepsilon \Gamma$, or both Γ, TX and Γ, FX are inconsistent.

Lemma 1: Let Γ be a consistent set of signed formulas, and P be a non-empty set of parameters containing at least every parameter used in Γ. Then Γ can be extended to a set Δ which is maximal consistent with respect to P.

Proof: P is countable, so we may enumerate all formulas with parameters from P: $X_1, X_2, X_3, \ldots$

Let $\Delta_0 = \Gamma$

Having defined Δ_n, consider X_{n+1}. If Δ_n, TX_{n+1} is consistent, let it be Δ_{n+1}. If not, but if Δ_n, FX_{n+1} is consistent, let it be Δ_{n+1}. If neither holds, let Δ_{n+1} be Δ_n.

Let $\Delta = \bigcup \Delta_n$

The conclusion of the lemma is now obvious.

Q.E.D.

Def: Let Γ be a set of signed formulas and P a set of parameters. We call Γ good with respect to P if

1) Γ is a maximal consistent with respect to P
2) $T(\exists x)X(x)\varepsilon\Gamma \Rightarrow TX(a)\varepsilon\Gamma$ for some $a\varepsilon P$

Lemma 2: Let Γ be a consistent set of signed formulas, and S be the set of parameters occurring in Γ. Let $\{a_1, a_2, a_3, \ldots\}$ be a countable set of distinct parameters not in S, and let $P = S \cup \{a_1, a_2, a_3, \ldots\}$. Then Γ can be extended to a set Δ which is good with respect to P.

Proof: P is countable, order the set of formulas with parameters from P; $X_1, X_2, X_3, \cdots$. We proceed.

1) let $\Delta_0 = \Gamma$
2) extend Δ_0 to a set Δ_1 maximal consistent with respect to S.
3) take the first X_i (in the above ordering) of the form $T(\exists x)X(x)$ such that $T(\exists x)X(x)\varepsilon\Delta_1$ but for no $\alpha\varepsilon S$ is $TX(\alpha)\varepsilon\Delta_1$. Let $\Delta_2 = \Delta_1, TX(a_1)$. Since

a_1 is "new", Δ_2 is consistent.

4) extend Δ_2 to a set Δ_3 maximal consistent with respect to $S \cup \{a_1\}$.

5) take the first X_i of the form $T(\exists x)X(x)$ such that $T(\exists x)X(x) \epsilon \Delta_3$ but for no $\alpha \epsilon S \cup \{a_1\}$ is $TX(\alpha) \epsilon \Delta_3$. Let $\Delta_4 = \Delta_3, TX(a_2)$. Again, Δ_4 is consistent.

6) extend Δ_4 to a set Δ_5 maximal consistent with respect to $S \cup \{a_1, a_2\}$ and so on.

Let $\Delta = \bigcup \Delta_n$. We claim Δ is good with respect to P.

First Δ is consistent since each Δ_n is consistent.

If X has all its parameters in P, then for some n, all the parameters of X are in $S \cup \{a_1, a_2, \ldots, a_n\}$. But in step 2n we extend Δ_{2n} to Δ_{2n+1}, a set maximal consistent with respect to $S \cup \{a_1, a_2, \ldots, a_n\}$. Thus TX or FX is in Δ_{2n+1} and hence in Δ, or neither can be added consistently. Thus Δ is maximal consistent with respect to P .

Finally, suppose $T(\exists x)X(x) \epsilon \Delta$. We note that the formula dealt with in step 5 is different than the one dealt with in step 3, and the one dealt with in step 7 is different again. Thus we must eventually reach $T(\exists x)X(x)$, and so, for some $\alpha \varepsilon P$, $TX(\alpha)\varepsilon\Delta$.

Thus Δ is good with respect to P.

Q.E.D.

Now let us order our countably many parameters as follows:

$$\begin{array}{lllll} S_1: & a_1^1, & a_2^1, & a_3^1, & \ldots \\ S_2: & a_1^2, & a_2^2, & a_3^2, & \ldots \\ S_3 & a_1^3, & a_2^3, & a_3^3. & \ldots \\ \vdots & & \vdots & \vdots & \end{array}$$

and let $P_n = S_1 \cup S_2 \cup \ldots \cup S_n$.

Let G be the collection of all sets of signed formulas which are good with respect to some P_n. We claim G is a Hintikka collection.

Suppose $\Gamma \varepsilon G$. Then Γ is good with respect to some P_i, say P_n. Then $\mathcal{P}(\Gamma)$ (the collection of all parameters of Γ) is P_n.

Suppose $TX \wedge Y \varepsilon \Gamma$ but $TX \notin \Gamma$. If $\Gamma, TX \wedge Y$ is consistent, so is $\Gamma, TX \wedge Y, TX$, and so Γ is not maximal. Thus $TX\varepsilon\Gamma$. Similarly, $TY\varepsilon\Gamma$. Hence

$$TX \wedge Y\varepsilon\Gamma \quad \Rightarrow \quad TX\varepsilon\Gamma \quad \text{and} \quad TY\varepsilon\Gamma$$

Similarly we may show

$$FX \vee Y\varepsilon\Gamma \quad \Rightarrow \quad FX\varepsilon\Gamma \quad \text{and} \quad FY\varepsilon\Gamma$$

$$TX \vee Y\varepsilon\Gamma \quad \Rightarrow \quad TX\varepsilon\Gamma \quad \text{or} \quad TY\varepsilon\Gamma$$

$$FX \wedge Y\varepsilon\Gamma \quad \Rightarrow \quad FX\varepsilon\Gamma \quad \text{or} \quad FY\varepsilon\Gamma$$

$$T{\sim}X\varepsilon\Gamma \quad \Rightarrow \quad FX\varepsilon\Gamma$$

$$TX \supset Y\varepsilon\Gamma \quad \Rightarrow \quad FX\varepsilon\Gamma \quad \text{or} \quad TY\varepsilon\Gamma$$

$$T(\forall x)X(x)\varepsilon\Gamma \quad \Rightarrow \quad TX(a)\varepsilon\Gamma \quad \text{for every} \quad a\varepsilon\mathcal{P}(\Gamma)$$

$$F(\exists x)X(x)\varepsilon\Gamma \quad \Rightarrow \quad FX(a)\varepsilon\Gamma \quad \text{for every} \quad a\varepsilon\mathcal{P}(\Gamma)$$

Moreover,

$$T(\exists x)X(x)\varepsilon\Gamma \quad \Rightarrow \quad TX(a)\varepsilon\Gamma \quad \text{for some} \quad a\varepsilon\mathcal{P}(\Gamma)$$

since Γ is good with respect to P_n.

Suppose $F{\sim}X\varepsilon\Gamma$. Since Γ is consistent, Γ_T, TX is consistent. Extend it to a set Δ which is good with respect to P_{n+1}. Then $\mathcal{P}(\Gamma) \subseteq \mathcal{P}(\Delta)$, and $\Gamma_T \subseteq \Delta$, so $\Gamma R\Delta$, and $TX\varepsilon\Delta$.

Similarly, if $FX \supset Y\varepsilon\Gamma$, there is a $\Delta\varepsilon G$ such that $\Gamma R\Delta$ and $TX\varepsilon\Delta$, $FY\varepsilon\Delta$.

Finally, if $F(\forall x)X(x)\varepsilon\Gamma$, since a_1^{n+1} does not occur in Γ, Γ_T, $FX(a_1^{n+1})$ is consistent. Extend it to a set Δ which is good with respect to P_{n+1}.

Again, $\Gamma R \Delta$ and $FX(a_1^{n+1}) \varepsilon \Delta$ for $a_1^{n+1} \varepsilon \mathcal{P}(\Delta)$.

Thus G is a Hintikka collection.

To complete the proof, suppose X is not provable. Then {FX} is consistent. Since it has only finitely many parameters, they must all lie in some P_n. Extend {FX} to a set Γ good with respect to P_n. Then $\Gamma \varepsilon G$ and $FX \varepsilon \Gamma$. This establishes completeness.

Remark: The model resulting from this Hintikka collection is a "universal" model in that it is a counter-model for every non-theorem. This is not the case for the model of section 5.

We will show later that, in a sense, this Hintikka collection is the analog of a classical truth set.

Section 7

An axiom system A_1

The following system was chosen to give a fairly quick completeness proof. It is very close to the system of [9, pg. 82].

Axiom schemas:

1. $X \supset (Y \supset X)$
2. $(X \supset Y) \supset ((X \supset (Y \supset Z)) \supset (X \supset Z))$
3. $((X \supset Z) \wedge (Y \supset Z)) \supset ((X \vee Y) \supset Z)$
4. $(X \wedge Y) \supset X$
5. $(X \wedge Y) \supset Y$
6. $X \supset (Y \supset (X \wedge Y))$
7. $X \supset (X \vee Y)$
8. $Y \supset (X \vee Y)$
9. $(X \wedge \sim X) \supset Y$
10. $(X \supset \sim X) \supset \sim X$
11. $X(a) \supset (\exists x)X(x)$
12. $(\forall x)X(x) \supset X(a)$

Rules:

13. $\dfrac{X(a) \supset Y}{(\exists x)X(x) \supset Y}$

14. $\dfrac{Y \supset X(a)}{Y \supset (\forall x)X(x)}$

15. $\dfrac{X, X \supset Y}{Y}$

In rules 13 and 14, the parameter a must not occur in Y. In a deduction from premises, the parameter a must not occur in the premises either. We use the usual notation, if X can be deduced from a finite subset of S, we write $S \vdash X$. We use $\vdash X$ for $\phi \vdash X$.

In the next three sections we establish the correctness and completeness of A_1. We introduce a second system A_2, equivalent to A_1 to aid in showing correctness. For use in showing completeness we need the following three lemmas.

Lemma 1: The deduction theorem holds for A_1.

Proof: The standard one. e.g. [9, section 21-22].

Lemma 2: $$\frac{\vdash (W \wedge Y) \supset X, \quad \vdash (W \wedge Z) \supset X, \quad \vdash W \supset (Y \vee Z)}{\vdash W \supset X}$$

Proof:

1)	$(W \wedge Y) \supset X$	by hypothesis, theorem
2)	$(W \wedge Z) \supset X$	by hypothesis, theorem
3)	$W \supset (Y \vee Z)$	by hypothesis, theorem
4)	W	premise
5)	$Y \vee Z$	3, 4, rule 15
6)	$W \supset (Y \supset (W \wedge Y))$	ax 6
7)	$Y \supset (W \wedge Y)$	4, 6, rule 15
8)	$W \supset (Z \supset (W \wedge Z))$	ax 6
9)	$Z \supset (W \wedge Z)$	4, 8, rule 15
10)	$Y \supset X$	via 1, 7
11)	$Z \supset X$	via 2, 9
12)	$(Y \vee Z) \supset X$	via 10, 11, ax 3
13)	X	5, 12, rule 15
14)	$W \supset X$	deduction theorem cancelling premise 4

Q.E.D.

Lemma 3: If a does not occur in W, $Y(x)$, or X,

$$\frac{\vdash (W \wedge Y(a)) \supset X, \quad \vdash W \supset (\exists x)Y(x)}{\vdash W \supset X}$$

Proof:

1)	$(W \wedge Y(a)) \supset X$	by hypothesis, theorems
2)	$W \supset (\exists x)Y(x)$	
3)	W	premise
4)	$(\exists x)Y(x)$	2, 3, rule 15
5)	$W \supset (Y(a) \supset (W \supset Y(a)))$	ax 6
6)	$Y(a) \supset (W \wedge Y(a))$	3, 5, rule 15
7)	$Y(a) \supset X$	via 1, 6
8)	$(\exists x)Y(x) \supset X$	7, rule 13
9)	X	4, 8, rule 15
10)	$W \supset X$	deduction theorem cancelling premise 3

Section 8

A second axiom system A_2

We introduce a second, very similar, axiom system, and prove equivalence.

A_2 has the same axioms as A_1, as well as rules 13 and 14. It does not have rule 15. It has rules

14a) $\dfrac{X(a)}{(\forall x)X(x)}$

15a) $\dfrac{(\forall x_1)\cdots(\forall x_n)X, \quad (\exists x_1)\cdots(\exists x_n)X \supset Y}{Y}$

provided all parameters of $(\forall x_1)\cdots(\forall x_n)X$ are also in Y. [n may be 0]

To show the two systems are equivalent, it suffices to show 14a) and 15a) are derived rules of A_1, and 15) is a derived rule of A_2.

To show 14a) is a derived rule of A_1, suppose in A_1 we have $X(a)$. Let T be any theorem of A_1 with no parameters. By axiom 1), $X(a) \supset (T \supset X(a))$, so by rule 15), $T \supset X(a)$. Since a is not in T, by rule 14), $T \supset (\forall x)X(x)$. But also T, so by rule 15), $(\forall x)X(x)$.

To show 15a) is a derived rule of A_1, suppose in A_1 we have $(\forall x_1)\cdots(\forall x_n)X(x_1,\cdots,x_n)$ and $(\exists x_1)\cdots(\exists x_n)X(x_1,\cdots,x_n) \supset Y$, and all parameters of $(\forall x_1)\cdots(\forall x_n)X(x_1,\cdots,x_n)$ are in Y. From $(\forall x_1)\cdots(\forall x_n)X(x_1,\cdots,x_n)$ by axiom 12), $X(a_1,\cdots,a_n)$. From axiom 11), $X(a_1,\cdots a_n) \supset (\exists x_1,\cdots,x_n)X(x_1,\cdots,xn)$ so by rule 15), $(\exists x_1)\cdots(\exists x_n)X(x_1,\cdots,x_n)$ and by rule 15) again, Y.

Finally to show rule 15) is a derived rule of A_2, suppose we have X and $X \supset Y$ in A_2. Let $a_1, a_2 \cdots, a_n$ be those parameters of X not in Y. Since we have $X(a_1, \ldots, a_n)$, by rule 14a), $(\forall x_1) \ldots (\forall x_n) X(x_1, \ldots, x_n)$. Similarly, since $X(a_1, \ldots, a_n) \supset Y$ and $a_1, \ldots, a_n$ do not occur in Y, by rule 13), $(\exists x_1) \ldots (\exists x_n) X(x_1, \ldots, x_n) \supset Y$. Now by rule 16a), Y.

Thus, A_1 and A_2 are equivalent. For use in the next section we state the straightforward.

Lemma: If in A_2 we can prove $X(a)$, there is a proof of the same length of $X(b)$ for any parameter b. [note: a does not occur in $X(b) = X(a) \binom{a}{b}$].

Section 9

Correctness of system A_2

Theorem: If X is provable in A_2, X is valid.

Proof: By induction on the length of the proof for X. If the proof is of length 1, X is an axiom and we leave the reader to show validity of the axioms.

Suppose the result is known for all formulas with proofs of length less than n steps, and X is provable in n steps. We investigate the steps involved in the proof of X. Axioms have been treated.

Rule 13), $X(a) \supset Y$ is provable in less than n steps where a is not in Y. Then $X(a) \supset Y$ is valid. Then $(\exists x)X(x) \supset Y$ is provable. We wish to show it is valid. Take any model $\langle G, R, \models, \mathcal{P} \rangle$ and any $\Gamma \varepsilon G$ and suppose $((\exists x)X(x) \supset Y) \varepsilon \hat{\mathcal{P}}(\Gamma)$. Suppose $\Gamma^* \models (\exists x)X(x)$. Then $\Gamma^* \models X(b)$ for some b. But $X(a) \supset Y$ is provable, so by the lemma of section 8, $(X(a) \supset Y) \binom{a}{b}$ is provable with a proof of the same length, hence by hypothesis, valid. Since a is not in Y, this is $X(b) \supset Y$. By validity, $\Gamma^* \models X(b) \supset Y$, hence $\Gamma^* \models Y$. Thus $\Gamma \models (\exists x)X(x) \supset Y$.

Rules 14) and 14a) are similar.

Rule 15a) Suppose $(\forall x_1)\ldots(\forall x_n)X$ and $(\exists x_1)\ldots(\exists x_n)X \supset Y$ are both provable and valid. Then Y is provable. We wish to show Y is valid. Let $\langle G, R, \models, \mathcal{P} \rangle$ be any model and $\Gamma \varepsilon G$. Suppose $Y \varepsilon \hat{\mathcal{P}}(\Gamma)$. Then $(\forall x_1)\ldots(\forall x_n)X$ and $(\exists x_1)\ldots(\exists x_n)X \supset Y$ are both in $\hat{\mathcal{P}}(\Gamma)$, and since they are valid, $\Gamma \models (\forall x_1)\ldots(\forall x_n)X$ and $\Gamma \models (\exists x_1)\ldots(\exists x_n)X \supset Y$. By the latter, either $\Gamma \not\models (\exists x_1)\ldots(\exists x_n)X$ or $\Gamma \models Y$. If $\Gamma \not\models (\exists x_1)\ldots(\exists x_n)X$, for some $a_1, \ldots, a_n \varepsilon \mathcal{P}(\Gamma)$,

$\Gamma \not\models X(a_1,\ldots,a_n)$, contradicting $\Gamma \models (\forall x_1)\ldots(\forall x_n)X$.

Hence $\Gamma \models Y$.

Q.E.D.

Section 10

Completeness of system A_1

The following Henkin type proof was discovered independently by Thomason [19] and the author.

We work in the system A_1. Let Γ be a set of unsigned formulas and P a collection of parameters. Suppose all the parameters of Γ are among those in P.

By the deductive completion of Γ with respect to P we mean the smallest set of formulas, Δ, involving only parameters of P, such that for any X over P, $\Gamma \vdash X \Rightarrow X \varepsilon \Delta$.

We call Γ deductively complete with respect to P if it is its own deductive completion with respect to P.

We say Γ has the O_r - property if
$X \vee Y \varepsilon \Gamma \Rightarrow X \varepsilon \Gamma$ or $Y \varepsilon \Gamma$.

We say Γ has the $\exists$-property if
$(\exists x)X(x) \varepsilon \Gamma \Rightarrow X(a) \varepsilon \Gamma$ for some parameter a.

We call Γ nice with respect to P if

1) Γ is deductively complete with respect to P
2) Γ has the O_r-property
3) Γ has the $\exists$-property
4) Γ is consistent

<u>Remark:</u> consistency here has its usual meaning.

<u>Lemma 1:</u> Let Γ be a set of formula and X a single formula. Let P be the set of all parameters of Γ or X. Let $\{a_1, a_2, a_3, \ldots\}$ be a countable collection of distinct parameters not in P, and let $Q = P \cup \{a_1, a_2, a_3 \ldots\}$. If $\Gamma \nvdash X$, then Γ can be extended to a set Δ which is nice with respect to Q such that $X \notin \Delta$.

<u>Proof:</u> Let $Z_1, Z_2, Z_3 \ldots$ be an enumeration of all formulas with parameters from Q of the form $Y \vee Z$ or $(\exists x)Y(x)$.

Since $\Gamma \nvdash X$, Γ is consistent. We define a sequence $\{\Gamma_n\}$.

Let Γ_0 be the deductive completion of Γ with respect to P. Then Γ_0 is consistent and $\Gamma_0 \nvdash X$.

Suppose we have defined Γ_n so that Γ_n is deductively complete with respect to $P \cup \{a_1, a_2, \ldots, a_n\}$ and $\Gamma_n \nvdash X$. Let $\Delta_n^0 = \Gamma_n$.

Suppose we have defined Δ_n^j $(j<n)$ so that it is consistent, $\Delta_n^j \nvdash X$. If $Z_j \notin \Delta_n^j$, let $\Delta_n^{j+1} = \Delta_n^j$. If $Z_j = Y \vee Z$, $Z_j \in \Delta_n^j$, and $Y \in \Delta_n^j$ or $Z \in \Delta_n^j$, let $\Delta_n^{j+1} = \Delta_n^j$. If $Z_j = (\exists x)Y(x)$, $Z_j \in \Delta_n^j$, and $Y(a) \in \Delta_n^j$ for some a, let $\Delta_n^{j+1} = \Delta_n^j$. This leaves the two key cases.

Suppose $Z_j \in \Delta_n^j$ and Z_j is $Y \vee Z$ but $Y \notin \Delta_n^j$, $Z \notin \Delta_n^j$. We claim we can add one of Y or Z to Δ_n^j so that the result still does not yield X. For otherwise, $\Delta_n^j, Y \vdash X$

$$\Delta_n^j, Z \vdash X$$

$$\Delta_n^j \vdash Y \vee Z$$

[since $Y \vee Z \in \Delta_n^j$]. But then by lemma 2, section 7, $\Delta_n^j \vdash X$, a contradiction. So, add to Δ_n^j one of Y or Z so that the result does not yield X. Call the result Δ_n^{j+1}.

Suppose $Z_j \in \Delta_n^j$ and Z_j is $(\exists x)Y(x)$, but $Y(a) \notin \Delta_n^j$ for any a. Take the first unused a_i of $\{a_1, a_2, \ldots\}$. We claim we can add $Y(a_i)$ to Δ_n^j

and the result will not yield X. This is as above but by lemma 3, section 7. Thus $\Delta_n^j, Y(a_i) \nvdash X$.

Let Δ_n^{j+1} be $\Delta_n^j, Y(a_i)$.

Thus, in any case, Δ_n^{j+1} is consistent, and $X \notin \Delta_n^{j+1}$.

Let Γ_{n+1} be the deductive completion of Δ_n^n with respect to $P\ \{a_1, a_2, \ldots, a_k\}$ where a_k is the last parameter used in Δ_n^n.

Let $\Delta = \bigcup \Gamma_n$

Δ uses exactly the parameters of Q.

$X \notin \Delta$ since $X \notin \Gamma_n$ for any n.

Δ is deductively complete with respect to Q.

Δ has the O_r-property, for if $Y \vee Z \varepsilon \Delta$, say $Y \vee Z = Z_n$, then $Y \vee Z \varepsilon \Delta_m$ for some m. We can take $m > n$. Then $Y \vee Z = Z_n \varepsilon \Delta_m^n$, so either Y or Z is in $\Delta_m^{n+1} \subseteq \Delta$.

Similarly, Δ has the $\exists$-property.

Q.E.D.

Lemma 2: If Γ is nice with respect to P,

1) $X \wedge Y \varepsilon \Gamma \iff X \varepsilon \Gamma$ and $Y \varepsilon \Gamma$
2) $X \vee Y \varepsilon \Gamma \iff X \varepsilon \Gamma$ or $Y \varepsilon \Gamma$
3) $\sim X \Rightarrow X \not\varepsilon \Gamma$
4) $X \supset Y \varepsilon \Gamma \Rightarrow X \not\varepsilon \Gamma$ or $Y \varepsilon \Gamma$
5) $(\exists x)X(x) \varepsilon \Gamma \iff X(a) \varepsilon \Gamma$ for some $a \varepsilon P$
6) $(\forall x)X(x) \varepsilon \Gamma \Rightarrow X(a) \varepsilon \Gamma$ for every $a \varepsilon P$

Proof: 1) is by axioms 4, 5 and 6, since Γ is deductively complete with respect to P.
$X \vee Y \varepsilon \Gamma \Rightarrow X \varepsilon \Gamma$ or $Y \varepsilon \Gamma$ since Γ has the O_r-property. The converse holds by axioms 7 and 8.

If $\sim X \varepsilon \Gamma$, $X \not\varepsilon \Gamma$ since Γ is consistent (using axiom 9).

If $X \supset Y \varepsilon \Gamma$, either $X \not\varepsilon \Gamma$ or $Y \varepsilon \Gamma$ since Γ is deductively complete with respect to P.

If $(\exists x)X(x) \varepsilon \Gamma$, $X(a) \varepsilon \Gamma$ for some $a \varepsilon P$ since Γ has the $\exists$-property. The converse is by axiom 11.

Property 6 is by axiom 12.

Q.E.D

Lemma 3: Suppose Γ is nice with respect to P, and $\{a_1, a_2, a_3 \ldots\}$ is a set of distinct parameters not in P.

Let $Q = P \cup \{a_1, a_2 . a_3 \ldots\}$. Then

1) If X has all its parameters in P but $\sim X \notin \Gamma$, Γ can be extended to a set Δ nice with respect to Q such that $X \varepsilon \Delta$.

2) If $X \supset Y$ has all its parameters in P but $X \supset Y \notin \Gamma$, Γ can be extended to a set Δ nice with respect to Q such that $X \varepsilon \Delta$ and $Y \notin \Delta$.

3) If $X(x)$ has all its parameters in P but $(\forall x) X(x) \notin \Gamma$, Γ can be extended to a set Δ nice with respect to Q such that for some $a \varepsilon Q$, $X(a) \notin \Delta$.

<u>Proof:</u>

1) since $\sim X \notin \Gamma$, Γ, X is consistent, for otherwise, $\Gamma, X \vdash \sim X$ so by the deduction theorem, $\Gamma \vdash X \supset \sim X$ and by axiom 10, $\Gamma \vdash \sim X$, so $\sim X \varepsilon \Gamma$. Since Γ, X is consistent, there is some Y such that $\Gamma, X \nvdash Y$. Now use lemma 1.

2) $\Gamma, X \nvdash Y$ for otherwise, by the deduction theorem, $\Gamma \vdash X \supset Y$ so $X \supset Y \varepsilon \Gamma$. Since $\Gamma, X \nvdash Y$, use lemma 1.

3) $a_1 \notin P$. We claim $\Gamma \nvdash X(a_1)$. Suppose $\Gamma \vdash X(a_1)$. For the conjunction, call it W, of some finite subset of Γ, $\vdash W \supset X(a_1)$. But a_1 does not occur in W. By rule 14, $\vdash W \supset (\forall x) X(x)$, so

$\Gamma \vdash (\forall x)X(x)$, $(\forall x)X(x) \varepsilon \Gamma$. Since $\Gamma \nvdash X(a_1)$, use lemma 1.

Q.E.D

Now we proceed to show completeness. We arrange the parameters as follows:

$$
\begin{array}{lllll}
S_1: & a_1^1, & a_2^1, & a_3^1, & \ldots \\
S_2: & a_1^2, & a_2^2, & a_3^2, & \ldots \\
S_3: & a_1^3, & a_2^3, & a_3^3, & \ldots \\
\vdots & & \vdots & &
\end{array}
$$

and let $P_n = S_1 \cup S_2 \cup \ldots \cup S_n$.

Let G be the collection of all nice sets with respect to any P_i.

If $\Gamma \varepsilon G$, Γ is nice with respect to, say, P_n. Let $\mathcal{P}(\Gamma) = P_n$. Let $\Gamma R \Delta$ if $\mathcal{P}(\Gamma) \subseteq \mathcal{P}(\Delta)$ and $\Gamma \subseteq \Delta$.

For any X, let $\Gamma \models X$ iff $X \varepsilon \Gamma$.

By lemmas 2 and 3, $\langle G, R, \models, \mathcal{P} \rangle$ is a model.

Finally, suppose $\nvdash X$. All the parameters are in, say, P_n. Since $\phi \nvdash X$, by lemma 1 we can extend ϕ to a set Γ, nice with respect to P_n such that $X \notin \Gamma$. Thus $\Gamma \varepsilon G$, $X \varepsilon \hat{\mathcal{P}}(\Gamma)$ and $\Gamma \not\models X$.

Remark: This is a "universal" model in the sense of section 6.

In section 4, chapter 6, we will show that the set of all theorems using only parameters of P_n is itself a nice set with respect to P_n. This would make the final use of lemma 1 above unnecessary.

Chapter 6

Additional First Order Results

Section 1

Compactness

We call an infinite set, S, of signed formulas realizable if there is a model, $\langle G, R, \models, \mathcal{P} \rangle$ and a $\Gamma \varepsilon G$ such that for any formula X,

$$TX \varepsilon S \Rightarrow X \varepsilon \hat{\mathcal{P}}(\Gamma) \text{ and } \Gamma \models X$$

$$FX \varepsilon S \Rightarrow X \varepsilon \hat{\mathcal{P}}(\Gamma) \text{ and } \Gamma \not\models X \; .$$

There is a similar concept for sets of unsigned formulas, $\mathcal{U}$. We say $\mathcal{U}$ is satisfiable if there is a model. $\langle G, R, \models, \mathcal{P} \rangle$ and a $\Gamma \varepsilon G$ such that for any formula X,

$$X \varepsilon \mathcal{U} \Rightarrow X \varepsilon \hat{\mathcal{P}}(\Gamma) \text{ and } \Gamma \models X.$$

Lemma 1: Let $\mathcal{U}$ be a set of unsigned formulas and define a set S of signed formulas to be $\{TX \mid X \varepsilon \mathcal{U}\}$. Then

1) $\mathcal{U}$ is satisfiable if and only if S is realizable

2) $\mathcal{U}$ is consistent if and only if S is consistent.

Proof: Part 1) is obvious.

To show part 2), suppose $\mathcal{U}$ is not consistent. Then some finite subset, $\{u_1, \dots , u_n\}$ is not consistent, so from it we can deduce any formula. Let A be an atomic formula having no predicate symbols or parameters in common with $\{u_1, \dots, u_n\}$. Then

$$\vdash_I (u_1 \wedge \dots \wedge u_n) \supset A$$

hence there is a closed tableau for

$$\{F(u_1 \wedge \ldots \wedge u_n) \supset A\}$$

so there is a closed tableau for

$$\{T(u_1 \wedge \ldots \wedge u_n) , FA\}$$

By the way we have chosen A, there must be a closed tableau for

$$\{T(u_1 \wedge \ldots \wedge u_n)\}$$

and hence, for

$$\{Tu_1, \ldots , Tu_n\} .$$

Thus S is not consistent.

The converse is trivial.

Q.E.D.

Because we have this lemma, we will only discuss realizability and consistency of sets of signed formulas.

Lemma 2: Let S be a set of signed formulas. If S is realizable, S is consistent.

Proof: If S is not consistent, some finite subset, Q, is not consistent. That is, there is a closed tableau, $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ in which $\mathcal{C}_1$ is {Q}. If Q were realizable, by the theorem of section 2 chapter 5, every $\mathcal{C}_i$ would be, but a closed configuration is not realizable.

Q.E.D.

Lemma 3: Let S be a finite set of signed formulas. If S is consistent, S is realizable.

Proof: Let S be $\{TX_1, \ldots, TX_n, FY_1, \ldots, FY_m\}$. S is consistent if and only if

$$\{F(X_1 \wedge \ldots \wedge X_n) \supset (Y_1 \vee \ldots \vee Y_m)\} \text{ is consistent.}$$

If this is consistent, $(X_1 \wedge \ldots \wedge X_n) \supset (Y_1 \vee \ldots \vee Y_m)$ is a non-theorem, so by the completeness theorem, there is a model $\langle G, R, \models, \mathcal{P}\rangle$ and a $\Gamma \varepsilon G$ such that $X_i \varepsilon \hat{\mathcal{P}}(\Gamma)$, $Y_j \varepsilon \hat{\mathcal{P}}(\Gamma)$, and $\Gamma \not\models (X_1 \wedge \ldots \wedge X_n) \supset (Y_1 \vee \ldots \vee Y_m)$. But then for some Γ^*,

$$\Gamma^* \models X_1 \wedge \ldots \wedge X_n \quad , \quad \Gamma^* \not\models Y_1 \vee \ldots \vee Y_m$$

so Γ^* realizes S.

Q.E.D.

This method does not work if S is infinite, but the lemma remains true, at least for sets with no parameters. The result can be extended to sets with some parameters, but we will not do so.

Lemma 4: Let S be an infinite set of signed formulas with no parameters. If S is consistent, S is realizable.

Proof: The proof can be based on either of the two tableau completeness proofs.

If we use the first proof, that of section 5 chapter 5, change step 0 to : " S is consistent. Extend it to a Hintikka element with respect to P_1. Call the result Γ_1". Continue the proof as written. The lemma is then obvious.

If we use the proof of section 6 chapter 5, the result is even easier. S is consistent, so by lemma 2 of that section, we can extend S to a set Γ which is good with respect to P_1. The result follows immediately.

Q.E.D.

Theorem: If S is any set of signed formulas with no parameters, S is consistent if and only if S is realizable.

Corollary: If every finite subset of S is realizable, so is S.

Corollary: If $\mathcal{U}$ is any set of unsigned formulas with no parameters, $\mathcal{U}$ is consistent if and only if $\mathcal{U}$ is satisfiable.

Remark: The last corollary could have been established directly by adopting the completeness proof of section 10 chapter 5.

Section 2

Concerning the excluded middle law

If S is a set of unsigned formulas, by $S \vdash_c X$ and $S \vdash_I X$ we mean classical and intuitionistic derivability respectively.

Let $X(\alpha_1, \ldots, \alpha_n)$ be a formula having exactly the parameters $\alpha_1, \ldots, \alpha_n$. By the closure of X we mean the formula $(\forall x_{i_1}) \ldots (\forall x_{i_n})\ X(x_{i_1}, \ldots, x_{i_n})$ [where x_{i_j} does not occur in $X(\alpha_1, \ldots, \alpha_n)$].

Let $\mathcal{M}$ be the collection of the closures of all formulas of the form $X \vee \sim X$. We wish to show

Theorem: If X has no parameters,

$$\vdash_c X \iff \mathcal{M} \vdash_I X.$$

We first show

Lemma: Let $\langle G, R, \models, \rho \rangle$ be a model, $\Gamma \varepsilon G$, and suppose $Y \varepsilon \mathcal{M} \Rightarrow \Gamma \models Y$. Then Γ can be included in a complete R-chain $\mathcal{C}$ such that $\bar{\mathcal{C}}$ is a truth set. [see section 6 chapter 4]

Proof: Enumerate all formulas beginning with a universal quantifier, $X_1, X_2, X_3, \ldots$.

Let $\Gamma_o = \Gamma$.

Having defined Γ_n, consider X_{n+1}. If $X_{n+1} \not\in \hat{\rho}(\Gamma_n^*)$ for any Γ_n^*, let $\Gamma_{n+1} = \Gamma_n$. Otherwise there is some $\Gamma_n{}^*$ such that $X_{n+1} \varepsilon \hat{\mathcal{P}}(\Gamma_n{}^*)$. Say X_{n+1} is $(\forall x)\, X(x)$. We have two cases. If $\Gamma_n{}^* \models (\forall x) X(x)$, let $\Gamma_{n+1} = \Gamma_n{}^*$. If $\Gamma_n{}^* \not\models (\forall x)\, X(x)$, there is a $\Gamma_n{}^{**}$ and an $\alpha \varepsilon \mathcal{P}(\Gamma_n{}^{**})$ such that $\Gamma_n{}^{**} \not\models X(\alpha)$. Let Γ_{n+1} be this $\Gamma_n{}^{**}$.

Let the R-chain $\mathcal{C}$ be $\{ \Gamma_0, \Gamma_1, \Gamma_2, \ldots \}$.

Since $Y \varepsilon \mathcal{M} \Rightarrow \Gamma \models Y$ and $\Gamma = \Gamma_0$, $\mathcal{C}$ is a complete R-chain, by definition of $\mathcal{M}$, and so $\bar{\mathcal{C}}$ is an almost-truth set. Thus we have only one more fact to show:

$Y(\alpha)\ \varepsilon\ \bar{\mathcal{C}}$ for every parameter α of $\bar{\mathcal{C}} \Rightarrow (\forall x)\, Y(x)\ \varepsilon\ \bar{\mathcal{C}}$.

Suppose $(\forall x)\ Y(x,\alpha_1, \ldots, \alpha_n) \not\varepsilon\ \bar{\mathcal{C}}$ [where $\alpha_1, \ldots, \alpha_n$ are all the parameters of Y]. If some α_i is not a parameter of $\bar{\mathcal{C}}$, we are done. So, suppose each α_i occurs in $\bar{\mathcal{C}}$. Then for some $\Gamma_n\ \varepsilon\ \mathcal{C}$, all $\alpha_i \varepsilon\ \mathcal{P}(\Gamma_n)$ and $\Gamma_n \not\models (\forall x)\ Y(x,\alpha_1, \ldots, \alpha_n)$. But by the construction of $\mathcal{C}$, there is a Γ_m $m \geq n$, such that $\Gamma_m \not\models Y(b, \alpha_1, \ldots, \alpha_n)$ for some $b \varepsilon\ \mathcal{P}(\Gamma_m)$. But,

$\Gamma \models (\forall x_1) \ldots (\forall x_n)(\forall x)\ [Y(x,x_1,\ldots,x_n) \vee \sim Y(x,x_1,\ldots,x_n)]$

and $\Gamma R \Gamma_m$, so

$\Gamma_m \models Y(b, \alpha_1, \ldots, \alpha_n) \vee \sim Y(b, \alpha, \ldots, \alpha_n)$,

thus $\Gamma_m \models \sim Y\ (b, \alpha_1, \ldots, \alpha_n)$.

$\sim Y(b, \alpha_1, \ldots, \alpha_n)\ \varepsilon\ \bar{\mathcal{C}}$, so $Y(b, \alpha, \ldots, \alpha_n) \not\varepsilon\ \bar{\mathcal{C}}$ for a parameter b of $\bar{\mathcal{C}}$.

Q.E.D.

Now to prove the theorem itself.

If $\mathfrak{M} \vdash_I X$ then for some finite subset $\{m_1, \ldots, m_n\}$ of $\mathfrak{M}$,

$$\vdash_I (m_1 \wedge \ldots \wedge m_n) \supset X.$$

By theorem 2, section 8 chapter 4 [and the completeness theorems]

$$\vdash_C (m_1 \wedge \ldots \wedge m_n) \supset X .$$

But $\vdash_C m_1 \wedge \ldots \wedge m_n$ hence $\vdash_C X$. Conversly, if $\mathfrak{M} \nvdash_I X$, let S be the set of signed formulas $\{FX\} \cup \{TY \mid Y \varepsilon \mathfrak{M}\}$.

Since $\mathfrak{M} \nvdash_I X$, S is consistent. Then by the results of the last section, S is realizable. Thus there is a model $\langle G, R, \models, \rho \rangle$ and a $\Gamma \varepsilon G$ such that

$$Y \varepsilon \mathfrak{M} \Rightarrow \Gamma \models Y$$

$$X \varepsilon \hat{\rho}(\Gamma) \text{ and } \Gamma \not\models X$$

But, X has no parameters, so $X \vee \sim X \varepsilon \mathfrak{M}$. Thus $\Gamma \models X \vee \sim X$. So, $\Gamma \models \sim X$. Now by the lemma, there is a truth set containing $\sim X$. Hence $\nvdash_C X$.

Section 3

Skolem - Löwenheim

By the domain of a model $\langle G, R, \models, \rho \rangle$ we mean $\bigcup_{\Gamma \varepsilon G} \rho(\Gamma)$. So far we have only considered models in which the domain was at most countable. Suppose now we have an uncountable number of parameters and we change the definitions of formula, model, and validity accordingly, but not the definition of proof.

<u>Theorem:</u> X is valid in all models if and only if X is valid in all models with countable domains.

<u>Proof:</u> Half is trivial.

Suppose there is a model $\langle G, R, \models, \rho \rangle$ with an uncountable domain in which X is not valid. The correctness proof of section 2 or section 9, chapter 5, is still applicable. Thus X is not provable. Since X is not provable, if we reduce the collection of parameters to a countable number, [including those of X] X still will not be provable. Then any of the completeness proofs will furnish a counter-model for X with a countable domain.

Q.E.D.

This method may be combined with that of section 1 to show

<u>Theorem:</u> If S is any countable set of signed formulas with no parameters, S is consistent if and only if S is realizable in a model with a countable domain.

Theorem: If $\mathcal{U}$ is any countable set of unsigned formulas with no parameters, $\mathcal{U}$ is consistent if and only if $\mathcal{U}$ is satisfiable in a model with a countable domain.

Remark: In part II, we will be using models with domains of arbitrarily high cardinality.

Section 4

Kleene tableaus

The system of this section is based on the intuitionistic system G3 of [9]. The modifications are due to Smullyan. The resulting system is like that of Beth except that sets of signed formulas never contain more than one F-signed formula. Explicitely, everything is as it was in section 1 chapter 2 and section 1 chapter 5 except that the reduction rules are replaced by the following, where S is a set of signed formulas with at most one F-signed formula.

KT∨ $\dfrac{S,\ TX\vee Y}{S,\ TX \mid S,\ TY}$

KF∨ $\dfrac{S_T,\ FX\vee Y}{S_T\ \ FX}$ $\dfrac{S_T,\ FX\vee Y}{S_T,\ FY}$

KT∧ $\dfrac{S,\ TX\wedge Y}{S,\ TX,\ TY}$

KF∧ $\dfrac{S_T,\ FX\wedge Y}{S_T,\ FX \mid S_T,\ FY}$

KT~ $$\frac{S,\ T{\sim}X}{S_T,\ FX}$$ KF~ $$\frac{S_T,\ F{\sim}X}{S_T,\ TX}$$

KT⊃ $$\frac{S,\ TX \supset Y}{S_T,\ FX \mid S, TY}$$ KF⊃ $$\frac{S_T,\ FX \supset Y}{S_T,\ TX,\ FY}$$

KT∃ $$\frac{S,\ T(\exists x)X(x)}{S,\ TX(a)}$$ KF∃ $$\frac{S_T,\ F(\exists x)X(x)}{S_T\ \ FX(a)}$$

KT∀ $$\frac{S,\ T(\forall x)X(x)}{S,\ TX(a)}$$ KF∀ $$\frac{S_T,\ F(\forall x)X(x)}{S_T,\ FX(a)}$$

where, in KT∃ and KF∀, the parameter a does not occur in S or X(x).

There are several ways of showing this is actually a proof system for intuitionistic logic. We choose to show it is directly equivalent to the Beth tableau system, that is, we give a proof translation proceedure.

We leave it to the reader to show the almost obvious fact that anything provable by Kleene tableaus is provable by Beth tableaus. To show the converse, we need

<u>Lemma:</u> If a Beth tableau for $\{TX_1, \ldots, TX_n, FY_1, \ldots, FY_m\}$ closes, then there is a closed Kleene tableau for

$$\{TX_1,\ \ldots,\ TX_n,\ F(Y_1 \vee \ldots \vee Y_m)\}$$

Proof: The proof is by induction on the length of the closed Beth tableau. If the tableau is of length 1, the result is obvious. Now suppose we know the result for all closed Beth tableaus of length less than n, and a closed tableau for the set in question is of length n. We have several cases depending on the first step of the tableau.

If the first step is an application of rule $F\wedge$, the Beth tableau begins

$$\{\{S_T, FX_1, \ldots, FX_n, F\, Y\wedge Z\}\}$$
$$\{\{S_T, FX_1, \ldots, FX_n, FY\}, \quad \{S_T, FX_1, \ldots, FX_n, FZ\}\}$$

and proceeds to closure. Now by the induction hypothesis, there are closed Kleene tableaus for

$$\{S_T, F(X_1 \vee \ldots \vee X_n \vee Y)\} \text{ and}$$
$$\{S_T, F(X_1 \vee \ldots \vee X_n \vee Z)\} .$$

We have two possibilities. If Y is not "used" in the first tableau, or if Z is not "used" in the second tableau, a Kleene tableau beginning

$$\{\{S_T, F(X_1 \vee \ldots \vee X_n \vee (Y\wedge Z))\}\}$$
$$\{\{S_T, F(X_1 \vee \ldots \vee X_n)\}\}$$

must close. If both Y and Z are "used", a Kleene tableau beginning

$$\{\{S_T, F(X_1 \vee \ldots \vee X_n \vee (Y\wedge Z))\}\}$$
$$\vdots$$
$$\{\{S_T, F(Y\wedge Z)\}\}$$
$$\{\{S_T, FY\}, \{S_T, FZ\}\}$$

must close.

The other cases are similar and are left to the reader.

Q.E.D.

Thus the two tableau systems are equivalent. Now we verify a remark made at the end of section 10 chapter 5.

Lemma: (Gödel, McKinsey and Tarski)

$\vdash_I X \vee Y$ iff $\vdash_I X$ or $\vdash_I Y$

Proof: Immediate from the Kleene tableau formulation.

Q.E.D.

Lemma: (Rasiowa and Sikorski)

If $\vdash_I (\exists x)\ X(x,a_1,\ldots,a_n)$ where $a_1,\ldots,a_n$ are all the parameters of X, then $\vdash_I X(b,a_1,\ldots,a_n)$ where b is one of the a_i. If X has no parameters, b is arbitrary and $\vdash_I (\forall x)\ X(x)$.

Proof: A Kleene tableau proof of $(\exists x)\ X(x,a_1,\ldots,a_n)$ begins

$$\{\{F(\exists x)\ X(x,a_1,\ldots,a_n)\}\}$$
$$\{\{\ FX(b,a_1,\ldots,a_n)\}\}$$

and proceeds to closure.

If b is some a_i, we are done. If not, we actually have a proof, except for a different first line, of

$$(\forall x)\ X(x, a_1, \ldots, a_n).$$

Q.E.D.

Section 5

Craig interpolation lemma

Theorem: If $\vdash_I X \supset Y$ and X and Y have a predicate symbol in common, then there is a formula Z involving only predicates and parameters common to X and Y such that $\vdash_I X \supset Z$ and $\vdash_I Z \supset Y$; if X and Y have no common parameters, either $\vdash_I \sim X$ or $\vdash_I Y$.

The classical version of this theorem was first proved by Craig, hence the name. The intuitionistic version is due to Schütte [16] . Essentially the same proof was given for a natural deduction system by Prawitz [14] . We give basically the same proof in the Kleene tableau system. For another proof in this system see [10] .

We find it convenient to temporarily introduce two symbols, t and f, into our collection of logical symbols, letting them be atomic formulas, and letting them combine according to the following rules.

$$X \vee t = t \vee X = t$$

$$X \vee f = f \vee X = X$$

$$X \wedge t = t \wedge X = X$$

$$X \wedge f = f \wedge X = f$$

$$\sim t = f \quad , \quad \sim f = t$$

$$X \supset t = f \supset X = t$$

$$t \supset X = X \qquad X \supset f = \sim X$$

$$(\exists x)\ t = (\forall x)\ t = t$$

$$(\exists x)\ f = (\forall x)\ f = f$$

By a block we mean a finite set of signed formulas containing at most one F-signed formula. When we call a block inconsistent, we mean there is a closed Kleene tableau for it. By an initial part of a block we mean any subset of the T-signed formulas. We make the convention that if S is the finite set of unsigned formulas $\{X_1, \ldots, X_n\}$ then TS is the set $\{TX_1, \ldots, TX_n\}$. We further make the convention that for a set S of formulas, S_1 and S_2 represent subsets such that $S_1 \cap S_2 = \phi$ and $S_1 \cup S_2 = S$. By $[S]$ we mean the set of predicates and parameters of formulas of S, together with t and f.

Now we define an interpolation formula X for the block $\{TS, FY\}$ [where S is a set of unsigned formulas and Y is a formula] with respect to the initial part TS_1, which we denote by $\{TS, FY\} / \{TS_1\}$, as follows.

[X may be t or f but we assume t and f are not part of S or Y]

X is an {TS, FY} / {TS_1} if

1) $[X] \subseteq [S_1] \cap [S_2, Y]$

2) {TS_1, FX} is inconsistent

3) {TX, TS_2, FY} is inconsistent

[we have temporarily added to the closure rules: closure of a set if it contains Tf or Ft].

Lemma: An inconsistent block has an interpolation formula with respect to every initial part.

Proof: We show this by induction on the length of the closed tableau for the block. If this is of length 1, the block must be of the form

{TS, TX, FX}

We have two cases.

case 1) The initial part is {TS_1, TX}. Then X is an interpolation formula.

case 2) The initial part is {TS_1} . Then {TS_2,TX,FX} is inconsistent and t is an interpolation formula.

Now suppose we have an inconsistent block, and the result is known for all inconsistent blocks with shorter closed tableaus. We have several cases depending on the first reduction rule used.

KT∨: The block is $\{TS, TX \vee Y, FZ\}$ and $\{TS, TX, FZ\}$ and $\{TS, TY, FZ\}$ are both inconsistent.

case 1) The initial part is $\{TS_1, TX \vee Y\}$. Then by induction hypothesis there are formulas U_1 and U_2 such that

U_1 is an $\{TS, TX, FZ\} / \{TS_1, TX\}$

U_2 is an $\{TS, TY, FZ\} / \{TS_1, TY\}$.

Then $U_1 \vee U_2$ is an $\{TS, TX \vee Y, FZ\} / \{TS_1, TX \vee Y\}$

case 2 The initial part is $\{TS_1\}$. Again, by hypothesis, there are U_1, U_2,

U_1 is an $\{TS, TX, FZ\} / \{TS_1\}$

U_2 is an $\{TS, TY, FZ\} / \{TS_1\}$

Then $U_1 \wedge U_2$ is an $\{TS, TX \vee Y, FZ\} / \{TS_1\}$

KF∨: The block is $\{TS, FX \vee Y\}$ and $\{TS, FX\}$ or $\{TS, FY\}$ is inconsistent.

Suppose the first. Let the initial part be $\{TS_1\}$. By hypothesis there is a U such that

U is an $\{TS, FX\} / \{TS_1\}$.

Then U is an $\{TS, FX \vee Y\} / \{TS_1\}$.

KT∧: The block is $\{TS, TX \wedge Y, FZ\}$ and $\{TS, TX, TY, FZ\}$ is inconsistent.

case 1) the initial part is $\{TS_1, TX \wedge Y\}$. By hypothesis there is a U such that

U is an $\{TS, TX, TY, FZ\}$ / $\{TS_1, TX, TY\}$.

Then U is an $\{TS, TX \wedge Y, FZ\}$ / $\{TS_1, TX \wedge Y\}$

<u>case 2)</u> The initial part is $\{TS_1\}$.

By hypothesis there is a U such that

U is an $\{TS, TX, TY, FZ\}$ / $\{TS_1\}$

Then U is an $\{TS, TX \wedge Y, FZ\}$ / $\{TS_1\}$.

<u>KF$\wedge$:</u> The block is $\{TS, FX \wedge Y\}$ and $\{TS, FX\}$ and $\{TS, FY\}$ are both inconsistent. Suppose the initial part is $\{TS_1\}$. By hypothesis there are U_1, U_2 such that

U_1 is an $\{TS, FX\}$ /$\{TS_1\}$

U_2 is an $\{TS, FY\}$ / $\{TS_1\}$.

Then $U_1 \wedge U_2$ is an $\{TS, FX \wedge Y\}$ /$\{TS_1\}$.

<u>KF~:</u> The block is $\{TS, F \sim X\}$ and $\{TS, TX\}$ is inconsistent. Suppose the initial part is $\{TS_1\}$. By hypothesis there is a U such that

U is an $\{TS, TX\}$ / $\{TS_1\}$

Then U is an $\{TS, F{\sim}X\}$ / $\{TS_1\}$

<u>KT~:</u> The block is $\{TS, T{\sim}X, FY\}$ and $\{TS, FX\}$ is inconsistent.

<u>case 1)</u> The initial part is $\{TS_1\}$. By hypothesis there is a U such that

U is an $\{TS, FX\}$ / $\{TS_1\}$

Then U is an $\{TS, T{\sim}X, FY\}$ / $\{TS_1\}$

<u>case 2)</u> The initial part is $\{TS_1, T{\sim}X\}$ By hypothesis there is a U such that

$$U \text{ is an } \{TS, FX\} \ / \ \{TS_2\}$$

We claim

$${\sim}U \text{ is an } \{TS, T{\sim}X, FY\} \ / \ \{TS_1\}$$

First we verify its predicates and parameters are correct.

By hypothesis, $[U] \subseteq [S_2] \cap [S_1, X]$ so immediately, $[{\sim}U] \subseteq [S_1, {\sim}X] \cap [S_2, Y]$

We have that the following two blocks are inconsistent,

$$\{TS_2, FU\}$$
$$\{TS_1, TU, FX\}$$

It follows that the following two blocks are also inconsistent,

$$\{TS_1, T{\sim}X, F{\sim}U\}$$
$$\{TS_2, T{\sim}U, FY\}$$

and we are done.

<u>KF⊃:</u> The block is $\{TS, FX \supset Y\}$ and $\{TS, TX, FY\}$ is inconsistent. Suppose the initial part is $\{TS_1\}$. By hypothesis there is a U such that

$$U \text{ is an } \{TS, TX, FY\} \ / \ \{TS_1\}$$

Then U is an $\{TS, FX \supset Y\} \ / \ \{TS_1\}$

<u>KT⊃:</u> The block is $\{TS, TX \supset Y, FZ\}$ and $\{TS, FX\}$ and $\{TS, TY, FZ\}$ are both inconsistent

<u>case 1)</u> The initial part is $\{TS_1\}$. By hypothesis there are U_1, U_2 such that

U_1 is $\{TS, FX\} / \{TS_1\}$

U_2 is an $\{TX, TY, FZ\} / \{TS_1\}$

Then $U_1 \wedge U_2$ is an $\{TS, TX \supset Y, FZ\} / \{TS_1\}$

<u>case 2)</u> The initial part is $\{TS_1, TX \supset Y\}$. By hypothesis there are U_1, U_2 such that

U_1 is an $\{TS, FX\} / \{TS_2\}$

U_2 is an $\{TS, TY, FZ\} / \{TS_1, TY\}$

We claim $U_1 \supset U_2$ is an

$$\{TS, TX \supset Y, FZ\} / \{TS_1, TX \supset Y\}.$$

By hypothesis,

$$[U_1] \subseteq [S_2] \cap [S_1, X]$$

$$[U_2] \subseteq [S_1, Y] \cap [S_2, Z]$$

so $$[U_1 \supset U_2] \subseteq [S_1, X \supset Y] \cap [S_2, Z]$$

We have that the following four blocks are inconsistent.

1) $\{TS_2, F U_1\}$

2) $\{T U_1, TS_1, FX\}$

3) $\{TS_1, TY, F U_2\}$

4) $\{T U_2, TS_2, FZ\}$

and we must show the following two blocks are inconsistent.

$$\{TS_1, TX \supset Y, F U_1 \supset U_2\}$$

$$\{T U_1 \supset U_2, TS_2, FZ\}.$$

The first follows from 2) and 3), and the second from 1) and 4).

KF∃ : The block is $\{TS, F(\exists x)\, X(x)\}$ and $\{TS, FX(a)\}$ is inconsistent. Suppose the initial part is $\{TS_1\}$. By hypothesis there is a U such that

$$U \text{ is an } \{TS, FX(a)\} \;/\; \{TS_1\}.$$

Then $[U] \subseteq [S_1] \cap [S_2, X(a)]$

case 1) $a \notin [U]$.

Then U is an $\{TS, F(\exists x)\, X(x)\} \;/\; \{TS_1\}$

case 2) $a \in [U]$, $a \in [S_2]$

Again U is an $\{TS, F(\exists x)\, X(x)\} \;/\; \{TS_1\}$

case 3) $a \in [U]$, $a \notin [S_2]$. Then $(\exists x)\, U\binom{a}{x}$ is an

$$\{TS, F(\exists x)\, X(x)\} \;/\; \{TS_1\}$$

KT∃ : The block is $\{TS, T(\exists x)\, X(x), FZ\}$ and $\{TS, TX(a), FZ\}$ is inconsistent, where $a \notin [S, X(x), Z]$.

case 1) The initial part is $\{TS_1, T(\exists x)\, X(x)\}$. By hypothesis there is a U such that

$$U \text{ is an } \{TS, TX(a), FZ\} \;/\; \{TS_1, TX(a)\}$$

Then U is an $\{TS, T(\exists x)\, X(x), FZ\} \;/\; \{TS_1, T(\exists x)\, X(x)\}$

case 2) The initial part is $\{TS_1\}$. By hypothesis there is a U such that

$$U \text{ is an } \{TS, TX(a), FZ\} \;/\; \{TS_1\}$$

Then U is an $\{TS, T(\exists x)\, X(x), FZ\} \;/\; \{TS_1\}$

KF∀ : The block is $\{TS, F(\forall x)\, X(x)\}$ and $\{TS, FX(a)\}$ is inconsistent where $a \notin [S, X(x)]$. Suppose the initial part is $\{TS_1\}$. By hypothesis there is a U such that

U is an $\{TS, FX(a)\} \ / \ \{TS_1\}$

Then U is an $\{TS, F(\forall x)\ X(x)\} \ / \ \{TS_1\}$

<u>KT∀ :</u> The block is $\{TS, T(\forall x)\ X(x), FZ\}$. and $\{TS, TX(a), FZ\}$ is inconsistent.

<u>case 1:</u> The initial part is $\{TS_1, T(\forall x)\ X(x)\}$. By hypothesis there is a U such that

U is an $\{TS, TX(a), FZ\} \ / \ \{TS_1, TX(a)\}$.

<u>case 1a:</u> $a \notin [U]$. Then U is an

$\{TS, T(\forall x)\ X(x), FZ\} \ / \ \{TS_1, T(\forall x)\ X(x)\}$.

<u>case 1b:</u> $a \varepsilon [U]$, $a \varepsilon [S_1, X(x)]$. Again

U is an $\{TS, T(\forall x)\ X(x), FZ\} \ / \ \{TS_1, T(\forall x)\ X(x)\}$.

<u>case 1c:</u> $a \varepsilon [U]$, $a \notin [S_1, X(x)]$.

Then $(\forall x)\ U\binom{a}{x}$ is an

$\{TS, T(\forall x)\ X(x), FZ\} \ / \ \{TS_1, T(\forall x)\ X(x)\}$

<u>case 2:</u> The initial part is $\{TS_1\}$.

By hypothesis there is a U such that

U is an $\{TS, TX(a), FZ\} \ / \ \{TS_1\}$.

<u>case 2a:</u> $a \notin [U]$. Then U is an

$\{TS, T(\forall x)\ X(x), FZ\} \ / \ \{TS_1\}$.

<u>case 2b:</u> $a \varepsilon [U]$, $a \varepsilon [S_2, X(x), Z]$. Again

U is an $\{TS, T(\forall x)\ X(x), FZ\} \ / \ \{TS_1\}$

<u>case 2c:</u> $a \varepsilon [U]$, $a \notin [S_2, X(x), Z]$.

Then $(\exists x)\ U\binom{a}{x}$ is an

$\{TS, T(\forall x)\ X(x), FZ\} \ / \ \{TS_1\}$.

Q.E.D.

Now to prove the original theorem.

Suppose $\vdash_I X \supset Y$. Then {TX, FY} is inconsistent. By the lemma, there is a U such that

U is an {TX, FY} / {TX}

We have three cases.

1) U = t. Then since {Tt, FY} is inconsistent, $\vdash_I Y$.

2) U = f . Then since {TX, Ff} is inconsistent, {F ~ X} is also inconsistent [f is not in X] . Thus $\vdash_I \sim X$

3) $U \neq t$, $U \neq f$. Then U is a formula not involving t or f, all the parameters and predicates of U are in X and Y, and since {TX, FU} and {TU, FY} are both inconsistent, $\vdash_I X \supset U$ and $\vdash_I U \supset Y$.

Section 6

Models with constant $\mathcal{P}$ function

In Part II we will be concerned with finding counter-models for formulas with no universal quantifiers, and we will confine ourselves to models with a constant $\mathcal{P}$ function. To justify this restriction, we show in this section

Theorem: If X is a formula with no universal quantifiers and $\nvdash_I X$, then there is a counter-model $\langle G, R, \models, \mathcal{P} \rangle$ for X in which ρ is a constant function.

<u>Def:</u> For this section only, let $a_1, a_2, a_3, \ldots$ be an enumeration of all parameters. We call a set Γ of signed formulas a Hintikka element if Γ is a Hintikka element with respect to some initial segment of $a_1, a_2, a_3, \ldots$ (See section 4 chapter 5).

<u>Lemma:</u> If S is a finite, consistent set of signed formulas with no universal quantifiers, S can be extended to a <u>finite</u> Hintikka element.

<u>Proof:</u> Suppose S is the set $\{X_1, X_2, \ldots, X_n\}$ where each X_i is a <u>signed</u> formula. We define the two sequences $\{P_k\}$, $\{Q_k\}$ as follows:

Let $P_0 = \phi$

$Q_0 = X_1, \ldots, X_n$

Suppose we have defined P_k and Q_k where

$P_k = Y_1, \ldots, Y_r$

$Q_k = W_1, \ldots, W_s$

and $P_k \cup Q_k$ (considered as a set) is consistent.

To define P_{k+1} and Q_{k+1} we have several cases depending on W_1.

<u>case atomic:</u> If W_1 is a signed atomic formula, let

$P_{k+1} = Y_1, \ldots, Y_r, W_1$

$$Q_{k+1} = W_2, \ldots, W_s.$$

case $T\vee$: If W_1 is $TX \vee Y$, either TX or TY is consistent with $P_k \cup Q_k$, say TX. Let

$$P_{k+1} = Y_1, \ldots, Y_r, TX \vee Y$$
$$Q_{k+1} = W_2, \ldots, W_s, TX.$$

case $F\vee$: If W_1 is $FX \vee Y$ then FX, FY is consistent with $P_k \cup Q_k$. Let

$$P_{k+1} = Y_1, \ldots, Y_r, FX \vee Y$$
$$Q_{k+1} = W_2, \ldots, W_s, FX, FY$$

cases $T\wedge$, $F\wedge$, $T\sim$, $T\supset$ are similar.

case $T\exists$: If W_1 is $T(\exists x)\, X(x)$, let a be the first in the sequence $a_1, a_2, \ldots$ not occuring in P_k or Q_k. Then $TX(a)$ is consistent with $P_k \cup Q_k$. Let

$$P_{k+1} = Y_1, \ldots, Y_r, T(\exists x)\, X(x)$$
$$Q_{k+1} = W_2, \ldots, W_s, TX(a).$$

case $F\exists$: If W_1 is $F(\exists x)\, X(x)$, let $\{a_{i_1}, \ldots, a_{i_t}\}$ be the set of parameters occuring in $P_k \cup Q_k$ such that no $FX(a_{i_j})$ occurs in $P_k \cup Q_k$. Then $\{FX(a_{i_1}), \ldots, FX(a_{i_t})\}$ is consistent with $P_k \cup Q_k$. Let

$$P_{k+1} = P_k$$
$$Q_{k+1} = W_2, \ldots, W_s, FX(a_{i_1}), \ldots, FX(a_{i_t}), F(\exists x)\, X(x)$$

After finitely many steps there will be no T-signed formulas left in the Q-sequence because each rule, $T\vee$, $T\wedge$,

$T\sim$, $T\supset$, $T\exists$ reduces degree, and no rule, $F\vee$, $F\wedge$, $F\exists$ introduces new T-signed formulas.

When no T-signed formulas are left in the Q-sequence, no new parameters can be introduced since rule $T\exists$ no longer applies.

After finitely many more steps we must reach an empty Q-sequence. The corresponding P-sequence is finite, consistent, and clearly a Hintikka element.

Q.E.D.

Remark: The above proof also shows the following which we will need later. Let R be a finite Hintikka element. Suppose we add (consistently) a finite set of F-signed formulas to R and extend the result to a finite Hintikka element S by the above method. Then

$$R_T = S_T.$$

Since $R \subseteq S$, certainly $R_T \subseteq S_T$. That $S_T \subseteq R_T$ also holds follows by an inspection of the above proof; no new T-signed formulas will be added.

Now we turn to the proof of the theorem itself. We have no universal quantifiers to consider, so we may use the definition of associated sets in section 4 chapter 2.

Suppose X is a formula with no universal quantifiers, and $\nvdash_I X$. Then $\{FX\}$ is consistent. Extend it to a finite Hintikka element, S_0^0.

Let $T_1, \ldots, T_n$ be the associated sets of S_0^0. Extend each to a finite Hintikka element, $S_1^0, \ldots, S_n^0$ respectively. Thus we have

$$S_0^0, S_1^0, \ldots, S_n^0.$$

For each parameter a of some S_i^0 and each formula of the form $F(\exists x)\, X(x)$ in S_0^0, adjoin $FX(a)$ to S_0^0 and extend the result to a Hintikka element S_0^1. Do the same for $S_1^0, \ldots, S_n^0$, producing $S_1^1, \ldots, S_n^1$ respectively. Thus we have now

$$S_0^1, S_1^1, \ldots, S_n^1.$$

Let $T_{n+1}, \ldots, T_m$ be the associated sets of $S_0^1, S_1^1, \ldots, S_n^1$. Extend each to a Hintikka element, $S_{n+1}^0, \ldots, S_m^0$ respectively. Thus we have now

$$S_0^1, S_1^1, \ldots, S_n^1, S_{n+1}^0, \ldots, S_m^0.$$

For each parameter a used so far, and for each formula of the form $F(\exists x)\, X(x)$ in S_0^1, adjoin $FX(a)$ to S_0^1 and extend the result to a finite Hintikka element S_0^2. Do the same for each. Thus we have now

$$S_0^2, S_1^2, \ldots, S_n^2, S_{n+1}^1, \ldots, S_m^1.$$

Again take the associated sets, and extend to finite Hintikka elements, producing now

$$S_0^2, S_1^2, \ldots, S_n^2, S_{n+1}^1, \ldots, S_m^1, S_{m+1}^0, \ldots, S_p^0.$$

Continue in this manner.

Let $S_0 = \bigcup_{k=0}^{\infty} S_0^k$, $S_1 = \bigcup_{k=0}^{\infty} S_1^k$, etc.

By the remark above, for each n,

$$S_{nT} = S^0_{nT} = S^1_{nT} = \ldots$$

Thus if S^k_n has as an associated set S^j_m, $S_{nT} \subseteq S_m$.

It now follows that $\{S_0, S_1, \ldots\}$ is a Hintikka collection. For example, suppose $F \sim Y \varepsilon S_j$. Let k be the least integer such that $F \sim Y \varepsilon S^k_j$. By the above construction, there is some set S^0_r such that S^0_r is an associated set of S^k_j and $TY \varepsilon S^0_r$. But then $S^k_{jT} \subseteq S^0_r$, so by the above, $S_{jT} \subseteq S_r$, and $TY \varepsilon S_r$. The other properties are shown similarly.

Moreover, $\mathcal{P}(S_n) = \mathcal{P}(S_m)$ for all m and n, as is easily seen. (Recall, $\mathcal{P}(S)$ is the collection of all parameters used in S.) Now as in section 3 chapter 5, there is a model for this Hintikka collection, and this model will have a constant $\mathcal{P}$ map, so the theorem is shown.

CHAPTER 7

Intuitionistic M_α Generalizations

Section 1

Introduction

Here and in the rest of part II we restrict our considerations to the following language: a countable collection of bound variables, x, y, z, ..., a collection of parameters (or constants) of arbitrarily high cardinality f, g, h, ..., one two-place predicate symbol, ε [we write $\varepsilon(x,y)$ as $(x \varepsilon y)$], and the usual connectives, quantifiers, and parantheses.

In all the models $\langle G, R, \models, \mathcal{P} \rangle$ which we will consider in part II, the map $\mathcal{P}$ will be constant, and so we will simply write the domain S of $\mathcal{P}$ instead of $\mathcal{P}$, thus, $\langle G, R, \models, S \rangle$ where $\mathcal{P}(\Gamma) = S$ for all $\Gamma \varepsilon G$.

We call a model $\langle G, R, \models, S \rangle$ an <u>intuitionistic ZF model</u> if classical equivalents of all the axioms of Zermello-Fraenkel set theory, expressed <u>without the use of the universal quantifier,</u> are valid in it.

As a special case, suppose $\langle G, R, \models, S \rangle$ is an intuitionistic ZF model and G has only one element, Γ. Then this is (isomorphically) a classical model for ZF.

If we define a truth function on all formulas over S by

$$v(X) = T \quad \text{if} \quad \Gamma \models X$$

$$v(X) = F \quad \text{if} \quad \Gamma \not\models X$$

v will be a classical truth function, and all the axioms of ZF map to T. Thus the notion of intuitionistic ZF model is a generalization of the classical notion.

Suppose $\langle G, R, \models, S \rangle$ were an intuitionistic ZF model such that $\sim$A.C. was valid in it, where A.C. is some classically equivalent form of the axiom of choice expressed without use of the universal quantifier. It follows that the axiom of choice is <u>classically</u> unprovable from the axioms of ZF. For otherwise,

$$ZF \vdash_c A.C.$$

so for some finite subset $A_1, \ldots, A_n$ of ZF,

$$A_1, \ldots, A_n \vdash_c A.C.$$

We may suppose $A_1, \ldots, A_n$ stated without the universal quantifier.

$$\vdash_c (A_1 \wedge \cdots \wedge A_n) \supset A.C.$$

So by the results of section 8, chapter 4,

$$\vdash_I \sim\sim((A_1 \wedge \ldots \wedge A_n) \supset A.C.)$$

equivalently,

$$\vdash_I (A_1 \wedge \ldots \wedge A_n) \supset \sim\sim A.C.$$

But $\langle G, R, \models, S\rangle$ is an intuitionistic model in which $A_1, \ldots, A_n, \sim$A.C. are valid, a contradiction.

Thus, to show the classical independence of the axiom of choice it suffices to construct an intuitionistic ZF model in which ~A.C. is valid. Similar results hold for the independence of the continuum hypothesis and of the axiom of constructability.

In this chapter we will define intuitionistic generalizations of the classical M_α sequence of Gödel [3], which provide intuitionistic generalizations of L, the class of constructable sets. We will show these generalizations are intuitionistic ZF models. In later chapters we will give specific intuitionistic generalizations of L establishing the independence of the axiom of choice, the continuum hypothesis, and the axiom of constructability.

The specific models constructed, and most of the general methods will be those of forcing, due to Cohen [2]. It is the point of view that is different. No classical models are constructed, complete sequences are not used, and countable ZF models are not required.

In [4], Gregorzyk noted the foundations of a connection between forcing and intuitionistic logic. In [12] Kripke discussed the relationship between forcing and his models.

Remark: For the rest of part II we shall distinguish informally between constants, bound variables, and free variables. We shall use $x, y, z, \ldots$ for both bound and free variables. This is an informal distinction. Formally, free variables and constants are both parameters in the sense of part I since free variables are simply place holders for arbitrary constants.

Section 2

The classical M_α sequence

Let V be a classical ZF model. In [3] Gödel defined over V the sequence M_α of sets as follows.

$$M_0 = \phi$$

$M_{\alpha+1}$ is the collection of all definable subsets of M_α.

$$M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha \quad \text{for limit ordinals, } \lambda.$$

Let the class L be $\bigcup_{\alpha \varepsilon V} M_\alpha$. Gödel showed that L was a classical ZF model.

As an introduction to the intuitionistic generalization, we re-state the Gödel construction using characteristic functions instead of sets. Now, of course, "ε" is to be considered as a formal predicate symbol, not as set membership.

Let M be some collection and let v be a truth function on the set of formulas with constants from M. We say a (characteristic) function, f, is definable over $\langle M,v\rangle$ if domain $(f) = M$, range $(f) \subseteq \{T,F\}$, and for some formula $X(x)$ with one free variable and all constants from M, for all $a \varepsilon M$,

$$f(a) = v(X(a))$$

Let M' be the elements of M together with all functions definable over $\langle M,v\rangle$.

We define a truth function, v', on the set of formulas with constants from M' by defining it for atomic formulas. If $f,g \varepsilon M'$ we have three cases.

1) $f,g \varepsilon M$. Let $v'(f \varepsilon g) = v(f \varepsilon g)$

2) $f \varepsilon M$, $g \varepsilon M'-M$. Let $v'(f \varepsilon g) = g(f)$

3) $f \varepsilon M'-M$ Let $X(x)$ be the formula which defines f over $\langle M,v\rangle$. If there is an $h \varepsilon M$ such that

$$v((\forall x)(x \varepsilon h \equiv X(x))) = T$$

and $v'(h \varepsilon g) = T$,

let $v'(f \varepsilon g) = T$

Otherwise, let $v'(f \varepsilon g) = F$.

[case 3 reduces the situation to case 1 or case 2]

We call the pair $\langle M', v'\rangle$ the derived model of $\langle M, v\rangle$.

Now, let $M_0 = \phi$ and let v_0 be the obvious truth function. Thus, we have $\langle M_o, v_o \rangle$.

Let $\langle M_{\alpha+1}, v_{\alpha+1} \rangle$ be the derived model of $\langle M_\alpha, v_\alpha \rangle$.

If λ is a limit ordinal, let $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$. Let $v_\lambda (f \varepsilon g) = T$ if for some $\alpha<\lambda$, $v_\alpha(f \varepsilon g) = T$. Otherwise let $v_\lambda (f \varepsilon g) = F$. Thus, we have $\langle M_\lambda, v_\lambda \rangle$.

Let $L = \bigcup_{\alpha \varepsilon V} M_\alpha$. Let $v(f \varepsilon g) = T$ if for some $\alpha \varepsilon V$, $v_\alpha(f \varepsilon g) = T$. Otherwise let $v(f \varepsilon g) = F$. Thus, we have the "class" model $\langle L, v \rangle$.

The reader may convince himself that this construction is essentially equivalent to Gödel's, so that if A is any axiom of ZF, $v(A) = T$. Thus, $\langle L, v \rangle$ is a classical ZF model, though not a standard one.

For a boolean generalization of this type of sequence see section 7, chapter 14.

Section 3

The intuitionistic M_α sequence

Suppose we have a model $\langle G, R, \models, S\rangle$. [recall, S is a set, the domain of the $\mathcal{P}$ map, and there is only one predicate symbol, ε.] For convenience, let P be the collection of all R-closed subsets of G.

We say a function f is definable over $\langle G, R, \models, S\rangle$ if domain (f) = S, range $(f) \subseteq P$, and for some formula X(x) with one free variable, all constants from S, and no universal quantifiers, for any $a \varepsilon S$,

$$f(a) = \{\Gamma \mid \Gamma \models X(a)\}$$

Let S' be the elements of S together with all functions definable over $\langle G, R, \models, S\rangle$.

We define a $\models'$ relation by giving it for atomic formulas over S'. If $f,g \varepsilon S'$ we have three cases.

1) $f,g \varepsilon S$. Then let $\Gamma \models' (f \varepsilon g)$ if $\Gamma \models (f \varepsilon g)$

2) $f \varepsilon S$, $g \varepsilon S' - S$. Let $\Gamma \models' (f \varepsilon g)$ if $\Gamma \varepsilon g(f)$.

3) $f \varepsilon S' - S$. Let X(x) be the formula which defines f over $\langle G, R, \models, S\rangle$. Let $\Gamma \models' (f \varepsilon g)$ if there is an $h \varepsilon S$ such that

$$\Gamma \models \sim(\exists x)\sim(x \varepsilon h \equiv X(x))$$

and $\Gamma \models' (h \varepsilon g)$.

[this reduces the situation to case 1 or case 2]

We call the model $\langle G, R, \models', S' \rangle$ the derived model of $\langle G, R, \models, S \rangle$.

Now let V be a <u>classical</u> (first order) model for ZF. We define a sequence of intuitionistic models in as follows.

Let $\langle G, R, \models_0, S_0 \rangle$ be any intuitionistic model satisfying the following five conditions.

1) $\langle G, R, \models_0, S_0 \rangle \varepsilon V$

2) S_0 is a collection of functions such that, if $f \varepsilon S_0$, domain $(f) \subseteq S_0$ and range $(f) \subseteq P$.

3) for $f, g \varepsilon S_0$, $\Gamma \models_0 (f \varepsilon g)$ iff $\Gamma \varepsilon g(f)$.

4) (extensionality) for $f, g, h \varepsilon S_0$, if $\Gamma \models_0 \sim(\exists\, x) \sim (x \varepsilon f \equiv x \varepsilon g)$ and $\Gamma \models_0 \sim(f \varepsilon h)$ then $\Gamma \models_0 \sim(g \varepsilon h)$.

5) (regularity) S_0 is well-founded with respect to the relation $x \varepsilon$ domain (y).

<u>Remark:</u> If we consider the symbols $\vee, \wedge, \sim, \supset, \forall, \exists, (\ ,\ ), \varepsilon,$ $x_1, x_2, x_3, \ldots$ to be suitable "code" sets, formulas are sequences of sets, and hence sets. It is in this sense that 1) is meant. See also section 14.

Next, let $\langle G, R, \models_{\alpha+1}, S_{\alpha+1} \rangle$ be the derived model of $\langle G, R, \models_\alpha, S_\alpha \rangle$.

If λ is a limit ordinal, let $S_\lambda = \bigcup_{\alpha < \lambda} S_\alpha$. Let $\Gamma \models_\lambda (f \varepsilon g)$ if for some $\alpha < \lambda$, $\Gamma \models_\alpha (f \varepsilon g)$. Thus, we have $\langle G, R, \models_\lambda, S_\lambda \rangle$.

Finally, let $S = \bigcup_{\alpha \varepsilon V} S_\alpha$. Let $\Gamma \models (f \varepsilon g)$ if for some $\alpha \varepsilon V$, $\Gamma \models_\alpha (f \varepsilon g)$. Thus we have the "class" model, $\langle G, R, \models, S \rangle$.

We will spend the rest of this chapter showing

<u>Theorem:</u> $\langle G, R, \models, S \rangle$ is an intuitionistic ZF model.

<u>Remark:</u> If as a special case we let S_0 be empty, and let $G = \{\Gamma\}$, and we identify T with $\{\Gamma\}$ and F with ϕ, the result is the characteristic function version of the M_α sequence in section 2. [The truth functions become $v_\alpha(X) = \{\Gamma | \Gamma \models_\alpha X\}$]

Thus as a special case of the above theorem, L is a classical ZF model.

<u>Notation:</u> Sometimes we will write $g_X \varepsilon S_{\alpha+1} - S_\alpha$ where by the subscript X we mean g is the function defined over the model $\langle G, R, \models_\alpha, S_\alpha \rangle$ by the formula X(x). Then part 2 of the definition of $\models'$ for the derived model may be restated.

If $f \varepsilon S$, $g_X \varepsilon S' - S$, then $\Gamma \models' (f \varepsilon g_X)$ if $\Gamma \models X(f)$

Section 4

Dominance

Def: Let $X(x_1,\ldots,x_n)$ be a formula with no constants and with all its free variables among $x_1,\ldots,x_n$. We call X dominant if, for any $\Gamma\varepsilon G$, and any $c_1,\ldots,c_n \varepsilon S_\alpha$,

$$\Gamma \models_\alpha X(c_1,\ldots,c_n) \iff \Gamma \models X(c_1,\ldots,c_n)$$

Def: Let

1) $(f \subseteq g)$ stand for $\sim(\exists x)\sim(x\varepsilon f \supset x\varepsilon g)$
2) $(f = g)$ stand for $(f \subseteq g)\wedge(g\subseteq f)$

Theorem: $(x\varepsilon y)$, $(x\subseteq y)$, and $(x = y)$ are dominant.

Proof: That $(x\varepsilon y)$ is dominant is obvious. If $(x\subseteq y)$ is dominant, so is $(x = y)$. That $(x\subseteq y)$ is dominant follows from the next three lemmas.

Lemma 1: If $f,g,\varepsilon S_\alpha$ and $\Gamma \models (f\subseteq g)$, then $\Gamma \models_\alpha (f\subseteq g)$

Proof: Suppose for some Γ^* and some $h\varepsilon S_\alpha$, $\Gamma^* \models_\alpha (h\varepsilon f)$. By dominance of $(x\varepsilon y)$, $\Gamma^* \models (h\varepsilon f)$. But $\Gamma^* \models \sim(\exists x)\sim(x\varepsilon f \supset x\varepsilon g)$ so by intuitionistic logic, $\Gamma^* \models \sim\sim(h\varepsilon g)$. By dominance again, $\Gamma^* \models_\alpha \sim\sim(h\varepsilon g)$. Thus $\Gamma \models_\alpha (\forall x)(x\varepsilon f \supset \sim\sim x\varepsilon g)$, which is equivalent to $\Gamma \models_\alpha \sim(\exists x)\sim(x\varepsilon f \supset x\varepsilon g)$.

Q.E.D.

Remark: The reader may show the two simple facts used above, and often later: X is dominant implies $\sim X$ is dominant and $\vdash_I (\forall x)(X(x) \supset \sim\sim Y(x)) \equiv$

$$\sim(\exists x)\sim(X(x) \supset Y(x))$$

Lemma 2: If $f, g \varepsilon S_\alpha$ and $\Gamma \models_\alpha (f \subseteq g)$ then $\Gamma \models_{\alpha+1} (f \subseteq g)$.

Proof: $\Gamma \models_\alpha (f \subseteq g)$. Suppose for some Γ^* and some $h \varepsilon S_{\alpha+1}$, $\Gamma^* \models_{\alpha+1} (h \varepsilon f)$. If $h \varepsilon S_\alpha$, by dominance, $\Gamma^* \models_\alpha (h \varepsilon f)$. But $\Gamma^* \models_\alpha (f \subseteq g)$ so as above $\Gamma^* \models_\alpha \sim\sim(h \varepsilon g)$ and by dominance, $\Gamma^* \models_{\alpha+1} \sim\sim(h \varepsilon g)$.

If $h \varepsilon S_{\alpha+1} - S_\alpha$, since $f \varepsilon S_\alpha$ and $\Gamma^* \models_{\alpha+1} (h \varepsilon f)$, it must be the case that h is h_X for some formula X over S_α, and there is some $k \varepsilon S_\alpha$ such that $\Gamma^* \models_{\alpha+1} (k \varepsilon f)$ and $\Gamma^* \models_\alpha \sim(\exists x)\sim(x \varepsilon k \equiv X(x))$. Since both $k, f \varepsilon S_\alpha$, by dominance, $\Gamma^* \models_\alpha (k \varepsilon f)$. Thus $\Gamma^* \models_\alpha \sim\sim(k \varepsilon g)$ and by dominance, $\Gamma^* \models_{\alpha+1} \sim\sim(k \varepsilon g)$. That is for any Γ^{**}, there is some Γ^{***} such that $\Gamma^{***} \models_{\alpha+1} (k \varepsilon g)$. But also $\Gamma^{***} \models_\alpha \sim(\exists x)\sim(x \varepsilon k \equiv X(x))$, $k \varepsilon S_\alpha$ so by definition, $\Gamma^{***} \models_{\alpha+1} (h_X \varepsilon g)$. Thus $\Gamma^* \models_{\alpha+1} \sim\sim(h_X \varepsilon g)$.

Hence, $\Gamma \models_{\alpha+1} (\forall x)(x \varepsilon f \supset \sim\sim x \varepsilon g)$ so $\Gamma \models_{\alpha+1} \sim(\exists x)\sim(x \varepsilon f \supset x \varepsilon g)$.

Q.E.D

Lemma 3: If $f, g \varepsilon S_\alpha$ and $\Gamma \models_\alpha (f \subseteq g)$, then $\Gamma \models (f \subseteq g)$

Proof: First, by transfinite induction, for any $\beta \geq \alpha$, $\Gamma \models_\beta (f \subseteq g)$. The successor ordinal step is given by lemma 2. Suppose λ is a limit ordinal, $\lambda > \alpha$, and the result is known for all β such that $\alpha \leq \beta < \lambda$. If $\Gamma \models_\lambda (h \varepsilon f)$, then for some $\beta < \lambda$, $\Gamma \models_\beta (h \varepsilon f)$. But $\Gamma \models_\beta (f \subseteq g)$ so $\Gamma \models_\beta {\sim\sim}(h \varepsilon g)$. By dominance, $\Gamma \models_\lambda {\sim\sim}(h \varepsilon g)$. So $\Gamma \models_\lambda (f \subseteq g)$.

Finally, that $\Gamma \models (f \subseteq g)$ follows just as in the limit ordinal case.

Q.E.D.

Section 5

A little about equality

Theorem: If $f \varepsilon S_\alpha$ and $g_X \varepsilon S_{\alpha+1} - S_\alpha$ then $\Gamma \models_\alpha {\sim}(\exists x){\sim}(x \varepsilon f \equiv X(x))$ if and only if $\Gamma \models_{\alpha+1} (f = g_X)$

This follows from the next two lemmas.

Lemma 1: If $f \varepsilon S_\alpha$, $g_X \varepsilon S_{\alpha+1} - S_\alpha$, and $\Gamma \models_{\alpha+1} (f = g_X)$, then $\Gamma \models_\alpha {\sim}(\exists x){\sim}(x \varepsilon f \equiv X(x))$

Proof: Suppose for some Γ^* and some $h\varepsilon S_\alpha$, $\Gamma^* \models_\alpha (h\varepsilon f)$. Then $\Gamma^* \models_{\alpha+1} (h\varepsilon f)$, so $\Gamma^* \models_{\alpha+1} {\sim\sim}(h\varepsilon g_X)$. Thus, for any Γ^{**} there is a Γ^{***} such that $\Gamma^{***} \models_{\alpha+1} (h\varepsilon g_X)$

But $h\varepsilon S_\alpha$, $g_X \varepsilon S_{\alpha+1} - S_\alpha$, so $\Gamma^{***}\varepsilon g_X(h)$, that is, $\Gamma^{***} \models_\alpha X(h)$. Thus, $\Gamma^* \models_\alpha {\sim\sim}X(h)$, so $\Gamma \models_\alpha (\forall x)(x\varepsilon f \supset {\sim\sim}X(x))$ or $\Gamma \models_\alpha {\sim}(\exists x){\sim}(x\varepsilon f \supset X(x))$ Similarly, $\Gamma \models_\alpha {\sim}(\exists x){\sim}(X(x) \supset x\varepsilon f)$. The result follows since ${\sim}(\exists x){\sim}X_1(x) \wedge {\sim}(\exists x){\sim}X_2(x) \vdash_I {\sim}(\exists x){\sim}(X_1(x) \wedge X_2(x))$

Q.E.D.

Lemma 2: If $f\varepsilon S_\alpha$, $g_X\varepsilon S_{\alpha+1} - S_\alpha$, and $\Gamma \models_\alpha {\sim}(\exists x){\sim}(x\varepsilon f \equiv X(x))$ then $\Gamma \models_{\alpha+1} (f = g_X)$.

Proof: $\Gamma \models_\alpha {\sim}(\exists x){\sim}(x\varepsilon f \equiv X(x))$. Suppose for some Γ^* and some $h\varepsilon S_{\alpha+1}$, $\Gamma^* \models_{\alpha+1} (h\varepsilon f)$.

If $h\varepsilon S_\alpha$, trivially $\Gamma^* \models_{\alpha+1} {\sim\sim}(h\varepsilon g_X)$

If $h\varepsilon S_{\alpha+1} - S_\alpha$, then since $f\varepsilon S_\alpha$ h must be h_Y for some formula Y over S_α, and there is some $k\varepsilon S_\alpha$ such that $\Gamma^* \models_{\alpha+1} (k\varepsilon f)$ and $\Gamma^* \models_\alpha {\sim}(\exists x){\sim}(x\varepsilon k \equiv Y(x))$. By dominance, $\Gamma^* \models_\alpha (k\varepsilon f)$, so $\Gamma^* \models_\alpha {\sim\sim}X(k)$. So, for every Γ^{**} there is a Γ^{***} such that $\Gamma^{***} \models_\alpha X(k)$. Thus, $\Gamma^{***} \models_{\alpha+1} (k\varepsilon g_X)$. But also $\Gamma^{***} \models_\alpha {\sim}(\exists x){\sim}(x\varepsilon k \equiv Y(x))$ so by definition, $\Gamma^{***} \models_{\alpha+1} (h_Y \varepsilon g_X)$. Thus, $\Gamma^* \models_{\alpha+1} {\sim\sim}(h\varepsilon g_X)$.

Hence $\Gamma \models_{\alpha+1} (f \subseteq g_x)$.

In a similar manner it can be shown that $\Gamma \models_{\alpha+1} (g_x \subseteq f)$.

Q.E.D.

For later use we show the following most useful corollary.

Theorem 2: If $\Gamma \models_\alpha (f \varepsilon g)$, then there is an $h \varepsilon$ domain (g) such that $\Gamma \models_\alpha (f = h) \wedge (h \varepsilon g)$.

Proof: By induction on α. If $\alpha = 0$, and $\Gamma \models_0 (f \varepsilon g)$, by definition f must be in the domain of g.

Suppose the result is known for α, and $\Gamma \models_{\alpha+1} (f \varepsilon g)$. We have three cases.

1) If $f, g \varepsilon S_\alpha$ the result is by induction hypothesis.

2) If $f \varepsilon S_\alpha$, $g \varepsilon S_{\alpha+1} - S_\alpha$ the result is trivial since $f \varepsilon$ domain (g).

3) If $f \varepsilon S_{\alpha+1} - S_\alpha$, by definition and theorem 1, for some $k \varepsilon S_\alpha$, $\Gamma \models_{\alpha+1} (k \varepsilon g) \wedge (k = f)$. Since $\Gamma \models_{\alpha+1} (k \varepsilon g)$, by case 1) or case 2) there is some $h \varepsilon$ domain (g) such that $\Gamma \models_{\alpha+1} (h \varepsilon g) \wedge (h = k)$. But trivially if $\Gamma \models_{\alpha+1} (h = k) \wedge (k = f)$, $\Gamma \models_{\alpha+1} (h = f)$.

The limit ordinal step is simple

Q.E.D.

Remark: By dominance of $(x \varepsilon g)$ and $(x = g)$, the result follows also for the class model.

Section 6

Weak substitutivity of equality

Theorem: Let $X(x)$ be a formula with one free variable and no universal quantifiers. If $\Gamma \models_\alpha (f = g)$ and $\Gamma \models_\alpha \sim X(f)$ then $\Gamma \models_\alpha \sim X(g)$. Similarly if $\Gamma \models (f = g)$ and $\Gamma \models \sim X(f)$ then $\Gamma \models \sim X(g)$.

Proof: Suppose the result is known in the model $\langle G, R, \models_\alpha, S_\alpha \rangle$ [or in $\langle G, R, \models, S \rangle$] for all atomic formulas $X(x)$. It then follows for all formulas $X(x)$ by the following intuitionistic theorems:

$$\begin{aligned} \sim X \equiv \sim Y \vdash_I \; & \sim(X \wedge Z) \equiv \sim(Y \wedge Z) \\ & \sim(X \vee Z) \equiv \sim(Y \vee Z) \\ & \sim(\sim X) \equiv \sim(\sim Y) \\ & \sim(X \supset Z) \equiv \sim(Y \supset Z) \\ & \sim(Z \supset X) \equiv \sim(Z \supset Y) \end{aligned}$$

$$(\forall x)\,[\sim X(x) \equiv \sim Y(x)] \vdash_I \sim(\exists x)X(x) \equiv \sim(\exists x)Y(x)$$

Thus we must show the result for atomic formulas.

Over $\langle G, R, \models_0, S_0\rangle$ an atomic formula must be either $(a\varepsilon x)$, $(x\varepsilon a)$, or $(a\varepsilon b)$, for $a, b\varepsilon S_0$. The case $(a\varepsilon b)$ is trivial. For the case $(a\varepsilon x)$, we are given: $\Gamma \models_0 \sim(\exists x)\sim(x\varepsilon f \equiv x\varepsilon g)$, and $\Gamma \models_0 \sim(a\varepsilon f)$. The result, $\Gamma \models_0 \sim(a\varepsilon g)$ follows by intuitionistic logic. For the case $(x\varepsilon a)$, the result is condition 4, on $\langle G, R, \models_0, S_0\rangle$ in section 3.

Suppose the result is known for all formulas over S_α. We show it for atomic formulas of $\langle G, R, \models_{\alpha+1}, S_{\alpha+1}\rangle$ Again, an atomic formula must be either $(a\varepsilon x)$, $(x\varepsilon a)$, or $(a\varepsilon b)$ for $a, b\varepsilon S_{\alpha+1}$. As above, $(x\varepsilon a)$ is the only difficult case. Thus, we are given $\Gamma \models_{\alpha+1}(f = g)$, and $\Gamma \models_{\alpha+1} \sim(f\varepsilon a)$. We have eight subcases:

1) $a, f, g\varepsilon S_\alpha$
2) $a, f\varepsilon S_\alpha$, $g\varepsilon S_{\alpha+1} - S_\alpha$
3) $a, g\varepsilon S_\alpha$, $f\varepsilon S_{\alpha+1} - S_\alpha$
4) $a\varepsilon S_\alpha$, $f, g\varepsilon S_{\alpha+1} - S_\alpha$
5) $a\varepsilon S_{\alpha+1} - S_\alpha$, $f, g\varepsilon S_\alpha$
6) $a, g\varepsilon S_{\alpha+1} - S_\alpha$, $f\varepsilon S_\alpha$
7) $a, f\varepsilon S_{\alpha+1} - S_\alpha$, $g\varepsilon S_\alpha$
8) $a, f, g\varepsilon S_{\alpha+1} - S_\alpha$.

We treat these cases separately..

Case 1) The result follows by dominance of $(x\varepsilon y)$ and $(x = y)$, and the induction hypothesis.

Case 2) Suppose $\Gamma \not\models_{\alpha+1} {\sim}(g\varepsilon a)$. Then for some Γ^*, $\Gamma^* \models_{\alpha+1} (g\varepsilon a)$. By theorem 2, section 5, there is an $h\varepsilon S_\alpha$ such that $\Gamma^* \models_{\alpha+1} (g = h) \wedge (h\varepsilon a)$. But $\Gamma^* \models_{\alpha+1} (f = g)$, hence $\Gamma^* \models_{\alpha+1} (f = h)$. By dominance $\Gamma^* \models_{\alpha} (f = h) \wedge (h\varepsilon a)$. By induction hypothesis, $\Gamma^* \models_{\alpha} {\sim}{\sim}(f\varepsilon a)$. By dominance, $\Gamma^* \models_{\alpha+1} {\sim}{\sim}(f\varepsilon a)$, so $\Gamma \not\models_{\alpha+1} {\sim}(f\varepsilon a)$.

Case 3) Suppose $\Gamma \not\models_{\alpha+1} {\sim}(g\varepsilon a)$. Then for some Γ^*, $\Gamma^* \models_{\alpha+1} (g\varepsilon a)$. But $\Gamma^* \models_{\alpha+1} (f = g)$. Now by theorem 1 section 5, and the definitions, $\Gamma^* \models_{\alpha+1} (f\varepsilon a)$, so $\Gamma \not\models_{\alpha+1} {\sim}(f\varepsilon a)$

Case 4) an elaboration of 2) and 3).

Case 5) a is $a_x \varepsilon S_{\alpha+1} - S_\alpha$. Suppose $\Gamma \not\models_{\alpha+1} {\sim}(g\varepsilon a_x)$ Then for some Γ^*, $\Gamma^* \models_{\alpha+1} (g\varepsilon a_x)$, so $\Gamma^* \models_{\alpha} X(g)$. But $\Gamma^* \models_{\alpha+1} (f = g)$ so by dominance, $\Gamma^* \models_{\alpha} (f = g)$. By hypothesis, $\Gamma^* \models_{\alpha} {\sim}{\sim}X(f)$ so it follows that $\Gamma^* \models_{\alpha+1} {\sim}{\sim}(f\varepsilon a_x)$. Hence $\Gamma \not\models_{\alpha+1} (f\varepsilon a_x)$.

Case 6) Suppose $\Gamma \not\models_{\alpha+1} {\sim}(g\varepsilon a)$. For some Γ^*, $\Gamma^* \models_{\alpha+1} (g\varepsilon a)$. By theorem 2 section 5, for some $h\varepsilon S_\alpha$, $\Gamma^* \models_{\alpha+1} (g = h) \wedge (h\varepsilon a)$. But $\Gamma^* \models_{\alpha+1} (f = g)$ so $\Gamma^* \models_{\alpha+1} (f = h)$. By dominance, $\Gamma^* \models_{\alpha} (f = h)$. Moreover, a must be $a_x \varepsilon S_{\alpha+1} - S_\alpha$. Since $\Gamma^* \models_{\alpha+1} (h\varepsilon a)$.

$\Gamma^* \models_{\alpha} X(h)$. By hypothesis, $\Gamma^* \models_{\alpha} {\sim\sim}X(f)$ and so $\Gamma^* \models_{\alpha+1} {\sim\sim}(f \varepsilon a_x)$. Thus, $\Gamma \not\models_{\alpha+1} {\sim}(f \varepsilon a_x)$.

Case 7) Suppose $\Gamma \not\models_{\alpha+1} {\sim}(g \varepsilon a)$. Then for some Γ^*, $\Gamma^* \models_{\alpha+1} (g \varepsilon a)$. But $\Gamma^* \models_{\alpha+1} (f = g)$, so by theorem 1 section 5, and the definitions, $\Gamma^* \models_{\alpha+1} (f \varepsilon a)$. Thus, $\Gamma \not\models_{\alpha+1} {\sim}(f \varepsilon a)$.

Case 8) an elaboration of 6) and 7). Thus, we have the result for successor models.

The result for atomic formulas in limit models, and in the class model is straightforward.

Q.E.D.

Section 7

More on dominance

Def: A formula X is called stable if $\vdash_I X \equiv {\sim\sim}X$

Def: A formula X (with no universal quantifiers) is said to have its quantifiers bounded if every subformula beginning with a quantifier is of the form

$$(\exists x)\ (\ (x \varepsilon v) \wedge Y\ (x)\ )$$

where v is a variable or a constant. Moreover, if Y is stable we say X has strongly bounded quantifiers.

<u>Theorem:</u> Let X be any formula with no constants, no universal quantifiers and all its quantifiers strongly bounded. Then, X is dominant.

<u>Proof:</u> By induction on the degree of X. If X is atomic the result is just the dominance of $(x \varepsilon y)$.

Suppose X is not atomic and the result is known for all formulas of lesser degree. The four cases X is $(Y \vee Z)$, $(Y \wedge Z)$, $\sim Y$, or $(Y \supset Z)$ are simple. Suppose $X(y, z, \ldots)$ is $(\exists x)[(x \varepsilon y) \wedge Y(x, y, z, \ldots)]$ where Y is stable, and by hypothesis, dominant. Suppose $a, b, \ldots \varepsilon S_\alpha$

If $\Gamma \models_\alpha X(a, b, \ldots)$ then $\Gamma \models_\alpha (\exists x)[(x \varepsilon a) \wedge Y(x, a, b, \ldots)]$. For some $f \varepsilon S_\alpha$, $\Gamma \models_\alpha (f \varepsilon a) \wedge Y(f, a, b, \ldots)$. By hypothesis, both of these are dominant, so $\Gamma \models (f \varepsilon a) \wedge Y(f, a, b, \ldots)$. $\Gamma \models (\exists x)[(x \varepsilon a) \wedge Y(x, a, b, \ldots)]$, $\Gamma \models X(a, b, \ldots)$.

Conversely, suppose $\Gamma \models X(a, b, \ldots)$. $\Gamma \models (\exists x)[(x \varepsilon a) \wedge Y(x, a, b, \ldots)]$. Then for some $f \varepsilon S$, $\Gamma \models (f \varepsilon a) \wedge Y(f, a, b, \ldots)$. $a \varepsilon S_\alpha$ so by theorem 2 section 5, there is a $g \varepsilon S_\alpha$ such that $\Gamma \models (f = g) \wedge (g \varepsilon a)$. By weak substitutivity of equality,

$\Gamma \models \sim\sim Y(g, a, b, \ldots)$. But Y is stable so

$\Gamma \models Y(g, a, b, \ldots)$. Now by dominance,

$\Gamma \models_\alpha (g \varepsilon a) \wedge Y(g, a, b, \ldots)$

$\Gamma \models_\alpha (\exists x) [(x \varepsilon a) \wedge Y(x, a, b, \ldots)]$

$\Gamma \models_\alpha X(a, b, \ldots)$

Q.E.D.

We define the following formula abbreviations.

$y = \phi$	for	$\sim(\exists x)\ (x \varepsilon y)$
$\phi \varepsilon y$	for	$(\exists x)\ (x \varepsilon y \wedge x = \phi)$
$y = x'$	for	$\sim(\exists w)\sim[w \varepsilon y \equiv (w \varepsilon x \vee w = x)]$
$x' \varepsilon y$	for	$(\exists w)\ (w \varepsilon y \wedge w = x')$
$\omega \subseteq y$	for	$\sim\sim(\phi \varepsilon y) \wedge (\exists x)\sim[x \varepsilon y \supset x' \varepsilon y]$
$x = \{y,z\}$	for	$\sim(\exists w)\sim[w \varepsilon x \equiv (w = y \vee w = z)]$
$x = \bigcup y$	for	$\sim(\exists z)\sim[z \varepsilon x \equiv (\exists w)(w \varepsilon y \wedge z \varepsilon w)]$

Theorem: The above formulas are dominant.

Proof: $y = \phi$ and $\phi \varepsilon y$ are directly by the above theorem.

$y = x'$ is equivalent to the conjunction of the following two formulas,

$$\sim(\exists w)\ [w \varepsilon y \wedge \sim(w \varepsilon x \vee w = x)\ ]$$

$$\sim(\exists w)\sim[\ (w \varepsilon x \vee w = x) \supset w \varepsilon y]$$

The dominance of the first is by the above theorem. That of the second is simple to show.

In a similar fashion the rest follows, making use of

$$\vdash_I \sim(\exists x)\sim[X(x) \supset Y(x)] \equiv \sim(\exists x)[X(x) \wedge \sim Y(x)]$$

and

$$\vdash_I \sim(\exists x)\sim[X(x) \equiv Y(x)] \equiv \sim(\exists x)\sim[X(x) \supset Y(x)] \wedge \sim(\exists x)\sim[Y(x) \supset X(x)]$$

Q.E.D.

Section 8

Axiom of extensionality

Theorem: The following is valid in $\langle G, R, \models, S \rangle$:

$$\sim(\exists x)(\exists y)\sim\{\sim(\exists w)\sim[w\varepsilon x \equiv w\varepsilon y] \supset \sim(\exists z)\sim[x\varepsilon z \equiv y\varepsilon z]\}.$$

In addition, it is valid in every model $\langle G, R, \models_\alpha, S_\alpha \rangle$.

Proof: For any $\Gamma\varepsilon G$ and any $f,g\varepsilon S$, if $\Gamma\models(f = g)$, by weak substitutivity of equality, $\Gamma\models\sim(f\varepsilon d) \equiv \sim(g\varepsilon d)$. But this holds for every $d\varepsilon S$, so $\Gamma\models(\forall z)[\sim(f\varepsilon z) \equiv \sim(g\varepsilon z)]$, and by intuitionistic logic, $\Gamma\models\sim(\exists z)\sim[f\varepsilon z \equiv g\varepsilon z]$. Thus the result follows. [The same proof also works for every α] Q.E.D.

Section 9

Null set axiom

Theorem: The following is valid in $\langle G, R, \models, S \rangle$, $(\exists x)\sim(\exists y)(y \varepsilon x)$. In addition, it is valid in any model $\langle G, R, \models_{\alpha}, S_{\alpha} \rangle$ for $\alpha > o$.

Proof: Suppose we show the formula is valid in $\langle G, R, \models_1, S_1 \rangle$. If $\Gamma \varepsilon G$, $\Gamma \models_1 (\exists x)\sim(\exists y)(y \varepsilon x)$ so for some $f \varepsilon S_1$, $\Gamma \models_1 \sim(\exists y)(y \varepsilon f)$ i.e. $\Gamma \models_1 f = \phi$.

The result then follows by dominance of $x = \phi$.

Let $X(x)$ be the formula $\sim(x = x)$. There is an $f_x \varepsilon S_1 - S_0$. We claim for any $\Gamma \varepsilon G$, $\Gamma \models_1 \sim(\exists y)(y \varepsilon f_x)$. Suppose otherwise, $\Gamma \not\models_1 \sim(\exists y)(y \varepsilon f_x)$. Then for some Γ^*, $\Gamma^* \models_1 (\exists y)(y \varepsilon f_x)$. For some $d \varepsilon S_1$, $\Gamma^* \models_1 (d \varepsilon f_x)$. By theorem 2 section 5, there is an $e \varepsilon S_0$ such that $\Gamma^* \models_1 (d = e) \wedge (e \varepsilon f_x)$. Since $\Gamma^* \models_1 (e \varepsilon f_x)$, by definition, $\Gamma^* \models_0 X(e)$, i.e. $\Gamma^* \models_0 \sim\sim(\exists x)\sim(x \varepsilon e \equiv x \varepsilon e)$ which is not possible by intuitionistic logic.

Q.E.D.

Section 10

Unordered pairs axiom

Theorem: The following is valid in the class model and in any limit model:

$$\sim(\exists x)(\exists y)\sim(\exists z)\sim(\exists w)\sim[w \varepsilon z \equiv (w = x \vee w = y)]$$

Proof: If we show that for any $f, g \varepsilon S_\alpha$ there is an $h \varepsilon S_{\alpha+1} - S_\alpha$ such that $h = \{f,g\}$ is valid in $\langle G, R, \models_{\alpha+1}, S_{\alpha+1}\rangle$, the result will follow by dominance of $x = \{y,z\}$.

Let $f, g \varepsilon S_\alpha$. Let $X(x)$ be the formula $(x = f) \vee (x = g)$. There is an $h_X \varepsilon S_{\alpha+1} - S_\alpha$. We show $h_X = \{f, g\}$ is valid in $\langle G, R, \models_{\alpha+1}, S_{\alpha+1}\rangle$. Let $\Gamma \varepsilon G$.

Suppose $\Gamma^* \models_{\alpha+1} (a \varepsilon h_X)$. Then there is some $b \varepsilon S_\alpha$ such that
$\Gamma^* \models_{\alpha+1} (a = b) \wedge (b \varepsilon h_X)$. Since
$\Gamma^* \models_{\alpha+1} (b \varepsilon h_X)$, $\Gamma^* \models_\alpha X(b)$.
$\Gamma^* \models_\alpha (b = f) \vee (b = g)$. By dominance
$\Gamma^* \models_{\alpha+1} (b = f) \vee (b = g)$. But $\Gamma^* \models_{\alpha+1} (a = b)$ so by intuitionistic logic
$\Gamma^* \models_{\alpha+1} (a = f) \vee (a = g)$. Thus,
$\Gamma \models_{\alpha+1} (\forall x)(x \varepsilon h_X \supset (x = f \vee x = g))$.

conversly, suppose
$\Gamma^* \models_{\alpha+1} (a = f) \vee (a = g)$. Then either

$\Gamma^* \models_{\alpha+1} (a = f)$ or $\Gamma^* \models_{\alpha+1} (a = g)$

Say $\Gamma^* \models_{\alpha+1} (a = f)$. It is trivial to show

$\Gamma^* \models_{\alpha+1} (f \varepsilon h_x)$ so by weak substitutivity of equality,

$\Gamma^* \models_{\alpha+1} \sim\sim(a \varepsilon h_x)$ Thus,

$\Gamma \models_{\alpha+1} (\forall x)((x = f \vee x = g) \supset \sim\sim x \varepsilon h_x)$

The result follows easily.

Q.E.D.

Section 11

Union Axiom

Theorem: The following is valid in the class model and in any limit model:

$$\sim(\exists x)\sim(\exists y)\sim(\exists z)\ [z \varepsilon y \equiv (\exists w)(z \varepsilon w \wedge w \varepsilon x)]$$

Proof: If we show that for any $f \varepsilon S_\alpha$ there is a $g \varepsilon S_{\alpha+1} - S_\alpha$ such that $g = \bigcup f$ is valid in $\langle G, R, \models_{\alpha+1}, S_{\alpha+1} \rangle$, the result will follow by dominance of $x = \bigcup y$.

Let $f \varepsilon S_\alpha$. Let $X(x)$ be the formula $(\exists w)(x \varepsilon w \wedge w \varepsilon f)$. There is a $g_x \varepsilon S_{\alpha+1} - S_\alpha$. We claim $g_x = \bigcup f$ is valid in $\langle G, R, \models_{\alpha+1}, S_{\alpha+1} \rangle$. Let $\Gamma \varepsilon G$.

Suppose $\Gamma^* \models_{\alpha+1} (\exists w)(h \varepsilon w \wedge w \varepsilon f)$

Then for some $k \varepsilon S_{\alpha+1}$

$\Gamma^* \models_{\alpha+1} (h \varepsilon k) \wedge (k \varepsilon f)$. Since

$\Gamma^* \models_{\alpha+1} (k \varepsilon f)$, there is some $t \varepsilon S_\alpha$ such that

$\Gamma^* \models_{\alpha+1} (k = t) \wedge (t \varepsilon f)$. By weak substitutivity of equality,

$\Gamma^* \models_{\alpha+1} \sim\sim(h \varepsilon t)$. Thus, for every Γ^{**} there is a

Γ^{***} such that $\Gamma^{***} \models_{\alpha+1} (h \varepsilon t)$. But $t \varepsilon S_\alpha$ so

there is an $s \varepsilon S_\alpha$ such that $\Gamma^{***} \models_{\alpha+1} (s = h) \wedge (s \varepsilon t)$.

But $\Gamma^{***} \models_{\alpha+1} (h \varepsilon k) \wedge (k \varepsilon f)$ and

$\Gamma^{***} \models_{\alpha+1} (s = h) \wedge (k = t)$ so

$\Gamma^{***} \models_{\alpha+1} \sim\sim[(s \varepsilon t) \wedge (t \varepsilon f)]$. Now, $s, t, f \varepsilon S_\alpha$ so by

dominance, $\Gamma^{***} \models_\alpha \sim\sim[(s \varepsilon t) \wedge (t \varepsilon f)]$

$\Gamma^{***} \models_\alpha (\exists w) \sim\sim[(s \varepsilon w) \wedge (w \varepsilon f)]$. By intuitionistic logic,

$\Gamma^{***} \models_\alpha \sim\sim(\exists w)\ [s \varepsilon w \wedge w \varepsilon f]$ That is

$\Gamma^{***} \models_\alpha \sim\sim X(s)$, so $\Gamma^{***} \models_{\alpha+1} \sim\sim(s \varepsilon g_x)$. But

$\Gamma^{***} \models_{\alpha+1} (s = h)$, so $\Gamma^{***} \models_{\alpha+1} \sim\sim(h \varepsilon g_x)$. Thus for every

Γ^{**} there is a Γ^{***} such that $\Gamma^{***} \models_{\alpha+1} \sim\sim(h \varepsilon g_x)$.

Then $\Gamma^* \models_{\alpha+1} \sim(h \varepsilon g_x)$. We have shown

$\Gamma \models_{\alpha+1} (\forall x)\ [(\exists w)(x \varepsilon w \wedge w \varepsilon f) \supset \sim\sim x \varepsilon g_x]$.

Conversely, suppose $\Gamma^* \models_{\alpha+1} (h \varepsilon g_x)$. There is

some $k \varepsilon S_\alpha$ such that $\Gamma^* \models_{\alpha+1} (h = k) \wedge (k \varepsilon g_x)$. So

$\Gamma^* \models_\alpha X(k)$ or $\Gamma^* \models_\alpha (\exists w)(k \varepsilon w \wedge w \varepsilon f)$. For some $t \varepsilon S_\alpha$,

$\Gamma^* \models_\alpha (k \varepsilon t) \wedge (t \varepsilon f)$. By dominance,

$\Gamma^* \models_{\alpha+1} (k \varepsilon t) \wedge (t \varepsilon f)$. $\Gamma^* \models_{\alpha+1} (\exists w)(k \varepsilon w \wedge w \varepsilon f)$.

$\Gamma^* \models_{\alpha+1} \sim\sim(\exists w)\ (h \varepsilon w \wedge w \varepsilon f)$.

We have shown

$\Gamma \models_{\alpha+1} (\forall x)\ [x \varepsilon g_x \supset \sim\sim(\exists w)\ (x \varepsilon w \wedge w \varepsilon f)\ ]$

The result follows easily.

Q.E.D.

Section 12

Axiom of infinity

Theorem: The following is valid in $\langle G, R, \models, S\rangle$ and in $\langle G, R, \models_\alpha, S_\alpha\rangle$ for $\alpha>\omega$:

$$(\exists x)\ [\phi \varepsilon x \wedge \sim(\exists y)\sim(y \varepsilon x \supset y' \varepsilon x)\ ]$$

Proof: If we show there is an $f \varepsilon S_{\omega+1} - S_\omega$ such that $\omega \subseteq f$ is valid in $\langle G, R, \models_{\omega+1}, S_{\omega+1}\rangle$ the result will follow by dominance of $\omega \subseteq x$.

Let $X(x)$ be the formula

$$\sim(\exists y) \sim \{[\sim(\exists z)\sim(z \varepsilon y \supset z' \varepsilon y) \wedge \phi \varepsilon y] \supset x \varepsilon y\}\ .$$

There is an $f_x \varepsilon S_{\omega+1} - S_\omega$. We claim $\omega \subseteq f_x$ is valid in $\langle G, R, \models_{\omega+1}, S_{\omega+1}\rangle$. This follows from the next four lemmas.

Lemma 1: If $\Gamma \models_\alpha f = \phi \wedge g = \phi$ then $\Gamma \models_\alpha f = g$.

Proof: $\Gamma \models_\alpha \sim(\exists x)(x \varepsilon f) \wedge \sim(\exists x)(x \varepsilon g)$ so by intuitionistic logic $\Gamma \models_\alpha \sim(\exists x)\sim(x \varepsilon f \equiv x \varepsilon g)$, $\Gamma \models_\alpha f = g$

Q.E.D.

Lemma 2: $\Gamma \models_{\omega+1} \phi \varepsilon f_X$

Proof: By the results of section 9, for some $g \varepsilon S_\omega$

$\Gamma \models_\omega g = \phi$. Suppose for some Γ^*,

$\Gamma^* \models_\omega \sim(\exists z)\sim(z \varepsilon k \supset z' \varepsilon k) \wedge \phi \varepsilon k$

Then $\Gamma^* \models_\omega \phi \varepsilon k$, that is $\Gamma^* \models_\omega (\exists w)(w = \phi \wedge w \varepsilon k)$

so for some $s \varepsilon S_\omega$, $\Gamma^* \models_\omega s = \phi \wedge s \varepsilon k$. By lemma 1

$\Gamma^* \models_\omega s = g$, so $\Gamma^* \models_\omega \sim\sim(g \varepsilon k)$. We have shown

$\Gamma \models_\omega (\forall x) \ \{[\sim(\exists z)\sim(z \varepsilon x \supset z' \varepsilon x) \wedge \phi \varepsilon x] \supset \sim\sim(g \varepsilon x)\}$

or equivalently,

$\Gamma \models_\omega \sim(\exists x)\sim\{[\sim(\exists z)\sim(z \varepsilon x \supset z' \varepsilon x) \wedge \phi \varepsilon x] \supset g \varepsilon x\}$

$\Gamma \models_\omega X(g)$

$\Gamma \models_{\omega+1} g \varepsilon f_X$

But $\Gamma \models_{\omega+1} g = \phi$ so by definition, $\Gamma \models_{\omega+1} \phi \varepsilon f_X$.

Q.E.D.

Lemma 3: If $g \varepsilon S_\alpha$, there is an $h \varepsilon S_{\alpha+1} - S_\alpha$ such that $h = g'$ is valid in $\langle G, R, \models_{\alpha+1}, S_{\alpha+1} \rangle$.

Proof: Let $Y(x)$ be the formula $(x \varepsilon g) \vee (x = g)$.

There is an $h_Y \varepsilon S_{\alpha+1} - S_\alpha$. We will show

$\Gamma \models_{\alpha+1} \sim(\exists w)\sim[w \varepsilon h_Y \equiv (w \varepsilon g \vee w = g)]$

Suppose for some Γ^*, $\Gamma^* \models_{\alpha+1} (s \varepsilon h_Y)$.

Then for some $t \varepsilon S_\alpha$,

$$\Gamma^* \models_{\alpha+1} (s = t) \wedge (t \varepsilon h_X)$$

$$\Gamma^* \models_{\alpha} Y(t)$$

$$\Gamma^* \models_{\alpha} (t \varepsilon g) \vee (t = g)$$

$$\Gamma^* \models_{\alpha+1} (t \varepsilon g) \vee (t = g)$$

$$\Gamma^* \models_{\alpha+1} \sim\sim((s \varepsilon g) \vee (s = g))$$

so $\Gamma \models_{\alpha+1} \sim(\exists\, w) \sim [w \varepsilon h_Y \supset (w \varepsilon g \vee w = g)]$

Conversely, suppose

$$\Gamma^* \models_{\alpha+1} (s \varepsilon g) \vee (s = g)$$

We have two cases.

If $\Gamma^* \models_{\alpha+1} (s \varepsilon g)$, since $g \varepsilon S_\alpha$ there is some $t \varepsilon S_\alpha$ such that

$$\Gamma^* \models_{\alpha+1} (s = t) \wedge (t \varepsilon g)$$

$$\Gamma^* \models_{\alpha} (t \varepsilon g)$$

$$\Gamma^* \models_{\alpha} (t \varepsilon g) \vee (t = g)$$

$$\Gamma^* \models_{\alpha} Y(t)$$

$$\Gamma^* \models_{\alpha+1} (t \varepsilon h_Y)$$

$$\Gamma^* \models_{\alpha+1} \sim\sim(s \varepsilon h_Y)$$

If $\Gamma^* \models_{\alpha+1} (s = g)$, since trivially

$$\Gamma^* \models_{\alpha+1} (g \varepsilon h_Y),$$

$$\Gamma^* \models_{\alpha+1} \sim\sim(s \varepsilon h_Y)$$

Thus we have

$$\Gamma \models_{\alpha+1} \sim(\exists\, w) \sim [(w \varepsilon g \vee w = g) \supset w \varepsilon h_Y]$$

Q.E.D.

Lemma 4: If $\Gamma \models_{\omega+1} (g \varepsilon f_x)$, $\Gamma \models_{\omega+1} (g' \varepsilon f_x)$.

Proof: $\Gamma \models_{\omega+1} (g \varepsilon f_x)$ so there is an $h \varepsilon S_\omega$ such that $\Gamma \models_{\omega+1} (g = h) \wedge (h \varepsilon f_x)$. Since $h \varepsilon S_\omega$, for some $\alpha < \omega$, $h \varepsilon S_\alpha$. By lemma 3, there is some $k \varepsilon S_{\alpha+1} - S_\alpha$ such that $\Gamma \models_{\alpha+1} k = h'$, so by dominance $\Gamma \models_\omega k = h'$. But also, $\Gamma \models_{\omega+1} (h \varepsilon f_x)$, $\Gamma \models_\omega X(h)$, so $\Gamma \models_\omega \sim(\exists y)\sim\{[\sim(\exists z)\sim(z \varepsilon y \supset z' \varepsilon y) \wedge \phi \varepsilon y] \supset h \varepsilon y\}$.

By intuitionistic logic it follows that

$\Gamma \models_\omega \sim(\exists y)\sim\{[\sim(\exists z)\sim(z \varepsilon y \supset z' \varepsilon y) \wedge \phi \varepsilon y] \supset k \varepsilon y\}$

that is $\Gamma \models_\omega X(k)$

$\Gamma \models_{\omega+1} (k \varepsilon f_x)$

but $\Gamma \models_{\omega+1} k = h'$

so by definition, $\Gamma \models_{\omega+1} h' \varepsilon f_x$

Q.E.D.

Section 13

Axiom of regularity

Theorem: The following is valid in all models:

$\sim(\exists x)\sim\{(\exists y)(y \varepsilon x) \supset (\exists y)\ [y \varepsilon x \wedge \sim(\exists z)(z \varepsilon x \wedge z \varepsilon y)]\}$

Proof: All the elements of the class S are functions. We have assumed S_0 is well-founded by the relation $x \varepsilon$ domain (y). It then follows that S is also well-founded by $x \varepsilon$ domain (y).

The formula

$\sim\{(\exists y)(y\varepsilon x)\supset(\exists y)[y\varepsilon x\wedge\sim(\exists z)(z\varepsilon x\wedge z\varepsilon y)]\}$ is equivalent to

$\sim\sim\{(\exists y)(y\varepsilon x)\wedge\sim(\exists y)[y\varepsilon x\wedge\sim(\exists z)(z\varepsilon x\wedge z\varepsilon y)]\}$

which is obviously dominant.

Suppose $f\varepsilon S_\alpha$ and $\Gamma\models_\alpha(\exists y)(y\varepsilon f)$. Then for some $g\varepsilon S_\alpha$, $\Gamma\models_\alpha(g\varepsilon f)$.

We claim

$\Gamma\models_\alpha\sim\sim(\exists y)[y\varepsilon f\wedge\sim(\exists z)(z\varepsilon f\wedge z\varepsilon y)]$. Suppose otherwise. Then there is some Γ^* such that

$\Gamma^*\models_\alpha\sim(\exists y)[y\varepsilon f\wedge\sim(\exists z)(z\varepsilon f\wedge z\varepsilon y)]$.

We define a set W to be

$\{x|x\varepsilon S_\alpha$ and for some Γ^{**}, $\Gamma^{**}\models_\alpha(x\varepsilon f)\}$

W is not empty since $g\varepsilon W$. The relation $x\varepsilon$ domain (y) well-founds W. Let s be a "smallest" element of W. That is, $s\varepsilon W$ but for no $t\varepsilon W$ is $t\varepsilon$ domain (s).

Since $s\varepsilon W$, for some Γ^{**}, $\Gamma^{**}\models_\alpha(s\varepsilon f)$. We claim

$\Gamma^{**}\models_\alpha\sim(\exists z)(z\varepsilon f\wedge z\varepsilon s)$. Suppose not. Then for some Γ^{***}, $\Gamma^{***}\models_\alpha(\exists z)(z\varepsilon f\wedge z\varepsilon s)$. Thus, for some $r\varepsilon S_\alpha$, $\Gamma^{***}\models_\alpha(r\varepsilon f)\wedge(r\varepsilon s)$. Since $\Gamma^{***}\models_\alpha(r\varepsilon s)$, there is some $t\varepsilon$ domain (s) such that $\Gamma^{***}\models_\alpha(r=t)\wedge(t\varepsilon s)$. But then $\Gamma^{***}\models_\alpha\sim\sim(t\varepsilon f)$, so for some Γ^{****}, $\Gamma^{****}\models_\alpha(t\varepsilon f)$, so $t\varepsilon W$, a contradiction.

Thus, $\Gamma^{**} \models_{\alpha} \sim(\exists z)(z \varepsilon f \wedge z \varepsilon s)$. But $\Gamma^{**} \models_{\alpha} (s \varepsilon f)$ so

$\Gamma^{**} \models_{\alpha} (\exists y)\,[y \varepsilon f \wedge \sim(\exists z)\,(z \varepsilon f \wedge z \varepsilon y)\,]$

and this contradicts

$\Gamma^{*} \models_{\alpha} \sim(\exists y)\,[y \varepsilon f \wedge \sim(\exists z)(z \varepsilon f \wedge z \varepsilon y)\,]$

Thus

$\Gamma \models_{\alpha} \sim\sim(\exists y)\,[y \varepsilon f \wedge \sim(\exists z)(z \varepsilon f \wedge z \varepsilon y)\,]$

But Γ was arbitrary. We have shown that for each $f \varepsilon S_{\alpha}$ the following is valid in $\langle G, R, \models_{\alpha}, S_{\alpha} \rangle$:

$$(\exists y)(y \varepsilon f) \supset \sim\sim(\exists y)\,[y \varepsilon f \wedge \sim(\exists z)(z \varepsilon f \wedge z \varepsilon y)\,]$$

The theorem now follows by the dominance of the formula mentioned earlier.

Q.E.D.

Section 14

Definability of the models

One of our initial assumptions was that $\langle G, R, \models_{0}, S_{0} \rangle \varepsilon V$. The definition of the sequence was an inductive definition. It should be clear that the definition can be carried out in V itself. That is, not only is $\langle G, R, \models_{\alpha}, S_{\alpha} \rangle \varepsilon V$ for each $\alpha \varepsilon V$ but moreover

<u>Theorem:</u> There is a formula $F(x, y)$ over V which defines the sequence of $\langle G, R, \models_\alpha, S_\alpha \rangle$. That is, for $x, y \in V$, $F(x, y)$ is true over V if and only if x is some ordinal, α, and y is $\langle G, R, \models_\alpha, S_\alpha \rangle$. [in fact, $F(x, y)$ can be absolute, as should be obvious]

Of course, $\langle G, R, \models, S \rangle$ is not in V since, in particular, S is not a set. But we do have

<u>Theorem:</u> Let $X(x_1, \ldots, x_n)$ be any formula with no constants and no universal quantifiers. There is a (classical) formula $R_X(z, x_1, \ldots, x_n)$ with constants from V such that for any $\Gamma \varepsilon G$ and $c_1, \ldots, c_n \varepsilon S$, $\Gamma \models X(c_1, \ldots, c_n)$ if and only if $R_X(\Gamma, c_1, \ldots, c_n)$ is true over V.

<u>Proof:</u> By induction on the degree of X. Suppose X is atomic, $(x \varepsilon y)$. Let $R_X(z, x, y)$ be the formula

$z \varepsilon G \wedge (\exists\, \alpha)\ (\text{ordinal}(\alpha) \wedge x \varepsilon S_\alpha \wedge y \varepsilon S_\alpha \wedge z \models_\alpha (x \varepsilon y))$

[Where we have used the obvious abbreviations allowed by the above theorem]

Suppose X is not atomic but the result is known for all formulas of lesser degree.

If $X(x_1, \ldots, x_n)$ is $Y(x_1, \ldots, x_n) \vee Z(x_1, \ldots, x_n)$ by hypothesis there are formulas $R_Y(w, x_1, \ldots, x_n)$ and

$R_Z(w, x_1, \ldots, x_n)$. Let $R_X(w, x_1, \ldots, x_n)$ be the formula $R_Y(w, x_1, \ldots, x_n) \vee R_Z(w, x_1, \ldots, x_n)$

The case X is $Y \wedge Z$ is similar.

Suppose $X(x_1, \ldots, x_n)$ is $\sim Y(x_1, \ldots, x_n)$. By hypothesis there is a formula $R_Y(z, x_1, \ldots, x_n)$. Let $R_X(z, x_1, \ldots, x_n)$ be the formula

$\sim(\exists w)\ (w \varepsilon G \wedge zRw \wedge R_Y\ (w, x_1, \ldots, x_n)\ )$

The case X is $Y \supset Z$ is similar.

Suppose $X(x_1, \ldots, x_n)$ is $(\exists y)Y(y, x_1, \ldots, x_n)$. By hypothesis there is a formula $R_Y(w, y, x_1, \ldots, x_n)$. Let $R_X(w, x_1, \ldots, x_n)$ be the formula $(\exists y)(\exists \alpha)[\text{ordinal } (\alpha) \wedge y \varepsilon S_\alpha \wedge R_Y(w, y, x_1, \ldots, x_n)]$.

Q.E.D.

Section 15

Power set axiom

We wish to show in this section that the power set axiom is valid in $\langle G, R, \models, S \rangle$.

Let c_0 be a fixed element of S. Then for some smallest ordinal α_0, $c_0 \varepsilon S_{\alpha_0}$. Thus α_0 is also fixed.

We first want to show that for a fixed $\Gamma\varepsilon G$ there is a β_0 such that for any $c\varepsilon S$, if $\Gamma \models (c \subseteq c_0)$, there is some $d\varepsilon S_{\beta_0}$ such that $\Gamma \models (c = d)$. After showing this we will show that in fact there is one β_0 which will do for all $\Gamma\varepsilon G$.

For the above fixed c_0, α_0, and Γ, for $c_1, c_2 \varepsilon S$ such that $\Gamma \models (c_1 \subseteq c_0) \wedge (c_2 \subseteq c_0)$, if for all Γ^* and for all $t\varepsilon S_{\alpha_0}$,

$$\Gamma^* \models ((t\varepsilon c_1) \equiv (t\varepsilon c_2))$$

then $\Gamma \models (c_1 = c_2)$

The proof is as follows.

Suppose for some Γ^* and some $h\varepsilon S$ $\Gamma^* \models (h\varepsilon c_1)$. Since $\Gamma \models (c_1 \subseteq c_0)$, $\Gamma^* \models \sim\sim(h\varepsilon c_0)$. Then for any Γ^{**} there is a Γ^{***} such that $\Gamma^{***} \models (h\varepsilon c_0)$. But $c_0 \varepsilon S_{\alpha_0}$ so there is some $t\varepsilon S_{\alpha_0}$ such that $\Gamma^{***} \models (h = t) \wedge (t\varepsilon c_0)$. Since $\Gamma^{***} \models (h\varepsilon c_1)$, $\Gamma^{***} \models \sim\sim(t\varepsilon c_1)$. Now by hypothesis, since $t\varepsilon S_{\alpha_0}$, $\Gamma^{***} \models \sim\sim(t\varepsilon c_2)$, so $\Gamma^{***} \models \sim\sim(h\varepsilon c_2)$. Thus, $\Gamma^* \models \sim\sim(h\varepsilon c_2)$. We have shown $\Gamma \models (\forall x)(x\varepsilon c_1 \supset \sim\sim x\varepsilon c_2)$ or $\Gamma \models (c_1 \subseteq c_2)$. Similarly, $\Gamma \models (c_2 \subseteq c_1)$.

Q.E.D.

Thus, (speaking intuitively) to decide if two subsets of c_0 are equal at Γ we can confine ourselves to elements of S_{α_0} provided we look at all Γ^*.

Now, let $\mathcal{P}$ be the collection of all elements $c \varepsilon S$ such that $\Gamma \models (c \subseteq c_0)$. We define (intuitively) a function $\mathcal{U}$ on $\mathcal{P}$ by

$$\mathcal{U}(c) = \{ \langle \Gamma^*, t \rangle \mid t \varepsilon S_{\alpha_0} \text{ and } \Gamma^* \models (t \varepsilon c) \}$$

By the above result, for $c_1, c_2 \varepsilon \mathcal{P}$, if $\mathcal{U}(c_1) = \mathcal{U}(c_2)$, $\Gamma \models (c_1 = c_2)$

Let B be the range of $\mathcal{U}$ on $\mathcal{P}$. $\mathcal{U} : \mathcal{P} \rightarrow B$ is a function but one-to-one. So, we cut down its domain to a new domain $\mathcal{P}'$ on which $\mathcal{U}$ is one-to-one. Thus, for $u \varepsilon B$, for $\mathcal{U}^{-1}(u)$, choose some single element x from the class of all $y \varepsilon \mathcal{P}$ such that $\mathcal{U}(y) = u$. Let $\mathcal{P}' = \{ \mathcal{U}^{-1}(u) \mid u \varepsilon B \}$. Let $\mathcal{U}'$ be $\mathcal{U}$ restricted to $\mathcal{P}'$. Then $\mathcal{U}'$ is an isomorphism between $\mathcal{P}'$ and B.

Suppose we could show for some $\beta_0 \varepsilon V$, $\mathcal{P}' \subseteq S_{\beta_0}$. Then if $c \varepsilon S$ and $\Gamma \models (c \subseteq c_0)$, $c \varepsilon \mathcal{P}$ so there is some $d \varepsilon \mathcal{P}'$ such that $\mathcal{U}(c) = \mathcal{U}(d)$, so $\Gamma \models (c = d)$, and $d \varepsilon S_{\beta_0}$. Thus, we would have the desired result. We now show $\mathcal{P}' \subseteq S_{\beta_0}$ for some $\beta_0 \varepsilon V$.

<u>Lemma 1:</u> There is a formula $F(x)$ over V such that $x \varepsilon \wp$ iff $F(x)$ is true over V.

<u>Proof:</u> Let $R_{\subseteq}(z, x, y)$ be the formula defining $z \models (x \subseteq y)$ as given in the last section. Let $F(x)$ be $R_{\subseteq}(\Gamma, x, c_0)$.

Q.E.D.

<u>Lemma 2:</u> There is a formula $G(x,y)$ over V such that $y \varepsilon \mathcal{U}(x)$ iff $G(x,y)$ is true over V.

<u>Proof:</u> Let $R_{\varepsilon}(Z, x, y)$ be the formula defining $Z \models (x \varepsilon y)$. Let $G(x, y)$ be

$F(x) \wedge (\exists\, r, s)\,[y = \langle r, s \rangle \wedge r \varepsilon G \wedge s \varepsilon S_{\alpha_0} \wedge \Gamma R r \wedge R_{\varepsilon}(r, s, x)]$

Q.E.D.

<u>Lemma 3:</u> For any $c \varepsilon S$, $\mathcal{U}(c) \varepsilon P(G \times S_{\alpha_0}) \varepsilon V$

[$P(x)$ is the power set of x in V]

<u>Proof:</u> $\mathcal{U}(c)$ is a subset of $G \times S_{\alpha_0} \varepsilon V$ [and is defined by $G(c, x)$]

Q.E.D.

Lemma 4: $B \varepsilon V$

Proof: By lemma 3, $\{\mathcal{U}(x) \mid x \varepsilon S\}$ is a subset of $P(G \times S_{\alpha_0}) \varepsilon V$. [It is a definable subset, defined by $(\exists \alpha)$ (ordinal $\alpha \wedge (\exists c)(c \varepsilon S_\alpha \wedge G(c, x)))$]

Q.E.D.

Lemma 5: There is a formula $H(x, y)$ such that $x \varepsilon y$, for y a subset of S, if and only if $H(x, y)$ is true over V. [that is, a choice function]

Proof: That S can be well ordered in V is straightforward.

Theorem: $\mathcal{P}' \subseteq S_{\beta_0}$ for some $\beta_0 \varepsilon V$

Proof: The function $\mathcal{U}^{-1}(u)$ can be defined by: $\mathcal{U}^{-1}(u)$ is that x such that $H(x, y)$ where $y = \{z \varepsilon \mathcal{P} \mid \mathcal{U}(z) = \mathcal{U}(u)\}$, which can be formalized. Now $\mathcal{P}'$ is the range of $\mathcal{U}^{-1}(u)$ on B. By the axiom of substitution in V, $\mathcal{P}' \varepsilon V$. Hence, $\mathcal{P}' \subseteq S_{\beta_0}$ for some $\beta_0 \varepsilon V$ since $\mathcal{P}' \subseteq S$ and S is a class.

Q.E.D.

Thus we have our first assertion. We have written it out fairly completely as illustration. From now on we will only indicate the steps.

Above, for fixed Γ we produced an appropriate β_0. But the procedure can itself be defined over V. Since $G\varepsilon V$, by the axiom of substitution again, there is a maximum $\beta_0 \varepsilon V$ which works for all $\Gamma\varepsilon G$. Thus, we have shown:

There is a $\beta_0 \varepsilon V$ such that for any $c\varepsilon S$ and any $\Gamma\varepsilon G$, if $\Gamma \models (c \subseteq c_0)$ then for some $d\varepsilon S_{\beta_0}$, $\Gamma \models (c = d)$

Now we can show the following, from which the power set axiom follows, since c_0 was arbitrary.

<u>Theorem:</u> The following is valid in $\langle G, R, \models, S\rangle$.
$(\exists y)\sim(\exists z)\sim[(z\varepsilon y) \equiv (z \subseteq c_0)]$

<u>Proof:</u> Let $X(x)$ be the formula $(x \subseteq c_0)$ $[c_0 \varepsilon S_{\alpha_0}]$. Let β_0 be as above, and let $\gamma = \max(\alpha_0, \beta_0)$. Then $\gamma\varepsilon V$. Consider $f_x \varepsilon S_{\gamma+1} - S_\gamma$. We claim $\sim(\exists z)\sim[(z\varepsilon f_x) \equiv (z \subseteq c_0)]$ is valid.

Let $\Gamma\varepsilon G$ and suppose $\Gamma^* \not\models \sim(h\varepsilon f_x)$.

Then for some Γ^{**}, $\Gamma^{**} \models (h \varepsilon f_x)$, so there is some $t \varepsilon S_\gamma$ such that $\Gamma^{**} \models (t = h) \wedge (t \varepsilon f_x)$. By dominance, $\Gamma^{**} \models_{\gamma+1} (t \varepsilon f_x)$, $\Gamma^{**} \models_\gamma X(t)$, so $\Gamma^{**} \models_\gamma (t \subseteq c_0)$, by permanence, $\Gamma^{**} \models (t \subseteq c_0)$. Thus $\Gamma^{**} \models \sim\sim(h \subseteq c_0)$. so $\Gamma^{*} \not\models \sim(h \subseteq c_0)$. We have shown $\Gamma \models (\forall x)\ [\sim(h \subseteq c_0) \supset \sim(h \varepsilon f_x)\ ]$ or equivalently, $\Gamma \models \sim(\exists\ x)\sim[(h \varepsilon f_x) \supset (h \subseteq c_0)\ ]$.

Conversely, suppose $\Gamma^{*} \not\models \sim(h \subseteq c_0)$. Then for some Γ^{**}, $\Gamma^{**} \models (h \subseteq c_0)$. There is some $t \varepsilon S_{\beta_0}$ such that $\Gamma^{**} \models (h = t)$. So

$\Gamma^{**} \models (t \subseteq c_0)$. [$x \subseteq y$ is stable] By dominance, $\Gamma^{**} \models_\gamma (t \subseteq c_0)$. $\Gamma^{**} \models_\gamma X(t)$. $\Gamma^{**} \models_{\gamma+1} (t \varepsilon f_x)$. $\Gamma^{**} \models (t \varepsilon f_x)$. $\Gamma^{**} \models \sim\sim(h \varepsilon f_x)$. Thus, $\Gamma^{*} \not\models \sim(h \varepsilon f_x)$.

We have shown

$\Gamma \models (\forall\ x)\ [\sim(h \varepsilon f_x) \supset \sim(h \subseteq c_0)\ ]$ or equivalently
$\Gamma \models \sim(\exists\ h)\sim[(h \subseteq c_0) \supset (h \varepsilon f_x)\ ]$ and the theorem follows.

Q.E.D.

Remark: Above we obtained β_0 by two applications of the axiom of substitution. These could have been combined into one step as in Cohen [2]. This proof was based on that one, which followed a suggestion of Solovay. We find this two step approach more intuitive, but the treatment in Cohen is more elegant.

Section 16

X - equivalence

Def: Let X be a formula with no universal quantifiers, and all constants in S_α. We call $\langle G, R, \models_\alpha, S_\alpha \rangle$ X - equivalent to $\langle G, R, \models, S \rangle$ if for every Y which is an instance of a subformula of X with all constants in S_α, for any $\Gamma \varepsilon G$,

$$\Gamma \models_\alpha Y \iff \Gamma \models Y.$$

Theorem: Let X be as above, with all its constants in S_α. There is an ordinal $\beta \varepsilon V$, $\alpha \leq \beta$, such that $\langle G, R, \models_\beta, S_\beta \rangle$ is X - equivalent to $\langle G, R, \models, S \rangle$.

We spend the rest of the section proving this.

Def: Let $\beta \varepsilon V$ and X be a formula with all its constants in S_β. We call [for this section only] X β-dominant if for any $\Gamma \varepsilon G$,

$$\Gamma \models_\beta X \iff \Gamma \models X$$

Lemma 1: Any atomic formula over S_β is β-dominant. If X and Y are β-dominant, so are $\sim X$, $(X \vee Y)$, $(X \wedge Y)$, and $(X \supset Y)$.

Proof: straightforward.

Lemma 2: Suppose for every $a \varepsilon S_\beta$ $X(a)$ is β-dominant. Then if $\Gamma \models_\beta (\exists x)X(x)$, $\Gamma \models (\exists x)X(x)$.

Proof: $\Gamma \models_\beta (\exists x)X(x)$ implies $\Gamma \models_\beta X(a)$ for some $a \varepsilon S_\beta$. By hypothesis, $\Gamma \models X(a)$, so $\Gamma \models (\exists x)X(x)$.

Q.E.D.

Now for the proof of the theorem. Recall X is a formula over S_α. There are only a finite number of formulas, $Y_1, Y_2, \ldots, Y_n$, with free variables but no constants, such that every subformula of X is an instance of some Y_i. By the theorem of section 14, there are formulas, $R_{Y_1}, R_{Y_2}, \ldots, R_{Y_n}$ over V such that $\Gamma \models Y_i(c_1, \ldots, c_k) \iff R_{Y_i}(\Gamma, c_1, \ldots, c_k)$ is true over V.

We define informally a sequence in V. Using the above R_{Y_i}, the sequence can be formally defined over V. We note again that there is a formula over V which well-orders the class S.

Let $D_0 = S_\alpha$

Suppose we have defined D_m, which is some S_β for $\beta\varepsilon V$. D_m can be well-ordered in V, so all subformulas of X with constants from D_m and of the form $(\exists x)Z(x)$ can be well-ordered (isomorphically) in V. If $(\exists x)Z(x)$ is a subformula of X and has all its constants from D_m, and if there is a $\Gamma\varepsilon G$ such that $\Gamma \models (\exists x)Z(x)$, for some $c\varepsilon S$, $\Gamma \models Z(c)$. Choose the smallest c in the well-ordering of S such that $\Gamma \models Z(c)$. Let K_{m+1} be D_m together with all such c. K_{m+1} can be defined as the range of a function, definable over V, whose domain is the collection of ordered pairs $\langle x,y \rangle$ where $x\varepsilon G$ and y is a formula of the form $(\exists x)Z(x)$, a subformula of X over D_m. This domain is a set, hence K_{m+1} is a set. But $K_{m+1} \subseteq S$. Thus, there is a least $\gamma\varepsilon V$ such that $K_{m+1} \subseteq S_\gamma$. Let $D_{m+1} = S_\gamma$.

In this way, we define the sequence $D_0, D_1, D_2, \ldots$ But this sequence can be defined formally over V. Thus $\cup D_n$ is an element of V. But by the definition, $\cup D_n$ must be some S_β for $\beta\varepsilon V$. $[D_k \subseteq D_{k+1}]$.

We have produced an $S_\beta \varepsilon V$, $\alpha \leq \beta$. We claim $\langle G, R, \models_\beta, S_\beta \rangle$ is X-equivalent to $\langle G, R, \models, S \rangle$. That is, for Y any subformula of X with constants from S_β, $\Gamma \models_\beta Y \iff \Gamma \models Y$. The proof is by induction on the degree of Y. All the cases but one are immediate by the above lemmas. The only non-trivial case is the following. Suppose $(\exists x)Z(x)$ is a subformula of X, has all its constants in S_β, and $\Gamma \models (\exists x)Z(x)$. All the constants of $(\exists x)Z(x)$ lie in $\bigcup D_n$, but there are only finitely many, so for some integer k, all the constants of $(\exists x)Z(x)$ lie in D_k. By definition, there is a $c \in D_{k+1} \subseteq S_\beta$ such that $\Gamma \models Z(c)$. By induction hypothesis, $\Gamma \models_\beta Z(c)$ so $\Gamma \models_\beta (\exists x)Z(x)$.

Section 17

Axiom of substitution

As we did for the power set axiom, we wish to show the axiom of substitution is valid over $\langle G, R, \models, S \rangle$. The proof is essentially that of [2].

Let $X(x, y)$ be a formula with no universal quantifiers, and constants from S, which defines a function at Γ, that is, such that

$$\Gamma \models \sim(\exists x)\sim(\exists ! y)X(x, y)$$

where $(\exists ! y)Z(y)$ abbreviates

$$(\exists y)\ [Z(y) \wedge \sim(\exists w)(Z(w) \wedge \sim(w = y)\ )\ ].$$

Let c_0 be a <u>fixed</u> element of S. Let α_0 be the smallest ordinal such that $c_0 \varepsilon S_{\alpha_0}$. We want to show there is some $f \varepsilon S$ such that
$\Gamma \models \sim(\exists x)\sim[x \varepsilon f \equiv (\exists w)(w \varepsilon c_0 \wedge X(w, x))]$.
That is, roughly, f is the range of X on c_0 at Γ.

By section 14, there is a formula $R_X(z, x, y)$ over V such that $\Delta \models \sim\sim X(x, y)$ iff $R_X(\Delta, x, y)$ is true over V.

Let $g(\Delta, c)$ be the smallest ordinal β such that for some $c' \varepsilon S_\beta$, $\Delta \models \sim\sim X(c, c')$ if there is such, and 0 otherwise, g is definable over V (using R_X).

Since $\alpha_0 \varepsilon V$, $G \times S_{\alpha_0} \varepsilon V$. By the axiom of substitution in V, the range of g on $G \times S_{\alpha_0}$ is a set in V. Thus, also $\bigcup(\text{range } g \text{ on } G \times S_{\alpha_0}) \varepsilon V$. Let β_0 be this union. Then β_0 is an ordinal, $\beta_0 \varepsilon V$.

<u>Lemma:</u> Suppose $\Gamma^* \models (\exists x)(x \varepsilon c_0 \wedge X(x, d))$. Then there is some $c' \varepsilon S_{\beta_0}$ such that $\Gamma^* \models (c' = d)$.

<u>Proof:</u> $\Gamma^* \models (\exists x)(x \varepsilon c_0 \wedge X(x, d))$ so for some $c \varepsilon S$,

$$\Gamma^* \models (c \varepsilon c_0) \wedge X(c, d).$$

$c_0 \varepsilon S_{\alpha_0}$ so there is some $t \varepsilon S_{\alpha_0}$ such that

$\Gamma^* \models (t \varepsilon c_0) \wedge (t = c)$. Hence $\Gamma^* \models \sim\sim X(t, d)$.

Now $\langle \Gamma^*, t \rangle$ ε domain g, so by definition, $g(\Gamma^*, t) \leq \beta_0$. Thus, there is some $c' \varepsilon S_{\beta_0}$ such that

$$\Gamma^* \models \sim\sim X(t, c')$$ But

$\Gamma^* \models \sim\sim X(t, d)$ and $\Gamma^* \models \sim(\exists x)\sim(\exists ! y) X(x, y)$

so by intuitionistic logic,

$\Gamma^* \models (c' = d)$ [(x = y) is stable]

Q.E.D.

Let $\varphi(x)$ be the formula $(\exists w)[w \varepsilon c_0 \wedge X(w, x)]$. There are only a finite number of constants in $\varphi(x)$ [recall, X may have constants], hence all lie in some S_γ (take $\gamma \geq \beta_0$). By the theorem of section 16, there is some $\delta \varepsilon V$, $\gamma \leq \delta$ such that $\langle G, R, \models_\delta, S_\delta \rangle$ is φ-equivalent to $\langle G, R, \models, S \rangle$.

Since φ is a formula over S_γ, φ is also a formula over S_δ. Thus, it defines a function $f_\varphi \varepsilon S_{\delta+1} - S_\delta$. We claim

$\Gamma \models \sim(\exists x)\sim[x \varepsilon f_\varphi \equiv (\exists w)(w \varepsilon c_0 \wedge X(w, x))]$

which is what we wanted. We now proceed with the proof.

Suppose $\Gamma^* \not\models \sim(c \varepsilon f_\varphi)$. Then for some Γ^{**}, $\Gamma^{**} \models (c \varepsilon f_\varphi)$. Since $f_\varphi \varepsilon S_{\delta+1} - S_\delta$, there is some $d \varepsilon S_\delta$ such that $\Gamma^{**} \models (c = d) \wedge (d \varepsilon f_\varphi)$. By dominance, $\Gamma^{**} \models_{\delta+1} (d \varepsilon f_\varphi)$ $\Gamma^{**} \models_\delta \varphi(d)$

But $\langle G, R, \models_\delta, S_\delta \rangle$ is φ-equivalent to $\langle G, R, \models, S \rangle$ hence

$$\Gamma^{**} \models \varphi(d)$$

$$\Gamma^{**} \models \sim\sim \varphi(c)$$

$$\Gamma^* \not\models \sim \varphi(c)$$

$$\Gamma^* \not\models \sim(\exists w)(w \varepsilon c_0 \wedge X(w, c))$$

Thus we have shown

$$\Gamma \models (\forall x)[\sim(\exists w)(w \varepsilon c_0 \wedge X(w, x)) \supset \sim(x \varepsilon f_\varphi)]$$

Conversely, suppose

$\Gamma^* \not\models \sim(\exists w)(w \varepsilon c_0 \wedge X(w, c))$

Then for some Γ^{**}

$$\Gamma^{**} \models (\exists w)(w \varepsilon c_0 \wedge X(w, c))$$

By the above lemma, there is some $c' \varepsilon S_{\beta_0}$ such that $\Gamma^{**} \models (c' = c)$. Hence $\Gamma^{**} \models \sim\sim(\exists w)(w \varepsilon c_0 \wedge X(w, c'))$ that is, $\Gamma^{**} \models \sim\sim \varphi(c')$.

But $c' \varepsilon S_{\beta_0} \subseteq S_\gamma \subseteq S_\delta$, and $\langle G, R, \models_\delta, S_\delta \rangle$ is φ-equivalent to $\langle G, R, \models, S \rangle$, hence

$$\Gamma^{**} \models_{\delta} \sim\sim \varphi(c')$$

$$\Gamma^{**} \models_{\delta+1} \sim\sim(c'\varepsilon f_{\varphi})$$

$$\Gamma^{**} \models \sim\sim(c'\varepsilon f_{\varphi})$$

but $\Gamma^{**} \models (c' = c)$ so

$$\Gamma^{**} \models \sim\sim(c\varepsilon f_{\varphi})$$

$$\Gamma^{*} \not\models \sim(c\varepsilon f_{\varphi})$$

We have shown

$$\Gamma \vdash (\forall x)\ [\sim(x\varepsilon f_{\varphi}) \supset \sim(\exists w)(w\varepsilon c_0 \wedge X(w, x))]$$

The assertion now follows.

Chapter 8

Independence of the Axiom of Choice

Section 1

The specific model

The model given here is adapted from the one of Cohen [2]. We have changed it from showing directly that there is an infinite subset with no countable subset to showing directly that there is a set with no choice function. The change was made because the notion of countability requires much more machinery in these models. See [2, Pg. 136] for a brief introduction to the model.

Following section 3 chapter 7, a sequence of models and a class model are defined if the 0th model is fixed. We now define a specific $\langle G, R, \models_0, S_0 \rangle$. All the work is relative to a classical model V .

Let e be some formal symbol. By a forcing condition we mean a finite consistent set Γ of statements of the form $(n \; e \; m)$ and $\sim (n \; e \; m)$ $[n \geq 0, \; m \geq 1]$ [$(n \; e \; m)$ can be some ordered triple in V , say $\langle n, 0, m \rangle$. Anything convenient. Similarly $\sim (n \; e \; m)$ can be some other triple, say $\langle n, 1, m \rangle$. We have written it like this for reading ease]

Let G be the collection of all forcing conditions, and let R be $\subseteq$, set inclusion.

Before defining S_0, we define the following partition of the integers.

$$I_0 = \{1, 3, 5, 7, \dots\}$$

$$I_1 = \{2, 6, 10, 14, \dots\}$$

$$I_2 = \{4, 12, 20, 28, \dots\}$$

etc.

in general,

$$I_n = \{2^n (1+2k) \mid k = 0, 1, 2, \dots\} .$$

This partition has the properties that each I_n is infinite and if $n \varepsilon I_m$, $n > m$.

Now we define S_0. It consists of the functions $\hat{0}, \hat{1}, \hat{2}, \dots, s_1, s_2, s_3, \dots, t_0, t_1, t_2, \dots, T$, whose definitions are the following.

For each integer n, the function $\hat{n}$ has domain $\{\hat{0}, \hat{1}, \dots, \widehat{n-1}\}$, and for $k < n$, $\hat{n}(\hat{k}) = G$.

Each s_n has as domain $\{\hat{0}, \hat{1}, \hat{2}, \dots\}$ and

$$s_n(\hat{m}) = \{\Gamma \varepsilon G \mid (m\,e\,n) \varepsilon \Gamma\}$$

Each t_n has as domain $\{s_1, s_2, s_3, \dots\}$ and

$$t_n(s_m) = \begin{cases} G \text{ if } m \varepsilon I_n \\ \phi \text{ otherwise} \end{cases}$$

T has as domain $\{t_0, t_1, t_2, \dots\}$ and $T(t_n) = G$.

From this technical definition, $\models_0$ for atomic formulas becomes

$$\Gamma \models_0 (\hat{m}\varepsilon\hat{n}) \quad \text{iff} \quad m < n$$

$$\Gamma \models_0 (\hat{m}\varepsilon s_n) \quad \text{iff} \quad (m\, \varepsilon\, n) \varepsilon \Gamma$$

$$\Gamma \models_0 (s_m\, \varepsilon t_n) \quad \text{iff} \quad m \varepsilon I_n$$

$$\Gamma \models_0 (t_n \varepsilon T)$$

We now examine the five properties of section 3 chapter 7. 1, 2, 3 and 5 are trivial. 4 is satisfied in the very strong sense that, for any $\Gamma\varepsilon G$ and any $a, b\varepsilon S_0$, if

$$\Gamma \models_0 \sim (\exists x) \sim [\, x\varepsilon a \equiv x\varepsilon b]$$

then a and b are the same function. This is proved by examining the various possible choices for a and b. We show only the most difficult case and leave the rest to the reader.

<u>Theorem:</u> If $m \neq n$, $\sim(s_m = s_n)$ is valid in $\langle G, R, \models_0, S_0 \rangle$.

<u>Proof:</u> We show, for any $\Gamma\varepsilon G$, $\Gamma \not\models_0 (s_m = s_n)$. Suppose $\Gamma \models_0 (s_m = s_n)$, for some $\Gamma\varepsilon G$. Since Γ is a forcing condition, it is finite, so we may choose an integer k such that neither $(k\, \varepsilon\, m)$, $\sim (k\, \varepsilon\, m)$, $(k\, \varepsilon\, n)$, $\sim (k\, \varepsilon\, n)$ belong to Γ. Let Δ be $\Gamma \cup \{(k\, \varepsilon\, m), \sim (k\, \varepsilon\, n)\}$.

Then $\Delta \varepsilon G$ and $\Gamma R \Delta$. By definition, $\Delta \models_0 (\hat{k} \varepsilon s_m)$. Since $\Delta \models_0 \sim (\exists x) \sim (x \varepsilon s_m \equiv x \varepsilon s_n)$, by intuitionistic logic, $\Delta \models_0 \sim\sim (\hat{k} \varepsilon s_n)$. Then for some Δ^*, $\Delta^* \models_0 (\hat{k} \varepsilon s_n)$, which means $(k\ e\ n)\ \varepsilon \Delta^*$. But $\sim (k\ e\ n)\ \varepsilon \Delta \subseteq \Delta^*$, a contradiction.

Q.E.D.

Thus all five conditions are met so the resulting class model $\langle G, R, \models, S \rangle$ is an intuitionistic ZF model.

Section 2

Symmetries

Let $\mathcal{G}$ be the collection of all permutations, Π, of integers such that Π permutes the elements of one I_n and is the identity on all I_m for $m \neq n$.

We may extend any $\Pi \varepsilon \mathcal{G}$ to S as follows.

$$\Pi(\hat{n}) = \hat{n}$$

$$\Pi(s_n) = s_{\pi(n)}$$

$$\Pi(t_n) = t_n$$

$$\Pi(T) = T$$

Let X be the formula $X(x, c_1, \ldots, c_n)$ where Π has been defined for $c_1, \ldots, c_n$. Let $\Pi(X)$ be $X(x, \Pi(c_1), \ldots, \Pi(c_n))$.

If $f_x \in S_{\alpha+1} - S_\alpha$, let $\Pi(f_x)$ be $f_{\pi(x)}$.

Thus Π is extended to S.

We also extend Π to G by

$$(n\ e\ m) \in \Gamma \iff (n\ e\ \Pi(m)) \in \Pi(\Gamma)$$

$$\sim (n\ e\ m) \in \Gamma \iff \sim (n\ e\ \Pi(m)) \in \Pi(\Gamma)$$

We note that $\Gamma \varepsilon G$ implies $\Pi(\Gamma) \varepsilon G$.

<u>Theorem:</u> For any formula X with all constants in S_α, with no universal quantifiers, any $\Gamma \varepsilon G$, and any $\Pi \varepsilon \mathcal{G}$

$$\Gamma \models_\alpha X \iff \Pi(\Gamma) \models_\alpha \Pi(X)$$

and $\Gamma \models X \iff \Pi(\Gamma) \models \Pi(X)$.

<u>Proof:</u> A straightforward induction on α and the degree of X.

<u>Def:</u> Let N be some collection of integers. By $\mathcal{G}_N$ we mean the subset of $\mathcal{G}$ leaving N invarient.

<u>Lemma:</u> Let $f \varepsilon S$. There is a <u>finite</u> set N of integers such that if $\Pi \varepsilon \mathcal{G}_N$, $\Pi(f) = f$.

<u>Proof:</u> If $f \varepsilon S_0$, we have two cases. If f is not some s_n, let $N = \phi$. If f is s_n, let $N = \{n\}$.

Suppose the result is known for all $g \varepsilon S_\alpha$. Let $f \varepsilon S_{\alpha+1} - S_\alpha$. Then f is f_X for some $X(x, c_1, \ldots, c_n)$

where $c_1, \ldots, c_n \in S_\alpha$. By hypothesis, there are finite sets, $N_1, \ldots, N_n$ of integers such that if $\Pi \in \mathcal{G}_{N_i}$, $\Pi(c_i) = c_i$. Let $N = N_1 \cup \ldots \cup N_n$. Then if $\Pi \in \mathcal{G}_N$, $\Pi(f_x) = f_{\pi(x)} = f_x$.

Q.E.D.

Section 3

Functions

We introduce the following formula abbreviations.

$x = \langle y,Z \rangle$ for $\sim\sim (\exists w)[w \varepsilon x \wedge w = \{y,Z\} \wedge x = \{y,w\}]$

$\langle x,y \rangle \varepsilon Z$ for $(\exists w)[w \varepsilon Z \wedge w = \langle x,y \rangle]$

ordpr (x) for $\sim(\exists y) \sim [y \varepsilon x \supset (\exists Z)(\exists w)(y = \langle Z,w \rangle)]$

relation (x) for $\sim(\exists y) \sim [y \varepsilon x \supset$ ordpr $(y)]$

function (x) for relation $(x) \wedge \sim (\exists y)(\exists Z)(\exists u)(\exists v) \sim [(\langle y,Z \rangle \varepsilon x \wedge \langle u,v \rangle \varepsilon x \wedge y = u) \supset Z = v]$

domain $(x) = y$ for $\sim (\exists Z)(\exists w) \sim [\langle Z,w\rangle \varepsilon x \supset Z\varepsilon y] \wedge \sim(\exists Z) \sim [Z\varepsilon y \supset (\exists w)(\langle Z,w\rangle \; \varepsilon x)]$.

<u>Theorem:</u> All the above formulas are dominant.

Section 4

Axiom of choice

Let A.C.(T) be the formula

$(\exists x)$ {function $(x) \wedge$ domain $(x) = T \wedge \sim(\exists y) \sim [y\varepsilon T \supset (\exists Z)(Z\varepsilon y \wedge \langle y, Z\rangle \varepsilon x)]$} .

That is, A. C. (T) says that T has a choice function.

In this section we show that $\sim$ A.C. (T) is valid in $\langle G, R, \models, S\rangle$. In fact, it is valid in $\langle G, R, \models_\alpha, S_\alpha\rangle$ for every α ; the same proof holds for each case.

We first show a preliminary

<u>Lemma:</u> If $f\varepsilon S$ and $\Gamma \models (f\varepsilon t_n)$ then for some $m\varepsilon I_n$, $\Gamma \models_\alpha (f = s_m)$.

<u>Proof:</u> $\Gamma \models (f\varepsilon t_n)$ so there is some $b\varepsilon$ domain (t_n) such that

$$\Gamma \models (f = b) \wedge (b\varepsilon t_n) \quad .$$

Q.E.D.

Now, suppose there is some $\Gamma \varepsilon G$ such that $\Gamma \models A.C.(T)$. Then for some $F \varepsilon S$,

$\Gamma \models$ function $(F) \wedge$ domain $(F) = T \wedge$

$\sim(\exists y) \sim [y \varepsilon T \supset (\exists Z)(Z \varepsilon y \wedge \langle y,Z \rangle \varepsilon F)]$

There is a finite set N of integers such that if

$\Pi \varepsilon \mathcal{G}_N$, $\Pi(F) = F$.

Let $n = 1 + \max N$.

$\Gamma \models \sim(\exists y) \sim [y \varepsilon T \supset (\exists Z)(Z \varepsilon y \wedge \langle y,Z \rangle \varepsilon F)]$ and

$\Gamma \models (t_n \varepsilon T)$ hence

$\Gamma \models \sim\sim (\exists Z)(Z \varepsilon t_n \wedge \langle t_n, Z \rangle \varepsilon F)$

Then for some Γ^*,

$\Gamma^* \models (\exists Z)(Z \varepsilon t_n \wedge \langle t_n, Z \rangle \varepsilon F)$.

For some $\alpha \varepsilon S$

$\Gamma^* \models (\alpha \varepsilon t_n) \wedge \langle t_n, \alpha \rangle \varepsilon F$

By the above lemma, for some $m \varepsilon I_n$,

$\Gamma^* \models (\alpha = s_m)$. Hence

$\Gamma^* \models \sim\sim (\langle t_n, s_m \rangle \varepsilon F)$

so for some Γ^{**},

$\Gamma^{**} \models \langle t_n, s_m \rangle \varepsilon F$.

Now $m \varepsilon I_n$ so $m > n = 1 + \max N$, hence $m \notin N$.

Choose an integer $k > n$ such that $k \neq m$ and neither $(p \varepsilon k)$ nor $\sim(p \varepsilon k)$ belongs to Γ^{**} for any integer p, but $k \varepsilon I_n$. [Γ^{**} is finite but I_n is infinite, so this is possible].

Let Π be the permutation $\Pi(m) = k$, $\Pi(k) = m$, on all other integers Π is the identity.

Since $m, k \notin N$, $\Pi \varepsilon \mathcal{G}_N$. Now

$$\Pi(\Gamma^{**}) \models \Pi(\langle t_n, s_m \rangle \ \varepsilon F)$$
$$\Pi(\Gamma^{**}) \models \langle \Pi(t_n), \ \Pi(s_m) \rangle \ \varepsilon \Pi(F)$$
$$\Pi(\Gamma^{**}) \models \langle t_n, s_k \rangle \ \varepsilon \ F$$

But $\Delta = \Gamma^{**} \cup \Pi(\Gamma^{**})$ is itself a forcing condition. It is finite, and since Γ^{**} and $\Pi(\Gamma^{**})$ must be the same except for statements involving m and k, and m is not (a second element of any statement) in $\Pi(\Gamma^{**})$ and k is not in Γ^{**}, $\Pi(\Gamma^{**})$ and Γ^{**} are compatible.

Thus $\Delta \varepsilon G$ and $\Gamma^{**} R \Delta$ and $\Pi(\Gamma^{**}) R \Delta$. So

$$\Delta \models \text{function } F \qquad (\text{since } \Gamma \models \text{function } F)$$
$$\Delta \models \langle t_n, s_m \rangle \ \varepsilon F$$
$$\Delta \models \langle t_n, s_k \rangle \ \varepsilon F$$

It then follows by intuitionistic logic that

$$\Delta \models \sim\sim(s_m = s_k)$$

or since $(x=y)$ is stable,

$$\Delta \models (s_m = s_k).$$

But $m \neq k$, contradicting the theorem of section 1. Thus, for all $\Gamma \varepsilon G$

$$\Gamma \not\models A.C.(T)$$

so

$$\Gamma \models \sim A.C.(T).$$

As we showed in section 1 chapter 7, the axiom of choice is now classically independent.

CHAPTER 9

Ordinals and Cardinals

Section 1

Definitions

Continuing section 3 chapter 8, we introduce the following formula abbreviations.

range $(x) = y$ for $\sim(\exists z)(\exists w)\sim[\langle z, w\rangle \varepsilon x \supset w \varepsilon y] \wedge \sim(\exists w)\sim[w \varepsilon y \supset (\exists z) \langle z, w\rangle \varepsilon x]$

$1 - 1(x)$ for $\sim(\exists y)(\exists z)(\exists u)(\exists v) \sim[(\langle y, z\rangle \varepsilon x \wedge \langle u, v\rangle \varepsilon x \wedge z = v) \supset y = u]$

trans (x) for $\sim(\exists y)(\exists z)\sim[(y \varepsilon x \wedge z \varepsilon y) \supset z \varepsilon x]$

ordered (x) for $\sim(\exists y)(\exists z)\sim[(y \varepsilon x \wedge z \varepsilon x) \supset (y = z \vee y \varepsilon z \vee z \varepsilon y)]$

welord (x) for ordered $x \wedge \sim(\exists y)\sim\{[y \subseteq x \wedge (\exists z)(z \varepsilon y)] \supset (\exists w)[w \varepsilon y \wedge \sim(\exists u)\sim(u \varepsilon y \supset (w \varepsilon u \vee w = u))]\}$

ordinal (x) for trans $(x) \wedge$ welord (x)

<u>Theorem:</u> All of the above formulas are dominant.

The proof is again primarily an application of section 7 chapter 7.

Section 2

Some properties of ordinals

In this section we establish some useful analogs of classical theorems. We use a method of proof which we call a classical-intuitionistic argument. Rather than stating it generally, we illustrate its use by writing out in full the first proof below.

<u>Theorem 1:</u> $\sim(\exists x)\sim(\text{ordered}(x) \equiv \text{welord}(x))$ is valid over $\langle G, R, \models, S\rangle$ [and by dominance, over any $\langle G, R, \models_\alpha, S_\alpha\rangle$]

<u>Proof:</u> It is a standard classical result that ZF, axiom of regularity $\vdash_c$

$$\sim(\exists x)\sim(\text{ordered}(x) \equiv \text{welord}(x))$$

So for some finite subset of ZF, with no universal quantifiers, $\vdash_c$ ($A_1 \wedge \ldots \wedge A_n \wedge$ axiom of regularity)

$$\supset \sim(\exists x)\sim(\text{ordered}(x) \equiv \text{welord}(x))$$

By the results of section 8 chapter 4, together with

$$\vdash_I \sim\sim(X \supset Y) \equiv (X \supset \sim\sim Y)$$

$$\vdash_I \sim\sim\sim X \equiv \sim X$$

We have

$$\vdash_I (A_1 \wedge \ldots \wedge A_n \wedge \text{ axiom of regularity}) \supset$$
$$\sim(\exists x)\sim(\text{ordered }(x) \equiv \text{ welord }(x)).$$

Since $\langle G, R, \models, S\rangle$ is an intuitionistic ZF model, $\sim(\exists x)\sim(\text{ordered }(x) \equiv \text{ welord }(x))$ is valid.

Q.E.D.

Theorem 2: If $\Gamma \models$ ordinal (f) and $\Gamma \models g\varepsilon f$ then $\Gamma \models$ ordinal (g).

Proof: By a classical-intuitionistic argument we have $\sim(\exists x)(\exists y)\sim[(\text{ordinal }(x) \wedge y\varepsilon x) \supset \text{ordinal }(y)]$ is valid in $\langle G, R, \models, S\rangle$. The result now follows by stability of ordinal (y).

Q.E.D.

Theorem 3: If $\Gamma \models$ ordinal (f) $\wedge$ ordinal (g) then $\Gamma \models \sim\sim(f\varepsilon g \vee f = g \vee g\varepsilon f)$.

Section 3

General ordinal representatives

We define inductively representatives for the classical ordinals. Later we discuss their existence and uniqueness.

Suppose we have defined general representatives in S for all ordinals $\beta<\alpha$. We call $f\varepsilon S$ a general representative of the ordinal α if

1) if g represents an ordinal $<\alpha$, $(g\varepsilon f)$ is valid in $\langle G, R, \models, S\rangle$
2) if $\Gamma\models(h\varepsilon f)$, there is some Γ^*, some $\beta<\alpha$, and some $g\varepsilon S$ which represents β, such that $\Gamma^*\models(g = h)$.

Theorem 1: If $f\varepsilon S$ is a general representative of some ordinal, ordinal (f) is valid in $\langle G, R, \models, S\rangle$.

Proof: Suppose f represents the ordinal α and the result is know[n] for all representatives of ordinals $\beta<\alpha$. We have three facts to show.

I. trans (f) is valid in $\langle G, R, \models, S\rangle$

Suppose $\Gamma\models(a\varepsilon f)\wedge(b\varepsilon a)$. Then for any Γ^*, $\Gamma^*\models(a\varepsilon f)\wedge(b\varepsilon a)$. By property 2) there is some $a'\varepsilon S$ which represents $\beta<\alpha$ and some Γ^{**} such that $\Gamma^{**}\models(a = a')$. Thus, $\Gamma^{**}\models\sim\sim(b\varepsilon a')$. There is some Γ^{***} such that $\Gamma^{***}\models(b\varepsilon a')$. Again by property 2) there is some $b'\varepsilon S$ which represents $\gamma<\beta$ and some Γ^{****} such that $\Gamma^{****}\models(b = b')$. By property 1) $\Gamma^{****}\models(b'\varepsilon f)$, hence $\Gamma^{****}\models\sim\sim(b\varepsilon f)$. Thus, for any Γ^* there is some $\Delta(=\Gamma^{****})$ such that $\Gamma^*R\Delta$ and $\Delta\models\sim\sim(b\varepsilon f)$. Thus, $\Gamma\models\sim\sim(b\varepsilon f)$. Since Γ was arbitrary, trans (f) is valid.

II. ordered (f) is valid in $\langle G, R, \models, S \rangle$.

Suppose $\Gamma \models (a \varepsilon f) \wedge (b \varepsilon f)$. For any Γ^*, $\Gamma^* \models (a \varepsilon f) \wedge (b \varepsilon f)$. By property 2), there is some Γ^{**} and some $a', b' \varepsilon S$ such that a' represents β and b' represents γ where $\beta<\alpha$, $\gamma<\alpha$. and $\Gamma^{**} \models (a = a') \wedge (b = b')$. By hypothesis, $\Gamma^{**} \models$ ordinal $(a') \wedge$ ordinal (b'). By theorem 3 section 2, $\Gamma^{**} \models \sim\sim(a' \varepsilon b' \vee a' = b' \vee b' \varepsilon a')$. So $\Gamma^{**} \models \sim\sim(a \varepsilon b \vee a = b \vee b \varepsilon a)$. Thus as above, $\Gamma \models \sim\sim(a \varepsilon b \vee a = b \vee b \varepsilon a)$. Again Γ is arbitrary, so ordered (f) is valid.

III. ordinal (f) is valid in $\langle G, R, \models, S \rangle$.

By the above, trans (f) $\wedge$ ordered (f) is valid. Then welord (f) is also valid by theorem 1 section 2 [welord (x) is stable] Thus, ordinal (f) is valid.

Q.E.D.

<u>Theorem 2:</u> If $f, g \varepsilon S$ are both general representatives of the same ordinal, $(f = g)$ is valid in $\langle G, R, \models, S \rangle$.

<u>Proof:</u> Suppose f and g both represent α. If $\Gamma \models (h \varepsilon f)$, for any Γ^*, $\Gamma^* \models (h \varepsilon f)$. By property 2, there is some Γ^{**}, some $\beta<\alpha$, and some k representing β, such that $\Gamma^{**} \models (h = k)$. Since g represents α and k represents β and $\beta<\alpha$,

by property 1, $\Gamma^{**} \models (k \varepsilon g)$. Thus, $\Gamma^{**} \models \sim\sim(h \varepsilon g)$, so $\Gamma \models \sim\sim(h \varepsilon g)$. Similarly if $\Gamma \models (h \varepsilon g)$, $\Gamma \models \sim\sim(h \varepsilon f)$. But Γ is arbitrary, so the result follows.

Q.E.D.

Section 4

Cannonical ordinal representatives

Again we postpone a discussion of existence.

We call $f \varepsilon S$ a cannonical representative of the ordinal α if

1) f is a general representative of α
2) for no $g \varepsilon$ domain (f) and for no $\Gamma \varepsilon G$ does $\Gamma \models (f = g)$
3) if $\Gamma \models \sim\sim(g \varepsilon f)$, $\Gamma \models (g \varepsilon f)$ for $g \varepsilon$ domain (f).

Theorem: Suppose $f \varepsilon S_{\alpha+1} - S_\alpha$ is a cannonical representative of some ordinal. Then f is f_X where $X(x)$ is the formula ordinal (x).

Proof: We must show for any $a \varepsilon S_\alpha$, $\Gamma \models_{\alpha+1} (a \varepsilon f)$ iff $\Gamma \models_\alpha$ ordinal (a).

Suppose $\Gamma \models_{\alpha+1} (a \varepsilon f)$. By theorem 1 section 3, $\Gamma \models$ ordinal (f), so by theorem 2 section 2, (and dominance), $\Gamma \models_\alpha$ ordinal (a).

Suppose $\Gamma \models_\alpha$ ordinal (a). By by theorem 1 section 3, $\Gamma \models$ ordinal (f). So by theorem 3 section 2 (and dominance), $\Gamma \models {\sim}{\sim}(a \varepsilon f \vee a = f \vee f \varepsilon a)$. Thus, for every Γ^* there is some Γ^{**} such that $\Gamma^{**} \models (a \varepsilon f) \vee (a = f) \vee (f \varepsilon a)$. If $\Gamma^{**} \models (f \varepsilon a)$, since $a \varepsilon S_\alpha$, there is some $g \varepsilon S_\alpha$ such that $\Gamma^{**} \models (f = g)$ contradicting part 2 of the above defintion. Similarly, $\Gamma^{**} \not\models f = a$. Thus, $\Gamma^{**} \models (a \varepsilon f)$. So, $\Gamma \models {\sim}{\sim}(a \varepsilon f)$, and by part 3 above, $\Gamma \models (a \varepsilon f)$, now by dominance, $\Gamma \models_{\alpha+1} (a \varepsilon f)$.

Q.E.D.

Section 5

Ordinalized models

We give a condition on our model [actually on $\langle G, R, \models_0, S_0 \rangle$] which will insure existence and uniqueness of cannonical representatives for the ordinals.

We call $\langle G, R, \models, S \rangle$ ordinalized if

1) no ordinal has more than one cannonical representative in S_0.

2) if $f \varepsilon S_0$ and $\Gamma \models$ ordinal f for some $\Gamma \varepsilon G$, then there is some Γ^* and some $h \varepsilon S_0$ which is a cannonical representative of an ordinal, such that $\Gamma^* \models (f = h)$.

Remark: By dominance, whether $\langle G, R, \models, S \rangle$ is ordinalized can be decided by considering only $\langle G, R, \models_0, S_0 \rangle$.

Theorem 1: If $\langle G, R, \models, S \rangle$ is ordinalized and $f, g \varepsilon S$ are both cannonical representatives for the same ordinal, f and g are identical.

Proof: Suppose first that $g \varepsilon S_\alpha$ and $f \varepsilon S_{\alpha+1} - S_\alpha$. By theorem 2 section 3, $(f = g)$ is valid, contradicting part 2) of the definition of cannonical representative. There is a similar contradiction, if $f \varepsilon S_\alpha$ and

$g \varepsilon S_{\alpha+1} - S_\alpha$. Thus, either $f, g \varepsilon S_0$, or for some α, $f, g \varepsilon S_{\alpha+1} - S_\alpha$. If $f, g \varepsilon S_0$, by part 1) of the above definition they are identical. If $f, g \varepsilon S_{\alpha+1} - S_\alpha$, they are identical by the theorem of section 4.

Q.E.D.

Thus, if an ordinal has any cannonical representatives, it has only one. From now on, by representative we will mean cannonical representative, and we will denote the representative of α, if it exists, by $\hat{\alpha}$.

We give the following temporary definition. We say $\beta \varepsilon V$ has the <u>representative property</u> provided: if α is the smallest ordinal not representable by an element of S_β, α is representable by an element of $S_{\beta+1}$. In other words, β has the representative property provided: if for all $\gamma < \alpha$, $\hat{\gamma} \varepsilon S_\beta$, but $\hat{\alpha} \notin S_\beta$, then $\hat{\alpha} \varepsilon S_{\beta+1} - S_\beta$.

<u>Lemma:</u> If $\langle G, R, \models, S \rangle$ is ordinalized and if all ordinals $< \beta$ have the representative property, so does β.

<u>Proof:</u> Let α be the smallest ordinal not representable in S_β. We must show $\hat{\alpha} \varepsilon S_{\beta+1} - S_\beta$.

Let $X(x)$ be the formula ordinal x, and let $f_x \varepsilon S_{\beta+1} - S_\beta$. We claim f_x is $\hat{\alpha}$.

Suppose $\Gamma \models (h \varepsilon f_x)$. Then there is some $g \varepsilon S_\beta$ such that $\Gamma \models (g = h) \wedge (g \varepsilon f_x)$. But then $\Gamma \models_\beta X(g)$, $\Gamma \models_\beta$ ordinal (g). We now have three cases.

Suppose $\beta = 0$. Since $\langle G, R, \models, S \rangle$ is ordinalized, there is some Γ^* and some $k \varepsilon S_0$ which is an ordinal representative (and by hypothesis, of an ordinal $<\alpha$) such that $\Gamma^* \models (k = g)$. Thus, $\Gamma^* \models (k = h)$.

Suppose β is a successor ordinal. By hypothesis, $\beta-1$ has the representative property. Let γ be the smallest ordinal not representable in $S_{\beta-1}$. Then $\hat{\gamma} \varepsilon S_\beta$. Now (theorem 1 section 3)

$$\Gamma \models \text{ordinal } (\hat{\gamma}) \wedge \text{ordinal } (g)$$

so by theorem 3 section 2,

$$\Gamma \models \sim\sim(g \varepsilon \hat{\gamma} \vee g = \hat{\gamma} \vee \hat{\gamma} \varepsilon g).$$

Then for some Γ^*,

$$\Gamma^* \models (g \varepsilon \hat{\gamma}) \vee (g = \hat{\gamma}) \vee (\hat{\gamma} \varepsilon g).$$

If $\Gamma^* \models (g \varepsilon \hat{\gamma})$, by definition of $\hat{\gamma}$, there is some Γ^{**} and some $\delta < \gamma$ such that $\Gamma^{**} \models (\hat{\delta} = g)$ and so $\Gamma^{**} \models (\hat{\delta} = h)$...

If $\Gamma^* \models (g = \hat{\gamma})$ then $\Gamma^* \models (h = \hat{\gamma})$

Finally, we can not have $\Gamma^* \models (\hat{\gamma} \varepsilon g)$ for, since $g \varepsilon S_\beta$ there is some $k \varepsilon S_{\beta-1}$ such that $\Gamma^* \models (\hat{\gamma} = k) \wedge (k \varepsilon g)$. But $\hat{\gamma} \varepsilon S_\beta - S_{\beta-1}$ and this contradicts part 2 of the definition in section 4.

Suppose β is a limit ordinal. Since $g \varepsilon S_\beta$, for some $\eta < \beta$, $g \varepsilon S_{\eta+1} - S_\eta$. Let γ be the smallest ordinal not representable in S_η. Then $\gamma \varepsilon S_{\eta+1} - S_\eta$. Now proceed as above.

Thus, in any case there is an ordinal $<\alpha$, a representative t of it, and a Δ such that $\Gamma R \Delta$ and $\Delta \models (h = t)$.

Thus, f_x is a general representative of α.

Next, suppose for some $g \varepsilon S_\alpha$, $\Gamma \models (g = f_x)$. Since f_x is a general representative of α, by theorem 1 section 3, $\Gamma \models$ ordinal (f_x). Thus, $\Gamma \models$ ordinal (g), so by dominance, $\Gamma \models_\alpha$ ordinal (g) $\Gamma \models_\alpha X(g)$. Thus, $\Gamma \models_{\alpha+1} (g \varepsilon f_x)$. Hence, $\Gamma \models_{\alpha+1} {\sim}{\sim}(g \varepsilon g)$, $\Gamma \models {\sim}{\sim}(g \varepsilon g)$, contradicting the validity of the axiom of regularity.

Finally, if $\Gamma \models {\sim}{\sim}(g \varepsilon f_x)$ for some $g \varepsilon S_\alpha$, then $\Gamma \models_{\alpha+1} {\sim}{\sim}(g \varepsilon f_x)$. For every Γ^* there is some Γ^{**} such that $\Gamma^{**} \models_{\alpha+1} (g \varepsilon f_x)$. Or, $\Gamma^{**} \models_\alpha X(g)$. $\Gamma^{**} \models_\alpha$ ordinal (g). Thus, $\Gamma \models_\alpha {\sim}{\sim}$ ordinal (g). But ordinal (x) is stable so $\Gamma \models_\alpha$ ordinal (g), $\Gamma \models_\alpha X(g)$, $\Gamma \models_{\alpha+1} (g \varepsilon f_x)$.

Thus f_x is a cannonical representative of α.

Q.E.D.

<u>Theorem:</u> Suppose $\langle G, R, \models, S \rangle$ is ordinalized. Then every ordinal in V is uniquely representable by an element of S.

<u>Proof:</u> immediate by the above lemma.

Q.E.D

<u>Remark:</u> Although it seems unlikely, it is conceivable that some ordinal not in V might be representable by an element of S. In fact, this can not happen. Suppose for some $\gamma \notin V$, $\rho \in S$. For some $\alpha \in V$, $\rho \in S_\alpha$. The class of elements of S which are ordinal representatives is definable over V. The intersection of this class with S_α is a set, i.e. an element of V. But the relation $\Gamma \models_\alpha (x \varepsilon y)$ well-orders this set, the relation is in V, and the order type must be γ (or greater). Hence $\gamma \varepsilon V$.

Thus, exactly the ordinals of V are representable in ordinalized $\langle G, R, \models, S \rangle$.

Section 6

Properties of ordinal representatives

Theorem: If $\langle G, R, \models, S \rangle$ is ordinalized and $\alpha, \beta \varepsilon V$ then if for some $\Gamma \varepsilon G$, $\Gamma \models (\hat{\alpha} = \hat{\beta})$, $\alpha = \beta$, and if $\alpha = \beta$, $(\hat{\alpha} = \hat{\beta})$ is valid.

Proof: If $\alpha < \beta$, by part 1 of the definition in section 3, $\Gamma \models \hat{\alpha} \varepsilon \hat{\beta}$, but if $\Gamma \models (\hat{\alpha} = \hat{\beta})$, $\Gamma \models \sim\sim(\hat{\alpha} \varepsilon \hat{\alpha})$ contradicting the axiom of regularity. Similarly if $\beta < \alpha$. Thus, if $\Gamma' \models (\hat{\alpha} = \hat{\beta})$, $\alpha = \beta$. The second half is by uniqueness of representatives.

Q.E.D.

Theorem 2: If $\langle G, R, \models, S \rangle$ is ordinalized and $\alpha, \beta \varepsilon V$, then if for some $\Gamma \varepsilon G$, $\Gamma \models (\hat{\alpha} \varepsilon \hat{\beta})$, $\alpha \varepsilon \beta$, and if $\alpha \varepsilon \beta$, $(\hat{\alpha} \varepsilon \hat{\beta})$ is valid.

Proof: If $\Gamma \models (\hat{\alpha} \varepsilon \hat{\beta})$, by part 2 of the definition in section 3, for some Γ^* and some $\gamma < \beta$, $\Gamma^* \models (\hat{\alpha} = \hat{\gamma})$. By theorem 1, $\alpha = \gamma$, and $\gamma \varepsilon \beta$. If $\alpha \varepsilon \beta$, by part 1 of the definition in section 3, $(\hat{\alpha} \varepsilon \hat{\beta})$ is valid.

Q.E.D.

Theorem 3: Suppose $\langle G, R, \models, S \rangle$ is ordinalized, and for some $\Gamma \varepsilon G$, $\Gamma \models$ ordinal (f). Then there is some Γ^* and some ordinal $\alpha \varepsilon V$ such that $\Gamma^* \models f = \hat{\alpha}$.

Proof: $f \varepsilon S$ so for some β, $f \varepsilon S_\beta$. Let γ be the smallest ordinal not representable in S_β [$S_\beta \varepsilon V$ so there must be one] Then $\hat{\gamma} \varepsilon S_{\beta+1} - S_\beta$ [see section 5] ~~for some $\delta > \beta$~~. But $\Gamma \models$ ordinal $(\hat{\gamma})$. Hence $\Gamma \models \sim\sim(f \varepsilon \hat{\gamma} \vee f = \hat{\gamma} \vee \hat{\gamma} \varepsilon f)$. For some Γ^*, $\Gamma^* \models (f \varepsilon \hat{\gamma}) \vee (f = \hat{\gamma}) \vee (\hat{\gamma} \varepsilon f)$. If $\Gamma^* \models f \varepsilon \hat{\gamma}$, we are done by part 2 of the definition in section 3. $\Gamma^* \not\models (f = \hat{\gamma})$ by part 2 of the definition in section 4. Finally, $\Gamma^* \models \hat{\gamma} \varepsilon f$ is not possible, for otherwise, since $f \varepsilon S_\beta$, there is some $g \varepsilon S_\beta$ such that $\Gamma^* \models (\hat{\gamma} = g)$. But $\hat{\gamma} \varepsilon S_{\beta+1} - S_\beta$ and this contradicts part 2 of the definition in section 4.

Q.E.D.

Section 7

Types of ordinals

We introduce the following formula abbreviations.

successor ordinal (x) for ordinal $(x) \wedge (\exists y)(y \varepsilon x \wedge x = y')$

limit ordinal (x) for ordinal $(x) \wedge \sim(\exists y)\sim(y \varepsilon x \supset y' \varepsilon x)$

integer (x) for ordinal $(x) \wedge \sim$ limit ordinal (x)

$\wedge \sim (\exists y)(y \varepsilon x \wedge$ limit ordinal (y))

x is ω for limit ordinal $(x) \wedge \sim (\exists y)(y \varepsilon x \wedge$ limit ordinal (y))

<u>Theorem:</u> The above formulas are dominant.

<u>Theorem:</u> If $\langle G, R, \models, S \rangle$ is ordinalized,

$\widehat{\alpha + 1} = \hat{\alpha}'$ is valid.

<u>Proof:</u> We must show for all $\Gamma \varepsilon G$,

$\Gamma \models \sim (\exists x) \sim [x \varepsilon \widehat{\alpha+1} \equiv (x \varepsilon \hat{\alpha} \vee x = \hat{\alpha})]$

Suppose $\Gamma \models f \varepsilon \widehat{\alpha+1}$. Then for every Γ^*, $\Gamma^* \models f \varepsilon \widehat{\alpha+1}$. There is some Γ^{**} and some $\beta < \alpha+1$, $\Gamma^{**} \models f = \hat{\beta}$. But $\beta \leq \alpha$ so $\Gamma^{**} \models (\hat{\beta} \varepsilon \hat{\alpha}) \vee (\hat{\beta} = \hat{\alpha})$. $\Gamma^{**} \models \sim\sim (f \varepsilon \hat{\alpha} \vee f = \hat{\alpha})$. Thus, $\Gamma \models \sim\sim (f \varepsilon \hat{\alpha} \vee f = \hat{\alpha})$. Similarly, if $\Gamma \models (f \varepsilon \hat{\alpha} \vee f = \hat{\alpha})$, then $\Gamma \models \sim\sim (f \varepsilon \widehat{\alpha+1})$. The result follows.

Q.E.D.

<u>Corollary:</u> If $\langle G, R, \models, S \rangle$ is ordinalized, successor ordinal $(\widehat{\alpha+1})$ is valid.

<u>Theorem:</u> If $\langle G, R, \models, S\rangle$ is ordinalized and for some $f \varepsilon S$ and some $\Gamma \varepsilon G$, $\Gamma \models$ successor ordinal f, then for some Γ^* and some $\alpha+1$, $\Gamma^* \models (f = \widehat{\alpha+1})$.

<u>Proof:</u> $\Gamma \models$ successor ordinal f, so for some $g \varepsilon S$, $\Gamma \models$ ordinal $g \wedge g \varepsilon f \wedge f = g'$. Since $\Gamma \models$ ordinal g, there is a Γ^* and an ordinal α such that $\Gamma^* \models g = \hat{\alpha}$. Then $\Gamma^* \models f = \hat{\alpha}'$, $\Gamma^* \models f = \widehat{\alpha+1}$.

Q.E.D.

In a similar manner we may show

<u>Theorem:</u> Suppose $\langle G, R, \models, S\rangle$ is ordinalized. Then

1) If λ is a limit ordinal, limit ordinal $(\hat{\lambda})$ is valid.

2) If $\Gamma \models$ limit ordinal (f), then for some Γ^* and some limit ordinal λ, $\Gamma^* \models (f = \hat{\lambda})$.

3) If n is an integer, integer $(\hat{n})$ is valid.

4) If $\Gamma \models$ integer (f), then for some Γ^* and some integer n, $\Gamma^* \models (f = \hat{n})$.

5) $\hat{\omega}$ is ω is valid.

6) If $\Gamma \models f$ is ω, then for some Γ^*, $\Gamma^* \models (f = \hat{\omega})$.

Section 8

Cardinalized models

Let cardinal (x) be an abbreviation for ordinal (x) $\wedge$ ~($\exists$ y)($\exists$ z)[yεx $\wedge$ function (z) $\wedge$ 1-1z $\wedge$ domain (z) = y $\wedge$ range (z) = x]

We remark that cardinal (x) is not dominant (probably) but it is stable.

Suppose $\langle G, R, \models, S \rangle$ is ordinalized. We call it cardinalized if for every $\alpha \varepsilon V$, if α is a cardinal of V, cardinal ($\hat{\alpha}$) is valid in $\langle G, R, \models, S \rangle$.

By the classical-intuitionistic technique of section 2,

~($\exists$ x)~[integer (x) $\supset$ cardinal (x)]

and ~($\exists$ x)~[x is ω $\supset$ cardinal (x)]

are both valid in $\langle G, R, \models, S \rangle$. But then by section 7, for any integer n, cardinal ($\hat{n}$) is valid. Also cardinal ($\hat{\omega}$) is valid.

Thus, the troublesome cardinals of V are the uncountable ones. In the next section we give a condition due to Cohen which will take care of such cardinals.

Remark: To say $\langle G, R, \models, S\rangle$ is cardinalized is to say the cardinals of V are among those of $\langle G, R, \models, S\rangle$. In fact, we will show in chapter 13 that the cardinals of $\langle G, R, \models, S\rangle$ are the same as the cardinals of L, the class of constructable sets of V.

Section 9

Countably incompatible G

The following argument is from [2]

Def: Two elements $\Gamma, \Delta \varepsilon G$ are called compatible if they have a common R-extension, that is, if some Γ^* is some Δ^*, Otherwise Γ and Δ are incompatible.

$G \varepsilon V$ is called countably incompatible if any subset of G of mutually incompatible Γ is at most countable in V.

Lemma: Suppose $\langle G, R, \models, S\rangle$ is ordinalized, G is countably incompatible, $\hat{\alpha}, \hat{\beta} \varepsilon S$, card α < card β and $\aleph_0$ < card β in V. Then $\sim(\exists f)$ [function $f \wedge$ 1-1 $f \wedge$ domain $f = \hat{\alpha} \wedge$ range $f = \hat{\beta}$] is valid in $\langle G, R, \models, S\rangle$.

<u>Proof:</u> Let f be some fixed element of S. We remarked earlier that the class of ordinal representatives was definable over V. Let $F(x)$ be the formula defining it. Let $A(x,y,z)$ be the formula $[x\varepsilon S \wedge y\varepsilon S \wedge F(y) \wedge z\varepsilon G \wedge z \models (\text{function } (f) \wedge \text{1-1 } f \wedge \text{domain } (f) = \hat{\alpha} \wedge \text{range } f = \hat{\beta} \wedge \langle x,y\rangle \varepsilon f)]$

Suppose for $\hat{\gamma}, \hat{\delta}, \Delta, \Delta', c,$ that $A(c,\hat{\gamma},\Delta)$ and $A(c,\hat{\delta},\Delta')$ are both true over V. If Δ and Δ' are compatible, some $\Delta^* \models \langle c,\hat{\gamma}\rangle \varepsilon f \wedge \langle c,\hat{\delta}\rangle \varepsilon f$. Hence $\Delta^* \models \hat{\gamma} = \hat{\delta}$ so $\gamma = \delta$. Thus, if $\gamma \neq \delta$, Δ and Δ' are incompatible. Thus, for any fixed $c\varepsilon S$, and any $\Delta\varepsilon G$, there are only countably many ordinals γ such that $A(c,\hat{\gamma},\Delta)$ is true over V, by the countable incompatibility hypothesis.

Let $B(x, y)$ be the formula $(\exists \Delta)(\Delta\varepsilon G \wedge A(x,y,\Delta))$. Then for fixed $c\varepsilon S$, the set defined by $B(c, y)$ is at most countable.

For an ordinal α, let α^0 be $\{\hat{\gamma} | \gamma<\alpha\}$. $\alpha^0 \varepsilon V$ for $\alpha\varepsilon V$.

Finally, let $C(x)$ be the formula $(\exists c)(c\varepsilon\alpha^0 \wedge B(c,x))$. Let C' be the class in V defined by $C(x)$, and let C be $\{\gamma | \hat{\gamma}\varepsilon C'\}$. Since C' is a definalbe subset of β^0, $C\varepsilon V$. For a bound on the cardinality of C we note that for any

$c\epsilon\alpha^0$, there are at most $\aleph_0$ x such that $B(c,x)$. Thus, card $C \leq \aleph_0 \cdot$ card $\alpha <$ card β so card $C <$ card β.

Next we show that if, for some $\Delta\epsilon G$, $\Delta \models$ (function $(f) \wedge$ 1-1 $f \wedge$ domain $(f) = \hat{\alpha} \wedge$ range $(f) = \hat{\beta}$ $\wedge \langle c,d \rangle$ ϵf) then there is some Δ^* and some $\delta\epsilon C$ such that $\Delta^* \models (d = \hat{\delta})$. For, since $\Delta \models \langle c,d \rangle$ ϵf, there must be some Δ^* such that $\Delta^* \models (d\epsilon\hat{\beta})$ and hence a Δ^{**} and a $\delta\epsilon\beta$ such that $\Delta^{**} \models (d = \hat{\delta})$. Thus, $\Delta^{**} \models$(function $(\dot{f}) \wedge$ 1-1 $(f) \wedge$ domain $(f) = \hat{\alpha} \wedge$ range $(f) = \hat{\beta} \wedge \langle c, \hat{\delta} \rangle$ ϵf).

So

$$A(c, \hat{\delta}, \Delta^{**}) \text{ is true over } V$$

$$B(c, \hat{\delta}) \text{ is true over } V$$

$$C(\hat{\delta}) \text{ is true over } V$$

$$\delta \,\epsilon C$$

Now, suppose there were some $\Gamma\epsilon G$ such that $\Gamma \models$ (function $(f) \wedge$ 1-1$(f) \wedge$ domain $(f) = \hat{\alpha} \wedge$range $(f) = \hat{\beta}$).

Since card $C <$ card β, but $C \subseteq \beta$, there is some $\gamma\epsilon\beta$, $\gamma\not\epsilon C$. Since $\gamma\epsilon\beta$, $\Gamma \models (\hat{\gamma}\epsilon\hat{\beta})$. Then since $\Gamma \models$ (range $(f) = \hat{\beta}$), for some Γ^*.

$\Gamma^* \models (\exists\, c)(c\epsilon\hat{\alpha} \wedge \langle c, \hat{\gamma} \rangle$ ϵf)

so for some $c\epsilon S$,

$\Gamma^* \models \langle c, \hat{\gamma} \rangle$ ϵf.

That is,

$\Gamma^* \models$ (function $(f) \wedge$ 1-1 $(f) \wedge$ domain $(f) = \hat{\alpha} \wedge$ range $(f) = \hat{\beta} \wedge \langle c, \hat{\gamma} \rangle \varepsilon f)$

By the above, there is some Γ^{**} and some $\delta \varepsilon C$ such that $\Gamma^{**} \models (\hat{\gamma} = \hat{\delta})$, but then $\gamma = \delta$ so $\gamma \varepsilon C$, a contradiction.

Since f is arbitrary, the result follows.

Q.E.D.

Theorem: Suppose $\langle G, R, \models, S \rangle$ is ordinalized, G is countably incompatible, and β is a cardinal of V. Then cardinal $(\hat{\beta})$ is valid in $\langle G, R, \models, S \rangle$.

Proof: By the last section we need only consider $\beta > \aleph_0 = \omega$. Suppose $\Gamma \not\models$ cardinal $(\hat{\beta})$. Then for some α, f, Γ^*, $\Gamma^* \models (\hat{\alpha} \varepsilon \hat{\beta} \wedge$ function $(f) \wedge$ domain $(f) = \hat{\alpha} \wedge$ range $(f) = \hat{\beta})$.

Since $\Gamma^* \models (\hat{\alpha} \varepsilon \hat{\beta})$, $\alpha \varepsilon \beta$ so

card $\alpha <$ card β [β is a cardinal].

Now, by the above lemma we are done.

Q.E.D

Remark: A simple corollary of this theorem (which should be obvious anyway) is the following. If L is the class of constructable sets of V, not only is L a classical

ZF model, but if α is a cardinal of V, α is a cardinal of L. This follows by noting that in the intuitionistic formulation of the classical M_α sequence [remark - section 3, chapter 7] G is trivially countably incompatible, since G is finite, and since $M_0 = \phi$, the model is ordinalized.

CHAPTER 10

Independence of the Continuum Hypothesis

Section 1

The Specific Model

Again the model is adapted from Cohen [2], with practically no change. We define a particular $\langle G, R, \models_0, S_0 \rangle$.

Recall V was some classical ZF model. Let $\delta \varepsilon V$ be that ordinal which is $\aleph_2$ in V. δ remains fixed for rest of this chapter.

As in chapter 8, let e be some formal symbol. By a forcing condition we mean a finite, consistent set of statements of the form $(n \, e \, \alpha)$ or $\sim(n \, e \, \alpha)$ where n is any integer and α is any ordinal $< \delta$.

Let G be the collection of all forcing conditions, and let R be $\subseteq$, set inclusion.

S_0 consists of functions which we write as $\hat{\alpha}$, a_α, $\{\hat{\alpha}\}$, $\{\hat{\alpha}, a_\alpha\}$, and $\langle \hat{\alpha}, a_\alpha \rangle$ for each $\alpha<\delta$ And W. The definitions are the following.

For each $\alpha<\delta$ the domain of $\hat{\alpha}$ is $\{\hat{\beta} \mid \beta<\alpha\}$ and for $\beta<\alpha$, $\hat{\alpha}(\hat{\beta}) = G$. a_α has domain $\{\hat{0}, \hat{1}, \hat{2}, \ldots\}$ and

$$a_\alpha(\hat{n}) = \{\Gamma\varepsilon G \mid (m\, e\, n)\varepsilon\Gamma\}.$$

$\{\hat{\alpha}\}$ has only $\hat{\alpha}$ in its domain, and $\{\hat{\alpha}\}(\hat{\alpha}) = G$. $\{\hat{\alpha}, a_\alpha\}$ has only $\hat{\alpha}$ and a_α in its domain and $\{\hat{\alpha}, a_\alpha\}\ (\hat{\alpha}) = G$, $\{\hat{\alpha}, a_\alpha\}(a_\alpha) = G$. $\langle\hat{\alpha}, a_\alpha\rangle$ has only $\{\hat{\alpha}\}$ and $\{\hat{\alpha}, a_\alpha\}$ in its domain and $\langle\hat{\alpha}, a_\alpha\rangle(\{\hat{\alpha}\}) = G$, $\langle\hat{\alpha}, a_\alpha\rangle(\{\hat{\alpha}, a_\alpha\}) = G$. Finally W has as domain all $\langle\hat{\alpha}, a_\alpha\rangle$ for $\alpha<\delta$, and $W(\langle\hat{\alpha}, a_\alpha\rangle) = G$.

From this, $\models_0$ for atomic formulas becomes

$$\Gamma \models_0 (\hat{\alpha}\varepsilon\hat{\beta}) \qquad \text{if } \alpha\varepsilon\beta$$

$$\Gamma \models_0 (\hat{n}\varepsilon a_\alpha) \qquad \text{if } (n\, e\, \alpha)\varepsilon\Gamma$$

$$\Gamma \models_0 (\hat{\alpha}\varepsilon\{\hat{\alpha}\})$$

$$\Gamma \models_0 (\hat{\alpha}\varepsilon\{\hat{\alpha}, a_\alpha\})$$

$$\Gamma \models_0 (a_\alpha\varepsilon\{\hat{\alpha}, a_\alpha\})$$

$$\Gamma \models_0 (\{\hat{\alpha}\}\varepsilon\langle\hat{\alpha}, a_\alpha\rangle)$$

$$\Gamma \models_0 (\{\hat{\alpha}, a_\alpha\}\varepsilon\langle\hat{\alpha}, a_\alpha\rangle)$$

$$\Gamma \models_0 (\langle\hat{\alpha}, a_\alpha\rangle\varepsilon W)$$

Thus $\langle G, R, \models_0, S_0\rangle$ is determined. We examine the five properties of section 3 chapter 7. 1, 2, 3 and 5 are trivial. 4 is satisfied in the same sense as in the model of chapter 8, that is, if $\Gamma \models_0 (a = b)$, a and b are identical. The proof is the same as in chapter 8.

Thus, $\langle G, R, \models, S\rangle$ is an intuitionistic ZF model.

That $\langle G, R, \models, S\rangle$ is ordinalized is straightforward. For $\alpha<\delta$, $\hat{\alpha}\varepsilon S_0$ is the representative of α, and if, for some $a\varepsilon S_0$, $\Gamma \models_0$ ordinal a, a must be $\hat{\alpha}$ for some $\alpha<\delta$.

Finally, in the next section we show $\langle G, R, \models, S\rangle$ is cardinalized.

Section 2

Countable incompatibility of G

Theorem: [Cohen] G is countably incompatible. [and hence $\langle G, R, \models, S\rangle$ is cardinalized]

Proof: We give the argument informally, but $G\varepsilon V$ and $R\varepsilon V$ so the argument can be formalized.

We note that, for this model, to say $\Gamma,\Delta\varepsilon G$ are compatible is to say $\Gamma\cup\Delta\varepsilon G$.

Let $H\subseteq G$. $[H\varepsilon V]$ and suppose any two elements of H are incompatible. We show H is countable.

Suppose H is not countable. For each $n>0$, let H_n be $\{\Gamma\varepsilon H | \Gamma$ contains $<n$ statements$\}$ Since $H = \bigcup H_n$, some H_n must be uncountable. Thus, let H_n be uncountable.

Let k be the largest integer such that for some $\Gamma\varepsilon H_n$, Γ has k statements and uncountably many $\Delta\varepsilon H_n$ are such that $\Gamma \subseteq \Delta$. [k must exist since $\phi\varepsilon H_n$ and there are uncountably many $\Delta\varepsilon H_n$ such that $\phi \subseteq \Delta$, and every $\Gamma\varepsilon H_n$ has $<n$ statements, so there is a largest k]

Pick some particular $\Gamma\varepsilon H_n$ such that Γ has k statements and Γ is a subset of uncountably many elements of H_n.

Let K be $\{\Delta\varepsilon H_n | \Gamma \subseteq \Delta\}$. We have the following facts:

1) any two elements of K are incompatible.

2) K is uncountable.

3) $\Delta\varepsilon K$ implies $\Gamma \subseteq \Delta$

4) Γ has k elements.

5) for any $\Omega\varepsilon K$ with more than k elements, there are only countably many $\Delta\varepsilon K$ such that $\Omega \subseteq \Delta$.

Now choose some $\Delta\varepsilon K$, $\Delta \neq \Gamma$. Then $\Delta - \Gamma = \{X_1, \ldots, X_m\}$. Since Δ is incompatible with all other elements of K, by 3), there must be uncountably many elements of K containing $\overline{X_i}$ for some X_i [$\overline{X_i}$ is $\sim(n \in \alpha)$ if X_i is $(n \in \alpha)$, and $\overline{X_i}$ is $(n \in \alpha)$ if X_i is $\sim(n \in \alpha)$]

Let $\Omega = \Gamma \cup \{\overline{X_i}\}$. Then $\Omega \in H_n$ since $X_i \notin \Gamma$. But there are uncountably many $\Delta \in H_n$ such that $\Omega \subseteq \Delta$ and Ω has $k+1$ statements, a contradiction.

Q.E.D.

Section 3

Cardinals and W

We now have that $\langle G, R, \models, S \rangle$ is a cardinalized model. We introduce the following abbreviations:

x is at least $\aleph_1$ for cardinal $x \wedge$
$(\exists y)(y \in x \wedge y \text{ is } \omega)$

x is at least $\aleph_2$ for cardinal $x \wedge$
$(\exists y)(y \in x \wedge y \text{ is at least } \aleph_1)$

Recall that in V, δ was $\aleph_2$. We wish to show ($\hat{\delta}$ is at least $\aleph_2$) is valid in $\langle G, R, \models, S \rangle$.

We showed in chapter 9, that ($\hat{\omega}$ is ω) is valid in $\langle G, R, \models, S \rangle$.

Let γ be the ordinal of V which is $\aleph_1$. Since γ is a cardinal, (cardinal $\hat{\gamma}$) is valid, and since $\omega\varepsilon\gamma$, $(\hat{\omega}\varepsilon\hat{\gamma})$ is valid. Thus ($\hat{\gamma}$ is at least $\aleph_1$) is valid in $\langle G, R, \models, S\rangle$. Finally, δ is a cardinal of V, so (cardinal $\hat{\delta}$) is valid, and $\gamma\varepsilon\delta$, so $(\hat{\gamma}\varepsilon\hat{\delta})$ is valid. Thus, ($\hat{\delta}$ is at least $\aleph_2$) is valid in $\langle G, R, \models, S\rangle$.

Now we list a few properties of W. The proofs are straightforward.

Lemma: $\langle\hat{\alpha}, a_\alpha\rangle = \langle\hat{\alpha}, a_\alpha\rangle$ is valid in $\langle G, R, \models, S\rangle$
[where the first of these expressions is the function in S_0, and the second is the expression of section 3 chapter 8]

Theorem: (function $W \wedge$ 1-1 $W \wedge$ domain $W = \hat{\delta}$) is valid in $\langle G, R, \models, S\rangle$.

Theorem: $\sim(\exists x)\sim[x\ \varepsilon$ range $(W) \supset \sim(\exists y)\sim(y\varepsilon x \supset$ integer $y)]$ is valid in $\langle G, R, \models, S\rangle$.

Section 4

Continuum hypothesis

Let (card $\mathcal{P}(\omega) \geq \aleph_2$) be an abbreviation for $(\exists x)$ {x is at least $\aleph_2 \wedge (\exists W)$ [function (W) $\wedge$ 1-1 (W) $\wedge$ domain (W) = x $\wedge$ $\sim(\exists y)\sim(y\varepsilon$ range (W) $\supset$ $\sim(\exists z)\sim(z\varepsilon y \supset$ integer (z)))]}

By the results of section 3, (card $\mathcal{P}(\omega) \geq \aleph_2$) is valid in $\langle G, R, \models, S\rangle$. Hence ~(continuum hypothesis) is valid in $\langle G, R, \models, S\rangle$.

Now, as we showed in section 1, chapter 7, the continuum hypothesis is classically independent of the axioms of ZF. Of course, we would also like that it is independent of ZF together with the axiom of choice. That the axiom of choice is valid in this model will be shown in chapter 13.

CHAPTER 11

Definability and Constructability

Section 1

Definitions

We introduce the following formula abbreviations.

partfun (f) for function (f) $\wedge$ ($\exists$ n)[integer (n) $\wedge$ domain (f) $\subseteq$ n]

partrel (R) for ~($\exists$ x)($\exists$ y)~[(xεR $\wedge$ yεR) $\supset$ (partfun (x) $\wedge$ partfun (y) $\wedge$ domain (x) = domain (y)]

n ε Domain (R) for ~($\exists$ x)~[(partfun (x) $\wedge$ xεR) $\supset$ nε domain (x)]

R is atomic (1) over X for ($\exists$ m)($\exists$ n) integer (m) $\wedge$ integer (n) $\wedge$ ~($\exists$ f)~[fεR ≡ (partfun (f) $\wedge$ domain (f) = {m, n} $\wedge$ f(m)εX $\wedge$ f(n)εX $\wedge$ f(m)εf(n))]}

R is atomic (2) over X for ($\exists$ n)($\exists$ a) integer n $\wedge$ ~~aεX $\wedge$ ~($\exists$ f)~[fεR ≡ (partfun f $\wedge$ domain f = {n} $\wedge$ f(n)εX $\wedge$ f(n)εa)]}

R is atomic (3) over X for $(\exists n)(\exists a)\{\text{integer}(n) \wedge$
$\sim\sim a\varepsilon X \wedge \sim(\exists f)\sim[f\varepsilon R \equiv (\text{partfun}(f) \wedge \text{domain}$
$(f) = \{n\} \wedge f(n)\varepsilon X \wedge a\varepsilon f(n))]\}$

R is atomic (4) over X for $(\exists a)(\exists b)\{\sim\sim a\varepsilon X \wedge \sim\sim b\varepsilon X$
$\wedge \sim(\exists f)\sim[f\varepsilon R \equiv (\text{partfun}(f) \wedge \text{domain}$
$(f) = \phi \wedge a\varepsilon b)]\}$

R is atomic over X for R is atomic (1) over X $\vee$ R is atomic (2) over X $\vee$ R is atomic (3) over X $\vee$ R is atomic (4) over X

R is not-S for partrel $S \wedge \sim(\exists x)\sim[x\varepsilon \text{ Domain } R \equiv x\varepsilon$
$\text{Domain } S] \wedge \sim(\exists f)\sim[f\varepsilon R \equiv \sim f\varepsilon S]$

$(f \upharpoonright \text{Domain } S)\varepsilon S$ for $(\exists g)[g\varepsilon S \wedge \sim(\exists x)\sim[x\varepsilon$
$\text{Domain } S \supset f(x) = g(x)]]$

R is S-and-T for partrel $S \wedge$ partrel $T \wedge \sim(\exists x)$
$\sim[x\varepsilon \text{ Domain } R \equiv (x\varepsilon \text{ Domain } S \vee x\varepsilon \text{ Domain } T)] \wedge$
$\sim(\exists f)\sim[f\varepsilon R \equiv ((f \upharpoonright \text{Domain } S)\varepsilon S \wedge (f \upharpoonright \text{Domain } T)$
$\varepsilon T)]$

R is S-or-T for partrel $S \wedge$ partrel $T \wedge \sim(\exists x)\sim[x\varepsilon$
$\text{Domain } R \equiv (x\varepsilon \text{ Domain } S \vee x\varepsilon \text{ Domain } T)] \wedge$
$\sim(\exists f)\sim[f\varepsilon R \equiv ((f \upharpoonright \text{Domain } S)\varepsilon S \vee (f \upharpoonright \text{Domain } T)$
$\varepsilon T)]$

R is S-implies - T for $\text{partrel } S \wedge \text{partrel } T \wedge$

$\sim(\exists x)\sim[x \varepsilon \text{ Domain } R \equiv (x \varepsilon \text{ Domain } S \vee x \varepsilon \text{ Domain } T)]$

$\wedge \sim(\exists f)\sim[f \varepsilon R \equiv ((f \upharpoonright \text{ Domain } S) \varepsilon S \supset (f \upharpoonright \text{ Domain } T)$

$\varepsilon T)]$

$f = g \upharpoonright$ Domain R for $\text{domain } (f) = \text{Domain } R \wedge \sim(\exists x)$

$\sim[x \varepsilon \text{ Domain } R \supset f(x) = g(x)]$

R is $(\exists n)S$ over X for $\text{partrel } S \wedge \text{integer } n \wedge$

$\sim(\exists x)\sim[x \varepsilon \text{ Domain } R \equiv (x \varepsilon \text{ Domain } S \wedge \sim x = n)]$

$\wedge \sim(\exists f)\sim[f \varepsilon R \equiv (\exists g)(g \varepsilon S \wedge f = g \upharpoonright \text{ Domain } R \wedge$

$g(n) \varepsilon X)]$

R is a definable relation over X for $(\exists F)(\exists n)$

$\{\text{function } (F) \wedge \text{integer } (n) \wedge \text{domain } (F) = n \wedge$

$\sim(\exists x)\sim[x \varepsilon n \supset F(x) \text{ is atomic over } X \vee (\exists y)(y \varepsilon x$

$\wedge F(x) \text{ is not - } F(y)) \vee (\exists y)(\exists z)(y \varepsilon x \wedge z \varepsilon x \wedge$

$F(x) \text{ is } F(y)\text{-and-}F(z))$

$\vee (\exists y)(\exists z)(y \varepsilon x \wedge z \varepsilon x \wedge F(x) \text{ is } F(y)\text{-or-}F(z))$

$\vee (\exists y)(\exists z)(y \varepsilon x \wedge z \varepsilon x \wedge F(x) \text{ is } F(y)\text{-implies-}F(z))$

$\vee (\exists y)(\exists k)(y \varepsilon x \wedge \text{integer } (k) \wedge F(x) \text{ is } (\exists k)$

$F(y) \text{ over } X)] \wedge (\exists m)(m \varepsilon n \wedge F(m) = R)\}$

X is definable over Y for

$(\exists R)(\exists n) \{ \text{partrel } R \wedge \text{integer } (n) \wedge R \text{ is a}$

$\text{definable relation over } Y \wedge \sim(\exists x)\sim[x \varepsilon \text{ Domain } R \equiv$

$x = n] \wedge \sim(\exists x)\sim[x \varepsilon X \equiv (x \varepsilon Y \wedge (\exists f)(f \varepsilon R \wedge f(n)=x))]\}$

Remark: In the above we have used a few additional minor but obvious abbreviations.

This approach to first order definability using partial relations is due to Smullyan. Intuitively, if we have the formula $X(x_2, x_4, x_5)$ which is true over the set Y for an instance $x_2 = a, x_4 = b, x_5 = c$, we can consider instead of the instance the partial function f with domain {2, 4, 5} such that f(2) = a, f(4) = b, f(5) = c. Instead of the formula X itself, we can consider the collection of all partial functions with domain {2, 4, 5} which represent true instances of X as above. This collection is called a partial relation.

We leave to the reader the verification of the fact that classically (X is definable over Y) does indeed represent first order definability. In the next sections we consider to what extent it represents it in our intuitionistic models. We also leave to the reader such elementary facts as

$ZF \vdash_I$ R is atomic over X $\supset$ partrel R

$ZF \vdash_I$ partrel S $\wedge$ R is not-S $\supset$ partrel R

$ZF \vdash_I$ partrel S $\wedge$ partrel T $\wedge$ R is S-and-T $\supset$ partrel R

etc,

Section 2

Adequacy of the definability formula

In this section we state two theorems of considerable use, whose classical analogs are reasonably intuitive. For the intuitionistic case the theorems are less obvious. The proofs are tedious and we relegate them to an appendix.

Theorem: Let $\langle G, R, \models, S\rangle$ be ordinalized and suppose for some $\Gamma \varepsilon G$ and some $g, f \varepsilon S$, $\Gamma \models f$ is definable over g. Then there is some Γ^* and some dominant formula $X(x)$ with no universal quantifiers such that

1) every quantifier of X is bound to g
2) if a is a constant of X other than a quantifier bound, $\Gamma^* \models (a \varepsilon g)$.
3) $\Gamma^* \models \sim(\exists x)\sim[x\varepsilon f \equiv (x\varepsilon g \wedge X(x))]$

Theorem 2: Let $\langle G, R, \models, S\rangle$ be ordinalized and $f, g \varepsilon S$. Suppose $X(x)$ is a formula with no universal quantifiers such that for some $\Gamma \varepsilon G$,

1) every quantifier of X is bound to g.
2) if a is a constant of X other than a quantifier bound $\Gamma \models \sim\sim(a\varepsilon g)$
3) Γ $\sim(\exists x)\sim[x\varepsilon f \equiv (x\varepsilon g \wedge X(x))]$

Then $\Gamma \models \sim\sim(f$ is definable over $g)$.

<u>Corollary:</u> (to theorem 1) Let $\langle G, R, \models, S\rangle$ be ordinalized, $g \varepsilon S_{\alpha}$, and $\Gamma \models f$ is definable over g. Then for some $k \varepsilon S_{\alpha+1} - S_{\alpha}$ and some Γ^*, $\Gamma^* \models (f = k)$.

<u>Proof:</u> $\Gamma \models f$ is definable over g, so there is a dominant formula $X(x)$ and a Γ^* as in theorem 1 above.

Suppose the constants of $X(x)$ other than g are $a_1, a_2, \ldots, a_n$.

$\Gamma^* \models (a_1 \varepsilon g)$ so there is an $h_1 \varepsilon S_{\alpha}$ such that $\Gamma^* \models (a_1 = h_1)$. Similarly we find $h_2, \ldots, h_n \varepsilon S_{\alpha}$ for $a_2, \ldots, a_n$. Let X' be

$$X\begin{pmatrix} a_1 & \ldots & a_n \\ h_1 & \ldots & h_n \end{pmatrix}$$

By weak substitutivity of equality,

$$\Gamma^* \models \sim(\exists x)\sim[X(x) \equiv X'(x)]$$

Let $Y(x)$ be $X'(x) \wedge x \varepsilon g$. Then all constants of Y are in S_{α}. Let $k_Y \varepsilon S_{\alpha+1} - S_{\alpha}$. We claim $\Gamma^* \models (k_Y = f)$. We leave the verification of this to the reader, after noting that by a classical-intuitionistic argument we have $\Gamma' \models f \subseteq g$ and $g \varepsilon S_{\alpha}$.

Section 3

ω-dominance

This definition of ω-dominance is not to be confused with that of section 16 chapter 7, which was used only that section.

We consider only ordinalized models. We call a formula $X(x_1, \ldots, x_n)$ with no constants ω-dominant if for any $\alpha \varepsilon V$ such that $\hat{\omega} \varepsilon S_\alpha$, and for any constants $c_1, \ldots, c_n \varepsilon S_\alpha$, $\Gamma \models_\alpha X(c_1, \ldots, c_n)$ iff $\Gamma \models X(c_1, \ldots, c_n)$.

We wish to show all the formulas of section 1 are ω-dominant.

Lemma: If $\langle G, R, \models, S \rangle$ is ordinalized, $\sim(\exists x) \sim [x \varepsilon \hat{\omega} \equiv \text{integer } (x)]$ is valid.

Proof: Suppose $\Gamma \models (a \varepsilon \hat{\omega})$. Then for any Γ^*, $\Gamma^* \models (a \varepsilon \hat{\omega})$. But $\Gamma^* \models$ ordinal a so there is some Γ^{**} and some ordinal α such that $\Gamma^{**} \models (a = \hat{\alpha})$. Then $\Gamma^{**} \models \sim\sim(\hat{\alpha} \varepsilon \hat{\omega})$. Then it must be that $\alpha \varepsilon \omega$, hence α is some integer n. Thus, $\Gamma^{**} \models (a = \hat{n})$. But $\Gamma^{**} \models$ integer $(\hat{n})$ so $\Gamma^{**} \models \sim\sim$ integer (a). Thus $\Gamma \models \sim\sim$ integer (a).

Conversely, if $\Gamma \models$ integer (a), for any Γ^*, $\Gamma^* \models$ integer (a). Then there is some Γ^{**} and some integer n such that $\Gamma^{**} \models (a = \hat{n})$. But $n \varepsilon \omega$ so $\Gamma^{**} \models (\hat{n} \varepsilon \hat{\omega})$. Thus $\Gamma^{**} \models \sim\sim(a \varepsilon \hat{\omega})$, $\Gamma \models \sim\sim(a \varepsilon \hat{\omega})$.

Since Γ is arbitrary, the result follows.

Q.E.D.

Now, replace in all the formulas of section 1, integer x by $x \varepsilon \hat{\omega}$. By the above lemma, the resulting formulas are weakly equivalent to the originals (i.e. their negations are equivalent) which is sufficient for our purposes.

We call a formula with constants dominant if the corresponding formula with free variables replacing the constants is dominant.

We leave it to the reader to show the formulas produced above are dominant. For example, partfun (f) is function (f) $\wedge$ $(\exists n)$(intger (n) $\wedge$ domain (f) $\subseteq$ n). This becomes function (f) $\wedge$ $(\exists n)(n \varepsilon \hat{\omega}$ $\wedge$ domain (f) $\subseteq$ n), and the corresponding formula with no constants is function (y) $\wedge$ $(\exists n)(n \varepsilon x$ $\wedge$ domain (y) $\subseteq$ n), which is dominant.

It then follows that the formulas of section 1 are ω-dominant.

Section 4

The M_α sequence

Let $(f \text{ is } M(\alpha))$ be an abbreviation for ordinal $(\alpha) \wedge \sim\sim(\exists F)\{\text{function }(F) \wedge \text{domain }(F) = \alpha' \wedge \sim(\exists x)\sim[x\varepsilon\alpha' \supset [(x = \phi \wedge F(x) = \phi) \vee (\exists y)(x = y' \wedge \sim(\exists z)\sim[z\varepsilon F(x) \equiv z \text{ is definable over } F(y)]) \vee (\text{limit ordinal }(x) \wedge \sim(\exists z)\sim[z\varepsilon F(x) \equiv (\exists w)(w\varepsilon x \wedge z\varepsilon F(w))])]] \wedge F(\alpha) = f\}$

Remark: by a classical-intuitionistic argument we have $ZF \vdash_I \sim(\exists x)(\exists y)(\exists z)\sim\{[x \text{ is } M(z) \wedge \sim(\exists w)\sim(w\varepsilon y \equiv w \text{ is definable over } x)] \supset y \text{ is } M(z')\}$.

Lemma 1: Suppose $\langle G, R, \models, S\rangle$ is ordinalized, $\hat{\omega}, \hat{\alpha}, f\varepsilon S_\beta$. and $(f \text{ is } M(\hat{\alpha}))$ is valid. Then there is some $g\varepsilon S_{\beta+2} - S_{\beta+1}$ such that $(g \text{ is } M(\widehat{\alpha+1}))$ is valid.

Proof: Let $X(x)$ be the formula $(x \text{ is definable over } f)$ and let $g_x \varepsilon S_{\beta+2} - S_{\beta+1}$. We claim $(g_x \text{ is } M(\widehat{\alpha+1}))$ is valid. Since $(x \text{ is } M(y))$ is stable, we must show $\sim\sim(g_x \text{ is } M(\widehat{\alpha+1}))$ is valid.

Using the above remark, it suffices to show $\sim(\exists w)\sim[w \varepsilon g_x \equiv w$ is definable over $f]$ is valid.

Suppose $\Gamma \models (c \varepsilon g_x)$. Since $g_x \varepsilon S_{\beta+2} - S_{\beta+1}$, $\Gamma \models (c = d) \wedge (d \varepsilon g_x)$, for some $d \varepsilon S_{\beta+1}$. So

$$\Gamma \models_{\beta+2} (d \varepsilon g_x)$$

$$\Gamma \models_{\beta+1} X(d)$$

$$\Gamma \models_{\beta+1} (d \text{ is definable over } f)$$

so by ω-dominance

$$\Gamma \models (d \text{ is definable over } f)$$

$$\Gamma \models \sim\sim(c \text{ is definable over } f).$$

Conversely, if $\Gamma \models (c$ is definable over $f)$, by the corollary in section 2, for some $d \varepsilon S_{\beta+1} - S_\beta$, $\Gamma \models (c = d)$. So $\Gamma \models \sim\sim(d$ is definable over $f)$.

and by ω-dominance,

$$\Gamma \models_{\beta+1} \sim\sim(d \text{ is definable over } f)$$

$$\Gamma \models_{\beta+1} \sim\sim X(d)$$

$$\Gamma \models_{\beta+2} \sim\sim(d \varepsilon g_x)$$

$$\Gamma \models \sim\sim(d \varepsilon g_x)$$

$$\Gamma \models \sim\sim(c \varepsilon g_x)$$

Since Γ is arbitrary, the result follows.

Q.E.D.

Lemma 2: Suppose $\langle G, R, \models, S\rangle$ is ordinalized. Let $\alpha\varepsilon V$, and let δ be the largest non-successor ordinal $\leq\alpha$. Then $\alpha = \delta+n$ for some integer $n\geq 0$. There is an $f\varepsilon S_{\delta+\omega+2n+1}$ such that (f is $M(\hat{\alpha})$) is valid in $\langle G, R, \models, S\rangle$.

Proof: By induction on α.

If $\alpha = 0$, the result becomes: there is an $f\varepsilon S_{\omega+1}$ such that (f is $M(\hat{0})$) is valid. But by a classical-intuitionistic argument,

$\sim(\exists x)\sim[\sim(\exists y)(y\varepsilon x)\supset x$ is $M(x)]$ is valid, and since $\hat{0}\varepsilon S_1$, we have $\sim\sim(\hat{0}$ is $M(\hat{0}))$ is valid, or by stability ($\hat{0}$ is $M(\hat{0})$).

Next, suppose the result is known for α. The result for $\alpha+1$ follows by lemma 1.

Finally, suppose α is a limit ordinal and the result is known for all ordinals $<\alpha$. [Here $\alpha = \delta$] We must show for some $f\varepsilon S_{\alpha+\omega+1}$, f is $M(\hat{\alpha})$ is valid. But it follows from the methods of chapter 9 that $\hat{\alpha}\varepsilon S_{\alpha+1}$, so $\hat{\alpha}\varepsilon S_{\alpha+\omega}$. Let $X(x)$ be the formula $(\exists y)(y\varepsilon\hat{\alpha}\wedge(\exists z)(z$ is $M(y)\wedge x\varepsilon z))$ and let $f_x\varepsilon S_{\alpha+\omega+1}-S_{\alpha+\omega}$. We claim ($f_x$ is $M(\hat{\alpha})$) is valid.

Since (limit ordinal $(\hat{\alpha})$) is valid, we must show $\sim(\exists x)\sim[x\varepsilon f_x \equiv (\exists y)(y\varepsilon\hat{\alpha}\wedge(\exists z)(z$ is $M(y)\wedge x\varepsilon z))]$ is valid. But this is ω-dominant, so we must show it is valid in $\langle G, R, \models_{\alpha+\omega+1}, S_{\alpha+\omega+1} \rangle$, but this follows from the validity of

$$\sim(\exists x)\sim[X(x) \equiv (\exists y)(y\varepsilon\hat{\alpha} \wedge (\exists z)(z \text{ is } M(y) \wedge x\varepsilon z))]$$

in $\langle G, R, \models_{\alpha+\omega}, S_{\alpha+\omega} \rangle$ [This is valid trivially because it is an identity].

Q.E.D

Theorem: Suppose $\langle G, R, \models, S \rangle$ is ordinalized and $\alpha\varepsilon V$. There is some $f\varepsilon S$ such that $(f \text{ is } M(\hat{\alpha}))$ is valid in $\langle G, R, \models, S \rangle$.

Section 5

Representatives of constructable sets

Somewhat as we did with ordinals in section 3 chapter 9, we associate with constructable sets elements of S which will represent them. We find it sufficient to work with general representatives, and do not single out cannonical ones.

We make the following preliminary definitions. We call a formula with no universal quantifiers E-stable if every subformula beginning with a quantifier is of the form $(\exists x)Y(x)$ where $Y(x)$ is stable.

Classically any formula is equivalent to many E-stable formulas. For a formula X, by X^y we mean the formula X with all quantifiers bound to y. That is, if a subformula of X is of the form $(\exists x)Y(x)$, the corresponding subformula of X^y has the form $(\exists x)[x \varepsilon y \wedge Y^y(x)]$. Clearly if X is E-stable, X^y has strongly bounded quantifiers and so by section 7 chapter 7, X^y is dominant.

Now suppose $\langle G, R, \models, S \rangle$ is ordinalized. Suppose we have defined representatives in S for all the elements of M_α. Let $C \varepsilon M_{\alpha+1} - M_\alpha$. Then C is a classically definable subset of M_α. Let $X(x)$ be any E-stable formula which defines C over M_α. Suppose the constants of X are $C_1, \ldots, C_n$. These are all in M_α. Let $\hat{C}_1, \ldots, \hat{C}_n$ be any representatives in S of $C_1, \ldots, C_n$ respectively, and let $\hat{X}$ be $X\begin{pmatrix} C_1 \ldots C_n \\ \hat{C}_1 \ldots \hat{C}_n \end{pmatrix}$. By the theorem of section 4, there is an $f \varepsilon S$ such that $(f \text{ is } M(\hat{\alpha}))$ is valid in $\langle G, R, \models, S \rangle$. Choose one such f. Let $Y(x)$ be the formula $[x \varepsilon f \wedge \hat{X}^f(x)]$. There are only finitely many constants in $Y(x)$. Let S_β contain them all. Consider $g_Y \varepsilon S_{\beta+1} - S_\beta$. We call g_Y a representative of the constructable set C. In this way we may associate representatives in S to every element of L, the class of constructable sets in V.

Representatives as defined are, of course, non-unique. They depend on the particular formula X chosen, on which f, on which representatives for the constants of X, and on which β. However, we will show later that if f and g both represent the same constructable set, (f = g) is valid in $\langle G, R, \models, S \rangle$.

We shall use the ambiguous notation that $\hat{C}$ is any one of the representatives of the constructable set C. Since an ordinal α is also a constructable set, $\hat{\alpha}$ is doubly ambiguous, but it will be clear from context whether we mean the ordinal or the constructable set representative. Moreover, as we show later, these two notions are closely connected.

Section 6

Properties of constructable set representatives

Let (x is constructable) be an abbreviation for the formula $(\exists z)(\exists y)(\text{ordinal } (z) \wedge y \text{ is } M(z) \wedge x \varepsilon y)$

In this section we show:

<u>Theorem 1:</u> Let $\langle G, R, \models, S\rangle$ be ordinalized and suppose for some $\Gamma \varepsilon G$, $\Gamma \models (\exists y)(y \text{ is } M(\hat{\alpha}) \wedge f \varepsilon y)$. Then there is some Γ^*, some $C \varepsilon M_\alpha$, and some $\hat{C}$ representing C such that $\Gamma^* \models (f = \hat{C})$.

<u>Corollary:</u> If $\langle G, R, \models, S\rangle$ is ordinalized and $\Gamma \models$ (f is constructable), then for some Γ^*, some constructable set C, and some representative, $\hat{C}$ of C, $\Gamma^* \models (f = \hat{C})$.

<u>Theorem 2:</u> If $\langle G, R, \models, S\rangle$ is ordinalized, $C \varepsilon M_\alpha$, and $\hat{C}$ is any representative of C, then $\sim\sim(\exists y)(y \text{ is } M(\hat{\alpha}) \wedge \hat{C} \varepsilon y)$ is valid in $\langle G, R, \models, S\rangle$.

<u>Corollary:</u> If $\langle G, R, \models, S\rangle$ is ordinalized, C is a constructable set, and $\hat{C}$ is any representative of C, $\sim\sim(\hat{C}$ is constructable) is valid in $\langle G, R, \models, S\rangle$.

<u>Proof of theorem 1:</u> By induction on α.

If $\alpha = 0$, since $M_0 = \phi$, it follows that $\sim(\exists y)(y \text{ is } M(\hat{0}) \wedge f \varepsilon y)$ is valid so the result is trivial.

Suppose the result is known for α and $\Gamma \models (\exists y)(y \text{ is } M(\widehat{\alpha+1}) \wedge f \varepsilon y)$. By a classical-intuitionistic argument, $ZF \vdash_I \sim(\exists f)(\exists \alpha)(\exists y)\sim[\text{successor ordinal } (\alpha) \wedge y \text{ is } M(\alpha) \wedge f \varepsilon y) \supset (\exists z)(\exists \beta)(\text{ordinal } (\beta) \wedge \alpha = \beta' \wedge$

z is $M(\beta) \wedge f$ is definable over $z)]$
Moreover, (successor ordinal $(\widehat{\alpha+1})$) is valid, so
$\Gamma \models \sim\sim(\exists z)(\exists \beta)(\text{ordinal }(\beta) \wedge \widehat{\alpha+1} = \beta' \wedge z$ is $M(\beta) \wedge$ f is definable over $z)$. It then follows that for some $g \varepsilon S$ and some Γ^* that $\Gamma^* \models g$ is $M(\hat{\alpha}) \wedge f$ is definable over g. But we have shown there is an $h \varepsilon S$ such that $(h$ is $M(\hat{\alpha}))$ is valid. Thus $\Gamma^* \models h$ is $M(\hat{\alpha})$ and by a classical-intuitionistic argument, $\Gamma^* \models (g = h)$. Thus $\Gamma^* \models \sim\sim(f$ is definable over $h)$. There is some Γ^{**} such that $\Gamma^{**} \models (f$ is definable over $h)$. Now by theorem 1 of section 2, there is some dominant formula $X(x)$ with only existential quantifiers, with all quantifiers bound to h, and some Γ^{***} such that if a is a constant of $X(x)$ other than a quantifier bound, $\Gamma^{***} \models (a \varepsilon h)$, and $\Gamma^{***} \models \sim(\exists x)\sim[x \varepsilon f \equiv (x \varepsilon h \wedge X(x))]$.

There are only a finite number of constants, $a_1, \ldots, a_n$ in X. Consider a_1. $\Gamma^{***} \models (a_1 \varepsilon h) \wedge (h \text{ is } M(\hat{\alpha}))$
By induction hypothesis, there is some Γ^{****} and a $C \varepsilon M_\alpha$ such that $\Gamma^{****} \models (a_1 = \hat{C}_1)$. Consider a_2 similarly, starting with Γ^{****}, and so on to a_n. Thus, we get some $\Gamma^{***\cdots*} = \Delta$ and some $C_1, \ldots, C_n \varepsilon M_\alpha$ such that $\Delta \models (a_1 = \hat{C}_1) \wedge \ldots \wedge (a_n = \hat{C}_n)$.

Now let X' be $X\binom{a_1 \ldots a_n}{\hat{c}_1 \ldots \hat{c}_n}$. Then by weak substitutivity of equality,

$\Delta \models \sim(\exists x)\sim[x\varepsilon f \equiv (x\varepsilon h \wedge X'(x))]$.

Let $Y(x)$ be the formula $x\varepsilon h \wedge X'(x)$. Let S_β contain all the constants of $Y(x)$, and f, and consider $g_Y \varepsilon S_{\beta+1} - S_\beta$. By definition, for some $C\varepsilon M_{\alpha+1}$, g_Y represents C. We claim

$\Delta \models (f = g_Y)$.

By dominance, we must show $\Delta \models_{\beta+1} (f = g_Y)$, or equivalently, $\Delta \models_\beta \sim(\exists x)\sim[x\varepsilon f \equiv Y(x)]$ or

$\Delta \models_\beta \sim(\exists x)\sim[x\varepsilon f \equiv (x\varepsilon h \wedge X'(x))]$. But this is dominant so we must show

$\Delta \models \sim(\exists x)\sim[x\varepsilon f \equiv (x\varepsilon h \wedge X'(x))]$ which we have.

If α is a limit ordinal, the result is trivial.

Q.E.D.

<u>Lemma for theorem 2:</u> Suppose $\langle G, R, \vdash, S\rangle$ is ordinalized. Suppose that for any $C\varepsilon M_\alpha$, for any representative $\hat{C}$ of C, $\sim\sim(\exists y)(y \text{ is } M(\hat{\alpha}) \wedge \hat{C}\varepsilon y)$ is valid in $\langle G, R, \models, S\rangle$. Then for any $C\varepsilon M_{\alpha+1}$, for any representative $\hat{C}$ of C, $\sim\sim(\exists y)(y \text{ is } M(\widehat{\alpha+1}) \wedge \hat{C}\varepsilon y)$ is valid.

Proof: Let $C \epsilon M_{\alpha+1}$ and let $\hat{C}$ represent C. Since $\hat{C}$ represents C, $\hat{C}$ is $f_Y \epsilon S_{\gamma+1} - S_\gamma$ where Y(x) is $(x\epsilon h \wedge \hat{X}^h(x))$ where X(x) is E-stable, X(x) defines C classically over M_α, and $(h \text{ is } M(\hat{\alpha}))$ is valid in $\langle G, R, \models, S \rangle$.

But $\sim(\exists x)\sim[x\epsilon\hat{C} \equiv (x\epsilon h \wedge \hat{X}^h(x))]$ is valid [remember, $\hat{X}^h(x)$ is dominant, and $h\epsilon S_\gamma$]. Moreover, suppose a is some constant of $\hat{X}^h(x)$ other than a quantifier bound. By definition, a must represent some element of M_α, so by hypothesis, $\sim\sim(\exists y)(y \text{ is } M(\hat{\alpha}) \wedge a\epsilon y)$ is valid. But again $(h \text{ is } M(\hat{\alpha}))$ is valid, so by a classical-intuitionistic argument, $\sim\sim(a\epsilon h)$ is valid. Now by theorem 2 section 2, $\sim\sim(\hat{C}$ is definable over h) is valid and $\widehat{\alpha+1} = \alpha'$ is valid so by another classical-intuitionistic argument, $\sim\sim(\exists y)(y \text{ is } M(\widehat{\alpha+1}) \wedge \hat{C}\epsilon y)$ is valid.

Q.E.D.

Now theorem 2 follows by a straightforward induction on α.

Section 7

The principal result

This section is devoted to showing the following:

Theorem: Let $\langle G, R, \models, S \rangle$ be ordinalized. Then

1) If $C, D \varepsilon L$, and $\hat{C}, \hat{D}$ are representatives of C, D respectively, then $C \varepsilon D$ iff $\sim\sim(\hat{C} \varepsilon \hat{D})$ is valid. and $C \not\varepsilon D$ iff $\sim(\hat{C} \varepsilon \hat{D})$ is valid.
2) If f and g both represent the same constructable set, $(f = g)$ is valid.
3) If f represents the ordinal α in an ordinal sense and g represents α in a constructable set sense, $(f = g)$ is valid.

We proceed with the proof.

Lemma: Let $\langle G, R, \models, S \rangle$ be ordinalized. Let X be an E-stable formula with no universal quantifiers, with all quantifiers bound to M_α, and with all constants other than quantifier bounds elements of M_α. By X' we mean (in this lemma) any formula which is like X except for having some representative, $\hat{C}$, in place of C, for every non-quantifier-bounding constant of X, and having all its quantifiers bound to h instead of M_α, where

$h \varepsilon S$ is such that $(h \text{ is } M(\hat{\alpha}))$ is valid. Then for the following to hold for all such formulas X, it is sufficient that they hold for atomic X:

X is true over M_α $\Rightarrow$
$\sim\sim X'$ is valid in $\langle G, R, \models, S\rangle$
X is false over M_α $\Rightarrow$
$\sim X'$ is valid in $\langle G, R, \models, S\rangle$

Proof: By induction on the degree of X. Suppose the result is known for all formulas of degree less than that of X. We have five cases.

Since $(Y \wedge Z)' = Y' \wedge Z'$
$(Y \vee Z)' = Y' \vee Z'$
$(\sim Y)' = \sim Y'$
$(Y \supset Z)' = Y' \supset Z'$

the four propositional cases follow easily.

Suppose X is $(\exists x)(x \varepsilon M_\alpha \wedge Y(x))$ [where $Y(x)$ is stable] and the result is known for Y. X' is $(\exists x)(x \varepsilon h \wedge Y'(x))$

X is true over M_α $\Rightarrow$

for some $C \varepsilon M_\alpha$, $Y(C)$ is true. But then by induction hypothesis, $\sim\sim Y'(\hat{C})$ is valid (for any representative $\hat{C}$). Since $C \varepsilon M_\alpha$, by theorem 2 section 6, $\sim\sim(\exists y)(y \text{ is } M(\hat{\alpha}) \wedge \hat{C} \varepsilon y)$ is valid. It follows that $\sim\sim(\hat{C} \varepsilon h)$ is valid. Thus $\sim\sim(\hat{C} \varepsilon h) \wedge \sim\sim Y'(\hat{C})$ is valid, which implies $\sim\sim(\exists x)(x\varepsilon h \wedge Y'(x))$ is valid, i.e. $\sim\sim X'$.

Conversely, X is false over M_α $\Rightarrow$ for every $C\varepsilon M_\alpha$ $Y(C)$ is false over M_α. Suppose for some Γ, $\Gamma \not\models \sim X'$. Then for some Γ^*, $\Gamma^* \models X'$

$$\Gamma^* \models (\exists x)(x\varepsilon h \wedge Y'(x))$$

For some $a\varepsilon S$

$$\Gamma^* \models (a\varepsilon h \wedge Y'(a))$$

But $\Gamma^* \models h$ is $M(\hat{\alpha})$ so by theorem 1 section 6, for some $C\varepsilon M_\alpha$ and some Γ^{**}, $\Gamma^{**} \models (a = \hat{C})$

$$\Gamma^{**} \models \sim\sim Y'(\hat{C})$$

But by hypothesis, $\sim Y'(\hat{C})$ is valid. Thus $\sim X'$ is valid.

Q.E.D.

Now we show part 1 of the theorem. The proof is by induction on the order of D [D is of order α if $D\varepsilon M_{\alpha+1}-M_\alpha$].

Suppose D is of order α and the result is known for all constructable sets of lower order. $D\varepsilon M_{\alpha+1}-M_\alpha$ so D is a definable subset of M_α. Let $\hat{D}$ be some corresponding element $f_Y \varepsilon S_{\beta+1}-S_\beta$, where

$Y(x)$ is the formula $(x\varepsilon h \wedge \hat{X}^h(x))$, where $(h \text{ is } M(\hat{\alpha}))$ is valid, and X defines D over M_α.

$C\varepsilon D$ iff $X(C)$ is true over M_α. By induction hypothesis, the conclusion of the above lemma is known for all atomic formulas over M_α, and hence for all formulas. Thus

$C\varepsilon D \Rightarrow X(C)$ is true over M_α

$\Rightarrow \sim\sim X'(\hat{C})$ is valid

But $C\varepsilon M_\alpha$ and $(h \text{ is } M(\hat{\alpha}))$ is valid so $\sim\sim(\hat{C}\varepsilon h)$ is valid. Thus $\sim\sim[\hat{C}\varepsilon h \wedge \hat{X}^h(\hat{C})]$ is valid. By dominance, $\sim\sim[\hat{C}\varepsilon h \wedge \hat{X}^h(\hat{C})]$ is valid in $\langle G, R, \models_\beta, S_\beta \rangle$, that is $\sim\sim Y(\hat{C})$. Then $\sim\sim(\hat{C}\varepsilon\hat{D})$ is valid in $\langle G, R, \models_{\beta+1}, S_{\beta+1} \rangle$ and hence in $\langle G, R, \models, S \rangle$.

The second half is similar, and the result follows.

Next we show part 2. Suppose f and g both represent the same constructable set $D\varepsilon M_{\alpha+1} - M_\alpha$. Suppose $\Gamma \models (a\varepsilon f)$. Since $D\varepsilon M_{\alpha+1}$, by theorem 2 section 6, $\Gamma \models \sim\sim(\exists y)(y \text{ is } M(\widehat{\alpha+1}) \wedge f\varepsilon y)$. By a classical-intuitionistic argument, $\Gamma \models \sim\sim(\exists y)(y \text{ is } M(\hat{\alpha}) \wedge a\varepsilon y)$. Then for any Γ^*, $\Gamma^* \models \sim\sim(\exists y)(y \text{ is } M(\hat{\alpha}) \wedge a\varepsilon y)$. Now by theorem 1 section 6, there is some $C\varepsilon M_\alpha$ and some Γ^{**} such that $\Gamma^{**} \models (a = \hat{C})$. But then $\Gamma^{**} \models \sim\sim(\hat{C}\varepsilon f)$, so by part 1 of the theorem, $C \varepsilon D$ is true [since f represents D] But since g also represents D, $\Gamma^{**} \models \sim\sim(\hat{C} \varepsilon g)$. So $\Gamma^{**} \models \sim\sim(a \varepsilon g)$, $\Gamma \models \sim\sim(a \varepsilon g)$. Since Γ is arbitrary and the argument with f and g is symmetric, part 2 holds.

Finally, to show part 3, we proceed by induction on the ordinal α.

Suppose the result is known for all $\beta < \alpha$. Let $O(\alpha)$ be some ordinal representative of α, and $C(\alpha)$ be some constructable set representative.

If $\Gamma \models a \varepsilon O(\alpha)$, for any Γ^*, $\Gamma^* \models a \varepsilon O(\alpha)$. But $\Gamma^* \models$ ordinal $O(\alpha)$ so $\Gamma^* \models$ ordinal a. Now by the results of chapter 9, there is an ordinal β and a Γ^{**} such that $\Gamma^{**} \models a = O(\beta)$. Thus $\Gamma^{**} \models O(\beta) \varepsilon O(\alpha)$ so it must be the case that $\beta \varepsilon \alpha$. But then, by part 1 above, $\Gamma^{**} \models C(\beta) \varepsilon C(\alpha)$, and by induction hypothesis, $\Gamma^{**} \models O(\beta) = C(\beta)$. Thus $\Gamma^{**} \models \sim\sim(O(\beta) \varepsilon C(\alpha))$ $\Gamma^{**} \models \sim\sim(a \varepsilon C(\alpha))$ so $\Gamma \models \sim\sim(a \varepsilon C(\alpha))$. Since Γ is arbitrary, $O(\alpha) \subseteq C(\alpha)$ is valid. The converse inclusion is similar.

Q.E.D.

CHAPTER 12

Independence of the Axiom of Constructability

Section 1

The specific model

Once again the model presented is adapted from Cohen [2]. Let e and a be formal symbols. By a forcing condition we mean any finite consistent set of statements of the form (nea) or $\sim(nea)$, for any integer n.

Let G be the collection of all forcing conditions, and let R be $\subseteq$, set inclusion.

S_0 consists of the functions $\hat{0}, \hat{1}, \hat{2}, \ldots,$ and a. The definitions are as follows: For each integer n, $\hat{n}$ has as domain $\{\hat{0}, \hat{1}, \ldots, \widehat{n-1}\}$, and if $m<n$, $\hat{n}(\hat{m}) = G$. a has as domain $\{\hat{0}, \hat{1}, \hat{2}, \ldots\}$, and

$$a(\hat{n}) = \{\Gamma \mid (n\varepsilon a)\varepsilon\Gamma\}.$$

Then $\models_0$ for atomic formulas is simply

$$\Gamma \models_0 (\hat{m}\varepsilon\hat{n}) \quad \text{if} \quad m\varepsilon n$$

$$\Gamma \models_0 (\hat{n}\varepsilon a) \quad \text{if} \quad (n\varepsilon a)\varepsilon\Gamma.$$

We leave to the reader the verification that $\langle G, R, \models_0, S_0 \rangle$ satisfies the five properties of section 3 chapter 7. Property 4 is shown just as in chapter 8 or 10.

Thus, $\langle G, R, \models, S \rangle$ is an intuitionistic ZF model. We also leave to the reader the straightforward verification that $\langle G, R, \models, S \rangle$ is ordinalized.

Section 2

Axiom of constructability

Theorem: $(\exists x) \sim [x$ is constructable] is valid in $\langle G, R, \models, S \rangle$.

Proof: We show in in particular that $\sim$(a is constructable) is valid.

Suppose for some $\Gamma \varepsilon G$, $\Gamma \models$ (a is constructable). By the corollary to theorem 1, section 6 chapter 11, for some constructable set $C \varepsilon V$ and some Γ^*, Γ^* $(a = \hat{C})$. We will show this is not possible.

Let Γ^{*+} be $\{n | (n \varepsilon a) \varepsilon \Gamma^*\}$. We have two cases.

Case 1: every integer of C is in Γ^{*+}. Choose some integer n such that $(n \varepsilon a)$ is not in Γ^*. [recall Γ^* is finite]. Let Γ^{**} be

$\Gamma^* \cup \{(n\varepsilon a)\}$. Then $\Gamma^{**}\varepsilon G$ and $\Gamma^* R \Gamma^{**}$. But $n \not\varepsilon C$ so $\Gamma^{**} \models \sim(\hat{n}\varepsilon\hat{C})$. Since $(n\varepsilon a)\varepsilon\Gamma^{**}$, $\Gamma^{**} \models (\hat{n}\varepsilon a)$, which is not possible.

<u>Case 2</u>: some integer of C is not in Γ^{*+}. Let n be such an integer. Let Γ^{**} be $\Gamma^* \cup \{\sim(n\varepsilon a)\}$. Again $\Gamma^{**}\varepsilon G$ and $\Gamma^* R \Gamma^{**}$. But $n\varepsilon C$ so $\Gamma^{**} \models \sim\sim(\hat{n}\varepsilon\hat{C})$. Since $\sim(n\varepsilon a)\varepsilon\Gamma^{**}$ it follows easily that $\Gamma^{**} \models \sim(\hat{n}\varepsilon a)$ which is again impossible.

Hence $\Gamma \not\models$ (a is constructable) and since Γ is arbitrary, the theorem follows.

Q.E.D.

Now we have classical independence by the results of section 1 chapter 7. In chapter 13 we will show that the axiom of choice and the generalized continuum hypothesis are both valid in this model, so the full independence is established.

CHAPTER 13

Additional Results

Section 1

S_α representatives

Def: We say $s \varepsilon S$ represents S_α if

1) $g \varepsilon S_\alpha$ implies $\sim\sim(g \varepsilon s)$ is valid in $\langle G, R, \models, S \rangle$

2) if $\Gamma \models (g \varepsilon s)$ then for some Γ^* and some $h \varepsilon S_\alpha$, $\Gamma^* \models (g = h)$

Lemma 1: Suppose $X(x_1, \ldots, x_n)$ is a formula with no universal quantifiers, and with all constants from S_α. Then for any $c_1, \ldots, c_n \varepsilon S_\alpha$ and any $\Gamma \varepsilon G$,

$$\Gamma \models_\alpha \sim X(c_1, \ldots, c_n) \quad \text{iff}$$
$$\Gamma \models \sim X^s(c_1, \ldots, c_n)$$

[X^s is X relativized to s]

Proof: A straightforward induction on the degree of X.

<u>Lemma 2:</u> Suppose s represents S_α. Then for any $f \varepsilon S$,

1) If $f \varepsilon S_{\alpha+1}$, $\sim\sim$(f is definable over s) is valid

2) If $\Gamma \models$ (f is definable over s) then for some Γ^* and some $h \varepsilon S_{\alpha+1}$, $\Gamma^* \models (f = h)$

<u>Proof:</u> Suppose $f \varepsilon S_{\alpha+1}$. If $f \varepsilon S_\alpha$, the result is simple. If $f \varepsilon S_{\alpha+1} - S_\alpha$, then f is f_X for some formula X over S_α. We claim

$$\sim(\exists x)\sim[x \varepsilon f_X \equiv (x \varepsilon s \wedge X^s(x))]$$

is valid in $\langle G, R, \models, S \rangle$. We leave this to the reader, using the above lemma. It then follows by theorem 2 section 2 chapter 11, that $\sim\sim$(f is definable over s) is valid.

Suppose conversely that

$$\Gamma \models (\text{f is definable over s})$$

By theorem 1 section 2 chapter 11, there is some Γ^* and a dominant formula $X(x)$ with no universal quantifiers, bound to s, with every non-quantifier-bounding constant a such that $\Gamma^* \models (a \varepsilon s)$ such that

$$\Gamma^* \models \sim(\exists x)\sim[x \varepsilon f \equiv (x \varepsilon s \wedge X(x))]$$

Now for any a of $X(x)$, $\Gamma^* \models (a \varepsilon s)$ so for some $a' \varepsilon S_\alpha$ and some Γ^{**}, $\Gamma^{**} \models (a = a')$. Similarly with all constants of $X(x)$ (other than s). Thus we have $\Delta = \Gamma^{**\cdots*}$ such that if b is any constant of $X(x)$

other than s, there is some $b' \varepsilon S_\alpha$ such that $\Delta \models (b = b')$. Now let X' be like X except for containing $a' \varepsilon S_\alpha$ for each a of X. Then it follows that

$$\Delta \models \sim(\exists x)\sim[x \varepsilon f \equiv (x \varepsilon s \wedge X'(x))]$$

Let X'' be like X' except for having unbounded quantifiers. Then X'' is a formula over S_α. Let $h_{X''} \varepsilon S_{\alpha+1} - S_\alpha$. We claim $\Delta \models (f = h_{X''})$.

This follows immediately by lemma 1.

Q.E.D.

Lemma 3: If s represents S_α and t represents $S_{\alpha+1}$, then

$$\sim(\exists x)\sim[x \varepsilon t \equiv x \text{ is definable over } s]$$

is valid if $\langle G, R, \models, S \rangle$.

Proof: By lemma 2 and the definition.

Lemma 4: If s represents S_α and $\sim(\exists x)\sim[x \varepsilon t \equiv x \text{ is definable over } s]$ is valid in $\langle G, R, \models, S \rangle$, then t represents $S_{\alpha+1}$

Proof: Again straightforward.

Remark: Every S_α is, of course, representable. Let X(x) be the formula $x = x$ and let $f_X \varepsilon S_{\alpha+1} - S_\alpha$.

Then f_x represents S_α.

Section 2

Definition functions

Let (F is a β length s function) be an abbreviation for function (F) $\wedge$ ordinal (β) $\wedge$ domain $F = \beta \wedge \sim(\exists\gamma)\sim\{\gamma\varepsilon\beta \supset [(\gamma = \phi \wedge F(\gamma) = s) \vee (\exists\delta)$ $[\delta\varepsilon\gamma \wedge \gamma = \delta' \wedge \sim(\exists x)\sim(x\varepsilon F(\gamma) \equiv x \text{ is definable over } F(\delta))] \vee [\text{limit ordinal } (\gamma) \wedge \sim(\exists x)\sim(x\varepsilon F(\gamma) \equiv (\exists\delta)$ $(\delta\varepsilon\gamma \wedge x\varepsilon F(\delta)))]\}$

The following is left to the reader.

Lemma: If $\Gamma \models [(\beta\varepsilon\gamma) \wedge F$ is a β length s function $\wedge$ G is a γ length s function] then $\Gamma \models (F \subseteq G)$.

For the rest of this section we assume our models are ordinalized.

Lemma: Let $s\varepsilon S_1 - S_0$ represent S_0. Then for any $\beta \geq 0$ there is an $F\varepsilon S_{\beta+3} - S_{\beta+2}$ such that [F is a $\widehat{\beta+1}$ length s function] is valid in $\langle G, R, \models, S\rangle$, and for any $\gamma < \beta$, if $\Gamma \models (h = F(\hat{\gamma}))$ then h represents S_γ.

Proof: By induction on β.

If $\beta = 0$, let $X(x)$ be the formula $x = \langle \hat{0}, s \rangle$ and consider $F_x \varepsilon S_3 - S_2$.

Suppose the result is known for β. Then there is an $F \varepsilon S_{\beta+3} - S_{\beta+2}$ satisfying the lemma. Let $f \varepsilon S_{\beta+2} - S_{\beta+1}$ represent $S_{\beta+1}$. Let $X(x)$ be the formula $x \varepsilon F \vee x = \langle \widehat{\beta+1}, f \rangle$ and let $G_x \varepsilon S_{\beta+4} - S_{\beta+3}$.

If β is a limit ordinal and the result is known for all lesser ordinals, let $X(x)$ be the formula $(\exists \gamma)(\exists F)(\gamma \varepsilon \hat{\beta} \wedge F$ is a γ length s function $\wedge x \varepsilon F)$ and let $G_x \varepsilon S_{\beta+3} - S_{\beta+2}$.

We leave verifications to the reader.

Q.E.D.

Theorem: Let $s \varepsilon S_1 - S_0$ represent S_0. Then $\sim(\exists x) \sim (\exists \beta)(\exists F)[F$ is a β' length s function $\wedge$ $x \varepsilon F(\beta)]$ is valid in $\langle G, R, \models, S \rangle$.

Section 3

Restriction on ordinals representable

We devote this section to a brief sketch of the proof of

<u>Theorem:</u> Suppose $\langle G, R, \models_\Omega, S_\Omega \rangle$ is itself an ordinalized intuitionistic Z-F model, where $\Omega>0$. Then exactly the ordinals $<\Omega$ are representable in S_Ω.

<u>Proof:</u> Trivially Ω must be a limit ordinal, so by the work of chapter 9, at least the ordinals $<\Omega$ are representable in S_Ω. We show now that $\hat{\Omega} \notin S_\Omega$.

Since $\Omega>0$ there is an $s \varepsilon S_1 - S_0$ (and hence $s \varepsilon S_\Omega$) such that s represents S_0 (see section 1). By the work in section 2, the following is valid in $\langle G, R, \models_\Omega, S_\Omega \rangle$:

$\sim(\exists x)\sim(\exists \beta)(\exists F)$ [F is a β' length s function $\wedge x \varepsilon F(\beta)$].

Suppose $\hat{\Omega} \varepsilon S_\Omega$. It then follows that

1) $\sim(\exists x)\sim(\exists \beta \varepsilon \hat{\Omega})(\exists F)$ [F is a β' length s function $\wedge$ $x \varepsilon F(\beta)$] is valid in $\langle G, R, \models_\Omega, S_\Omega \rangle$.

Moreover, β-length s functions form a chain, that is, the following is valid in $\langle G, R, \models_\Omega, S_\Omega \rangle$:

$\sim(\exists \alpha \varepsilon \hat{\Omega})(\exists \beta \varepsilon \hat{\Omega})(\exists F)(\exists G)\sim[(\alpha \varepsilon \beta \wedge$ F is an α length s function $\wedge$ G is a β length s function) $\supset F \subseteq G$]

(see section 2)

It then follows that the following is valid in $\langle G, R, \models_\Omega, S_\Omega \rangle$ (using obvious abbreviations)

2) $\sim\sim(\exists y)(y = \bigcup\{F | F$ is a β'-length s function, for $\beta \varepsilon \hat{\Omega}\})$

From 1) and 2) the validity of $\sim\sim(\exists z)\sim(\exists x)\sim(x \varepsilon z)$ follows, which is not possible.

Q.E.D.

Section 4

A classical connection

The result of section 7 chapter 11 may be extended to

Theorem 1: Suppose $\langle G, R, \models, S\rangle$ is ordinalized. Let X be any formula with no universal quantifiers, no free variables, and all constants from L. Let X' be like X except for having constants $\hat{C}$ where X has C, and having all its quantifiers bound to the formula (x is constructable).
Then

X is true over L iff $\sim\sim X'$ is valid in $\langle G, R, \models, S\rangle$.

X is false over L iff $\sim X'$ is valid in $\langle G, R, \models, S\rangle$.

Proof: By induction on the degree of X. If X is atomic, the result is the theorem of section 7 chapter 11.

Suppose the result is known for all formulas of degree less than that of X. The four cases X is $Y \supset Z$, $\sim Y$, $Y \vee Z$, or $Y \wedge Z$ are simple.

Suppose X is $(\exists x)Y(x)$. Then X' is $(\exists x)($ x is constructable $\wedge\ Y'(x))$. If X is true over L, for some $C \varepsilon L$, $Y(C)$ is true over L. By induction hypothesis, $\sim\sim Y'(\hat{C})$ is valid. But by corollary theorem 2 section 6 chapter 11, $\sim\sim(\hat{C}$ is constructable) is also valid. Hence $(\exists x)(\sim\sim x$ is constructable $\wedge\ \sim\sim Y'(x))$ is valid. But this implies $\sim\sim(\exists x)(x$ is constructable $\wedge\ Y'(x))$ is vlaid, i.e. $\sim\sim X'$.

Conversely, suppose X is false over L. Then $Y(C)$ is false over L for every $C \varepsilon L$. By induction hypothesis, $\sim Y'(\hat{C})$ is valid, for every $C \varepsilon L$. Now suppose for some $\Gamma \varepsilon G$, $\Gamma \not\models \sim X'$. Then for some Γ^*, $\Gamma^* \models X'$ or $\Gamma^* \models (\exists x)(x$ is constructable $\wedge\ Y'(x))$. For some $a \varepsilon S$, $\Gamma^* \models (a$ is constructable $\wedge\ Y'(a))$. By corollary theorem 1 section 6 chapter 11, for some Γ^{**} and some $C \varepsilon L$, $\Gamma^{**} \models (a = \hat{C})$, so $\Gamma^{**} \models \sim\sim Y'(\hat{a})$, a contradiction.

Q.E.D.

<u>Remark:</u> Suppose $\langle G, R, \models_\Omega, S_\Omega \rangle$ were itself an ordinalized intuitionistic ZF model. We showed in section 3 that exactly the ordinals $<\Omega$ are representable in S_Ω. It then follows that for any $C \varepsilon M_\Omega$, $\hat{C} \varepsilon S_\Omega$ and conversely. This may be shown by adapting the methods of chapter 11. Now the above theorem may be restricted to

<u>Theorem 2:</u> Suppose $\langle G, R, \models_\Omega, S_\Omega \rangle$ is an ordinalized intuitionistic ZF model. Let X and X' be as above, save that X has constants only from M_Ω Then

X is true over M_Ω iff

$\sim\sim X'$ is valid in $\langle G, R, \models_\Omega, S_\Omega \rangle$

X is false over M_Ω iff

$\sim X'$ is valid in $\langle G, R, \models_\Omega, S_\Omega \rangle$

<u>Proof:</u> This may be shown exactly as theorem 1 was shown. It is simple to establish that the theorem of section 7 chapter 11, relativizes to $\langle G, R, \models_\Omega, S_\Omega \rangle$ in the obvious manner.

Q.E.D.

Section 5

Sets which are models

Classically, certain of the M_α themselves may be Z-F models. For example, M_Ω, where Ω is the first inaccessible cardinal, is such a model. We now examine the intuititonistic counterpart.

Theorem 1: Suppose M_α is a classical Z-F model, and $\langle G, R, \models_0, S_0\rangle \in M_\alpha$. Then $\langle G, R, \models_\alpha, S_\alpha\rangle$ is an intuitionistic Z-F model.

Proof: In the proofs of chapter 7, V was any arbitrary classical ZF model. If we take V to be M_α, all the results still hold. But now, the class model $\langle G, R, \models, S\rangle$ with respect to M_α is actually $\langle G, R, \models_\alpha, S_\alpha\rangle$.

Q.E.D.

Theorem 2: Suppose $\langle G, R, \models_\alpha, S_\alpha\rangle$ is an ordinalized intuitionistic ZF model. Then M_α is a classical ZF model.

Proof: Let X be any ZF axiom stated with no universal quantifiers. Since X has no constants, X' as in theorem 2 section 4, is simply X relativized to the constructable sets. It is shown in the course of the Gödel consistency proofs that $ZF \vdash_C X'$ (for example, see [2]). Hence, as usual, $ZF \vdash_I \sim\sim X'$. Thus, $\sim\sim X'$ is valid in $\langle G, R, \models_\alpha, S_\alpha \rangle$. Now if X were not true over M_α, by theorem 2 section 4, $\sim X'$ would be valid in $\langle G, R, \models_\alpha, S_\alpha \rangle$. Hence X is true over M_α.

Q.E.D.

Section 6

Restriction on cardinals representable

In section 8 chapter 9, we called $\langle G, R, \models, S \rangle$ cardinalized if all the cardinals of V were cardinals of S. We now want to verify the remark made there that the cardinals of S were the same as the cardinals of L. More precisely,

Theorem: Suppose $\langle G, R, \models, S \rangle$ is ordinalized and for some $\alpha \varepsilon V$ and some $\Gamma \varepsilon G$, $\Gamma \models (\text{cardinal } (\hat{\alpha}))$. Then α is a cardinal of L, the class of constructable sets of V.

Proof: Suppose α is not a cardinal of L. Then for some $\beta \varepsilon \alpha$ and some $F \varepsilon L$ the following formula is true over L: $[\text{function } (F) \wedge 1\text{-}1(F) \wedge \text{domain } (F) = \beta \wedge \text{range } (F) = \alpha]$. But $\beta \varepsilon \alpha$ so $\sim\sim(\hat{\beta} \varepsilon \hat{\alpha})$ is valid in

$\langle G, R, \models, S \rangle$. By theorem 1 section 4, $\sim\sim$[function (F) $\wedge$ 1-1 (F) $\wedge$ domain (F) = $\beta \wedge$ range (F) = α]' is valid in $\langle G, R, \models, S \rangle$. But this is $\sim\sim[\text{function}^L (\hat{F}) \wedge \text{1-1}^L (\hat{F}) \wedge \text{domain}^L \hat{F} = \hat{\beta} \wedge \text{range}^L \hat{F} = \hat{\alpha}]$ where the superscript L means the formula has been relativized to (x is constructable). But classically,

$$ZF \vdash_C \sim(\exists x)\sim[(x \text{ is constructable} \wedge \text{function}^L (x)) \supset \text{function } (x)]$$

and similarly for 1-1, domain,and range. By corollary theorem 2 section 6 chapter 11, $\sim\sim(\hat{F}$ is constructable) $\wedge \sim\sim(\hat{\alpha}$ is constructable) $\wedge \sim\sim(\hat{\beta}$ is constructable) is valid. Hence $\sim\sim$[function $(\hat{F}) \wedge$ 1-1 $(\hat{F}) \wedge$ domain $(\hat{F}) = \hat{\beta} \wedge$ range $(\hat{F}) = \hat{\alpha}$] is valid. This contradicts $\Gamma \models$ (cardinal $(\hat{\alpha})$)

Q.E.D.

Remark: In the above it does not matter whether $\hat{\alpha}$ and $\hat{\beta}$ are ordinal or constructable set representatives. See theorem section 7 chapter 11.

Section 7

Axiom of choice

By $F(X)$ we mean the collection of all classically definable subsets of the set X. Suppose we can define classically a sequence of sets as follows:

$$S_0 = X$$
$$S_{\alpha+1} = F(S_\alpha)$$
$$S_\lambda = \bigcup_{\alpha<\lambda} S_\alpha \quad \text{[limit ordinals } (\lambda)\text{]}$$

and let the class $S = \bigcup S_\alpha$. If X can be well ordered by some relation R, then it is easy to show there is a class which well orders S, or, any set in S can be well ordered. Formally, we have

$$\begin{aligned} ZF \vdash_C {\sim}{\sim}(\exists X){\sim}(\exists x){\sim}(\exists \beta)(\exists F)[(&F \text{ is a } \beta' \text{ length } X \text{ function} \wedge \\ &x \varepsilon F(\beta)) \wedge (\exists R)(R \text{ well orders } X)] \supset \\ &{\sim}(\exists y){\sim}(\exists t)(t \text{ well orders } y) \end{aligned}$$

Now by a classical-intuitionistic argument we have

Theorem: Let $\langle G, R, \Vdash, S\rangle$ be ordinalized. Suppose $s \varepsilon S_1 - S_0$ represents S_0. Then if $\Gamma \models (\exists R)(R \text{ well orders } s)$ then $\Gamma \models$ axiom of choice.

Now we consider the specific models constructed earlier.

In the model of chapter 12, if $X(x)$ is the formula $x = x$ and $s_x \varepsilon S_1 - S_0$, s_x represents S_0. We wish to show $(\exists R)(R \text{ well orders } s_x)$ is valid in $\langle G, R, \models, S \rangle$.

Let $Y(x)$ be the formula

$(\exists y)(\exists z)\{[\text{integer } y \wedge \text{integer } (z) \wedge y \varepsilon z \wedge$
$x = \langle y,z \rangle\ ] \vee [\text{integer } (y) \wedge z = a \wedge x = \langle y,z \rangle\ ]\}$

and let $R_Y \varepsilon S_{\omega+3} - S_{\omega+2}$. Then $(R_Y \text{ well orders } s_x)$ is valid. Thus the axiom of choice is valid in the model of chapter 12.

In the model of chapter 10, as above, s_x represents S_0. A reasonable well-ordering of S_0 would be (schematically) $\hat{0}, \hat{1}, \hat{2}, \ldots, a_0, a_1, a_2, \ldots,$ $\{\hat{0}\}, \{\hat{1}\}, \{\hat{2}\}, \ldots, \{\hat{0},a_0\}, \{\hat{1},a_1\}, \{\hat{2},a_2\}, \ldots,$ $\langle \hat{0},a_0 \rangle, \langle \hat{1},a_1 \rangle, \langle \hat{2},a_2 \rangle, \ldots, W.$

We leave it to the reader to show that this well ordering can be expressed in the model. The only non-trivial part of the well-ordering is $a_0, a_1, a_2, \ldots,$ since the subscripts are not part of the model. But W itself provides this ordering.

Thus the axiom of choice is valid in the model of chapter 10.

Section 8

Continuum hypothesis

In this section we show that the generalized continuum hypothesis is valid in the model of chapter 12. More generally, we show the following.

Theorem: Suppose $\langle G, R, \models, S \rangle$ is ordinalized, $\langle G, R, \models_0, S_0 \rangle \varepsilon L$, and G and S_0 are countable in L. Then the generalized continuum hypothesis is valid in $\langle G, R, \models, S \rangle$.

We devote the rest of this section to the proof.

We remarked in section 14 chapter 7, that the definition of the sequence of intuitionistic models is absolute. If L is the class of constructable sets of V, since $\langle G, R, \models_0, S_0 \rangle \varepsilon L$, the construction of the sequence is the same over V or over L. Thus, in this case we may assume in all the preceeding work, V was L. [We use the continuum hypothesis in L].

Trivially, card $S_{\alpha+1} = \aleph_0 \cdot$ card S_α in L. Since $\langle G, R, \models, S \rangle$ is ordinalized and S_0 is countable in L, it follows by the work of chapter 9, that for any ordinal α of L, if $\alpha \geq \omega$, and if β is the least ordinal such that $\hat{\alpha} \varepsilon S_\beta$, then card α = card S_β in L.

We use $\mathcal{P}(x)$ to denote the power set operation both in L and in $\langle G, R, \models, S \rangle$ in an obvious way.

Lemma: Under the conditions of the theorem, if $\alpha, \beta \varepsilon L$ and card $\alpha \geq \aleph_0$ in L, and if, for some $\Gamma \varepsilon G$,

$$\Gamma \models (\text{card } \mathcal{P}(\hat{\alpha}) = \text{card } \hat{\beta})$$

then card $\mathcal{P}(\alpha) \geq$ card β in L.

Proof: As we showed in section 15 of chapter 7, for fixed α there is some $\gamma \varepsilon L$ such that if $\Gamma \models (f \subseteq \hat{\alpha})$, there is some $g \varepsilon S_\gamma$ such that $\Gamma \models (f = g)$. Assume Γ is fixed.

$S_\gamma \varepsilon L$. We have the axiom of choice in L so we can define a set $P \varepsilon L$ such that $P \subseteq S_\gamma$ and if $\Gamma \models (f \subseteq \hat{\alpha})$, there is some $g \varepsilon P$ such that $\Gamma \models (f = g)$, and if $f, g \varepsilon P$ and $f \neq g$, $\Gamma \not\models (f = g)$.

Now as in section 15 chapter 7, the following is definable (as a class) over L: the function U such that for $u \varepsilon P$, $U(u) =$

$\{\langle \Gamma^*, t \rangle \mid t \varepsilon S_{\alpha_0} \wedge \Gamma^* \models (t \varepsilon u)\}$

where α_0 is the least ordinal such that $\hat{\alpha} \varepsilon S_{\alpha_0}$.

In this case since $P \varepsilon L$, U is a set in L, i.e. $U \varepsilon L$.

As we showed in chapter 7, for $u, v \varepsilon P$, if $U(u) = U(v)$, then $\Gamma \models (u = v)$ and hence $u = v$ here. Thus, $u = v$ if and only if $U(u) = U(v)$, for $u, v \varepsilon P$. Thus, if R is the range of U on P, since U is 1-1, card P = card R in L.

But $R \subseteq \mathcal{P}(G \times S_{\alpha_0})$ so card $R \leq$ card $\mathcal{P}(G \times S_{\alpha_0})$.

Since card $(G \times S_{\alpha_0})$ = card G · card S_{α_0}

$$= \aleph_0 \cdot \text{card } \alpha$$

$$= \text{card } \alpha$$

then card $R \leq$ card $\mathcal{P}(\alpha)$

card $P \leq$ card $\mathcal{P}(\alpha)$

We have $\Gamma \models (\text{card } \mathcal{P}(\hat{\alpha}) = \text{card } \hat{\beta})$ so for some $F \varepsilon S$,

$\Gamma \models [\text{function } F \wedge \text{1-1 } F \wedge \text{domain } F = \hat{\beta} \wedge \text{range } F = \mathcal{P}(\hat{\alpha})]$.

We can thus define a function $G \varepsilon L$ to satisfy domain $G = \beta$ and for $\delta < \beta$, $G(\delta)$ is that element e of P such that $\Gamma \models (F(\hat{\delta}) = e)$ [there is only one such element e for each δ]

G is a function in L, range $G \subseteq P$, and it is easy to see G is 1-1. Thus, card $\beta \leq$ card P in L. so card $\beta \leq$ card $\mathcal{P}(\alpha)$ in L.

Q.E.D.

Now we show the theorem itself.

Suppose for some $\Gamma \varepsilon G$,

$\Gamma \not\models$ generalized continuum hypothesis. Then for some $\alpha, \beta, \gamma \varepsilon L$ and some Γ^*, $\Gamma^* \models$ cardinal $\hat{\alpha} \wedge$ cardinal $\hat{\beta}$ $\wedge$ cardinal $\hat{\gamma} \wedge \hat{\alpha} \varepsilon \hat{\beta} \wedge \hat{\beta} \varepsilon \hat{\gamma} \wedge (\hat{\omega} \varepsilon \hat{\alpha} \vee \hat{\omega} = \hat{\alpha}) \wedge$ card $\mathcal{P}(\hat{\alpha}) =$ card $\hat{\gamma}$

Then by section 3, α, β, and γ are cardinals of L.

Moreover, $\alpha \varepsilon \beta$, $\beta \varepsilon \gamma$, $\omega \varepsilon \alpha$ or $\omega = \alpha$, so card $\alpha \geq \aleph_0$ in L.

By the above lemma,

card $\mathcal{P}(\alpha) \geq$ card γ in L.

Thus β is a cardinal in L between α and $\mathcal{P}(\alpha)$ contradicting the continuum hypothesis in L.

Q.E.D.

Section 9

Classical counter models

In the foregoing we have obtained independence results in set theory without constructing any classical models. In more traditional treatments of forcing, classical models are constructed by a method due to Cohen; for example, see [2], but countable classical ZF models are required. Essentially this method was used in section 7 chapter 4 to prove the theorem there. It is possible, using an ultralimit construction, to construct suitable non-standard classical models without countability requirements. The following method is from Vopěnka [20] and is simply translated from the topological intuitionistic models used there to the Kripke semantic models we use. It can be applied in more general settings but we only give it in a form which applies directly to intuitionistic ZF models.

Let $\langle G, R, \models, S \rangle$ be a class model over the classical model V and suppose the axiom of choice is true over V. As we showed in section 6 chapter 1, if $\mathcal{P}$ is the collection R-closed subsets of G, $\langle \mathcal{P}, \subset \rangle$ is a pseudo-boolean algebra. Let F be any maximal filter in $\mathcal{P}$. See [15, pgs. 44, 66].

Define the class $\bar{S}$ to be the collection of all functions f such that domain $f \varepsilon F$, range $f \subseteq S$. Define $\varepsilon \subseteq \bar{S} \times \bar{S}$ by: $f \varepsilon g$ is true if and only if

$$\{\Gamma \varepsilon G \mid \Gamma \varepsilon \text{ dom } f,\ \Gamma \varepsilon \text{ dom } g,\ \Gamma \models (f(\Gamma) \varepsilon g(\Gamma))\} \ \varepsilon\ F$$

We claim that for any formula $X(x_1, \ldots, x_n)$ with no universal quantifiers, $X(f_1, \ldots, f_n)$ is true over $\bar{S}$ if and only if

$$\{\Gamma \varepsilon G \mid \Gamma \varepsilon \text{ dom } f_1 \cap \ldots \cap \text{dom } f_n,\ \Gamma \models X(f_1(\Gamma), \ldots, f_n(\Gamma))\} \ \varepsilon\ F$$

The proof is by induction on the degree of X. We have the result for atomic formulas by definition. The propositional cases are straightforward, using the various properties of maximal filters. We show the existential quantifier case. Suppose X is $(\exists x)\, Y(x, f_1, \ldots, f_n)$ and the result is known for formulas of lesser degree.

Suppose $(\exists x) Y(x, f_1, \ldots, f_n)$ is true over $\bar{S}$. Then for some $g \varepsilon \bar{S}$, $Y(g, f_1, \ldots, f_n)$ is true over $\bar{S}$. By inductive hypothesis,

$$\{\Gamma \mid \Gamma \varepsilon \text{ dom } g \cap \text{dom } f_1 \cap \ldots \cap \text{dom } f_n,\ \Gamma \models Y(g(\Gamma), f_1(\Gamma), \ldots, f_n(\Gamma))\} \ \varepsilon\ F$$

But this set is contained in

$$\{\Gamma \mid \Gamma \varepsilon \text{ dom } f_1 \cap \ldots \cap \text{dom } f_n,\ \Gamma \models (\exists x) Y(x, f_1(\Gamma), \ldots, f_n(\Gamma))\}$$

so this is an element of F.

Conversely, suppose

$$\{\Gamma \mid \Gamma \varepsilon \text{ dom } f_1 \cap \ldots \cap \text{dom } f_n,$$
$$\Gamma \models (\exists x)Y(x, f_1(\Gamma), \ldots, f_n(\Gamma))\} \varepsilon F$$

Let this set be A. We define a function g on $A \varepsilon F$ as follows. Suppose $\Gamma \varepsilon A$. Then

$$\Gamma \models (\exists x)Y(x, f_1(\Gamma), \ldots, f_n(\Gamma))$$

so for some $a \varepsilon S$,

$$\Gamma \models Y(a, f_1(\Gamma), \ldots, f_n(\Gamma)).$$

choose one such a, and let $g(\Gamma) = a$. Thus, by definition, for $\Gamma \varepsilon A$,

$$\Gamma \models (\exists x)Y(x, f_1(\Gamma), \ldots, f_n(\Gamma))$$

iff $\Gamma \models Y(g(\Gamma), f_1(\Gamma), \ldots, f_n(\Gamma))$.

Thus A =

$$\{\Gamma \mid \Gamma \varepsilon \text{ dom } f_1 \cap \ldots \cap \text{dom } f_n \cap \text{dom } g,$$
$$\Gamma \models Y(g(\Gamma), f_1(\Gamma), \ldots, f_n(\Gamma))\} \varepsilon F$$

So by hypothesis, $Y(g, f_1, \ldots, f_n)$ is true over $\bar{S}$, so $(\exists x)Y(x, f_1, \ldots, f_n)$ is true over $\bar{S}$.

As a special case we have: If X has no universal quantifiers and no constants, X is true over $\bar{S}$ iff $\{\Gamma \mid \Gamma \models X\} \varepsilon F$.

Since the unit element of $\langle \mathcal{P}, \subseteq \rangle$ is G, we have $G \varepsilon F$. Thus, if X has no universal quantifiers and no constants, and X is valid in $\langle G, R, \models, S \rangle$, X is true over $\bar{S}$.

CHAPTER 14

Additional Classical Model Generalizations

Section 1

Introduction

All of the preceeding work in part II has been with intuitionistic M_α generalizations, but other kinds of generalizations are possible. In this chapter we briefly examine some of them.

Classically two particular models have proved of great use; the model of constructable sets, and the model of sets with rank. We have discussed an intuitionistic generalization of the first. In a similar fashion, an intuitionistic generalization of the R_α sequence is possible.

Scott and Solovay have developed what they call boolean valued models for set theory [17]. These are really boolean valued generalizations of the classical R_α sequence, in a sense to be given later. A similar boolean valued generalization of the M_α sequence is possible.

Section 2

Boolean valued logics

This section is intended as a preliminary to boolean valued models for set theory. The subject is treated completely in [15]. Also, see section 5 chapter 1.

In a pseudo boolean algebra, if $-a$, the pseudo-compliment of a, has the property $a \cup -a = V$, then $-a$ is called the compliment of a. A pseudo boolean algebra in which every element has a compliment is called a boolean algebra.

Let B be a boolean algebra and let v be a map from W, the set of formulas, to B. v is called a (propositional) homomorphism if

$$v(X \wedge Y) = v(X) \cap v(Y)$$

$$v(X \vee Y) = v(X) \cup v(Y)$$

$$v(\sim X) = -v(X)$$

$$v(X \supset Y) = v(X) \Rightarrow v(Y) = -v(X) \cup v(Y)$$

In addition, v is called a (Q) homomorphism if

$$v((\exists x)X(x)) = \bigcup_{a \in T} v(X(a))$$

$$v((\forall x)X(x)) = \bigcap_{a \in T} v(X(a))$$

where T is the collection of all parameters. The infinite sups and infs corresponding to quantifiers are assumed to exist.

It can be shown that for X a formula with no parameters, X is a theorem of classical logic if and only if $v(X) = \bigvee$ for any Q homomorphism into any boolean algebra.

One way of generating a theory [a collection of formulas called true, closed under modus ponens, containing all valid formulas] is to give a boolean algebra B and a Q homomorphism v, and to call a formula X true in the theory being described if $v(X) = \bigvee$.

Section 3

Boolean valued R_α generalizations

This generalization is from [17], though the particular formulation of it is different.

As usual, V is a classical ZF model. Let B be a complete boolean algebra such that $B \varepsilon V$ [B is complete if all sups and infs exist. Any boolean algebra can be imbedded in a complete one. See [15]]. We define a transfinite sequence R_α^B as follows:

$$R_0^B = \phi$$

$$R_{\alpha+1}^B = B^{R_\alpha^B} \cup R_\alpha^B$$

$$R_\lambda^B = \bigcup_{\alpha<\lambda} R_\alpha^B \quad \text{for limit ordinals}$$

and let

$$R^B = \bigcup_{\alpha \varepsilon V} R_\alpha^B$$

Thus R^B is a class of boolean valued functions. [If B is the two element algebra {0,1} this sequence is homorphically the classical R_α sequence].

Simultaneously we define a sequence of homomorphisms v_α from W_α^B to B where W_α^B is the collection of all formulas with constants from R_α^B, and a final homomorphism v from W^B to B. Note that to define a homomorphism it is sufficient to define it for atomic formulas. This we do as follows.

v_0 is trivial, there are no atomic formulas.

Suppose v_α is known, and $f,g\ R_{\alpha+1}^B$.

1) if $f,g \varepsilon R_\alpha^B$ let

$$v_{\alpha+1}(f \varepsilon g) = v_\alpha(f \varepsilon g)$$

2) if $f \varepsilon R_\alpha^B$ and $g \varepsilon R_{\alpha+1}^B - R_\alpha^B$ let

$$v_{\alpha+1}(f \varepsilon g) =$$

$$\bigcup_{h \varepsilon \text{ dom } g} \{g(h) \cap \bigcap_{x \varepsilon R^B_\alpha} (f(x) \Longleftrightarrow v_\alpha(x \varepsilon h))\}$$

Remark: If an equality symbol is defined in the usual way, condition 3 is the same as $v_{\alpha+1}(f \varepsilon g) =$

$$\bigcup_{h \varepsilon \text{ dom } g} \{g(h) \cap v_{\alpha+1}(f = g)\}$$

If λ is a limit ordinal and v_α is defined for all $\alpha<\lambda$, and if $f,g \varepsilon R^B_\lambda$, then for some $\alpha<\lambda$, $f,g \varepsilon R^B_\alpha$.
Let $v_\lambda(f \varepsilon g) = v_\alpha(f \varepsilon g)$.

If $f,g\ R^B$, for some $\alpha \varepsilon V$, $f,g \varepsilon R^B_\alpha$. Let $v(f \varepsilon g) = v_\alpha(f \varepsilon g)$.

Thus, we have a class, R^B, and a Q homomorphism v from W^B to B. As we remarked in the last section, all the classically valid formulas map to V. In [17] moreover, it is shown that all the axioms of ZF [as well as the axiom of choice, if true in V] map to V. Thus R^B is called a boolean valued model for ZF.

Finally, in [17], a specific model of this kind is produced in which the continuum hypothesis does not map to V, which establishes independence. Similarly for the axiom of constructability.

Section 4

Intuitionistic R_α generalizations

Let V be a classical ZF model. We define a (class of) transfinite sequence of intuitionistic models $\langle G, R, \models_\alpha, R^G_\alpha \rangle$, and a class model $\langle G, R, \models, R^G \rangle$ as follows.

Let G be some non-empty element of V, and let R be some arbitrary reflexive, transitive relation on G, also a member of V.

Let $\mathcal{P}$ be the collection of all R-closed subsets of G. As we showed in section 6 chapter 1, $\mathcal{P}$ under the ordering $\subseteq$ is a pseudo-boolean algebra. An element $a \varepsilon \mathcal{P}$ is called regular if $--a = a$. We call a function with range $\mathcal{P}$ regular if every member of the range is regular.

We define a sequence R^G_α as follows:

$$R^G_0 = \phi$$

$R^G_{\alpha+1}$ is R^G_α together with all regular functions from R^G_α to $\mathcal{P}$.

$$R^G_\lambda = \bigcup_{\alpha<\lambda} R^G_\alpha$$

and let $R^G = \bigcup_{\alpha \varepsilon V} R^G_\alpha$

Remark: The restriction to regular functions is not necessary, but no power is lost, and it simplifies matters. Similarly in chapter 7, in defining $S_{\alpha+1}$ from S_α we could have confined ourselves to formulas $X(x)$ over S_α which were stable.

Next we define the sequence of $\models_\alpha$ relations.

$\models_0$ holds for no atomic formulas.

If $\models_\alpha$ is defined, $\Gamma \varepsilon G$, and $f, g \varepsilon R^G_{\alpha+1}$ then $\Gamma \models_{\alpha+1} (f \varepsilon g)$ if

1) $f, g \varepsilon R^G_\alpha$ and $\Gamma \models_\alpha (f \varepsilon g)$
2) $f \varepsilon R^G_\alpha$, $g \varepsilon R^G_{\alpha+1} - R^G_\alpha$ and $\Gamma \varepsilon g(f)$
3) $f \varepsilon R^G_{\alpha+1} - R^G_\alpha$ and for some $h \varepsilon$ domain g, $\Gamma \varepsilon g(h)$ and $\Gamma \varepsilon (f(x) \iff \{\Delta \mid \Delta \models_\alpha (x \varepsilon h)\})$

for every $x \varepsilon R^G_\alpha$

Remark: the expression in part 3 is an element of the pseudo-boolean algebra $\mathcal{P}$, $\iff$ is the operation of $\mathcal{P}$. The definition could have been stated without such a use of $\mathcal{P}$, but less concisely.

If λ is a limit ordinal, $f, g \varepsilon R^G_\lambda$, then $\Gamma \models_\lambda (f \varepsilon g)$ if for some $\alpha < \lambda$, $\Gamma \models_\alpha (f \varepsilon g)$.

Finally, $\Gamma \models (f \varepsilon g)$ if for some $\alpha \varepsilon V$, $\Gamma \models_\alpha (f \varepsilon g)$.

Thus, we have a sequence of models $\langle G, R, \models_\alpha, R^G_\alpha \rangle$ and a class model $\langle G, R, \models, R^G \rangle$, determined by specifying G and R. In the next section we show, by translation to a boolean valued R_α sequence, that $\langle G, R, \models, R^G \rangle$ is an intuitionistic ZF model.

Section 5

$\langle G, R, \models, R^G \rangle$ is an intuitionistic ZF model

As we remarked in the last section, $\mathcal{P}$, the collection of all R-closed subsets of G, is a pseudo boolean algebra. Moreover, it is complete, i.e. all sups and infs exist. This follows since, in this case a sup is an infinite union, and the union of R-closed subsets is an R-closed subset, and similarly for infs.

The results of section 6 chapter 1, concerning the relationship of $\mathcal{P}$ and $\langle G, R, \models_\alpha, R^G_\alpha \rangle$ may be stated as: for any formulas X and Y,

$\{\Gamma | \Gamma \models_\alpha X\} \in \mathcal{P}$ and

$$\{\Gamma | \Gamma \models_\alpha X\} \cup \{\Gamma | \Gamma \models_\alpha Y\} = \{\Gamma | \Gamma \models_\alpha X \vee Y\}$$

$$\{\Gamma | \Gamma \models_\alpha X\} \cap \{\Gamma | \Gamma \models_\alpha Y\} = \{\Gamma | \Gamma \models_\alpha X \wedge Y\}$$

$$\{\Gamma | \Gamma \models_\alpha X\} \Rightarrow \{\Gamma | \Gamma \models_\alpha Y\} = \{\Gamma | \Gamma \models_\alpha X \supset Y\}$$

$$-\{\Gamma | \Gamma \models_\alpha X\} = \{\Gamma | \Gamma \models_\alpha \sim X\}$$

In this case, the relationship extends to

$$\bigcup_{f \in R^G_\alpha} \{\Gamma | \Gamma \models_\alpha X(f)\} = \{\Gamma | \Gamma \models_\alpha (\exists x)X(x)\}$$

$$\bigcap_{f \in R^G_\alpha} \{\Gamma | \Gamma \models_\alpha X(f)\} = \{\Gamma | \Gamma \models_\alpha (\forall x)X(x)\}$$

Similar results hold between the class models.

Now we construct a boolean valued R_α sequence as in section 2.

An element $a \in \mathcal{P}$ is called dense if $-a = \wedge$ or equivalently, if $--a = \vee$. Let F be the collection of all dense elements of $\mathcal{P}$. F is a filter and [15, pg. 132-5.8] $\mathcal{P}/F = B$ is a boolean algebra. Moreover, $B \in V$. [$\mathcal{P}/F$ is the collection of all equivalence classes of $\mathcal{P}$ where a and b are equivalent if $(a \Rightarrow b)\varepsilon F$ and $(b \Rightarrow a)\varepsilon F$.] In fact, denoting the equivalence class of $a\varepsilon \mathcal{P}$ by $|a|\varepsilon B$ we have

$$|a| \cup |b| = |a \cup b|$$
$$|a| \cap |b| = |a \cap b|$$
$$|a| \Rightarrow |b| = |a \Rightarrow b|$$
$$-|a| = |-a|$$

and the unit of B is $|V| = |G|$.

Furthermore, B is complete and for any index set T,

$$\bigcup_{x\varepsilon T} |a_x| = |\bigcup_{x\varepsilon T} a_x|$$

<u>Remark:</u> This relation does not extend generally to $\bigcap$ but since in a boolean algebra, $\bigcap$ is equivalent to $- \bigcup -$, the above is sufficient for completeness.

We include the proof of this last statement as it is so useful.

<u>Lemma 1:</u> For $a,b\varepsilon \mathcal{P}$,

$$--(a \Rightarrow b) = (a \Rightarrow --b)$$

<u>Proof:</u> By [15, pg. 62, -37]

$$--(a \Rightarrow b) \leq (a \Rightarrow --b)$$

conversely, $--(--c \Rightarrow c) = V$ [15 pg. 132-5.7]

and $a \cap . --b \leq --b$, so

$$--[(a \cap --b) \Rightarrow b] = V$$

[15 pg. 60-14] $--[(a \cap (a \Rightarrow --b)) \Rightarrow b] = V$

[15 pg. 60-18] $--[(a \Rightarrow --b) \Rightarrow (a \Rightarrow b)] = V$

[15 pg. 60-37] $(a \Rightarrow --b) \Rightarrow --(a \Rightarrow b) = V$

$$(a \Rightarrow --b) \leq --(a \Rightarrow b)$$

Q.E.D.

Lemma 2: In $\mathcal{P}$, for any index set T,

$$\bigcap_{x \in T} --(a_x \Rightarrow b) = --\bigcap_{x \in T} (a_x \Rightarrow b)$$

Proof: $--\bigcap_{x \in T} (a_x \Rightarrow b) =$ [15 pg. 136-7]

$--(\bigcup_{x \in T} a_x \Rightarrow b) =$ (lemma 1)

$\bigcup_{x \in T} a_x \Rightarrow --b =$ [15 pg. 136-7]

$\bigcap_{x \in T} (a_x \Rightarrow --b) =$ (lemma 1)

$\bigcap_{x \in T} --(a_x \Rightarrow b)$

Q.E.D.

<u>Theorem:</u> $$\bigcup_{x\varepsilon T} |a_x| = |\bigcup_{x\varepsilon T} a_x|$$

<u>Proof:</u> In $\mathcal{P}$, for any $x \varepsilon T$,

$$a_x \leq \bigcup_{x\varepsilon T} a_x$$

so $$--(a_x \Rightarrow \bigcup_{x\varepsilon T} a_x) = V$$

$$(a_x \Rightarrow \bigcup_{x\varepsilon T} a_x) \varepsilon F$$

so $$|a_x| \leq |\bigcup_{x\varepsilon T} a_x| \quad \text{for all } x\varepsilon T$$

Conversely, suppose for some $b \varepsilon \mathcal{P}$,

$$|a_x| \leq |b| \quad \text{for all } x\varepsilon T$$

Then $--(a_x \Rightarrow b) = V$ for all $x\varepsilon T$

and since $\mathcal{P}$ is complete,

$$\bigcap_{x\varepsilon T} --(a_x \Rightarrow b) = V$$

$$-- \bigcap_{x\varepsilon T} (a_x \Rightarrow b) = V$$

[15 pg. 136-7] $$--(\bigcup_{x\varepsilon T} a_x \Rightarrow b) = V$$

so $\left| \bigcup_{x \varepsilon T} a_x \right| \leq |b|$

Q.E.D.

Thus $B = \mathcal{P}/F$ is a complete boolean algebra. As shown in section 2, this determines the sequence R^B_α, the homomorphisms v_α, and the class model R^B and v. We now wish to investigate the relationship between this and the intuitionistic model from which it arose.

First, we claim there is an isomorphism between R^G_α and R^B_α [and between R^G and R^B] of a rather substantial kind. We show this by induction on α. R^G_0 and R^B_0 are identical.

Suppose we have a mapping between R^G_α and R^B_α [Pairing $f \varepsilon R^G_\alpha$ with $f' \varepsilon R^B_\alpha$]

Let $g \varepsilon R^G_{\alpha+1} - R^G_\alpha$. Let $g' \varepsilon R^B_{\alpha+1} - R^B_\alpha$ be the function whose value at $f' \varepsilon R^B_\alpha$ is

$$g'(f') = |g(f)|$$

This map from $R^G_{\alpha+1}$ to $R^B_{\alpha+1}$ is one to one, for suppose $g, h \varepsilon R^G_{\alpha+1} - R^G_\alpha$ are distinct functions. If g and h are different, there must be some

$f \varepsilon R_\alpha^G$ such that $g(f) \neq h(f)$. If $|g(f)| = |h(f)|$ then by definition,

$$g(f) \Rightarrow h(f) \varepsilon F$$

or
$$--(g(f) \Rightarrow h(f)) = V$$

or by lemma 1

$$(g(f) \Rightarrow --h(f)) = V$$

but h is a regular function, so

$$(g(f) \Rightarrow h(f)) = V$$

$$g(f) \leq h(f)$$

Similarly $h(f) \leq g(f)$, so

$$g(f) = h(f)$$

Secondly, this map from $R_{\alpha+1}^G$ to $R_{\alpha+1}^B$ is onto. For, let $h \varepsilon R_{\alpha+1}^B - R_\alpha^B$. Let s be any function from R_α^G to P defined by :

for $f \varepsilon R_\alpha^G$, $s(f)$ is some particular element of $h(f')$.

Let g be the function defined by $g(x) = --s(x)$. Then g is regular, with domain R_α^G, so $g \varepsilon R_{\alpha+1}^G - R_\alpha^G$. Moreover, for $f \varepsilon R_\alpha^G$, $g'(f') =$ $|g(f)| = |--s(f)| = --|s(f)| = |s(f)| = h(f')$ and so h is g' for $g \varepsilon R_{\alpha+1}^G - R_\alpha^G$.

Next we establish the essential identity of the two models.

<u>Theorem</u>: Let X be a formula over R^G_α with no universal quantifiers. Then $X = X(f_1, \ldots, f_n)$ for $f_1, \ldots, f_n \varepsilon R^G_\alpha$. Let $X' = X(f_1', \ldots, f_n')$ where $f_i' \varepsilon R^B_\alpha$ is the image of f_i as above. Then

$$v_\alpha(X') = |\{\Gamma | \Gamma \models_\alpha X\}|$$

[similarly for the class models]

<u>Corollary 1</u>: If X is any formula with no universal quantifiers and no constants, X is valid in the boolean model R^B_α [that is, $v_\alpha(X) = \bigvee$] if and only if $\sim\sim X$ is valid in $\langle G, R, \models_\alpha, R^G_\alpha \rangle$ [and similarly for the class models]

<u>Proof:</u> The unit element of B is $|G|$ so

$$v_\alpha(X) = \bigvee \quad \text{iff}$$
$$v_\alpha(X) = |G| \quad \text{iff}$$
$$|\{\Gamma | \Gamma \models_\alpha X\}| = |G| \quad \text{iff}$$
$$--\{\Gamma | \Gamma \models_\alpha X\} = --G \quad \text{iff}$$
$$\{\Gamma | \Gamma \models_\alpha \sim\sim X\} = G$$

Q.E.D.

<u>Corollary 2:</u> $\langle G, R, \models, R^G \rangle$ is an intuitionistic ZF model [and the axiom of choice is valid if it is true over V]

Proof: By corollary 1 and the results reported in section 2.

We now turn to the proof of the theorem.

Suppose the result is known for atomic formulas over R^G_α. In then follows for all formulas over R^G_α by induction on the degree. For example, suppose X is $\sim Y$ and the result is known for Y. Then

$$
\begin{aligned}
v_\alpha(X') &= v_\alpha(\sim Y') \\
&= -v_\alpha(Y') \\
&= -|\{\Gamma|\Gamma \models_\alpha Y\}| \\
&= |-\{\Gamma\{\Gamma \models_\alpha Y\}| \\
&= |\{\Gamma|\Gamma \models_\alpha \sim Y\} \\
&= |\{\Gamma|\Gamma \models_\alpha X\}|
\end{aligned}
$$

Also, suppose the result is known for all formulas $Y(f)$, and X is $(\exists x)Y(x)$. Then

$$
\begin{aligned}
v_\alpha(X') &= v_\alpha((\exists x)Y'(x)) \\
&= \bigcup_{f' \varepsilon R^B_\alpha} v_\alpha(Y'(f')) \\
&= \bigcup_{f' \varepsilon R^B_\alpha} |\{\Gamma|\Gamma \models_\alpha Y(f)\}| \\
&= \bigcup_{f \varepsilon R^G_\alpha} |\{\Gamma|\Gamma \models_\alpha Y(f)\}| \\
&= \Big|\bigcup_{f \varepsilon R^G_\alpha} \{\Gamma|\Gamma \models_\alpha Y(f)\}\Big|
\end{aligned}
$$

$$= \quad |\{\Gamma \mid \Gamma \models_\alpha (\exists x)Y(x)\}|$$
$$= \quad |\{\Gamma \mid \Gamma \models_\alpha X\}|$$

The other cases are similar.

Thus, we must show the result holds for atomic formulas. Suppose the result holds for all formulas over R^G_α. Let $f,g \varepsilon R^G_{\alpha+1}$. We have three cases.

Case 1: $f,g \varepsilon R^G_\alpha$. The result is then trivial.

Case 2: $f \varepsilon R^G_\alpha$, $g \varepsilon R^G_{\alpha+1} - R^G_\alpha$. Then

$$\begin{aligned} v_{\alpha+1}(f' \varepsilon g') &= g'(f') \\ &= |g(f)| \\ &= |\{\Gamma \mid \Gamma \models_{\alpha+1} f \varepsilon g\}| \end{aligned}$$

Case 3: $f \varepsilon R^G_{\alpha+1} - R^G_\alpha$

We first note that the following holds in any complete pseudo boolean algebra:

$$\bigcap_{x \varepsilon T} (-a_x \iff -b_x) = - \bigcup_{x \varepsilon T} - (a_x \iff b_x)$$

Now, for any $h \varepsilon$ domain g, let

$$P_h = \{\Gamma \mid \Gamma \varepsilon g(h) \quad \text{and}$$

$$\Gamma \varepsilon \bigcap_{x \varepsilon R^G_\alpha} (f(x) \iff \{\Delta \mid \Delta \models_\alpha \sim\sim x \varepsilon h\})\}$$

Then $\bigcup_{h \varepsilon \text{ dom } g} \mathcal{P}_h = \{\Gamma | \Gamma \models_{\alpha+1} f \varepsilon g\}$

But also, $\mathcal{P}_h =$

$$g(h) \cap \bigcap_{x \varepsilon R^G_\alpha} (f(x) \iff --\{\Delta | \Delta \models_\alpha x \varepsilon h\})$$

so, since f is regular, $\mathcal{P}_h =$

$$g(h) \cap -\bigcup_{x \varepsilon R^G_\alpha} -(f(x) \iff \{\Delta | \Delta \models_\alpha x \varepsilon h\})$$

Thus $|\mathcal{P}_h| =$

$$|g(h)| \cap -\bigcup_{x \varepsilon R^G_\alpha} -(|f(x)| \iff |\{\Delta | \Delta \models_\alpha x \varepsilon h\}|)$$

$$= g'(h') \cap \bigcap_{x' \varepsilon R^B_\alpha} (f'(x') \iff v_\alpha(x' \varepsilon h'))$$

and so $v_{\alpha+1}(f' \varepsilon g') =$

$$\bigcup_{h' \varepsilon \text{ dom } g'} |\mathcal{P}_h| = |\bigcup_{h \varepsilon \text{dom } g} \mathcal{P}_h|$$

$$= |\{\Gamma | \Gamma \models_{\alpha+1} f \varepsilon g\}|$$

The case of limit ordinals, and of the class models, is straightforward.

Q.E.D.

Section 6

Equivalence of the R_α generalizations

In the last section we showed that for any intuitionistic R_α generalization there is a corresponding equivalent boolean valued R_α generalization. In this section we show, under restricted conditions, a converse.

Let B be a complete boolean algebra. A maximal (= prime) filter F is called a Q-filter if, whenever $\bigcup_{x \varepsilon T} a_x \varepsilon F$, $a_t \varepsilon F$ for some $t \varepsilon T$, for any index set T. We say B has property (1) if every non-zero element of B belongs to some Q-filter. [15 pgs. 86-88].

Suppose we have a boolean valued R_α sequence as in section 3, and suppose the algebra B has property (1).

Let G be the collection of all Q-filters of B, and let R be $\subseteq$ [which is actually equality, since all Q-filters are maximal]. As we showed in section 3, this determines an intuitionistic R_α sequence. We now proceed to show these two models are equivalent.

Let s be the function from B to [R-closed] subsets of G defined by: s(a) is the collection of all Q-filters with a as an element. Since B has property (1), s is an isomorphism between B and the power set of G [any subset is R-closed], where the boolean operations in G are the ordinary set-theoretic ones [15 pg. 87].

We define a reasonable isomorphism between R^B_α and R^G_α.

R^B_0 and R^G_0 are identical.

Suppose an isomorphism has been defined between R^B_α and R^G_α [pairing $f \varepsilon R^B_\alpha$ with $f' \varepsilon R^G_\alpha$]

Suppose $g \varepsilon R^B_{\alpha+1} - R^B_\alpha$. Let g' be that element of $R^G_{\alpha+1} - R^G_\alpha$ defined by

$$g'(f') = s(g(f))$$

This defines an isomorphism between $R^B_{\alpha+1}$ and $R^G_{\alpha+1}$.

Now we give the key theorem.

<u>Theorem:</u> Let X be a formula over R^B_α. Then $X = X(f_1, \ldots, f_n)$ for $f_1, \ldots, f_n \varepsilon R^B_\alpha$. Let $X' = X(f_1', \ldots, f_n')$ where $f_i' \varepsilon R^G_\alpha$ is the image of f_i as above. Then

$$\{\Gamma \mid \Gamma \models_\alpha X'\} \quad = \quad s(v_\alpha(x))$$

[similarly for the class models]

<u>Proof:</u> Suppose the result is known for all atomic formulas over R^B_α. It then follows for all formulas X by induction on the degree of X. Suppose the result is known for all formulas of degree less than that of X.

If X is $\sim Y$, $\{\Gamma \mid \Gamma \models_\alpha X'\}$ =

$$\{\Gamma \mid \Gamma \models_\alpha \sim Y'\} \quad = \quad -\{\Gamma \mid \Gamma \models_\alpha Y\}$$

[where this the compliment in the boolean algebra of all subsets of G. Since $\Gamma R \Delta$ implies $\Gamma = \Delta$, it follows that either $\Gamma \models_\alpha Y'$ or $\Gamma \models_\alpha \sim Y'$, so this follows]

$$= \quad -s(v_\alpha(Y)) \quad = \quad s(-v_\alpha(Y))$$

$$= \quad s(v_\alpha(\sim Y)) \quad = \quad s(v_\alpha(X))$$

Similarly, if X is $(\exists x)Y(x)$,

$$\{\Gamma \mid \Gamma \models_\alpha X'\} \quad = \quad \{\Gamma \mid \Gamma \models_\alpha (\exists x)Y'(x)\}$$

$$= \quad \bigcup_{f' \; R^G_\alpha} \{\Gamma \mid \Gamma \models_\alpha Y'(f')\}$$

$$= \quad \bigcup_{f \in R^B_\alpha} s(v_\alpha(Y(f)))$$

$$= \quad s\Big(\bigcup_{f \in R^B_\alpha} v_\alpha(Y(f))\Big)$$

$$= \quad s(v_\alpha((\exists x)Y(x)))$$

$$= \quad s(v_\alpha(x))$$

The other cases are similar.

Thus, we must show the result for atomic formulas.

Suppose the result holds for all formulas over R^B_α. Let $f, g \varepsilon R^B_{\alpha+1}$. We have three cases.

Case 1: $f, g \varepsilon R^B_\alpha$. Then the result is trivial.

Case 2: $f \varepsilon R^B_\alpha$, $g \varepsilon R^B_{\alpha+1} - R^B_\alpha$. Then

$$\{\Gamma | \Gamma \models_{\alpha+1} f' \varepsilon g'\} = g'(f') = s(g(f)) = s(v_{\alpha+1}(f \varepsilon g)).$$

Case 3: $f \varepsilon R^B_{\alpha+1} - R^B_\alpha$. Then

$$s(v_{\alpha+1}(f \varepsilon g)) =$$

$$s\Big(\bigcup_{h \varepsilon \text{ dom } g} \big(g(h) \cap \bigcap_{x \varepsilon R^B_\alpha} (f(x) \iff v_\alpha(x \varepsilon h))\big)\Big) =$$

$$\bigcup_{h \varepsilon \text{ dom } g} \big(s(g(h)) \cap \bigcap_{x \varepsilon R^B_\alpha} (s(f(x)) \iff s(v_\alpha(x \varepsilon h)))\big) =$$

$$\bigcup_{h'\varepsilon dom\ g'} \left(g'(h') \cap \bigcap_{x'\varepsilon R^G_\alpha} (f'(x') \iff \{\Gamma \mid \Gamma \models_\alpha x'\varepsilon h'\}\right))$$

$$= \{\Gamma \mid \Gamma \models_{\alpha+1} f'\varepsilon g'\}$$

The limit ordinal and class cases are straightforward.

Q.E.D.

From this theorem, the essential equivalence of the two models follows.

As a special case, suppose V, the underlying classical ZF model, is countable. Then [15 pg 87-9.3] if $B\varepsilon V$ is a complete boolean algebra, B also has property (1). Thus, if we assume there is a countable ZF model, the two R_α generalizations are equal in power.

The following results would be interesting, but are, as yet, undone.

1) A direct proof that $\langle G, R, \models, R^G\rangle$ is an intuitionistic ZF model.

2) A more general set of circumstances under which a boolean valued R_α sequence has a corresponding

equivalent intuitionistic R_α sequence.

3) A direct proof that there are intuitionistic R_α generalization providing counter models for the continuum hypothesis, or the axiom of constructability. [preferably not using countability of V]

Section 7

Boolean valued M_α generalizations

Let V be a classical ZF model, and let $B\varepsilon V$ be a complete boolean algebra. We define simultaneously a sequence M_α^B of boolean valued functions, and a sequence v_α of homomorphisms from M_α^B to B. This is a direct generalization of the sequence of section 2 chapter 7.

Let M_0^B be some arbitrary collection of functions with domains subsets of M_0^B and ranges subsets of B. We assume M_0^B is well-founded with respect to the relation $x\varepsilon$ domain y. We assume $M_0^B \varepsilon V$. v_0 is defined by the condition: for $f, g\varepsilon M_0^B$,

$$v_0(f\varepsilon g) = g(f).$$

We require that M_0^B and v_0 satisfy the equality condition

$$v_0((\forall x)(x\varepsilon f \equiv x\varepsilon g)) \cap v_0(f\varepsilon h) \leq v_0(g\varepsilon h)$$

for any $f, g, h\varepsilon M_0^B$.

Suppose we have defined M_α^B and v_α. If $X(x)$ is any formula over M_α^B with one free variable, by f_X we mean the function whose domain is M_α^B, whose range is B, and which is defined by

$$f_X(x) = v_\alpha(X(x))$$

for all $x\varepsilon M_\alpha^B$.

Let $M_{\alpha+1}^B$ be M_α^B together with all f_X for all formulas $X(x)$ over M_α^B. We define $v_{\alpha+1}$ for atomic formulas as follows. If $f, g\varepsilon M_{\alpha+1}^B$,

1) if $f, g\varepsilon M_\alpha^B$, let

$$v_{\alpha+1}(f\varepsilon g) = v_\alpha(f\varepsilon g)$$

2) if $f\varepsilon M_\alpha^B$, $g\varepsilon M_{\alpha+1}^B - M_\alpha^B$. let

$$v_{\alpha+1}(f\varepsilon g) = g(f)$$

3) if $f_X\varepsilon M_{\alpha+1}^B - M_\alpha^B$, let $v_{\alpha+1}(f\varepsilon g)$

$$= \bigcup_{h\varepsilon M_\alpha^B} \{v_{\alpha+1}(h\varepsilon g) \cap \bigcap_{x\varepsilon M_\alpha^B} (f(x) \Longleftrightarrow v_\alpha(x\varepsilon h))\}$$

[where $v_{\alpha+1}(h\varepsilon g)$ has been defined in case 1 or case 2]

If λ is a limit ordinal, let

$M_\lambda^B = \bigcup_{\alpha<\lambda} M_\alpha^B$. If $f,g\varepsilon M_\lambda^B$, then for some $\alpha<\lambda$, $f,g\varepsilon M_\alpha^B$. Let

$v_\lambda(f\varepsilon g) = v_\alpha(f\varepsilon g)$

Finally, let $M^B = \bigcup_{\alpha\varepsilon V} M_\alpha^B$ If $f,g\varepsilon M^B$, for some $\alpha\varepsilon V$, $f,g\varepsilon M_\alpha$. Let $v(f\varepsilon g) = v_\alpha(f\varepsilon g)$.

Thus we have a boolean valued generalization of the M_α sequence, and of L.

Section 8

Equivalence of the M_α generalizations

Let $\langle G, R, \models_\alpha, S_\alpha \rangle$ be any intuitionistic M_α generalization, satisfying the conditions of chapter 1. We proceed almost as we did in section 5.

If $f, g \varepsilon S_{\alpha+1} - S_\alpha$ call f and g equivalent if $(f = g)$ is valid in $\langle G, R, \models_{\alpha+1}, S_{\alpha+1} \rangle$. Let S_α' be some subset of S_α containing only one from each collection of equivalent elements.

$\mathcal{P}$ is the collection of all R-closed subsets of G. $\mathcal{P}$ under $\subseteq$ is a pseudo boolean algebra. If F if the filter of all dense elements of $\mathcal{P}$, $B = \mathcal{P}/F$ is a boolean algebra. Define M_0^B from S_0 by induction on the well-founded relation $x \varepsilon$ domain y, so that for $f, g \varepsilon S_0$ the corresponding elements $f', g' \varepsilon M_0^B$ satisfy.

$$g'(f') = |g(f)|$$

Under this definition, M_0^B and S_0' are isomorphic, by induction on the well founded relation $x \varepsilon$ domain y. For if $g' = h'$, then for all $f' \varepsilon$ dom g' = dom h', $g'(f') = h'(f')$ so $|g(f)| = |h(f)|$. It follows that for all $\Gamma \varepsilon G$, $\Gamma \models_0 \sim\sim(f \varepsilon g) \equiv \sim\sim(f \varepsilon h)$ and so $\Gamma \models_0 \sim(\exists x)\sim(x \varepsilon g \equiv x \varepsilon h)$, so $\Gamma \models_0 g = h$. Then if g, h are in S_0', g is h.

Next we may show S'_α and M^B_α are isomorphic, and the mapping still satisfies $g'(f') = |g(f)|$

Then following the procedure of section 5, we may show

Theorem: If X is any formula with no universal quantifiers and no constants, X is valid in the boolean valued model M^B if and only if $\sim\sim X$ is valid in $\langle G, R, \models, S\rangle$.

Similarly, following the procedure of section 6, we may show

Theorem: Let B be a complete boolean algebra satisfying property (1), and let M^B_0 and v_0 satisfy the conditions in section 6. Then there is an intuitionistic sequence such that if X is any formula with no constants, X is valid in M^B if and only if X is valid in $\langle G, R, \models, S\rangle$.

Again the following results would be interesting.

1) A direct proof that M^B is a boolean valued ZF model.

2) A more general set of circumstances under which a boolean valued M_α sequence has a corresponding equivalent intuitionistic M_α sequence.

3) A direct proof that there are boolean valued M_α sequences which establish the various set theory independence results.

APPENDIX

[to section 2 chapter 11]

Section 1

Corresponding formulas

Def: Suppose $\Gamma \models$ partrel R. We say R corresponds to the formula X over g with respect to Γ if there is a Γ^* and a finite set of integers $\{i_1, \ldots i_n\}$ such that X is $X(x_{i_1}, \ldots, x_{i_n})$ and

1) X is dominant
2) all the quantifiers (existential only) are bound to g.
3) for any constant a of X not a quantifer bound, $\Gamma^* \models (a \varepsilon g)$
4) $\Gamma^* \models \sim(\exists x)\sim[x \varepsilon \text{ Domain } R \equiv (x = \hat{i}_1 \vee \cdots \vee x = \hat{i}_n)]$
5) $\Gamma^* \models \sim(\exists x_{i_1})\ldots(\exists x_{i_n})\sim[X(x_{i_1}, \ldots, x_{i_k}) \equiv (\exists f)(f \varepsilon R \wedge f(\hat{i}_1) = x_{i_1} \wedge \ldots \wedge f(\hat{i}_n) = x_{i_n})]$

Lemma: Suppose $\langle G, R, \models, S \rangle$ is ordinalized. If $\Gamma \models$ (R is atomic over g) then R corresponds to an atomic formula over g with respect to Γ.

Proof: There are four cases, all treated similarly. We show only one. Thus, suppose $\Gamma \models$ (R is atomic (2) over g). Then for some $a, b \varepsilon S$,

$$\Gamma \models [\text{integer } (b) \wedge \sim\sim(a \varepsilon g) \wedge \sim(\exists f)\sim(f \varepsilon R \equiv (\text{partfun } (f) \wedge \text{domain } (f) = \{b\} \wedge f(b) \varepsilon a))]$$

Since $\Gamma \models$ integer (b), there is some Γ^* and some integer n such that $\Gamma^* \models (b = \hat{n})$. Since $\Gamma^* \models \sim\sim(a \varepsilon g)$, there is some Γ^{**} such that $\Gamma^{**} \models (a \varepsilon g)$. Let $\Delta = \Gamma^{**}$.

Then

$$\Delta \models [\text{integer } (\hat{n}) \wedge a \varepsilon g \wedge \sim(\exists f)\sim(f \varepsilon R \equiv (\text{partfun } (f) \wedge \text{domain } (f) = \{\hat{n}\} \wedge f(\hat{n}) \varepsilon a)]$$

Now we claim R corresponds to the formula $(x_n \varepsilon a)$ over g. If we take the set of integers to be $\{n\}$, properties 1-4 are immediate. Property 5 becomes

$$\Delta \models \sim(\exists x_n)\sim[x_n \varepsilon a \equiv (\exists f)(f \varepsilon R \wedge f(\hat{n}) = x_n)]$$

We show this in two parts.

Suppose $\Delta^* \models (\exists f)(f \varepsilon R \wedge f(\hat{n}) = b)$. Then for some $f \varepsilon S$, $\Delta^* \models (f \varepsilon R \wedge f(\hat{n}) = b)$.

Since $\Delta^* \models (f \varepsilon R)$, by the above,

$\Delta^* \models \sim\sim f(\hat{n}) \varepsilon a$. But also

$\Delta^* \models f(\hat{n}) = b \wedge$ function (f), so

$\Delta^* \models \sim\sim(b \varepsilon a)$. Thus

$$\Delta \models \sim(\exists x)\sim[(\exists f)(f \varepsilon R \wedge f(\hat{n}) = x) \supset x \varepsilon a]$$

Conversely, suppose $\Delta^* \models (b \varepsilon a)$. Let $Z(x)$ be the formula $x = \langle \hat{n}, b \rangle$ and let w_z be in some suitable $S_{\alpha+1} - S_\alpha$. The reader may verify

$\Delta^* \models$ [partfun $(w_z) \wedge$ domain $(w_z) = \hat{n} \wedge w_z(\hat{n}) = b$]

But $\Delta^* \models b \varepsilon a$, so $\Delta^* \models \sim\sim(w_z \varepsilon R)$. Thus

$\Delta^* \models (\exists f)(\sim\sim f \varepsilon R \wedge f(\hat{n}) = b)$.

$\Delta^* \models \sim\sim(\exists f)(f \varepsilon R \wedge f(\hat{n}) = b)$

$\Delta \models \sim(\exists x)\sim[x \varepsilon a \supset (\exists f)(f \varepsilon R \wedge f(\hat{n}) = x)]$

Q.E.D.

<u>Lemma:</u> Suppose $\langle G, R, \models, S \rangle$ is ordinalized. If S corresponds to a formula X over g with respect to Γ, and $\Gamma \models (R \text{ is not} - S)$ then R corresponds to the formula $\sim X$ over g with respect to Γ.

<u>Proof:</u> Suppose without loss of generality that the finite set of integers for S is $\{1, 2, \ldots, n\}$. We keep the same set for R. By hypothesis, X is dominant, hence so is $\sim X$, thus property 1. Properties 2, 3, and 4 are immediate. Property 5 becomes

$$\Gamma^* \models \sim(\exists x_1)\ldots(\exists x_n)\sim[\sim X(x_1,\ldots,x_n) \equiv (\exists f)(f \varepsilon R \wedge f(\hat{1}) = x_1 \wedge \ldots \wedge f(\hat{n}) = x_n)].$$

But we are given

$$\Gamma^* \models \sim(\exists x_1)\ldots(\exists x_n)\sim[X(x_1,\ldots,x_n) \equiv (\exists f)(f \varepsilon S \wedge f(\hat{1}) = x_1 \wedge \ldots \wedge f(\hat{n}) = x_n)]$$

and $\Gamma \models$ (R is not - S). We show property 5 in two parts. Suppose $\Gamma^* R \Delta$.

If $\Delta \models (\exists f)(f\varepsilon R \wedge f(\hat{1}) = c_1 \wedge \ldots \wedge f(\hat{n}) = c_n)$ then for some $f\varepsilon S$,
$\Delta \models (f\varepsilon R \wedge f(\hat{1}) = c_1 \wedge \ldots \wedge f(\hat{n}) = c_n)$ But
$\Gamma \models \sim(\exists f)\sim [f\varepsilon R \equiv \sim f\varepsilon S]$ so
$\Delta \models \sim(f\varepsilon S)$. We claim from this follows
$\Delta \models \sim X(c_1, \ldots, c_n)$, for otherwise, for some
Δ^*, $\Delta^* \models X(c_1, \ldots, c_n)$. Then
$\Delta^* \models \sim\sim(\exists f)(f\varepsilon S \wedge f(\hat{1}) = c_1 \wedge \ldots \wedge f(\hat{n}) = c_n)$ so
for some $g\varepsilon S$,
$\Delta^* \models \sim\sim(g\varepsilon S) \wedge g(\hat{1}) = c_1 \wedge \ldots \wedge g(\hat{n}) = c_n$
But $\Delta^* \models \sim\sim(g\varepsilon S) \wedge (f\varepsilon R)$ and
$\Delta^* \models \sim(\exists x)\sim[x\varepsilon \text{ Domain } R \equiv x\varepsilon \text{ Domain } S]$
so it follows that $\Delta^* \models$ domain (f) = domain (g)
$\Delta^* \models$ domain (f) $= \{\hat{1}, \ldots, \hat{n}\}$. And
$\Delta^* \models f(\hat{1}) = g(\hat{1}) \ \ldots \ f(\hat{n}) = g(\hat{n})$. Thus
$\Delta^* \models f = g$. But $\Delta^* \models \sim(f\varepsilon S) \wedge \sim\sim(g\varepsilon S)$
a contradiction. Hence $\Delta \models \sim X(c_1, \ldots, c_n)$. Thus
$\Gamma^* \models \sim(\exists x_1)\ldots(\exists x_n)\sim[(\exists f)(f\varepsilon R \wedge f(\hat{1}) = x_1 \wedge \ldots \wedge f(\hat{n}) = x_n)$
$\supset \sim X(x_1, \ldots, x_n)]$

Suppose conversely, $\Delta \models \sim X(c_1, \ldots, c_n)$.
Then $\Delta \models \sim(\exists f)(f\varepsilon S \wedge f(\hat{1}) = c_1 \wedge \ldots \wedge f(\hat{n}) = c_n)$.

Let $Y(x)$ be the formula

$x = \langle \hat{1}, c_1 \rangle \vee \ldots \vee x = \langle \hat{n}, c_n \rangle$ and consider

g_Y in some suitable $S_{\alpha+1} - S_\alpha$. The reader may verify that

$$\Delta \models [\text{partfun}\ (g_Y) \wedge \text{domain}\ (g_Y) = \{\hat{1}, \ldots, \hat{n}\}$$

$$\wedge\ g_Y(\hat{1}) = c_1 \wedge \ldots \wedge g_Y(\hat{n}) = c_n]$$

It follows that $\Delta \models \sim(g_Y \varepsilon S)$. Hence

$\Delta \models \sim\sim(g_Y \varepsilon R)$ That is

$$\Delta \models \sim\sim(g_Y \varepsilon R) \wedge g_Y(\hat{1}) = c_1 \wedge \ldots \wedge g_Y(\hat{n}) = c_n.$$

$$\Delta \models \sim\sim(\exists f)[f \varepsilon R \wedge f(\hat{1}) = c_1 \wedge \ldots \wedge f(\hat{n}) = c_n]$$ So

$$\Gamma \models \sim(\exists x_1) \ldots (\exists x_n) \sim[\sim X(x_1, \ldots, x_n) \supset$$

$$(\exists f)(f \varepsilon R \wedge f(\hat{1}) = x_1 \wedge \ldots \wedge f(\hat{n}) = x_n)]$$

Q.E.D

We may in a similar fashion show

<u>Lemma:</u> Suppose $\langle G, R, \models, S \rangle$ is ordinalized. Suppose S corresponds to a formula X over g and T corresponds to a formula Y over g with respect to Γ. Then

1) If $\Gamma \models R$ is S-and-T, R corresponds to to $X \wedge Y$ over g.
2) If $\Gamma \models R$ is S-or-T, R corresponds to $X \vee Y$ over g.
3) If $\Gamma \models R$ is S-implies-T, R corresponds to $X \supset Y$ over g.

Finally we show

Lemma: Suppose $\langle G, R, \models, S\rangle$ is ordinalized. Suppose S corresponds to a formula $X(x_1, \ldots, x_n)$ over g with respect to Γ, and $\Gamma \models R$ is $(\exists j)S$ over g. Then R corresponds to the formula

$(\exists x_j)\,[(x_j \varepsilon g) \wedge \sim\sim X(x_1, \ldots, x_n)]$

over g with respect to Γ.

Proof: The finite set of integers for S is $\{1, \ldots, n\}$. We may take j to be 1. Then let the set of integers for R be $\{2, \ldots, n\}$. Now property 1 follows by the theorem of section 7 chapter 7. Properties 2 and 3 are immediate, and 4 is straightforward. Property 5 becomes

$$\Gamma^* \models \sim(\exists x_2)\ldots(\exists x_n)\sim[(\exists x_1)(x_1 \varepsilon g \wedge \sim\sim X(x_1, \ldots, x_n))$$
$$\equiv (\exists f)(f \varepsilon R \wedge f(\hat{2}) = x_2 \wedge \ldots \wedge f(\hat{n}) = x_n)]$$

We are given

$$\Gamma^* \models \sim(\exists x_1)\ldots(\exists x_n)\sim[X(x_1, \ldots, x_n) \equiv$$
$$(\exists f)(f \varepsilon S \wedge f(\hat{1}) = x_1 \wedge \ldots \wedge f(\hat{n}) = x_n)$$

We show property 5 in two parts

Let $\Gamma^* R \Delta$.

Suppose $\Delta \models (\exists f)(f \varepsilon R \wedge f(\hat{2}) = c_2 \wedge \ldots \wedge f(\hat{n}) = c_n)$

Then for some $f \varepsilon S$,

$\Delta \models f \varepsilon R \wedge f(\hat{2}) = c_2 \wedge \ldots \wedge f(\hat{n}) = c_n.$

But $\Delta \models R$ is $(\exists 1)S$ over g, so

$\Delta \models \sim\sim(\exists h)(h \varepsilon S \wedge f = h \upharpoonright \text{Domain } R \wedge h(\hat{1}) \varepsilon g)$

Then for any Δ^* there is a Δ^{**} such that

$\Delta^{**} \models h \varepsilon S \wedge f = h \upharpoonright \text{Domain } R \wedge h(\hat{1}) \varepsilon g$.

For some $a \varepsilon S$, $\Delta^{**} \models h(\hat{1}) = a \wedge a \varepsilon g$.

It now follows that

$\Delta^{**} \models h(\hat{1}) = a \wedge h(\hat{2}) = c_2 \wedge \ldots \wedge h(\hat{n}) = c_n$.

So

$\Delta^{**} \models \sim\sim X(a, c_2, \ldots, c_n)$

$\Delta^{**} \models (\exists x_1) [\sim\sim X(x_1, c_2, \ldots, c_n) \wedge x_1 \varepsilon g]$

$\Delta^{**} \models \sim\sim(\exists x_1) [X(x_1, c_2, \ldots, c_n) \wedge x_1 \varepsilon g]$

$\Delta \models \sim\sim(\exists x_1) [X(x_1, c_2, \ldots, c_n) \wedge x_1 \varepsilon g]$

This establishes half.

Conversely suppose

$\Delta \models (\exists x_1)[x_1 \varepsilon g \wedge \sim\sim X(x_1, c_2, \ldots, c_n)]$

then for some $a \varepsilon S$

$\Delta \models a \varepsilon g \wedge \sim\sim X(a, c_2, \ldots, c_n)$. Thus

$\Delta \models \sim\sim(\exists f)(f \varepsilon S \wedge f(\hat{1}) = a \wedge f(\hat{2}) = c_2 \wedge \ldots \wedge f(\hat{n}) = c_n)$

so for any Δ^* there is a Δ^{**} such that

$\Delta^{**} \models (\exists f)(f \varepsilon S \wedge f(\hat{1}) = a \wedge f(\hat{2}) = c_2 \wedge \ldots \wedge f(\hat{n}) = c_n)$

$\Delta^{**} \models f \varepsilon S \wedge f(\hat{1}) = a \wedge f(\hat{2}) = c_2 \wedge \ldots \wedge f(\hat{n}) = c_n$.

Let $Y(x)$ be the formula

$x = \langle \hat{2}, c_2 \rangle \vee \ldots \vee x = \langle \hat{n}, c_n \rangle$

and let h_Y be in some $S_{\alpha+1} - S_\alpha$.

The reader may show

$\Delta^{**} \models \text{partfun}(h_Y) \wedge h_Y = f \upharpoonright \text{Domain } R$

$\wedge f(\hat{1}) \varepsilon g$

So $\Delta^{**} \models h_Y \varepsilon R$

$\Delta^{**} \models (h_Y \varepsilon R \wedge h_Y(\hat{2}) = c_2 \wedge \ldots \wedge h_Y(\hat{n}) = c_n)$

$\Delta^{**} \models (\exists h)(h \varepsilon R \wedge h(\hat{2}) = c_2 \wedge \ldots \wedge h(\hat{n}) = c_n)$

$\Delta \vdash \sim\sim(\exists h)(h \varepsilon R \wedge h(\hat{2}) = c_2 \wedge \ldots \wedge h(\hat{n}) = c_n)$

This establishes the second half.

Q.E.D.

<u>Theorem:</u> Suppose $\langle G, R, \models, S \rangle$ is ordinalized and $\Gamma \models$ (R is a definable relation over g). Then R corresponds to a dominant formula X over g with respect to Γ.

<u>Proof:</u> $\Gamma \models$ (R is a definable relation over g) so for some $F \varepsilon S$, some integer n, and some Γ^*,

$\Gamma^* \models$ function (F) $\wedge$ integer ($\hat{n}$) $\wedge$ domain (F) = $\hat{n}$ $\wedge$

$\sim(\exists x)\sim[x \varepsilon \hat{n} \supset F(x)$ is atomic over g $\vee$

$(\exists y)(y \varepsilon x \wedge F(x)$ is not-$F(y)) \vee \cdots \vee$

$(\exists y)(\exists k)(y \varepsilon x \wedge$ integer $(k) \wedge$

$F(x)$ is $(\exists k)F(y)$ over X)] $\wedge$

$(\exists m)(m \varepsilon \hat{n}) \wedge F(m) = R)$

Now n is some particular integer. We examine 0, 1, ..., n-1. That is $\Gamma^* \models \hat{0} \varepsilon \hat{n}$, so

$\Gamma^* \models \sim\sim[F(\hat{0})$ is atomic over g $\vee$

$(\exists y)(y \varepsilon \hat{0} \wedge F(\hat{0})$ is not $F(y)) \vee \cdots]$

so for some Γ^{**}

Γ^{**} $F(\hat{0})$ is atomic over $g \vee \ldots$

In fact, since $\Gamma^{**} \models \sim(\exists y)(y \varepsilon \hat{0})$,

$\Gamma^{**} \models F(\hat{0})$ is atomic over g.

Next, $\Gamma^{**} \models \hat{1} \varepsilon \hat{n}$, so similarly there is a Γ^{***} such that $\Gamma^{***} \models F(\hat{1})$ is atomic over $g \vee$

$(\exists y)(y \varepsilon \hat{1} \wedge F(\hat{1})$ is not-$F(y)) \vee \ldots$

and also $\Gamma^{***} \models F(\hat{0})$ is atomic over g.

We proceed similarly for each $m<n$. Thus we have some $\Delta = \Gamma^{**\cdots *}$ such that for each $m<n$,

$\Delta \models F(\hat{m})$ is atomic over $g \vee$

$(\exists y)(y \varepsilon \hat{m} \wedge F(\hat{m})$ is not-$F(y)) \vee \ldots$

Now by the above lemmas, $F(\hat{0})$ corresponds to a dominant formula over g with respect to Δ (hence to Γ) So $F(\hat{1})$ corresponds to a dominant formula over g with respect to Δ (Γ) and so on, to $F(\widehat{n-1})$. Finally,

$\Delta \models (\exists m)(m \varepsilon \hat{n} \wedge F(m) = R)$ so in some

Δ^*, $\Delta^* \models \hat{m} \varepsilon \hat{n} \wedge F(\hat{m}) = R)$

Q.E.D.

Section 2

Completeness of the definability formula

Theorem: Suppose $\langle G, R, \models, S\rangle$ is ordinalized and for some $\Gamma \varepsilon G$, $f, g \varepsilon S$,

$$\Gamma \models (f \text{ is definable over } g)$$

Then there is some Γ^* and some dominant formula $X(x)$ with one free variable, no universal quantifier, all quantifiers bound to g, such that if a is a constant of $X(x)$ not a quantifier bound, $\Gamma^* \models (a\varepsilon g)$ and

$\Gamma^* \models \sim(\exists x)\sim[x\varepsilon f \equiv (x\varepsilon g \wedge X(x))]$

Proof: $\Gamma \models$ (f is definable over g) so for some Γ^*, $R\varepsilon S$, integer n, $\Gamma^* \models$ partrel $R \wedge$ integer $\hat{n} \wedge$

R is a definable relation over $g \wedge$

$\sim(\exists x)\sim[x\varepsilon \text{ Domain } R \equiv x = \hat{n}] \wedge$

$\sim(\exists x)\sim[x\varepsilon f \equiv (x\varepsilon g \wedge (\exists h)(h\varepsilon R \wedge h(\hat{n}) = x))]$

By the theorem of section 1, R corresponds to a permanent formula X over g with respect to Γ. X must be one-placed, $X = X(x_n)$. Moreover, X is dominant, has no universal quantifiers, and has all quantifiers bound to g. There is some Γ^{**} such that for any a of X not a quantifier bound $\Gamma^{**} \models a\varepsilon g$. And

$\Gamma^{**} \models \sim(\exists x_n)\sim[X(x_n) \equiv (\exists f)(f\varepsilon R \wedge f(\hat{n}) = x_n)]$

Now if $\Gamma^{**} R \Delta$ and $\Delta \models c\varepsilon f$ then

$\Delta \models \sim\sim(c\varepsilon g \wedge (\exists h)(h\varepsilon R \wedge h(\hat{n}) = c))$ so

$\Delta \models \sim\sim(c\varepsilon g \wedge X(c))$.

Conversely, if $\Delta \models c\varepsilon g \wedge X(c)$ then

$\Delta \models c\varepsilon g \wedge \sim\sim(\exists f)(f\varepsilon R \wedge f(\hat{n}) = c)$

$\Delta \models \sim\sim[c\varepsilon g \wedge (\exists f)(f\varepsilon R \wedge f(\hat{n}) = c)]$ so $\Delta \models \sim\sim c\varepsilon f$.

Thus, $\Gamma^{**} \models \sim(\exists x)\sim[x\varepsilon f \equiv (x\varepsilon g \wedge X(x))]$

Q.E.D.

Thus we have established theorem 1 of section 2 chapter 11.

Section 3

Adequacy of the definability formula

The proof of theorem 2 section 2 chapter 11 is rather like that of theorem 1, so we only sketch the general steps.

Def: Suppose $X(x_{i_1}, \ldots, x_{i_n})$ is a formula with no universal quantifiers, with all quantifiers bound to $g\varepsilon S$, and such that if a is a constant of X other than a quantifier bound, $\Gamma \models \sim\sim(a\varepsilon g)$. We say X corresponds to the partial relation R with respect to Γ if

1) $\Gamma \models \sim(\exists x)\sim[x \varepsilon \text{ Domain } R \equiv (x = \hat{i}_1 \vee \ldots \vee x = \hat{i}_n)]$

2) $\Gamma \models \sim(\exists x_{i_1}) \ldots (\exists x_{i_n})\sim[X(x_{i_1}, \ldots, x_{i_n}) \equiv (\exists f)(f \varepsilon R \wedge f(\hat{i}_1) = x_{i_1} \wedge \ldots \wedge f(\hat{i}_n) = x_{i_n})]$

3) $\Gamma \models \sim\sim$(R is a definable relation over g)

We wish to show

Theorem: Suppose $\langle G, R, \models, S \rangle$ is ordinalized and X is a formula with no universal quantifiers, with all quantifiers bound to $g \varepsilon S$, and such that for $\Gamma \varepsilon G$, for any constant a of X other than a quantifier bound $\Gamma \models \sim\sim(a \varepsilon g)$. Then X corresponds to some partial relation R with respect to Γ.

To show this we must show a sequence of lemmas similar to those of section 1. For example.

Lemma: If $\langle G, R, \models, S \rangle$ is ordinalized, $g, a \varepsilon S$, and $\Gamma \models \sim\sim(a \varepsilon g)$. Then the formula $x_n \varepsilon a$ corresponds to a partial relation R with respect to Γ such that $\Gamma \models R$ is atomic (2) over g.

Proof: Let Y(x) be the formula partfun $(x) \wedge$ domain $(x) = \{\hat{n}\} \wedge x(\hat{n}) \varepsilon a$. Let $R_Y \varepsilon S_{\alpha+1} - S_\alpha$ [where $a, \hat{n} \varepsilon S_\alpha$]. Then $\Gamma \models R_Y$ is atomic (2) over g, and $x_n \varepsilon a$ corresponds to R_Y.

Q.E.D.

Similarly, we may show the analogs of the other lemmas of section 1.

Finally, to show the theorem stated at the beginning of this section, in a sense we reverse the procedure of the proof in section 1. We proceed through subformulas of X, using the lemmas referred to above, concluding with X.

Given this theorem, theorem 2 of section 2 chapter 11 is straightforward.

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