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ASYMPTOTIC RELATIONS BETWEEN DIFFERENTIAL AND DIFFERENCE EQUATIONS

by

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Chapter 1. Introduction

Over the past two hundred years there have been many theorems relating sums to integrals. Perhaps the simplest of these is the following:

Theorem 1.1. Let $f(x)$ be of bounded variation on the interval $[0,n]$. Then

$$\left|\int_0^n f(x)dx - \Sigma_1^n f(k)\right| \leq \mathrm{var}[0,n]f(x).$$

The theorem that is perhaps the most generally used is the following one due to Maclaurin (published first by Euler who waived claim to priority):

Theorem 1.2. Let $f(x)$ be $2k+1$ times continuously differentiable in $[1,n]$. Then

$$\Sigma_1^n f(\nu) = \int_1^n f(x)dx + \tfrac{1}{2}(f(1) + f(n)) + \frac{B_2}{2!}(f'(n) - f'(1))$$

$$k + \frac{B_4}{4!}(f'''(n) - f'''(1)) + \ldots + \frac{B_{2k}}{(2k)!}(f^{(2k-1)}(n) - f^{(2k-1)}(1))$$

$$+ \int_1^n P_{2k+1}(k) f^{(2k+1)}(x)dx$$

where the B_n are Bernoulli numbers and

$$P_1(x) = x-[x]-\tfrac{1}{2},\ P'_{r+1}(x) = P_r(x),\ \int_0^1 P_k(x) = 0 \text{ for any } k.$$

Thus it is possible to find a complete asymptotic expansion for $\Sigma_1^n f(\nu)$ if a satisfactory esimtate of the error term can be found.

We shall concern ourselves here largely with the first term of such an expansion, the validity of $\Sigma_1^n f(k) \sim \int_1^n f(t)dt$ and generalizations thereof.

In the first part of the paper we deal with the following type of question: if $\Sigma_1^n f(k) \sim f(n)$ what can we say about $\Sigma_1^n G(f(k))$? More generally, if $f(x) \sim k(x)$, what can we say about $G(f(x))$?

The second part of the paper considers the formula $x_n = \Sigma_1^n f(k) \sim \int f(t)dt$ from the point of view of the calculus of finite difference. The given relation is equivalent to the following statement:

If x_n is a solution to the difference equation $x_{n+1} - x_n = f(n)$ and $\mu(n)$ is a solution to the differential equation $\frac{d\mu}{dn} = f(n)$ (where n is assumed to be continuous) then $x_n \sim \mu_n$. This leads naturally to the question: when are the solutions of $x_{n+1} - x_n = f(x_n,n)$ asymptotic to those of $\frac{d\mu}{dn} = f(\mu,n)$?

In the third part we consider briefly some higher order non-linear difference equations.

Chapter II. Asymptotic Preserving Operations

Let us begin by defining four relations between functions.

i) $f(x) = o(g(x))$ (f is little 0 of g) $(x \to a)$ if $\lim (x \to a)\ f(x)/g(x) = 0$.

ii) $f(x) = O(g(x))$ (f is big 0 of g) $(x \to a)$ if $f(x)/g(x)$ remains bounded as $x \to a$.

iii) $f(x) \approx g(x)$ (f is of the same order as g) $(x \to a)$ if $f(x) = O(g(x))$ $(x \to a)$ and $g(x) = O(f(x))$ $(x \to a)$.

iv) $f(x) \sim g(x)$ (f is asymptotic to g) $(x \to a)$ if $\lim_{x \to a} f(x)/g(x)=1$.

When "$x \to a$" is understood from context, it may be omitted.

It is obvious that all four relations are transitive; that the last three are reflexive and that the first is irreflexive the last two are symmetric and that the first is asymmetric. Further, $f \sim g$ implies $f \approx g$ which in turn implies that $f = O(g)$ and that $f = o(g)$ is false.

These relations are all well-defined with respect to multiplication e.g. $f_1 = O(g_1)$ and $f_2 = O(g_2)$ imply $f_1 f_2 = O(g_1 g_2)$.

The last two are well-defined with respect to division as well as if we adopt the conventive that $0/0 = 1$ (or just ignore the points at which the denominator vanishes).

With respect to addition, the situation is not so good. We have $x + 1 \sim x$ $(x \to \infty)$ and $-x \sim -x$ $(x \to \infty)$ but $1 \sim 0$ $(x \to \infty)$ is obviously false. If we restrict ourselves to positive functions, these by virtue of the inequality

$$\max\left(\frac{a}{b}, \frac{c}{d}\right) \geq \frac{a + c}{b + d} \geq \min\left(\frac{a}{b}, \frac{c}{d}\right),$$

all four

relations are well-defined for addition. Henceforth, all functions will be assumed positive unless the contrary is stated. When we consider limits, most of these properties fail. For example, if $f_n(x) = \Sigma_o^n \frac{x^k}{k!}$ and $g(x) = e^{x/2}$ then $f_n(x) = o(g(x))$ $(x \to \infty)$ for each n and $f(x) = \lim (n \to \infty)\ f_n(x)$ exists $(= e^x)$ and $f(x) \neq o(g(x))$. In fact $g(x) = o(f(x))$.

Let $h(\vec{y}) = h(y_1,\ldots,y_n)$ be a function of n variables defined for $y_i > 0$, $i=1,\ldots,n$. We shall say that h is <u>asymptotic preserving</u> if whenever

$$f_i(x) \to \infty,\ f_i(x) \sim g_i(x) \quad (x \to \infty) \quad i=1,\ldots,n$$

we have $h(f_1(x),\ldots,f_n(x)) \sim h(g_1(x),\ldots,g_n(x))$. We say that h is <u>order preserving</u> if in the above definition we replace "$\sim$" by "$\approx$". One class of functions with both properties is represented by $h(\vec{y}) = y_1^{\alpha_1} y_2^{\alpha_2} \ldots y_n^{\alpha_n}$. We observe that

$$\frac{\vec{y}\cdot\vec{\nabla}h(\vec{y})}{h(\vec{y})} = \frac{(y_1,\ldots,y_n)\cdot(\alpha_1 y_1^{\alpha_1-1} y_2^{\alpha_2}\ldots y_n^{\alpha_n},\ldots,\alpha_n y_1^{\alpha_1} y_2^{\alpha_2}\ldots y_n^{\alpha_n-1})}{y_1^{\alpha_1} y_2^{\alpha_2}\ldots y_n^{\alpha_n}}$$

$$= \alpha_1 + \alpha_2 + \ldots + \alpha_n$$

is bounded and prove our first theorem which extends and refines a result by R.C. Entringer [2].

Theorem 2.1. A function $h(\vec{y})$ is asymptotic preserving iff for each $\epsilon > 0$, h can be expressed as a product $h(\vec{y}) = h_1(\vec{y})h_2(\vec{y})$ where $h_1(\vec{y})$ is continuously differentiable and

$$\frac{\vec{y}\cdot\vec{\nabla}h_1(\vec{y})}{h_1(\vec{y})} \text{ is bounded and } |h_2(\vec{y}) - 1| < \epsilon \text{ as } \vec{y} \to \infty$$

where $\vec{y} \to \infty$ means $y_i \to \infty$, $i=1,\ldots,n$.

If we let $k(y_1,\ldots,y_n) = \log h(\exp y_1,\ldots,\exp y_n)$ and define k to be <u>approximation preserving</u> if whenever

$$f_i(x) \to \infty,\ f_i(x) - g_i(x) \to 0\ (x \to \infty)\ i=1,\ldots,n$$

we have $k(f_1(x),\ldots,f_n(x)) - k(g_1(x),\ldots,g_n(x)) \to 0$ as $x \to \infty$,

Theorem 2.1 is equivalent to

Theorem 2.2. A function $k(\vec{y})$ is approximation preserving iff for each $\epsilon > 0$, k can be expressed as a sum $k(\vec{y}) = k_1(\vec{y}) + k_2(\vec{y})$ where $k_1(\vec{y})$ is continuously differentiable and $\|\vec{\nabla}k_1(\vec{y})\|$ is bounded and $|k_2(\vec{y})| < \epsilon$ for $\vec{y}$ sufficiently large.

We prove this result by means of two lemmas.

Lemma 1. A function $k(\vec{y})$ is approximation preserving iff it can be expressed as a sum $k(\vec{y}) = k_1(\vec{y}) + k_2(\vec{y})$ where $k_1(\vec{y})$ is uniformly continuous and $k_2(\vec{y}) \to 0$ as $x \to \infty$.

Proof. Suppose k is approximation preserving. Let

$$k_1(y_1,\ldots,y_n) = y_1\cdots y_n \int_{y_1}^{y_1+1/y_1} \cdots \int_{y_n}^{y_n+1/y_n} k(u_1,\ldots,u_n)\,du_1\ldots du_n.$$

If $k_2(\vec{y}) = k(\vec{y}) - k_1(\vec{y})$,

$$|k_2(\vec{y})| = y_1 \cdots y_n \left| \int_{y_1}^{y_1+1/y_1} \cdots \int_{y_n}^{y_n+1/y_n} (k(\vec{y}) - k(\vec{u}))\, du_1 \cdots du_n \right|$$

$$\leq y_1 \cdots y_n \int_{y_1}^{y_1+1/y_1} \cdots \int_{y_n}^{y_n+1/y_n} |k(\vec{y}) - k(\vec{u})|\, du_1 \cdots du_n$$

Since $|y_i - u_i| < 1/y_i$ $i=1,\ldots,n$, it follows that for $\vec{y}$ sufficiently large, the integrand, and hence the integral, is arbitrarily small. Hence $k_2(\vec{y}) \to 0$. Suppose $k_1(\vec{y})$ is not uniformly continuous. There are an $\epsilon > 0$ and sequences $\vec{y}_m \to \infty$, $\vec{\delta}_m \to 0$, such that $|k_1(\vec{y}_m + \vec{\delta}_m) - k_1(\vec{y}_m)| > \epsilon$ for all m. But

$$|k_1(\vec{y}_m + \vec{\delta}_m) - k_1(\vec{y}_m)|$$

$$\leq |k_1(\vec{y}_m+\vec{\delta}_m) - k(\vec{y}_m+\vec{\delta}_m)| + |k(\vec{y}_m+\vec{\delta}_m) - k(\vec{y}_m)| + |k(\vec{y}_m) - k_1(\vec{y}_m)|,$$

where the first and third terms are values of k_2 and the second approaches zero by hypothesis. This contradiction proves that k_1 is uniformly continuous.

Conversely, a uniformly continuous function is certainly approximation preserving, and the addition of a null function cannot affect this property.

Lemma 2. A function $k(\vec{y})$ is uniformly continuous iff for each $\epsilon > 0$ it can be expressed as a sum $k(\vec{y}) = k_1(\vec{y}) + k_2(\vec{y})$ where $k_1(\vec{y})$ is continuously differentiable and its differential is bounded (in norm) and $|k_2(\vec{y})| < \epsilon$.

Proof. Suppose $k(\vec{y})$ is uniformly continuous. Then for any $\epsilon > 0$ there is a $\delta > 0$ such that for $|y_i - u_1| < \delta, i=1,\ldots,n$ we have $|k(\vec{y}) - k(\vec{u})| < \epsilon$. Let

$$k_1(y_1,\ldots,y_n) = \frac{1}{\delta^n}\int_{y_1}^{y_1+\delta} \cdots \int_{y_n}^{y_n+\delta} k(u_1,\ldots,u_n)du_1\ldots du_n.$$

If $k_2(\vec{y}) = k(\vec{y}) - k_1(\vec{y})$, then $|k_2(\vec{y})| < \epsilon$. Also

$$\frac{\partial}{\partial y_1} k_1(y_1,\ldots,y_n) =$$

$$= \frac{1}{\delta^n} \int_{y_2}^{y_2+\delta} \cdots \int_{y_n}^{y_n+\delta} (k(y_1+\delta,u_2,\ldots,u_n) - k(y_1,u_2,\ldots,u_n))du_2\ldots du_n$$

which is continuous and less in absolute value than

$$\frac{1}{\delta^n} \int_{y_2}^{y_2+\delta} \cdots \int_{y_n}^{y_n+\delta} \epsilon \, du_2\ldots du_n = \epsilon/\delta.$$

Similarly, the other partials are continuous and bounded. Conversely, suppose that for each $\epsilon > 0$, k can be expressed as a sum $k(\vec{y}) = k_1(\vec{y}) + k_2(\vec{y})$ where $k_1(\vec{y})$ is uniformly continuous and $|k_2(\vec{y})| < \epsilon$. Let $\epsilon > 0$ be given. Choose $k_1(\vec{y})$ and $k_2(\vec{y})$ to satisfy the hypotheses with $\epsilon/3$. We can find a $\delta > 0$ such that for $\|\vec{y}-\vec{u}\| < \delta$, $|k_1(\vec{y}) - k_1(\vec{u})| < \epsilon/3$. Then for $\|\vec{y}-\vec{u}\| < \delta$,

$$|k(\vec{y})-k(\vec{u})| \leq |k_1(\vec{y})-k_1(\vec{u})| + |k_2(\vec{y})-k_2(\vec{u})| < \epsilon/3 + 2\epsilon/3 = \epsilon.$$

Hence $k(\vec{y})$ is uniformly continuous.

Combining the two lemmas, we have the theorem. That we cannot strengthen the second lemma, and hence the theorem, to

have $k_2(\vec{y}) \to 0$ is obvious upon considering $k(y) = \sqrt{|\sin y|}$ where a function that approximates to within ϵ must have slopes of the order of $1/\epsilon$.

While it is not obvious from the definition that an asymptotic preserving function is order preserving, the following theorem shows this to be true.

Theorem 2.3. A function $h(\vec{y})$ is order preserving iff it can be expressed as a product $h(\vec{y}) = h_1(\vec{y})h_2(\vec{y})$ where $h_1(\vec{y})$ is continuously differentiable and $\dfrac{\vec{y}\cdot\vec{\nabla}h_1(\vec{y})}{h_1(\vec{y})}$ is bounded and $h_2(\vec{y}) \approx 1$ as $\vec{y} \to \infty$.

If we define $k(\vec{y})$ as before and define k to be <u>error preserving</u> if whenever

$$f_i(\vec{x}) \to \infty,\ f_i(x) - g_i(x) \text{ is bounded as } x \to \infty \quad i=1,\ldots,n$$

it follows that $k(f_1(x),\ldots,f_n(x)) - k(g_1(x),\ldots,g_n(x))$ is bounded, then Theorem 2.3 is equivalent to

Theorem 2.4. A function $k(\vec{y})$ is error preserving iff it can be expressed as a sum $k(\vec{y}) = k_1(\vec{y}) + k_2(\vec{y})$, where $k_1(\vec{y})$ is continuously differentiable and $\|\vec{\nabla}k_1(\vec{y})\|$ and $k_2(\vec{y})$ are bounded.

Proof. Let

$$k_1(y_1,\ldots,y_n) = \int_{y_1}^{y_1+1} \cdots \int_{y_n}^{y_n+1} k(u_1,\ldots,u_n)du_1\ldots du_n \text{ and } k_2(\vec{y})=k(\vec{y})-k_1(\vec{y}).$$

The result follows by arguments similar to preceeding ones.

Functions which are asymptotic to themselves when their arguments are merely of the same order were investigated by

Karamata [4] who called them <u>slowly increasing</u>. He obtained the single variable version of the following theorem whose proof is similar to the previous ones.

Theorem 2.5. A function $h(\vec{y})$ is slowly increasing iff it can be expressed as a produce $h(\vec{y}) = h_1(\vec{y})h_2(\vec{y})$ where $h_1(\vec{y})$ is continuously differentiable and

$$\frac{\vec{y}\cdot\vec{\nabla}h_1(\vec{y})}{h_1(\vec{y})} \to 0 \quad \text{and} \quad h_2(\vec{y}) \to 1 \quad \text{as} \quad \vec{y} \to \infty.$$

Note that this class of functions is smaller than that of asymptotic preserving functions. On the other hand

Theorem 2.6. Any function which is the same order as itself when its arguments are asymptotic is necessarily order preserving.

Proof. We make the usual transformation and show that if

$$k(y_1+e_1(y_1),\ldots,y_n+e_n(y_n)) - k(y_1,\ldots,y_n)$$

is bounded whenever $\vec{e}(\vec{y}) \to 0$, as $\vec{y} \to \infty$, then

$$k(y_1+M_1(y_1),\ldots,y_n+M_n(y_n)) - k(y_1,\ldots,y_n)$$

is bounded whenever $\vec{M}(\vec{y})$ is bounded as $\vec{y} \to \infty$.

Let $\vec{M}(\vec{y})$ be given with $\|\vec{M}(\vec{y})\| < M$ and suppose $k(\vec{y}+\vec{M}(\vec{y}))-k(\vec{y})$ is not bounded. Then for some sequence $\vec{y}_m \to \infty$, $y_{m+1,i} - y_{m,i} > 2M$ $i=1,\ldots,n$ we have

$$|k(\vec{y}_m+\vec{M}(\vec{y}_m)) - k(\vec{y}_m)| > m^2, \quad m=1,2,\ldots$$

Define $e_i(y_i) = M_i(y_i)/m$ for y_i between $y_{m,i}$ and $y_{m,i} + M_i(y_{m,i})$ and zero otherwise, $i=1,\ldots,n$. Obviously, $\vec{e}(\vec{y}) \to 0$ as $\vec{y} \to \infty$. Then for each m, one of the m numbers

$$|k(\vec{y}_m + j\vec{e}(\vec{y}_m)) - k(\vec{y}_m + (j-1)\vec{e}(\vec{y}_m)| \quad j=1,\ldots,m$$

must exceed m, which is a contradiction.

Karamata's proof uses the weaker hypothesis that for each $b > 0$, $G(by) \sim G(y)$ $(y \to \infty)$ and he concludes that their relation holds uniformly for $0 < b_1 \leq x \leq b_2$ Korevaar [5] proves the uniformity assuring measurability (and not integrability) and then proceeds as above.

Landau [6] notes that if G is monotone and $G(2x) \sim G(x)$, then $G(Ax) \sim G(x)$ for every $A > 0$. Another result of Karamata gives a partial answer to our first question.

Theorem: If $f(x)$ is slowly increasing, there exist numbers k and $a_k > 0$ such that

$$(1) \quad \int_0^x f(t)t^k dt \sim a_k \, f(x) \int_0^x t^k dt$$

Conversely, if there are such numbers k and a_k for which 1) holds

then $f(x) = x^a L(x)$ where $L(x)$ is slowly increasing and $a_k = \frac{a+k+1}{a+1}$. In particular, $f(x)$ is slowly increasing iff for some k, $\int_0^x f(t)t^k dt \sim f(x) \int_0^x t^k dt$. We prove the following generalization.

Theorem 2.7. If $f(x)$ is slowly increasing, $G(x) = \int_0^\infty g(t)dt \uparrow \infty$ and $G^{-1}(x)$ is order preserving, then

$$\int_0^x f(t)g(t)dt \sim f(x) \int_0^x g(t)dt \quad (x \to \infty).$$

We first prove the following

Lemma. If $G(x) = \int_0^x g(t)dt \uparrow \infty$ and $f(G^{-1}(A(x)G(x))) \sim f(x)$ $(x \to \infty)$ for every $A(x) \approx 1$, then $\int_0^x f(t)g(t)dt \sim f(x) \int_0^x g(t)dt$.

Proof. By hypothesis, whenever $G(x)$ increases by a bounded factor, $f(x)$ increases by a factor approaching 1. If $G(x)$ is bounded, $f(x)$ must be identically one and the result is trivial. Let $\epsilon > 0$ be given and suppose for $t > x'$, $\left|\frac{f(x)}{f(t)} - 1\right| < \epsilon$ when $G(x) \leq 2G(t)$. Let $n = \max\{K : G(x)/2^k > G(x')\}$. Let $x_i = G^{-1}(G(x)/2^{n-i})$

$$\int_0^x f(t)g(t)dt = \int_0^{x_0} f(t)g(t)dt + \Sigma_{i=1}^n \int_{x_{i-1}}^{x_i} f(t)g(t)dt$$

which lies between

$$K + \Sigma_1^n \left(\frac{1\pm\epsilon}{2}\right)^i f(x)G(x)$$

$$= K + \frac{1\pm\epsilon}{1\mp\epsilon}\left(1 - \left(\frac{1\pm\epsilon}{2}\right)^n\right) f(x)G(x)$$

Now as x increases, $G(x)$ increases without bound and whenever it doubles, $f(x)G(x)$ is increased by a factor of at least $\frac{2}{1+\epsilon}$ and n increases by 1. Then for x sufficiently large, the sum is between

$(1\pm 4\epsilon)\ f(x)G(x)$. Since ϵ is arbitrary, we have the result. The converse is also true but we shall not prove it.

Now for the theorem. $G^{-1}(x)$ order preserving means that for each $A'(y) \approx 1$ there is a $B'(y) \approx 1$ such that $G^{-1}(A'(y)y) = B'(y)G^{-1}(y)$. Let $y = G(x)$, $A(x) = A'(G(x))$, $B(x) = B'(G(x))$ and we have $G^{-1}(A(x)G(x)) = B(x)x$. Hence $f(G^{-1}(A(x)G(x))) = f(B(x)x) \sim f(x)$ and the result follows by the lemma.

We observe for $k > -1$, $G(x) = \int_0^x x^\alpha dt = \frac{x^{\alpha+1}}{\alpha+1}$, $G^{-1}(x) = [(\alpha+1)x]^{\frac{1}{\alpha+1}}$ is order preserving (in fact, asymptotic preserving) which implies part of Karamata's result. It is intuitively obvious that if $g(x)$ is growing faster, the class of functions that can be factored out is larger. A simple result of this kind is the following

Theorem 2.8. If $G(x) = \int_0^x g(t)dt \uparrow \infty$ and $G^{-1}(x)$ is slowly increasing, and $f(x)$ is asymptotic preserving, then $\int_0^x f(t)g(t) \sim f(x) \int_0^x g(t)dt$. The proof is simple. For each $A'(x) \approx 1$ we have $G^{-1}(A'(y)y) \sim G^{-1}(y)$. Let $y = G(x)$, $A(x) = A'(G(x))$. Hence $G^{-1}(A(x)G(x)) \sim x$ and $f(g^{-1}(a(x)G(x)) \sim f(x)$.

Thus, for example $\int_0^x t^\alpha e^t dt \sim x^\alpha \int_0^x e^t dt \sim x^\alpha e^x$ α real.

Now let us examine what happens on the boundary of the conditions described in Theorem 4. It is certainly not true that $\int_0^x \frac{\log t}{t}\, dt \sim \log x \int_1^x \frac{dt}{t} = \log^2 x$. In fact the integral is equal to $\frac{\log^2 x}{2}$. Although $\log t$ is slow,

$H(x) = G^{-1}(x) = (\int_0^x \frac{dt}{t})^{-1} = e^x$ and $\frac{dH(x)}{H(x)} = 1 \neq o(\frac{1}{x})$. There are two things we <u>can</u> say: The first is that if $f(e^x)$ is slow, we <u>can</u> factor it out: $\int_0^x \frac{f(t)}{t}\,dt = \int_0^{\log x} f(e^y)dy$ $\sim f(e^{\log x}) \int_0^{\log x} dy = f(x) \log x$. The second is that if $f(x)$ is slow and if $g(x) = \int_1^x \frac{f(t)}{t}\,dt$ diverges, $g(x)$ is slow itself. Since $f(t) > 0$, $g(x)$ is monotone and we need only show $g(2x) \sim g(x)$ or $g(2x) - g(x) = o(g(x))$. Let $\epsilon > 0$ be given. Choose x so large that for $y > x/2^n$ (n to be chosen later), $|f(y_1)/f(y_2) - 1| < \epsilon$ when $\frac{1}{2} \leq y_1/y_2 \leq 2$. Then

$$0 \leq g(2x) - g(x) = \int_x^{2x} (f(t)/t)dt \leq (1+\epsilon)f(x) \int_x^{2x} \frac{dt}{t} = (1+\epsilon)\, f(x) \log 2$$

$$g(x) \geq \Sigma_1^n \int_{x/2^{n-k+1}}^{x/2^{n-k}} \frac{f(t)}{t}\,dt \geq \Sigma_1^n \frac{f(x)}{(1+\epsilon)^n} \int_{x/2^{n-k+1}}^{x/2^{n-k}} \frac{dt}{t} =$$

$= \Sigma_1^n \frac{f(x) \log 2}{(1+\epsilon)^n} \to f(x) \log 2$ as $n \to \infty$; for n sufficiently large $0 < (g(2x) - g(x))/g(x) \leq 2\epsilon$. The next question is: what happens when $g(x) = \int_1^x \frac{f(t)}{t}\,dt$ converges, say to C e.g. $f(x) = 1/\log^2 t$. $g(x)$ is then slow but in a trivial sense. Less trivial is the fact that $h(x) = C - g(x) = \int_x^\infty \frac{f(t)}{t}\,dt$ is also slow. The proof is simple:

$$0 \leq 1 - \frac{\int_{2x}^\infty \frac{f(t)}{t}\,dt}{\int_x^\infty \frac{f(t)}{t}\,dt} = \frac{\int_x^{2x} \frac{f(t)}{t}\,dt}{\int_x^\infty \frac{f(t)}{t}\,dt} \leq \frac{(1+\epsilon)\log^2 f(x)}{\log^2 f(x)\, \Sigma_1^\infty \frac{1}{(1+\epsilon)}} = \epsilon(1+\epsilon)$$

In this line, a simple variation of the previous methods enables us to prove: If $f(x)$ is slow, $G(x) = \int_x^\infty g(t)dt \downarrow 0$ and $G^{-1}(x)$ is order preserving, then $\int_x^\infty f(t)g(t)dt \sim f(t) \int_x^\infty g(t)dt$ $(x \to \infty)$.

Let us turn from functions to operators. The statement that if $f(x) \sim g(x)$ $(x \to \infty)$ and $\int_0^x g(t)dt \to \infty$ implies $\int_0^x f(t)dt \sim \int_0^x g(t)dt$, is just a rewording of l'Hospital's rule. So we have: integration preserves asymptotic relations. In fact, fractional integration of any order preserves asymptotic relations. If $f(x) \sim g(x)$ $(x \to \infty)$ and and $g_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^\alpha g(t)dt \to \infty$, then $F_\alpha(x) \sim G_\alpha(x)$ $(\alpha > 0)$

$$\left|\frac{F_\alpha(x)}{G_\alpha(x)} - 1\right| = \left|\frac{\int_0^x (x-t)^{\alpha-1}(f(t)-g(t))dt}{\int_0^x (x-t)^{\alpha-1}g(t)dt}\right|$$

$$\leq \frac{\int_0^{x_0} (x-t)^{\alpha-1}(f(t) + g(t))dt}{\int_0^x (x-t)^{\alpha-1}g(t)dt} + \frac{\int_{x_0}^x (x-t)^{\alpha-1}|f(t)-g(t)|dt}{\int_{x_0}^x (x-t)^{\alpha-1}g(t)dt}$$

We can choose x_0 so large that the second term is less then $\epsilon/2$ by the hypothesis. Now consider $0 < \alpha \leq 1$. In the first term the numerator is non increasing with x, the denominator approaches infinity so for x sufficiently large, it too $< \epsilon/2$. If $\alpha > 1$, $F_\alpha(x) = \int_0^x F_{\alpha-1}(t)dt$ and the result follows by induction on the integer part of α. We can replace $(x-t)^{\alpha-1}$ with an arbitrary kernel $k(x,t)$ provided for each t, $k(x,t)$ remain bounded as $x \to \infty$. The same applies to sum i.e. if $k(x,n)$ is bounded as $x \to c^-$, $a_n \sim b_n$ $(n \to \infty)$, $\Sigma_1^\infty a_n k(x,n)$, $\Sigma_1^\infty b_n k(x,n)$ exist for $x < C$ and $\lim (x \to c^-)$ $\Sigma_1^\infty b_n k(x,n) = \infty$, thus $\Sigma_1^\infty a_n k(x,n) \sim \Sigma_1^\infty b_n k(x,n)$ $(x \to \infty)$. Differentiation, in general, does not preserving asymptotic relations as the following example shows.

$f(x) = x + e^{\sin x} \sim x$ but $f'(x) = 1 + e^{\sin x} \cos x \not\sim 1$.

Sufficient conditions have been supplied by Obreskov [1] who proved if $\varphi(x) = x^{\alpha} L(x)$ where $L(x)$ is slow and $\varphi^{(n)}(x) > -\mu x^{-n}\varphi(x)$, then for $1 \leq i \leq n-1$, $\varphi^{(i)}(x) \sim \alpha(\alpha-1)\ldots(\alpha-i+1)x^{\alpha-i}L(x)$. The result holds for $i-n$ if $\varphi^{(n)}(x)$ is monotonic. For an exhaustive study of the ways in which power series preserve asymptotic relations see Hardy [3].

Chapter III. Difference Equations and Differential Equations

Before proceeding with some examples of difference equations, we present a useful extension of Bernoulli's inequality which does not seem to be in any collection we have examined.

Lemma. Let $F(k,a,x) = 1+ax + \frac{a(a-1)}{2!} x^2 + \ldots + \binom{a}{k} x^k$ be the kth partial sum of the binomial series for $(1+x)^a$ where $x > -1$. If the first term omitted, i.e. $\binom{a}{k+1} x^{k+1}$, is positive, then $(1+x)^a > F(k,a,x)$; if the first term omitted is zero, then $(1+x)^a = F(k,a,x)$; if the first term omitted is negative, then $(1+x)^a < F(k,a,x)$.

Proof. The result is trivial if $k=0$. For $k=1$ it is Bernoulli's inequality. Suppose the theorem is true for all (real) a, all $x > -1$ and all positive integers less than k. Let $G(k,a,x) = (1-x)^a - F(k,a,x)$ $G(k,a,x) = a\int_0^x G(k-1,a-1,t)dt = aI$. We first consider the cases of equality. If a is a non-negative integer $\leq k$, the series has terminated; because if $x = 0$. We now consider cases.

I. The first omitted term is positive:

$$\binom{a}{k+1}x^{k+1} = \frac{a(a-1)\ldots(a-k)}{(k+1)!} x^{k+1} > 0$$

A. $x > 0$. 1. If $a > 0$, $\binom{a-1}{k} = \frac{(a-1)\ldots(a-k)}{k!}$ is also positive. Thus $G(k-1,a-1,t) > 0$, $I > 0$, $aI > 0$.

2. If $a < 0$ $\binom{a-1}{k} x^k < 0$, $G(k-1,a-1)t) < 0$, $I < 0$, $aI > 0$.

B. $x < 0$. 1. If $a > 0$, $\binom{a-1}{k}$ has the same sign as $\binom{a}{k+1}$, x^k has sign opposite from x^{k+1}, $\binom{a-1}{k} x^k < 0$, $G(k-1,a-1,t) < 0$, $I > 0$, $aI > 0$.

2. $a < 0$. $\binom{a-1}{k}$ has opposite sign from $\binom{a}{k+1}$, x^k has opposite sign from x^{k+1}, $\binom{a-1}{k}x^k > 0$, $G(k-1,a-1,t) > 0, I < 0, aI > 0$.

II. The first omitted term is negative. The analysis is the same.

We introduce the concept of oscillation and extend Theorem 1.1 to Riemann-Stieltjes integrals.

$\operatorname{osc}[k-1,k]f(t) = \sup [k-1,k]f(t) - \inf[k-1,k]f(t)$,

$\operatorname{osc}[0,n]f(t) = \sum_{k=1}^{n} \operatorname{osc}[k-1,k]f(t)$.

Theorem 3.0. $\left|\int_0^n G(x)dF(x) - \sum_1^n G(k)(F(k) - F(k-1))\right| \leq \sum_1^n \operatorname{osc}[k-1,k]\, G(x) \operatorname{var}[k-1,k]F(x)$.

Proof: $\left|\int_{k-1}^{k} G(x)dF(x) - G(k)(F(k)-F(k-1))\right|$

$$= \left|\lim_{\delta \to 0} \Sigma\, (G(x_i)-G(k))(F(x_i)-F(x_{i-1}))\right|$$

$$\leq \operatorname{osc}[k-1,k]G(x)\cdot \lim_{\delta\to 0} \Sigma\, |F(x_i)-F(x_{i-1})|$$

$= \operatorname{osc}[k-1,k]G(x)\cdot \operatorname{var}[k-1,k]F(x)$, where the x_i represent points of a partition of $[k-1,k]$ and δ represents the norm. The result follows upon summing. If $F(x) = x$, this reduces to the refinement of Theorem 1.1.

If $\operatorname{osc}[0,n]G(x)=o(\sum_1^n G(k))$ or $o(\int_0^n G(t)dt)$ then $\sum_1^n G(k) \sim \int_0^n G(t)dt$. The gain is illustrated by $G(x) = \frac{1}{x^2} \sin \frac{\pi}{\{x\}^3}$ ($\{x\}$ means the fractional part of x). For any k, $\operatorname{var}[k,k+1]G(x) = \infty$ while $\operatorname{osc}[0,\infty]\, G(x)$ is finite.

What conditions can we place on a function to guarantee this behavior? Without monotonicity we can find bounded real analytic functions which vanish at the integers but not in be-between eg $\sin^2 \pi x$. On the other hand, there are functions all of whose derivatives are everywhere positive and yet because of their rapid growth violate this condition. For example,

$\int_0^n e^t dt = e^n - 1$ while $\Sigma_1^n e^k = \frac{e^{n+1}-1}{e-1}$ which differs by a factor of less than two. Combining the two types of conditions and switching to sequences, whose oscillations are the same by monotonicity, we have

Theorem 3.1. If r is an integer ≥ 1, $\Delta^r a_n$ is eventually non-negative and $a_n = O(n^{2r-1})$, then $a_n = o(\Sigma_1^n a_k)$ and this is best possible.

Proof. Let P(r) be: if $a_n \neq o\, \Sigma_1^n a_1$ and $\Delta^r a_n \geq 0$, then $\Delta^k a_n$ is eventually positive for $0 \leq k < r$ and $a_n \neq O(n^{2r-1})$. Let R(k) be: If $0 \leq k < r$, $a_n \neq o(\Sigma_1^n a_1)$ and $\Delta^r a_n \geq 0$, then $\Delta^k a_n > 0$, $\Delta^k a_n$ is unbounded and $a_n \neq O(r^{2k-1})$. Suppose $a_n \neq o(\Sigma_1^n a_1)$ and $\Delta^r a_n \geq 0$. For some subsequences $\{a_{nk}\}$ and some $\epsilon > 0$, $a_{nk} > \epsilon\, \Sigma_1^{n_k} a_r \geq \epsilon\, a_1, a_{nk} > \epsilon\, \Sigma_1^{n_k} a_r \geq \epsilon \Sigma_1^k \epsilon a_1 = k\epsilon^2 a_1$. a_{nk} is unbounded, and R(0) and P(1). Suppose P(r-1). Suppose R(r-1). Since $\Delta^{k-1} a_n > 0$ and $\Delta^r a_n \geq 0$, $\Delta^k a_n$ cannot change sign more than r-k times (Rolle's theorem for sequences). $\Delta^{k-1} a_n$ is unbounded, the final sign must be positive. By P(k), $a_n \neq O(n^{2k-1})(k \geq 1)$, $\Delta^k a_n$ is unbounded, R(k) for $k = 0,1,\ldots,n-1$. We need yet another induction to finish. For the subsequence $\{a_{ni}\}$ of above, let S(k) be:

$\Delta^k a_{k_i} > \frac{\epsilon}{2} \Delta^{k-1} a_{n_i} = \frac{\epsilon}{2} \Sigma_1^{n_i} \Delta^k a_m$. Then S(0). Suppose S(k-1). Suppose $\Delta^{k-1} a_{n_i-1} \geq \frac{n_i-1}{n_i} \Delta^{k-1} a_{k_i}$. Since $\Delta^{k-1} a_n$ is increasing

$\Delta^{k-1} a_m \geq \frac{m}{n_i} \Delta^{k-1} a_{n_i}$, $\Delta^{k-1} a_{n_i} > \epsilon\, \Sigma_1^{n_i} \Delta^{k-1} a_m$

$> \frac{\epsilon}{n_i} \Delta^{k-1} a_{n_i} \Sigma_1^{n_i} m > \frac{\epsilon}{3} \Delta^{k-1} a_{n_i} \cdot n_i$ which is impossible for $n_i > \epsilon/3$.

Thus $\Delta^{k-1}a_{n_i-1} < \frac{n_i-1}{n_i}\Delta^{k-1}a_{n_i}$. Hence $\Delta^{k-1}a_{n_i} > \epsilon\Sigma_1^{n_i}\Delta^{k-1}a_1$

$$> \frac{\epsilon}{2}\cdot\frac{(\Delta^{k-1}a_{n_i})^2}{\Delta^{k-1}a_{n_i}-\Delta^{k-1}a_{n_i-1}}, \quad \Delta^k a_{n_i} > \frac{\epsilon}{2}\Delta^{k-1}a_{n_i}, \quad S(k), k=0,\ldots,r.$$

Now by $P(r-1)$, $a_n \neq O(n^{2r-3})$, $\Delta^{r-1}a_n$ is unbounded. It is also monotone; for n sufficiently large, $\Delta^{r-1}a_n = M$, $a_n > \frac{Mn^r}{r!}$ By $S(r)$, $\Delta^{r-1}a_{n_i} > (\epsilon/2)^r a_{n_i} > (\epsilon/2)^r Mn_i^{\,r}/r!$ Thus

$$a_{2n_i} > \frac{(2n_i-n_i)^{r-1}}{r!}\Delta^{r-1}a_{n_i} > \frac{n_i^{\,r-1}}{r!}\frac{Mn_i^{\,r}}{r!}(\epsilon/2)^r =$$

$$= \frac{M}{(r!)^2}(\epsilon/2)^r n_i^{\,2r-1}.$$

By choice of M, the first factor is arbitrarily large. Conversely, for any sequence $a_n \neq O(n)$, $\Delta^r a_n \geq 0$, we can find a sequence $b_n \leq a_n$ such that $\Delta^r b_n \geq 0$ and $b_n \neq O(\Sigma_1^n b_k)$. We do not give the details here.

This completes our discussion of the difference equation $x_{n+1} = x_n + f(n)$ and we turn our attention to

1) $x_{n+1} = x_n + f(x_n)$

and the corresponding differential equation

2) $\frac{d\mu}{dn} = f(\mu)$

We have the following result about the convergence of x_n. Let $f(x)$ be a real or complex-valued function of a real or complex variable. a) If there is a number L such that $f(L) = 0$ and $f'(L)$ exists and $|1+f'(L)| < 1$ (f need not be even continous elsewhere), then for x_1 in a sufficiently small neighborhood of L, $x_n \to L$.

b) If in addition $f(x)$ is continuous, the set of all x_o which result in convergence to L is open.

c) If $|1 + f'(L)| > 1$, the sequence cannot converge to L unless $x_n = L$ for some n.

Proof: a) Let $g(x_n) = x + f(x)$. Then $|g'(L)| = 1-\epsilon$ for some $\epsilon > 0$ and $\left|\frac{x_{n-1}-L}{x_n - L}\right| = \left|\frac{g(x_n)-g(L)}{x_n - L}\right| < 1-\epsilon/2$ for x_n in some neighborhood $\mathcal{N}$ of L by definition of derivative. Hence for x_o in $\mathcal{N}$

$$|x_n-L| < |x_o-L|\ (1-\epsilon/2)^n.$$

b) Let $f_1(x) = f(x)$, $f_{n+1}(x) = f(f_n(x))$. Then for each n, $f_n(x)$ is continuous and $x_n = f_n(x_o)$. If x_o results in convergence to L, for some n, $x_n = f_n(x_o) \in \mathcal{N}$ and hence for all x in some neighborhood of x_o, $f_n(x) \in \mathcal{N}$ and they too result in convergence.

c) For all x in some deleted neighborhood of L,

$$\left|\frac{x_{n+1}-x_n}{x_n - L}\right| = \left|\frac{g(x_n)-g(L)}{x_n - L}\right| > 1+\epsilon/2.$$

The nature of the convergence, or what is the same thing, divergence to infinity, was first discussed by Lublin [7] who obtained a series for the nth term which is rapidly convergent as well as asymptotic in the case where $g(x)$ is a polynomial of degree ≥ 2. DeBruijn [1,ch.8] gives a complete discussion of

convergence in the case where g is analytic. He includes the case $g(x) = a_1 x + \Sigma_2^\infty a_i x^i$ where $|a_1| < 1$ and the very slow case $g(x) = x + \Sigma_2^\infty a_i x^i$ a_i real, $i=2,3,\ldots$. In many cases considered, the solution of 2) either converges or has a pole for some positive value of n. Our main theorems are concerned with cases that lie between these two extremes. To begin, let us consider the difference equation

3) $x_{n+1} = x_n + x_n^{\alpha}$ $\alpha < 1$.

The differential equation is

4) $\frac{d\mu}{dn} = \mu^{\alpha}$

The solutions of 4) are $\mu = [(1-\alpha)(n+c)]^{1/1-\alpha} \sim [(1-\alpha)n]^{1/1-\alpha}$

Thus any two solutions of 4) are asymptotic to each other. We show that they are asymptotic to any solutions of 3) as well.

$$x_{n+1}^{1-\alpha} = x_n^{1-\alpha}(1+\frac{1}{x_n^{1-\alpha}})^{1-\alpha} = x_n^{1-\alpha}(1 + \frac{1-\alpha}{x_n^{1-\alpha}} + o(\frac{1}{x_n^{1-\alpha}})) =$$

$$x_n^{1-\alpha} + 1-\alpha + o(1).x_n^{1-\alpha} = (1-\alpha)n+o(n). \quad x_n \sim [(1-\alpha)n]^{1/1-\alpha}$$

since powers are asymptotic preserving.

Lemma I. Let x_o be non-negative and let $f(x)$ be defined and positive on $[x_o,\infty]$ and let $f(x\,y)/f(x) \leq y$ for $y \geq 1$. Then any solution of 1) or 2) is unbounded.

$x_{n+1} = x_n + f(x_n) > x_n$, $x_1 > x_o \geq 0$. Thus if x_n is bounded, it must converge to a positive number A and $f(A) > 0$.

$x_n = x_o + f(x_o) + \Sigma_1^{n-1} f(x_k) \geq x_o + f(x_o) + \Sigma_1^{n-1} x_k\, f(A)/A$

$\geq x_o + f(x_o) + (n-1)\, x_1\, f(A)/A$, which is unbounded.

The argument for 2) is similar. We now prove

Theorem 3.2. Let x_n be any solution of 1) and $\mu(n)$ be any solution of 2). Let $f(x) > 0$ for $x \geq x_o$, $f(xy)/f(x) \leq y$ for $y \geq 1$ and $f(x) = o(\frac{x}{\log x})$. Then $x_n \sim \mu(n)$.

Suppose first $\mu(0) = x_o > 0$

We have $\frac{d\mu}{f(\mu)} = dn$, $n = \int_{\mu_o}^{\mu} \frac{dt}{f(t)}$

Suppose $\mu(n) = x_n y_n$ with $y_n > 1$ (for some n).

$$\frac{x_n(y_n - 1)}{f(x_n)y_n} \leq \int_{x_n}^{x_n y_n} \frac{1}{f(x_n)\, t/x_n} \leq \int_{x_n}^{\mu(n)} \frac{dt}{f(t)} = n - \int_{x_o}^{x_n} \frac{dt}{f(t)} =$$

$$\Sigma_o^{n-1} [1 - \int_{x_k}^{x_{k+1}} \frac{dt}{f(t)}] \leq \Sigma_o^{n-1} [1 - \int_{x_k}^{x_{k+1}} \frac{dt}{f(x_k)\ t/x_k}] \leq$$

$$\Sigma_o^{n-1} [1 - \frac{x_{k+1} - x_k}{f(x_k)x_{k+1}/x_k}] = \Sigma_1^{n-1} [1 - \frac{x_k}{x_{k+1}}] \leq$$

$\log x_n - \log x_o$.

Hence $\frac{y_n - 1}{y_n} \leq \frac{f(x_n)}{x_n/[\log x_n - \log x_o]} \to 0$. $\frac{\mu(n)}{x_n} = y_n \to 1$ for those n for which $\mu(n) \geq x_n$. We interrupt the proof to show that the hypothesis cannot in general be weakened, then complete the proof with a weaker hypothesis. Consider

$$\frac{d\mu}{dn} = \frac{\mu}{\log \mu},\ \frac{d\mu}{\mu} \log \mu = dn,\ \frac{\log^2 \mu}{2} = n + c,\ \mu = e^{\sqrt{2n+c}} = e^{\sqrt{2n}+o(1)} \sim$$

$\sim e^{\sqrt{2n}}$. Now for x_n. $x_{n+1} = x_n + \frac{x_n}{\log x_n} = x_n(1 + \frac{1}{\log x_n})$,

$$\log x_{n+1} = \log x_n + \frac{1}{\log x_n} - \frac{1}{2 \log^2 x_n} + O(\frac{1}{\log^3 x_n}),$$

$$\log^2 x_{n+1} = \log^2 x_n + 2 - \frac{1}{\log x_n} + O(\frac{1}{\log^2 x_n}) < \log^2 x_n + 2 \text{ for } n$$

large.

$\log^2 x_n \leq 2n$, $\log^2 x_{n+1} \leq \log^2 x_n + 2 - \frac{1}{\sqrt{2n}} + O(\frac{1}{n})$. Thus

$\log^2 x_n < 2n - \sqrt{2n} + O(\log n)$ and

$x_n \leq e^{\sqrt{2n-\sqrt{2n}}} = e^{\sqrt{2n}-1+O(1)} \sim e^{\sqrt{2n}}/e$. We could prove

$x_n \sim e^{\sqrt{2n}}/e$ but this is enough.

We suppose now $x_n > \mu(n)$ in which case $f(x) = o(x)$ is sufficient! (This is no real gain; the only way $\mu(n)$ can lay behind x_n is if $f(x)$ is decreasing most of the time.) Since $\mu(n)$ is divergent, for any $\epsilon > 0$ we can find n so large that $x_m/x_{m-1} < 1+\epsilon$ whenever $x_m \geq \mu(n)$. For some non-negative integer k, $x_{n-k}/1+\epsilon < x_{n-1-k} < \mu(n) \leq x_{n-k}$. We show that for all $m \geq n$, $\mu_m > x_{m-k}/1+\epsilon$. For m=n, we have the result. Suppose true for integers less than m. If $\mu_m > x_{m-1-k}$, then $\mu_{m-k}/1+\epsilon$. If

$$\mu_m \leq x_{m-1-k}, \quad \frac{\mu_m}{x_{m-k}} \geq \frac{\mu_{m-1} + \min[\mu_{m-1},\mu_m]\cdot f(x)}{x_{n-1-k} + f(x_{m-1-k})}$$

$$\geq \frac{\mu_{m-1} + f(x_{m-1-k})(\frac{\mu_{m-1}}{x_{m-1-k}})}{x_{n-1-k} + f(x_{m-1-k})} = \frac{\mu_{m-1}}{x_{m-1-k}} .$$

Now consider $x_{n+1} = x_n + f(x_n,n)$. We know that if f is independent of x, we must have a) osc $[0,n]$ $f(x,n) = o\Sigma_1^n f(x,k)$ and hence uniformly in x, and if f is independent of n, b) $f(xy/n)/f(x,n) \leq y$ and $\frac{f(x,n)}{f(1,n)} = o(\frac{x}{\log x})$ to insure that $\mu(n) \sim x_n$.

We give an example to show that if we do not strengthen a) we must strengthen b) and that the slightest strengthening of the first

condition of a) eliminates the need for the second and then we prove our main theorem.

Consider $x_{n+1} = x_n + \frac{x_n n^2}{\log^2 x_n}$, $\frac{d\mu}{dn} = \frac{\mu n^2}{\log \mu}$ which satisfies a) and b).

We have $\frac{d\mu}{\mu} \log^2\mu = n^2 dn$, $\frac{\log^3\mu}{3} = \frac{n^3+c}{3}$, $\mu = e^{n+o(n^{-2})}$

Let $x_n = e^{k_n n}$. $e^{k_{n+1}(n+1)} = e^{k_n n}(1 + \frac{n^2}{k^2 n^2}) = e^{k_n n}(1 + \frac{1}{k^2})$.

The sequence k_n has a fixed point for $e^k = 1 + \frac{1}{k^2}$ where $k \approx .85$ so $x_n = O(e^{.85n})$.

Lemma II. If $\frac{f(xy)}{f(x)} \leq k(y)$ for all $x,y \geq 1$, then

$\overline{\lim_{x\to\infty}} \frac{\log f(x)}{\log x} \leq \inf \frac{\log k(y)}{\log y}$ which implies that if for a single value of y, $k(y) < y$, then for some $\alpha < 1$, $\frac{f(xy)}{f(x)} < y^\alpha$ and $f(x) = O(x^\alpha)$.

Proof. Let $f(x) = \exp h(\log x)$ and the condition becomes $\exp h(\log x + \log y)/\exp k(\log x) \leq k(y)$. Take logs, let $\log x = u$, $\log y = v$ and $v\, y(v) = \log k(y)$. Then

$k(u + v) - k(u) < vy(v)$

$k(u + 2v) - k(u + v) < vy(v)$

...

$k(u + nv) - k(u + (n-1)v) < vy(v)$

Hence $k(u + nv) < h(v) + nv\, y(v)$

and $\frac{k(u + nv)}{u + nv} < \frac{k(v) + nv\, y(v)}{u + nv}$. Let $n \to \infty$

$\overline{\lim} \frac{h(u)}{u} \leq g(v)$, $\lim \frac{h(u)}{u} \leq \inf g(v)$ whence the result.

We omit the easy proof of

Lemma III. a) If $f(x)$ is non-decreasing and for $a < b$, $f(a) > a$ and $f(b) < b$, then for some $c \in (a,b)$, $f(c) = c$.

Thus, if $f(x)$ is monotone, $f(x) = x$ has a unique smallest "solution".

Let $f(n,x) > 0$ and $f(n,xy)/f(n,x) < y$ for $y \geq 1$.
We cannot say, as before, that x_n is divergent e.g. $f(n,x) = \frac{1}{n^2}$.
We do have the following result:

Lemma IV. a) If $\Sigma_{n=1}^{\infty} f(n,x)$ converges for some $x > 0$, it converges for every larger x and x_n also converges.

b) If $\Sigma_{n=1}^{\infty} f(n,x)$ diverges for x arbitrarily large, it diverges for all $x > 0$ and x_n also diverges.

Proof. a) Suppose $\Sigma_1^{\infty} f(k,x_o)$ converges to S. Then for $x > x_o$, $\Sigma_1^n f(k,x) \leq \frac{x}{x_o} \Sigma_1^n f(k,x_o) < \frac{xS}{x_o}$. Then for n sufficiently large, $\Sigma_n^{\infty} f(k,x_o) < \frac{1}{2x_o}$. Suppose x_n diverges. Then for m sufficiently large, $k_{n+m} > 2x_o$. But

$$x_{n+m}-x_n = \Sigma_n^{n+m-1} f(k,x_k) \leq \Sigma_n^{n+m-1} f(k,x_o) \frac{x_k}{x_o} \leq \frac{x_{n+m}}{x_o} f(k,x_o) <$$

$< \frac{1}{2} x_{n+m}$, a contradiction.

b) The first part follows from a). Suppose x_n converged to L. Then $x_1 \leq x_n < L$.

$$x_n = x_o + \Sigma_1^n f(k,x_k) \geq x_o + \Sigma_1^n f(k,L) \frac{x_k}{L} > x_o + \frac{x_1}{L} \Sigma_1^n f(k,L)$$

which diverges. A corresponding result holds for $\mu(n)$.

A quick corollary. Let $\Sigma_1^{\infty} f(k)$ be divergent and let $h(x)$ be decreasing. Then if $x_{n+1} - x_n = f(n)h(x_n)$,

$x_{n+1} - x_o = \Sigma_1^n f(k)h(x_k)$ is divergent i.e. for any divergent series there is a slower divergent series. In fact, in the variables separate case weaker hypotheses suffice.

Let $x_{n+1} = x_n + f(n)\, g(x_n)$

a) If $g(x) = O(x)$, then the convergence of $\Sigma_1^n f(k)$ implies that of $x_{n+1} = x_1 + \Sigma_1^n f(k)\, g(x_k)$.

b) If $g(x)$ is bounded away from zero on each finite interval, the divergence of $\Sigma_1^n f(k)$ implies that of x_n.

Proof. a) Suppose $\Sigma f(k)$ converges and x_n diverges. There is an M for which $g(x_n) < Mx_n$. For $n > N_o$, $\Sigma_n^\infty f(k) < \frac{1}{2M}$ and $x_{m+n} > 2x_m$.

$$x_{n+m} - x_n = \Sigma_m^{m+n-1} f(k)\, g(x_k) \leq [\Sigma_m^{m+n-1} f(k)]\ \mathrm{Max}[x_m \leq x \leq x_{n+m}]$$

$$< \frac{1}{2m} \cdot M\, x_{m+n} = \tfrac{1}{2}\, x_{m+n} \cdot \quad x_{m+n} < \tfrac{1}{2}\, x_n, \text{ a contradiction.}$$

b) Suppose $\Sigma f(k)$ diverges and x_n converges to A and on $[0,A]$ $g(x) \geq c$. Then $x_n = x_1 + \Sigma_1^{n-1} f(k)\, g(x_k) \geq c\Sigma_1^{n-1} f(k)$ which is divergent.

We now come to our main theorem.

Theorem 3.3. Let $f(x,n) > 0$ for $x > 0$, $n > 0$ and suppose that

a) osc $[0,n]$ $f(x,k) = o(\Sigma_1^n f(x,k))$ uniformly in x

b) For some $\alpha < 1$ and all $x > 0$, $y \geq 1$, $n \geq 0$

$f(xy,n)/f(x,n) \leq y^\alpha$ and $f(x,n) = f(x^-,n)$.

Let x_n be any solution of

5) $x_{n+1} = x_n + f(x_n,n)$

with $x_o > 0$, and let $\mu(n)$ be any solution of

6) $\frac{d\mu}{dn} = f(\mu,n)$

with $\mu(0) > 0$. Then x_n and $\mu(n)$ are both unbounded and

$$\mu(n) \sim x_n \qquad (n \to \infty).$$

For any function which satisfies the first condition of b), the limits $f(x^+,n)$ and $f(x^-,n)$ exist so that the second condition is not restrictive. Also, if the condition is satisfied for $\alpha < 0$, it is satisfied for $\alpha = 0$. We shall assume $\alpha \geq 0$. The first conclusion is immediate. Suppose for some x, $\Sigma_1^\infty f(x,k)$ is convergent. Then osc $[0,n]$ $f(x,k)$ must be identically zero and $f(x,k)$ must be constant in k. The constant must be zero which contradicts the hypothesis. The result follows by Lemma IV. The rest of the proof is long so we have broken it into sections.

I. We show that if $\{l_n\}$ is any sequence with $n \leq l_n \leq n+1$, then $f(x,l_n) = o(\Sigma_1^{n-1} f(x,l_k))$ uniformly in x

For n large, osc $[0,n]f(x,k) < \epsilon\, \Sigma_1^n f(x,k)$

$$= \epsilon\, \Sigma_1^{n-1} f(x,k) + \epsilon\, f(x,n)$$

$$< \epsilon\, \Sigma_1^{n-1} f(x,k) + \epsilon[f(x,n-1) + \text{osc } [n-1,n]f(x,k)]$$

$$< 2\epsilon\, \Sigma_1^{n-1} f(x,k) + \epsilon \text{ osc}[1,n]\, f(x,k). \quad \text{Therefore}$$

$$\text{osc } [0,n]\, f(x,k) < \frac{2\epsilon}{1-\epsilon}\, \Sigma_1^{n-1} f(x,k).$$

$$\Sigma_1^{n-1} f(x,k) < \Sigma_1^{n-1}[f(x,l_k) + \text{osc}[k,k+1]f(x,l)]$$

$$= \Sigma_1^{n-1} f(x,l_k) + \text{osc}[1,n]f(x,k)$$

$$< \Sigma_1^{n-1} f(x,l_k) + \epsilon\Sigma_1^{n-1} f(x,l_k)$$

for n sufficiently large. Thus

$$\Sigma_1^{n-1} f(x,k) < \frac{1}{1-\epsilon} \Sigma_1^{n-1} f(x,1_k)$$

Let N be so large that for $n > N$, $\operatorname{osc}[0,n]f(x,k) < \frac{\epsilon}{2} \Sigma_1^{n-1} f(x,k)$ and $\frac{1}{N-1} < \frac{\epsilon}{2}$. Then for each $n > N$, either

a) $f(x,0) \leq 2\operatorname{osc}[0,n]f(x,k) = 2\,\frac{\epsilon}{2}\,\Sigma_1^{n-1} f(x,k) = \epsilon\,\Sigma_1^{n-1} f(x,k)$

or b) $\operatorname{osc}[0,n]\, f(x,k) < \frac{1}{2} f(x,0)$ in which case

$f(x,k) \geq f(x,0) - \operatorname{osc}[0,k]f(x,1) > \frac{1}{2} f(x,0)$ for all $k \leq n$

and $f(x,0) < \frac{2}{n-1} \Sigma_1^{n-1} < \epsilon\, \Sigma_1^{n-1} f(x,k)$

Then $f(x,n) < f(x,0) + \operatorname{osc}[0,n]f(x,k) < \frac{3}{2}\,\epsilon\, \Sigma_1^{n-1} f(x,k)$.

$$< \frac{3\epsilon}{2(1-\epsilon)} \Sigma_1^{n-1} f(x,1_k)$$

and
$$\begin{aligned}
f(x,1_n) &< f(x,n) + \operatorname{osc}[n,n+1]f(x,k) \\
&< \epsilon\, \Sigma_1^{n-1} f(x,k) + \operatorname{osc}[0,n+1]f(x,k) \\
&< \epsilon\, \Sigma_1^{n-1} f(x,k) + \epsilon\, \Sigma_1^{n} f(x,k) \\
&= 2\epsilon\, \Sigma_1^{n-1} f(x,k) + \epsilon\, f(x,n) \\
&< [2\epsilon + \frac{3\epsilon^2}{2}]\, \Sigma_1^{n-1} f(x,k) \\
&< (\frac{2\epsilon + (3/2)\epsilon^2}{1-\epsilon})\, \Sigma_1^{n-1} f(x,1_k) \\
&< 6\epsilon\, \Sigma_1^{n-1} f(x,1_k) \quad \text{for } \epsilon < \tfrac{1}{2}
\end{aligned}$$

It follows that there is an absolute constant A_1 such that for all x,

$f(x,0) < A_1 \Sigma_1^{n-1} f(x,k)$ and $f(x,n) < A_1\, \Sigma_1^{n-1} f(x,k)$ $n > 1$.

II. We show that $x_{n+1} \sim x_n$ uniformly for $x_o \geq c > 0$.

$$x_1 - x_o = f(x_o,0) \leq f(c,0)(x_o/c)^\alpha \leq f(c,0)(x_o/c)$$

$$x_1 < (1 + f(c,0)/c)\, x_o < A_3 x_o, \quad A_3 \text{ to be chosen later.}$$

Assume $x_{k+1} \leq A_3 x_k$ for $k < n$. Then

$$\begin{aligned} x_{n+1} - x_n &= f(x_n,n) \\ &< A_1 \Sigma_1^{n-1} f(x_n,k) \\ &\leq A_1 \Sigma_1^{n-1} f(x_k,k)(x_n/x_k)^\alpha \\ &= A_1 x_n^\alpha \sum_1^{n-1} (x_{k+1}-x_k)/x_k^\alpha \\ &< A_1 A_3^\alpha x_n^\alpha \sum_1^{n-1} (x_{k+1}-x_k)/x_{k+1}^\alpha \quad \text{by induction hypothesis} \\ &\leq A_1 A_3^\alpha x_n^\alpha (x_n^{1-\alpha} - x_1^{1-\alpha})/(1-\alpha) \quad \text{by comparison with integral,} \\ &\leq A_1 A_3^\alpha x_n/1-\alpha \end{aligned}$$

Thus $x_{n+1} \leq (1 + A_1 A_3^\alpha/(1-\alpha))x_n < A_3 x_n$ for A_3 sufficiently large.

We have also shown

7) $\Sigma_1^{n-1} f(x_n,k) \leq \dfrac{A_3^\alpha}{1-\alpha} x_n$

Now choose N so large that for $n \geq N$, $f(x,n) < \dfrac{\epsilon(1-\alpha)}{A_3^\alpha} \Sigma_1^{n-1} f(x,k)$.

Repeating the argument for $n \geq N$ with $\dfrac{\epsilon(1-\alpha)}{A_3^\alpha}$ in place of A_2, we have $0 < x_{n+1} - x_n < \epsilon x_n$.

III. Let u_n and v_n be any two solutions of 5). We show that $u_n \sim v_n$. Suppose $u_n > v_n$. Let N be so large that $v_{n+1}/v_n < 1+\epsilon$

a) If $u_{n+1} < v_{n+1}$, we have $v_n < u_n < u_{n+1} < v_{n+1} < (1+\epsilon)v_n$

$1 < u_n/v_n < 1+\epsilon$. Furthermore,

$1 < v_{n+1}/u_{n+1} < v_{n+1}/v_n < 1+\epsilon$, so whenever the order changes, both ratios are less than $1+\epsilon$.

b) Suppose $u_n > v_n$ for all n.

$$\frac{u_{n+1}}{v_{n+1}} = \frac{u_n + f(u_n,n)}{v_{n+1}} \leq \frac{u_n + (u_n/v_n)^{\alpha} f(v_n,n)}{v_{n+1}} = \frac{u_n + (u_n/v_n)^{\alpha}(v_{n+1}-v_n)}{v_{n+1}}$$

$$= \frac{u_n}{v_n}\left[\frac{v_n + (v_n/u_n)^{1-\alpha}(v_{n+1}-v_n)}{v_{n+1}}\right] < \frac{u_n}{v_n}$$

Thus if $u_n/v_n < 1+\epsilon$ for any n, the same is true for all succeeding n.

Suppose $u_n/v_n > 1+\epsilon$ for all n. Rearranging the last expression we obtain

$$\frac{u_{n+1}}{v_{n+1}} \leq \frac{u_n}{v_n}\left[1 - \frac{(v_{n+1}-v_n)[1-(v_n/u_n)^{1-\alpha}]}{v_{n+1}}\right]$$

$$< \frac{u_n}{v_n}\left[1 - \frac{c(v_{n+1}-v_n)}{v_{n+1}}\right], \text{where } 1 > c > 1 - \left(\frac{1}{1+\epsilon}\right)^{1-\alpha} > 0.$$

$$\frac{u_{n+m}}{v_{n+m}} = \frac{u_n}{v_n}\prod_{k=n}^{n+m}\left[1 - \frac{c(v_{k+1}-v_k)}{v_{k+1}}\right]$$

But the product diverges to zero since the product

$$\Pi_{k=n}^{n+m} 1 - \frac{v_{k+1}-v_k}{v_{k+1}} = \Pi_{k=n}^{n+m} \frac{v_k}{v_{k+1}} = \frac{v_n}{v_{n+m}}$$

diverges to zero. (The convergence of Π 1-cx, Σ cx, Σ x, Π 1-x are equivalent provided $0 < cx < 1$, $0 < x < 1$.) Thus for m large, $u_{n+,}/v_{n+m} < 1+\epsilon$.

IV. We establish bounds for the solutions of 5) and 6). The previous step allows us to assume $x_o = \mu(0) > 0$. Let $z_o = y_o = \mu(0)$ also. Define

$$y_{n+1} = \inf\{y : y = y_n + \sup\{f(x,k) : y_n \leq x \leq y,\ n \leq k \leq n+1\}$$

$$z_{n+1} = \inf\{z : z = z_n + \inf\{f(x,k) : z_n \leq x \leq z,\ n \leq k \leq n+1\}$$

The existence of y_{n+1} is assured by the lemma on monotone functions and the fact that $f(x,n) = O(x^{\alpha})$ for a closed and bounded n-interval. That of z_{n+1} is assured by continuity from the left in x.

Extend these definitions to $[0,\infty]$ by making y and z polygonal functions of n.

$\mu(0) = y(0)$. Suppose $\mu \leq y$ on $[0,n]$. If $\mu(n+1) \leq y(n)$, then $\mu \leq y$ on $[n,n+1]$. Suppose $\mu(n+\eta) = y(n)$ $0 \leq \eta < 1$ and for some θ, $\eta < \theta < 1$ $\mu(n+\theta) > y(n+\theta)$. Then for some ξ in (η,θ)

$$f(\xi) = \mu'(\xi) = \frac{\mu(n+\theta)-\mu(n+\eta)}{\theta - \eta} > \frac{y(n+\theta)-y(n,\eta)}{\theta - \eta}$$

$$\geq \frac{y(n+\theta)-y(n)}{\theta} = y(n+1)-y(n),$$ which contradicts the definitions of y_{n+1}. A similar argument shows $\mu \geq z$.

$x_o \geq z_o$. Suppose $x_n \geq z_n$. If $x_n \geq z_{n+1}$, then $x_{n+1} > z_{n+1}$.
If $x_n < z_{n+1}$, $x_{n+1} = x_n + f(x_n,n) \geq z_n + f(x_n,n) \geq z_{n+1}$.
$x_o \leq y_o$. If $x_n \leq y_n$, either $x_{n-1} \leq y_{n+1}$ or $x_{n+1} < (1+\epsilon)x_n \leq$
$\leq (1+\epsilon)y_n < (1+\epsilon)y_{n+1}$.

If $x_n > y_n$, $\dfrac{x_{n+1}}{y_{n+1}} \leq \dfrac{x_n + f(x_n,n)}{y_n + f(y_n,n)} < \dfrac{x_n + (x_n/y_n)f(y_n,n)}{y_n + f(y_n,n)} = \dfrac{x_n}{y_n}$

Thus, although y_n is not an upper bound, it is asymptotic to x_n whenever it is not.

We show z_n diverges. Suppose z_n converges to A.

$$z_{n+1} = z_n + \mathrm{Min}[z_n, z_{n+1}; n, n+1]f(z,k) = z_n + f(v_n, m_n)$$

$$\geq z_n + f(A,m_n)\left(\frac{v_n}{A}\right)^{\alpha} > z_n + [f(A,n+1) - \mathrm{osc}[n,n+1]f(A,k)]\left(\frac{z_n}{A}\right)^{\alpha}$$

$$> z_n + \tfrac{1}{2}[f(A,m_n) - \mathrm{osc}[n,n+1]f(A,k)] \text{ for n sufficiently large.}$$

$$z_{n+N} - z_n > \tfrac{1}{2}[\Sigma_{n+1}^{n+N} f(A,k) - \mathrm{osc}[n,n+k]f(A,k)]$$

$$> \tfrac{1}{2}[\Sigma_{n+1}^{n+N} f(A,k) - \tfrac{1}{2}\Sigma_{1}^{n+N} f(A,k)] \text{ for n sufficiently large}$$

$$= \tfrac{1}{4}\Sigma_{n+1}^{n+N} f(A,k) - \tfrac{1}{2}\Sigma_{1}^{n} f(A,k)$$

which diverges. This proves the assertion that $\mu(n)$ diverges.

V. We show that $y_{n+1} \sim y_n$ and $z_{n+1} \sim z_n$

$$y_{n+1} - y_n = \mathrm{Max}[y_n, y_{n+1}; n, n+1] f(x,k) = f(w_n, l_n)$$

$$\leq A_2 \Sigma_1^{n-1} f(w_n, l_k)$$

$$\leq A_2 \Sigma_1^{n-1} f(y_{k+1}, l_k)(w_n/y_{k+1})^\alpha$$

$$\leq A_2 \Sigma_1^{n-1} f(w_k, l_k)(w_n/y_{k+1})^\alpha$$

$$\leq A_2 y_{n+1}^\alpha \Sigma_1^{n-1} (y_{k+1} - y_k)/y_{k+1}^\alpha$$

$$\leq A_2 y_{n+1}^\alpha (y_n^{1-\alpha} - y_1^{1-\alpha})/1-\alpha$$

$$< A_2 y_{n+1}^\alpha y_n^{1-\alpha}/1-\alpha$$

If $y_{n+1} = c_n y_n$, we have $(c-1)y_n \leq A_2 c^\alpha y_n/(1-\alpha)$ or

$\dfrac{c_{n-1}}{c_n^\alpha} \leq \dfrac{A_2}{1-\alpha} \cdot c_n$ is uniformly bounded. Further, for n sufficiently large, we can replace A_2 by ϵ and get c_n as close to 1 as we please.

It follows that $\Sigma_1^{n-1} f(y_n, n) < A_4 y_n$ for some $A_4 \cdot z_1 < Bz_o$ for some B. Suppose $z_{k+1} < Bz_k$ for $k < n$.

$$z_{n+1} - z_n = \mathrm{Min}[z_n, z_{n+1}; n, n+1] f(x,k) = f(v_n, m_n) \leq f(z_n, m_n)$$

$$\leq A_2 \Sigma_1^{n-1} f(z_n, m_k)$$

$$\leq A_2 \Sigma_1^{n-1} f(v_k, m_k)(z_n/v_k)^\alpha$$

$$\leq A^2 z_n^\alpha \Sigma_1^{n-1} (z_{k+1} - z_k)/z_k^\alpha$$

$$\leq A_2 B^\alpha \Sigma_1^{n-1} (z_{k+1} - z_k)/z_{k+1}^\alpha \text{ by induction hypothesis,}$$

$$\leq A_2 B^\alpha z_n^\alpha (z_n^{1-\alpha} - z_1^{1-\alpha})/1-\alpha$$

$$< A_2 B^\alpha z_n/1-\alpha < Bz_n \quad \text{for B sufficiently large.}$$

Now choose N so large that for $n \geq N$, A_2 can be replaced by $\epsilon(1-\alpha)/B^{\alpha}$ and repeat the argument.

VI. We now show $y_n \sim z_n$

Suppose for $n > N_o$, $\mathrm{osc}[0,n]f(x,k) < \eta \Sigma_1^n f(x,k)$, $y_{n+1}/y_n < 1+\eta$ and $z_{n+1}/z_n < 1+\eta$. Since y_n and z_n are monotone, it suffices to show the asymptotic relation for a subsequence $\{n(j)\}$ defined as follows:

$n(0) = N_o$, $n(j+1)$ is the first integer which $z_{n(j+1)} \geq y_{n(j)}$.

It can only worsen the relation to assume that equality holds in each case, and we shall do so. Hence

$$R_j = y_{n(j)}/z_{n(j)} = y_{n(j)}/y_{n(j-1)} = z_{n(j+1)}/z_{n(j)}$$

Let $V = \mathrm{osc}[n(j),n(j+1)]f(y_{n(j)},k)$, $C = \Sigma_1^{n(j)} f(y_{n(j)},k)$,

$$= \Sigma_{n(j)+1}^{n(j+1)} f(y_{n(j)},k)$$

$$\begin{aligned}
D + V &= \Sigma_{k=n(j)+1}^{n(j+1)} [f(y_{n(j)},k) + \mathrm{osc}[k+1,k]f(y_{n(j)},k)] \\
&\geq \Sigma_{k=n(j)+1}^{n(j+1)} \mathrm{Max}\,[k-1,k]\, f(y_{n(j)},1) \\
&\geq \Sigma_{k=n(j)+1}^{n(j+1)} \mathrm{Max}[y_{k-1},y_k;k-1,k]f(x,1)(y_{n(j)}/y_k)^{\alpha} \\
&= y_{n(j)}^{\alpha} \Sigma_{n(j)+1}^{n(j+1)} (y_k-y_{k-1})/y_k^{\alpha} \\
&\geq y_{n(j)}^{\alpha} \Sigma_{n(j)+1}^{n(j+1)} (y_k-y_{k-1})/y_{k-1}^{\alpha}(1+\eta)^{\alpha} \\
&\geq y_{n(j)}^{\alpha} (y_{n(j+1)}^{1-\alpha}-y_{n(j)}^{1-\alpha})/(1-\alpha)(1+\eta)^{\alpha} \\
&= y_{n(j)} [(\frac{y_{n(j+1)}}{y_{n(j)}})^{1-\alpha} -1]/(1-\alpha)(1+\eta)^{\alpha}
\end{aligned}$$

$$D-V = \Sigma_{k=n(j)+1}^{n(j+1)} [f(y_{n(j)},k) - \mathrm{osc}[k-1,k]f(y_{n(j)},1)]$$

$$\leq \Sigma_{k=n(j)+1}^{n(j+1)} \mathrm{Min}[k-1,k]f(y_{n(j)},1)$$

$$\leq \Sigma_{k=n(j)+1}^{n(j+1)} \mathrm{Min}[z_{k-1},z_k;k-1,k]f(x,1)(y_{n(j)}/z_{k-1})^{\alpha}$$

$$= y_{n(j)}^{\alpha} \Sigma_{n(j)+1}^{n(j+1)} (z_k - z_{k-1})/z_{k-1}^{\alpha}$$

$$\leq y_{n(j)}^{\alpha} (1+\eta)^{\alpha} \Sigma_{n(j)+1}^{n(j+1)} (z_k - z_{k-1})/z_k^{\alpha}$$

$$\leq y_{n(j)}^{\alpha} (1+\eta)^{\alpha} (z_{n(j+1)}^{1-\alpha} - z_{n(j)}^{1-\alpha})/(1-\alpha)$$

$$= y_{n(j)}(1+\eta)^{\alpha} [1 - (z_{n(j)}/z_{n(j+1)})^{1-\alpha}]/(1-\alpha)$$

$V < \eta(C+D)$, $C < A_4 y_{n(j)}$

Let $y_{n(j)}/z_{n(j)} = R_j$

$$D + \eta(A_4 y_{n(j)} + D) > D+V > y_{n(j)}[(y_{n(j+1)}/y_{n(j)})^{1-\alpha} - 1]/(1-\alpha)(1+\eta)^{\alpha}$$

$$D - \eta(A_4 y_{n(j)} + D) < D-V < y_{n(j)}[1 - (z_{n(j)}/z_{n(j+1)})^{1-\alpha}](1+\eta)^{\alpha}/(1-\alpha)$$

$$\frac{1}{1+\eta}\left(\frac{R_{j+1}^{1-\alpha} - 1}{(1+\eta)^{\alpha}(1-\alpha)} - \eta A_4\right) \leq \frac{D}{y_{n(j)}} < \frac{1}{1-\eta}\left(\frac{(1+\eta)^{\alpha}(1 - 1/R_j^{1-\alpha})}{1-\alpha} + \eta A_4\right)$$

Let $R_j^{1-\alpha} = 1+r_j$.

$$r_{j+1} < \left[\frac{1+\eta}{1-\eta}\left(\frac{(1+\eta)^{\alpha}(1 - \frac{1}{1+r_j})}{1-\alpha} + \eta A_4\right) + \eta A_4\right](1+\eta)^{\alpha}(1-\alpha)$$

$$< (1+\epsilon)\left(\frac{r_j}{1+r_j}\right) + \epsilon^2 \text{ for } \eta \text{ sufficiently small.}$$

Now the sequence S_n defined by $S_{n+1} = (1+\epsilon)\frac{S_n}{1+S_n} + \epsilon^2$ converges to the fixed point $\frac{1}{2}[\epsilon + \epsilon^2 + \sqrt{5\epsilon^2 + 2\epsilon^3 + \epsilon^4}]$ so that for n sufficiently large, $r_n < S_n < 2\epsilon$. This completes the proof.

It is apparent from the proof that the difference equations $x_{n+1}-x_n = f(x_n,m_n)$ and $x_{n+1}-x_n = f(x_{n+1},m_n)$, $n \leq m_n \leq n+1$ also have solutions asymptotic to the solutions of 5).

It also follows by slight modifications of the proof that if, instead of conditions a), f satisfies condition

a') $\operatorname{osc}[0,n]\ f(x,k) = O(\Sigma_1^{n-1} f(x,k))$ uniformly in x,

then $x_n \approx \mu(n)$.

We cannot expect a similar result for systems of equations even when the functions involved are decreasing, for even in the simple system $\frac{d\mu}{dn} = \frac{1}{v}$, $\frac{dv}{dn} = \frac{1}{\mu}$, the asymptotic behavior of the solution

$$\mu = c\sqrt{n+d},\ v = \sqrt{n+d}/c,\ c = \sqrt{\mu(0)/v(0)},\ d = \mu(0)v(0)$$

depends on the initial conditions.

We give an example which illustrates what happens as α approaches and finally exceeds 1.

Consider

9) $x_{n+1} = x_n + x_n^{\alpha}/n^{\alpha}$

For $x_1 = 1$ and each α, $x_n = n$ is a solution

If $\alpha < 0$,

$$x_{n+1}^{1-\alpha} = x_n^{1-\alpha}\left(1 + \frac{1}{n^{\alpha} x_n^{1-\alpha}}\right)^{1-\alpha} \geq x_n^{1-\alpha}\left(1 + \frac{1}{n^{\alpha} x_n^{1-\alpha}}\right) = x_n^{1-\alpha} + (1-\alpha)n^{-\alpha}$$

Summing, $x_{n+1}^{1-\alpha} \geq n^{1-\alpha} + O(n^{-\alpha})$. Hence the second order terms in the binimal $= O\left(\frac{1}{n^{2\alpha} x_n^{1-\alpha}}\right) = O\left(\frac{1}{n^{1+\alpha}}\right)$ which sums to $O(n^{-\alpha})$.

Hence $x_n^{1-\alpha} = n^{1-\alpha} + O(n^{-\alpha})$,

$$x_n = n\left(1 + O\left(\frac{1}{n}\right)\right)^{\frac{1}{1-\alpha}} = n + O(1).$$

If $\alpha = 0$, $x_{n+1} = x_n + 1$, $\quad x_n = n + (x_1 - 1)$.

If $0 < \alpha < 1$, $x_{n+1}^{1-\alpha} = x_n^{1-\alpha}(1 + \frac{1}{n^\alpha x_n^{1-\alpha}})^{1-\alpha} \leq$

$\leq x_n^{1-\alpha}(1 + \frac{1-\alpha}{n^\alpha x_n^{1-\alpha}}) = x_n^{1-\alpha} + \frac{1-\alpha}{n^\alpha}$. Summing,

$x_n^{1-\alpha} \leq n^{1-\alpha} + O(1)$. $x_{n+1}^{1-\alpha} \geq x_n^{1-\alpha}(1 + \frac{1-\alpha}{n^\alpha x_n^{1-\alpha}} - \frac{\alpha(1-\alpha)}{2n^{2\alpha} x_n^{2(1-\alpha)}})$

$= x_n^{1-\alpha} + \frac{1-\alpha}{n^\alpha} - \frac{\alpha(1-\alpha)}{2n^{2\alpha} x_n^{1-\alpha}} \geq x_n^{1-\alpha} + \frac{1-\alpha}{n^\alpha} - \frac{\alpha(1-\alpha)}{2n^{2\alpha} x_1^{1-\alpha}}$. Summing,

$x_{n+1}^{1-\alpha} \geq n^{1-\alpha} + O(1)$. Therefore,

$x_n^{1-\alpha} = n^{1-\alpha} + O(1)$, and

$x_n = n(1 + O(n^{\alpha-1}))^{1-\alpha} = n(1 + O(n^{\alpha-1})) = n + O(n^\alpha)$

The corresponding differential equation

10) $\frac{d\mu}{dn} = \frac{\mu^\alpha}{n^\alpha}$

has the solution $\mu^{1-\alpha} = n^{1-\alpha}(\mu_1^{1-\alpha} - 1)$

These are the cases covered by the theorem: x_n and $\mu(n)$ agree and their growth is independent of initial conditions.

$\alpha=1$. $x_{n+1} = x_n + x_n/n = x_n(1 + \frac{1}{n})$ $\quad x_n = nx_1$

$\frac{d\mu}{dn} = \frac{\mu}{n}$, $\log \mu - \log \mu_1 = \log n$, $\mu = n\mu_1$

Hence the two solutions agree if $x_1 = \mu_1$ but this is not always the case for $\alpha=1$.

Consider $x_{n+1} = x_n + x_n/n^\beta = x_n(1 + 1/n^\beta)$, $\frac{1}{3} < \beta < \frac{1}{2}$.

$$\log x_n = \log x_1 + \Sigma_1^{n-1} \log(1 + 1/k^\beta) = \frac{n^{1-\beta}}{1-\beta} - \frac{n^{1-2\beta}}{1-2\beta} + C + o(1)$$

whereas $\frac{d\mu}{dn} = \frac{\mu}{n^\beta}$, $\log \mu - \log \mu_1 = \frac{n^{1-\beta}-1}{1-\beta}$. Hence x_n is smaller by a factor of $e^{\frac{n^{1-2\beta}}{1-2\beta}}$.

Now consider $\alpha > 1$. We show that for $x_1 < 1$, x_n and $\mu(n)$ converge at similar rates and for $x_1 > 1$, $\log x_n \approx \alpha^n$ while $\mu(n)$ has a pole at some finite positive number.

We have $\mu_n = [(\mu_1^{1-\alpha} - 1) + n^{1-\alpha}]^{-\frac{1}{\alpha-1}}$ which shows that if

$\mu_1 < 1$, $\mu_n \uparrow (\mu_1^{1-\alpha} - 1)^{-\frac{1}{\alpha-1}}$, and if $\mu_1 > 1$, there is a pole

at $n = (1 - \mu_1^{1-\alpha})^{-\frac{1}{\alpha-1}}$. Suppose $x_1 < 1$. We show there exist

constants $c > 0$ and $2^{\alpha-1} > B > 1$ such that

$$x_n \leq (c + \frac{B^{\alpha/\alpha-1}}{n^{\alpha-1}})^{-\frac{1}{\alpha-1}} = f(n) \text{ and a fortiori } x_n < c^{-\frac{1}{\alpha-1}}.$$

Suppose true for n. Then

$$x_{n+1} = x_n + (x_n/n)^\alpha \leq f(n) + (f(n)/n)^\alpha \leq f(n+1) \text{ if}$$

$$f(n+1) - f(n) \geq f'(n+1) = (\frac{B}{c(n+1)^{\alpha-1}+B^{\alpha/\alpha-1}})^{\alpha/\alpha-1} \geq (\frac{f(n)}{n})^\alpha = (cn^{\alpha-1}+B^{\alpha/\alpha-1})^{-\alpha/\alpha-1}$$

That is, if $B \geq \frac{c(n+1)^{\alpha-1} + B^{\alpha/\alpha-1}}{c\, n^{\alpha-1} + B^{\alpha(\alpha-1)}}$, which is true for all n if

$B \geq \frac{c\, 2^{\alpha-1} + B^{\alpha/\alpha-1}}{c + B^{\alpha/\alpha-1}}$ or $c = \frac{B^{\alpha/\alpha-1}(B-1)}{2^{\alpha-1} - B}$. Now choose B so that

$$x_1 = \frac{1}{(c+B^{\alpha/\alpha-1})^{1/\alpha-1}} = \frac{1}{B^\alpha}\left(\frac{2^{\alpha-1} - B}{2^{\alpha-1} - 1}\right)^{1/\alpha-1}.$$

Suppose $x_1 > 1$. $x_{n+1} > x_n^\alpha/n^\alpha$, $\log x_{n+1} > \alpha \log x_n - \alpha \log n$

Let $\alpha^n y_n = \log x_n$. $\alpha^{n+1} y_{n+1} > \alpha.\alpha^n y_n - \alpha \log n$, $y_{n+1} > y_n - \frac{\log n}{\alpha^n}$

$$y_n > y_k - \Sigma_k^n \frac{\log i}{\alpha^i}, \quad \frac{\log x_n}{\alpha^n} > \frac{\log x_k}{\alpha^k} - \Sigma_k^n \frac{\log i}{\alpha^i} >$$

$$> \frac{\log x_k}{\alpha^k} - \Sigma_k^\infty \frac{\log i}{\alpha^i}$$

If for some k, $\frac{\log x_k}{\alpha^k} - \Sigma_k^\infty \frac{\log i}{\alpha^i}$ $c > 0$, then $\log x_n > c\,\alpha^n$.

We need $\log x_k > \alpha^k \Sigma_k^\infty \frac{\log i}{\alpha^i} = \Sigma_0^\infty \frac{\log (i+k)}{\alpha^i} = \Sigma_0^k \frac{\log(i+k)}{\alpha^i} +$

$+ \Sigma_k^\infty \frac{\log (i+k)}{\alpha^i}$ which is less than $\log 2k\, \Sigma_0^k \frac{1}{\alpha^i} + \Sigma_k^\infty \frac{\log 2i}{\alpha^i} <$

$< \frac{\alpha}{\alpha-1} \log k + \log D$. Let $x_n/n = z_n$, $x_1 = z_1$. Then 9) becomes

$$(n+1)z_{n+1} = nz_n + z_n^\alpha, \quad z_{n+1} = z_n\left(\frac{n + z_n^{\alpha-1}}{n+1}\right)$$

i) $z_{n+1} > z_n$: $z_2 = z_1\left(\frac{1+z_1^{\alpha-1}}{1+1}\right) > z_1$ since $z_1 = x_1 > 1$

Assume true for integers $< n$. $z_{n+1} = z_n\left(\frac{1+z_n^{\alpha-1}}{1+1}\right) > z_n\left(\frac{1+1}{1+1}\right) = z_n$

ii) Let $z_1^{\alpha-1} = 1+\epsilon$. $z_n^{\alpha-1} > 1+\epsilon$. We say

$z_n > \Pi_{k=1}^n (1 + \frac{\epsilon}{k}) \sim \frac{n^\epsilon}{\Gamma(\epsilon)}$. Assume true for n.

$$Z_{n+1} > \Pi_1^n (1 + \frac{\epsilon}{k})\left(\frac{n+(1+\epsilon)}{n+1}\right) = \Pi_1^{n-1} (1+ \frac{\epsilon}{k})(1 + \frac{\epsilon}{n+1}) = \Pi_1^{n+1}(1 + \frac{\epsilon}{k})$$

iii) For $n > n_o$, $z_n^{1-\alpha} > \frac{\alpha}{\alpha-1} \cdot \quad z_{n_o+n} > z_{n_o} \prod_{n_o+1}^{n_o+n}(1 - \frac{\alpha/\alpha-1}{k}) >$

$> E\, n^{\alpha/\alpha-1}$, $E > 0$

$x_n > nE\, n^{\alpha/\alpha-1} > Dn^{\alpha/\alpha-1}$ for n sufficiently large. Thus

$\alpha^n = O(\log x_n)$. Next we show that for some $F > 0$, $\log x_n < F\, \alpha^n$.

$x_{n+1} < x_n + x_n^{\alpha} < 2x_n^{\alpha}$, $\log x_{n+1} < \alpha \log x_n + \log 2$.

$y_{n+1} < y_n + \log 2/\alpha^n$

$$x_n/\alpha^n = y_n < y_1 + \Sigma_1^n \frac{\log 2}{\alpha^n} < y_1 + \Sigma_1^{\infty} \frac{\log 2}{\alpha^n} = \frac{x_1}{\alpha^1} + \frac{\alpha \log 2}{1-\alpha} .$$

We note in passing that it is possible to violate the hypothesis of the theorem and still have $x_n = \mu_n$. Consider

$x_{n+1} = x_n + 1 + 2^{[n]} \sin 2\pi n$. Then $x_n = \mu_n = x_o + n$.

We also note that the solutions of 9) and 10) for $\alpha \leq 0$ are more than just asymptotic. In fact, their differences are bounded. Sufficient conditions for this are given in Theorem 3.4. Let $f(x,n) \geq 0$ be non-increasing in x and n. Let x_n be any solution of 5) and $\mu(n)$ be any solution of 6). Then

$$|x_n - \mu(n)| \leq \mathrm{Max}[f(x_o,0), f(\mu(0),0), |\mu(0) - x_o|] + f(\mu(0),0).$$

First, if x_n and y_n are any two solution of 5),

11) $|x_n - y_n| \leq \mathrm{Max}\ (f(x_o,0), f(y_o,0), |x_o - y_o|)$.

Chapter 4. Some Higher Order Non-linear Difference Equations

As a first example of a non-linear difference equation of higher order, we consider $x_{n+1} = x_n + 1/\Sigma_1^n x_k$. The corresponding integral equation is $\frac{d\mu}{dn} = 1/\int_1^n \mu(t)dt$. Differentiating and eliminating the integral, we get $\frac{d^2\mu}{dn^2} + (\frac{d\mu}{dn})^2 = 0$. Let

$\rho = \frac{d\mu}{dn}$ to get $\rho \frac{d\rho}{d\mu} + \rho\mu^2 = 0$. Reject $\rho = 0$ and find

$\rho = ce^{-\mu^2/2} = \frac{d\mu}{dn}$, $\quad n = \frac{1}{c}\int_{\mu_o}^{\mu} e^{t^2/2}\, dt$. Let

$v = t^2/2, t = \sqrt{2v}$, $dt = \frac{dv}{\sqrt{2v}}$

$$n = \frac{1}{c}\int_{\mu_o^2/2}^{\mu^2/2} e^v \frac{dv}{\sqrt{2v}} \sim \frac{1}{c} e^v/\sqrt{2v}\Big|^{\mu^2/2} = \frac{1}{c\mu} e^{\mu^2/2}$$

$n\, c\, \mu \sim e^{\mu^2/2}$, $\mu \sim \sqrt{2 \log (n\, c\, \mu)} \sim \sqrt{2 \log n}$ since μ is obviously $o(n)$. We prove a more general result for the difference equation. Let $f(n)$ be slowly oscillating. Let

$x_{n+1} = x_n + 1/\Sigma_1^n f(k)x_k$. Then $x_n \sim \sqrt{2\Sigma_1^n \frac{1}{k\, f(k)}}$. To simplify the argument, we omit some factors of $(1+\epsilon)$.

$x_{n+1} > x_n$. $\quad x_{n+1} - x_n > 1/x_n\, \Sigma_1^n f(k) \sim 1/n\, f(n)\, x_n$. Hence

$x_n > \sqrt{2\Sigma_1^n \frac{1}{k\, f(k)}}$. Thus $x_{n+1} - x_n < 1/\Sigma_1^n f(k) \sqrt{2\Sigma_1^k 1/j\, f(j)}$ but

$\Sigma_1^k 1/j\, f(j)$ is itself slow and the product with $f(k)$ is also slow. Hence the last expression is $\sim 1/n\, f(n) \sqrt{2\Sigma_1^n 1/k\, f(k)}$.

Summing, $x_{N+1} < \Sigma_1^N \left(2\Sigma_1^n \frac{1}{kf(k)}\right)^{-\frac{1}{2}} \frac{1}{nf(n)}$. Let $G(n) = \Sigma_1^n \frac{1}{kf(k)}$.

Then $x_{N+1} < \Sigma_1^N (2G(n))^{-\frac{1}{2}} (G(n)-G(n-1)) \sim \sqrt{2G(N)} = \sqrt{2\Sigma_1^N \frac{1}{nf(n)}}$

by theorem 3.0, since $G(n)$ is monotone and slow, if it diverges.

If $G(n)$ is bounded, so is x_n.

If $f(n) = 1$, $\sqrt{2\Sigma_1^N 1/n} \sim \sqrt{2 \log N}$.

We consider an extension of the previous problem.

1) $x_{n+1} - x_n = 1/\Sigma \frac{f(k)}{k^\alpha} x_k$ where $0 < \alpha < 2$ and $f(n)$ is slow.

$$\frac{1}{x_{n+1}-x_n} = \frac{f(n)}{n^\alpha} x_n + \Sigma_1^{n-1} \frac{f(k)}{k^\alpha} x_k = \frac{f(n)}{n^\alpha} x_n + \frac{1}{x_n - x_{n-1}}$$

Let $x_o = 0$, $z_n = 1/(x_n - x_{n-1})$. Then $x_n = \Sigma_1^n 1/z_k$,

$z_{n+1} = z_n + \frac{f(n)}{n^\alpha} \Sigma_1^n \frac{1}{z_k}$, $z_n \uparrow$. Thus

$z_{n+1} \geq z_n + \frac{f(n)}{n^\alpha} \frac{n}{z_n} = z_n + \frac{f(n)n^{1-\alpha}}{z_n}$. Squaring,

$z_{n+1}^2 \geq z_n^2 + 2f(n)n^{1-\alpha}$. Summing,

$z_n^2 \geq 2f(n)n^{2-\alpha}/2-\alpha$. Let $\beta = 1-\alpha/2, 1-\beta = \alpha/2$. Then

$z_n \geq \sqrt{f(n)/\beta}\, n^\beta$. Using this bound in 1),

$$z_{n+1} - z_n \leq \frac{f(n)}{n^\alpha} \Sigma_1^n 1/\sqrt{f(n)/\beta}\, n^\beta \sim \frac{f(n)}{n^\alpha} \frac{n^{\alpha/2}}{\alpha/2} \sqrt{\beta/f(n)}$$

$$= \frac{\sqrt{\beta f(n)}}{(\alpha/2)n^{\alpha/2}}, \text{ since } \beta < 1.$$

Summing, $z_n \leq \frac{\sqrt{\beta f(n)}}{\alpha/2} \frac{n^\beta}{\beta} = \frac{1}{\alpha/2} \sqrt{f(n)/\beta}\, n^\beta$.

The geometric mean of our bounds, $\bar{z}_n = \frac{\sqrt{f(n)/\beta}}{\sqrt{\alpha/2}} n^\beta$, satisfies 1) and the associated integral equation. The bound can be improved

to show that any solution is asymptotic to $\bar{z}_n$. We illustrate with $f(n) = 1$, $\alpha=1$. Then $z_{n+1} = z_n + \frac{1}{n}\Sigma_1^n \frac{1}{z_k}$. We have as bounds $2\sqrt{2n} \geq z_n \geq \sqrt{2n}$ and we claim $z_n \sim 2\sqrt{n}$. It is easy to see that an upper bound of the form $2\sqrt{n}a$, $a > 1$, determines a lower bound $2\sqrt{n}/a$. $\sqrt{2}$ is a value of a. Suppose the best possible value of a exceeds 1.

$$z_{n+1} = z_n + \frac{1}{z_n}\left(1 + \frac{1}{n}\Sigma \frac{z_n}{z_k} - 1\right))$$

$$1 + \frac{1}{n}\Sigma_1^n \left(\frac{z_n}{z_k} - 1\right) > 1 + \frac{1}{n}\Sigma_1^{n/a^2}\left(\frac{z_n}{z_k} - 1\right) \geq 1 + \frac{1}{n}\Sigma_1^{n/a^2}\left(\frac{2\sqrt{n}/a}{2\sqrt{k}\ a} - 1\right)$$

$$\to 1 + \frac{2}{a^3} - \frac{1}{a^2} = c > \frac{2}{a^2},$$

since $1 + \frac{2}{a^3} - \frac{3}{a^2} = (1 - \frac{1}{a})^2 (1 + \frac{2}{a}) > 0$.

Thus $z_{n+1} \geq z_n + c/z_n$, $z_n \geq \sqrt{c/2}\ 2\sqrt{n}$ with $\sqrt{c/2} > 1/a$. Substituting, $z_{n+1} - z_n \leq \frac{1}{n}\Sigma_1^n \frac{1}{\sqrt{c/2}\ 2\sqrt{k}} \sim \frac{1}{\sqrt{c/2}\ \sqrt{n}}$. Summing, $z_n \leq \sqrt{2/c}\ 2\sqrt{n}$ with $\sqrt{2/c} < a$. This contradiction proves that $z_n \sim 2\sqrt{n}$.

Summing in the general case, we find that $x_n \sim \frac{\sqrt{1-\alpha/2}}{\sqrt{f(n)\alpha/2}}\ n^{\alpha/2}$.

$\alpha = 2$ is a boundary case. We consider it in more detail.

Let $x_{n+1} = x_n + 1/\Sigma_1^n \frac{x_k}{k^2 \log^\alpha k}$, $\alpha < 1$.

$\frac{1}{x_{n+1}-x_n} = \frac{x_n}{n^2 \log^\alpha n} + \frac{1}{x_n - x_{n-1}}$. By the same substitution,

$z_{n+1} = \frac{1}{n^2 \log^\alpha n}\ \Sigma_1^n \frac{1}{z_k} + z_n \geq z_n + \frac{1}{n \log^\alpha n\ z_n}$. Apply Theorem 3.4.

$$\frac{d\mu}{dn} = \frac{1}{\mu n \log^{\alpha} n} \qquad \mu d\mu = \frac{dn}{n \log^{\alpha} n} \quad \frac{\mu^2}{2} \sim \frac{\log^{1-\alpha} n}{1-\alpha}$$

Let $\beta = \frac{1-\alpha}{2}$, $\gamma = \frac{1+\alpha}{2}$, so $\alpha+\beta = \gamma$, $1-\gamma = \beta$. $z_n \geq \log^{\beta} n/\beta^{\frac{1}{2}}$.

$$z_{n+1} - z_n \leq \frac{1}{n^2 \log^{\alpha} n} \Sigma_1^n \beta^{\frac{1}{2}}/\log^{\beta} k$$

$$\sim \frac{1}{n^2 \log^{\alpha} n} \cdot \frac{n\beta^{\frac{1}{2}}}{\log^{\beta} n} = \frac{\beta^{\frac{1}{2}}}{n \log^{\gamma} n} . \quad \text{Summing,}$$

$$z_{n+1} \leq \Sigma_1^n \frac{\beta^{\frac{1}{2}}}{k \log^{\gamma} k} \sim \frac{\beta^{\frac{1}{2}}}{1-\gamma} \log^{1-\gamma} n \text{ by monotonicity,}$$

$= \log^{\beta} n/\beta^{\frac{1}{2}}$. Therefore $z_n \sim \log^{\beta} n/\beta^{\frac{1}{2}}$. Summing again,

$x_n \sim n \log^{\beta} n/\beta^{\frac{1}{2}}$. Since

the constant depends on α, we cannot expect a result for a general slow function. Now suppose $x_{n+1} = x_n + 1/\Sigma_1^n \frac{1}{k^2 \log k}$

$$\frac{1}{x_{n+1}-x_n} = \frac{x_n}{n^2 \log n} + \frac{1}{x_n - x_{n-1}} ,$$

$$z_{n+1} = z_n + \frac{1}{n^2 \log n} \Sigma_1^n \frac{1}{z_k} \geq z_n + \frac{1}{n \log n \, z_n} . \quad \text{Applying Theorem 3.3,}$$

$$\frac{d\mu}{dn} = \frac{1}{n \log n \, \mu} , \quad \mu d\mu = \frac{dn}{n \log n}, \quad \frac{\mu^2}{2} \sim \log \log n.$$

Thus $z_n \geq \sqrt{2 \log \lg n}$.

$$z_{n+1} - z_n \leq \frac{1}{n^2 \log n} \Sigma_1^n \frac{1}{\sqrt{2 \log \log k}} \sim \frac{1}{n \log n \sqrt{2 \log \log n}}$$

$$z_{n+1} \leq \Sigma_1^n \frac{1}{k \log k \sqrt{2 \log \log k}} \sim \int_1^n \frac{d(\log \log t)}{\sqrt{2 \log \log t}}$$

$$= \sqrt{2 \log \log n}$$

$z_n \sim \sqrt{2 \log \log n}$ and $x_n \sim n \sqrt{2 \log \log n}$.

Finally, suppose $x_{n+1} = x_n + 1/\Sigma_1^n a_k x_k$ where $\Sigma\, k a_k$ converges.

$$z_{n+1} - z_n = a_n \Sigma_1^n \frac{1}{z_k} < \frac{n\, a_n}{z_1}$$

$z_{n+1} < \frac{1}{z_1} \Sigma_1^n k\, a_k < c_1$, z_n converges to some c_2 and

$x_n \sim c_2 n$.

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