ABSTRACT

WEIGHTED $L^2$ APPROXIMATION OF ENTIRE FUNCTIONS AND RELATED TOPICS
by
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In Chapter 1, sufficient conditions for polynomials to be dense in the space of entire functions of $L^2(dm)$ are examined, where dm is a positive, absolutely continuous measure defined on the complex plane. Let S be the space of entire functions such that $\|f(z)\|^2 = \iint |f(z)|^2 dm(z) < \infty$. Write dm(z) as $K(z)dxdy = K(r,\theta)rdrd\theta$. The main theorems are: 1) Suppose $\ln \inf_\theta K(r,\theta)$ is asymptotic to $\ln \sup_\theta K(r,\theta)$ (together with other mild restrictions). Then polynomials are dense in S. 2) Let $K(z) = e^{-\phi(z)}$ where $\phi(z)$ is a convex function of z such that $e^{tz}$ belongs to S for all complex t. Then the exponentials are complete in S. (Corollary: Polynomials are dense in S.) 1) is extended to the several-variable case. 2) was recently proven by B. A. Taylor for the many-variable situation. Our proof does not extend beyond the case of one variable, but for this it is simpler and more direct than Taylor's. Examples of spaces in which polynomials are not dense are also given.

In the second chapter we discuss the existence of entire solutions f(Z) to the equation $\bar{P}(D)P(Z)f(Z) = 0$ (more generally, to $\sum_{i=1}^{n} \bar{P}_i(D)P_i(Z)f(Z) = 0$), where Z is the vector $(z_1, \ldots z_k)$, P(Z) a polynomial, and $\bar{P}(D)$ is the differential operator obtained from $\bar{P}(Z)$. A summary of known results is given. The main theorem is the following: Let $P(Z) = \sum b_j Z^{N_j}$ where $N_j = (n_{j_1}, \ldots n_{j_k})$. Suppose there are positive

constants $a_1, a_2, \ldots a_k, M$ such that $\sum_{i=1}^{k} a_i n_{j_i} = M$ for all j. Then the only entire solution to $\bar{P}(D)P(Z)f(Z) = 0$ is the trivial one. In fact, we show that under these conditions no non-trivial formal power series $g(Z)$ satisfies $\bar{P}(D)P(Z)g(Z) = 0$.

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by

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I thank my advisor, Professor Donald J. Newman, for suggesting these problems, for guiding my research, and for showing me that good mathematics is and ought to be naturally elegant.

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This work is dedicated to my parents.

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## 0. Introduction

In this paper we study natural generalizations of two questions raised by D. J. Newman and H. S. Shapiro in [7]. There they discuss the (Fischer) space $\mathfrak{F}_Z$ $[Z = (z_1, z_2, \dots z_k),\ |Z|^2 = |z_1|^2 + |z_2|^2 + \dots |z_k|^2]$ which is defined to be the space of entire functions $f(Z)$ in complex Euclidean K-space ($C_K$) normed by $\|f(Z)\|^2 = \frac{1}{\pi^K} \int |f(Z)|^2 e^{-|Z|^2} dA_Z$. Here integration is over all of $C_K$ and $dA_Z$ is the Lebesgue measure with respect to $C_K$. If $\phi(Z) = O[e^{\frac{|Z|^2}{2} - A|Z|}]$ for all $A > 0$, clearly $\phi(Z)$ is in $\mathfrak{F}_Z$ and multiplication by $\phi(Z)$ is a well-defined operator on $\mathfrak{F}_Z$, its domain being the set of functions $f(Z)$ in $\mathfrak{F}_Z$ for which $\phi(Z)f(Z)$ is in $\mathfrak{F}_Z$. For such $\phi(Z)$ they define the operator $\bar{\phi}(D)$ by

$$\bar{\phi}(D)f(Z) = \frac{1}{\pi^K} \int \bar{\phi}(\xi) f(\xi) e^{Z \cdot \xi} e^{-|\xi|^2} dA_\xi \text{ for all } f(Z) \text{ in } \mathfrak{F}_Z .$$

$\bar{\phi}(D)$ then represents a formal adjoint to the operator "multiplication by $\phi(Z)$". If $\phi(Z)$ is a polynomial, let $\bar{\phi}(Z)$ denote the polynomial obtained from $\phi(Z)$ by replacing each coefficient by its complex conjugate. It turns out that for polynomial $\phi$, the formal adjoint $\bar{\phi}(D)$ is the differential operator obtained from $\bar{\phi}(Z)$ by replacing each $Z_i$ by $\frac{\partial}{\partial Z_i}$. Given $\phi(Z)$ satisfying the above growth condition, the authors show that the following two questions are equivalent.

1. Are polynomials dense in the Hilbert space of entire functions with measure $|\phi(Z)|^2 e^{-|Z|^2} dA_Z$ ?

2. Is the only solution in $\mathfrak{F}_Z$ to $\bar{\phi}(D)f(Z) = 0$ where $f(Z) = \phi(Z)g(Z)$, $g(Z)$ entire, the trivial one?

Moreover, they show that when $\phi(Z) = \sum_{\tau} P_{\tau}(Z)e^{\tau\cdot Z}$, i.e., when $\phi(Z)$ is an exponential polynomial, the result is in the affirmative, i.e., polynomials are dense in the Hilbert space of entire functions with measure $|\phi(Z)|^2 e^{-|Z|^2} dA_Z$ or equivalently, there is no non-trivial solution $f(Z)$ to $\bar{\phi}(D)f(Z) = 0$ where $f(Z) \in \mathcal{F}_Z$ and $f(Z) = \phi(Z)g(Z)$, $g(Z)$ entire.

Clearly both of the above questions have meaningful extensions beyond the context of the Fischer space.

1. Given a positive measure dm, one might ask when the analytic polynomials are dense in the entire functions of $L^2(dm)$. In Chapter 1, we give some sufficient conditions on the measure dm which ensure that they will be.

2. As we remarked above, when $\phi$ is a polynomial, $\bar{\phi}(D)f(Z)$, $[D = (\frac{\partial}{\partial Z_1}, \frac{\partial}{\partial Z_2} \ldots \frac{\partial}{\partial Z_k})]$ is meaningful outside the Fischer space and we may then ask whether there exists a non-trivial entire function $f(Z)$ (not necessarily in the Fischer space) such that $f(Z) = \phi(Z)g(Z)$, $g(Z)$ entire, and satisfies $\bar{\phi}(D)f(Z) = 0$. In Chapter 2 we investigate this and some closely related questions. Our results here are rather limited. For a restricted class of polynomials we have shown that no such entire function can exist. The general case is still open.

# I. Polynomial Approximation to Entire Functions in $L^2(dm)$

In this chapter we consider an analogue of the Bernstein problem of weighted polynomial approximation to a continuous function on the real line (see [6]). Instead of continuous functions on the real line, we consider entire functions in the plane and extensions to several variables. More specifically we let $C_K$ denote complex Euclidean K-space. For points $Z = (z_1, \ldots z_k)$ and $W = (w_1, \ldots w_k)$ of $C_K$ we denote their inner product by $\sum_1^k z_i \bar{w}_i$ by $\langle Z,W\rangle$ and write $r^2 = |Z|^2 = \langle Z,Z\rangle$. We denote by $\bar{Z}$ the K-tuple $(\bar{z}_1, \ldots \bar{z}_k)$. We let $E_K$ denote the set of entire functions in $C_K$. Let m be a positive measure defined on the Borel sets of $C_K$. We define $S = L^2(dm)$ to be the set of entire functions f in $E_K$ such that

$$\|f\|^2 = \int |f|^2 dm(Z) < \infty \tag{1}$$

Here the integration is over all of $C_K$. Initially all we require of the measure is that all polynomials belong to the space S. We then ask what conditions on dm are sufficient for polynomials to be dense in S with the metric imposed by (1). Henceforth when we use the phrase "dense in S" we will mean dense in the entire functions in the metric $L^2(dm)$. We consider only the case where dm(Z) is absolutely continuous with respect to the Lebesgue measure in $C_K$, i.e., we may write

$$dm(Z) = K(X,Y)dV = K(r,\Theta)dV$$

where

$$dV = \prod_{m=1}^{k} dx_m dy_m = \prod_{m=1}^{k} r_m dr_m d\theta_m = \prod dA_{z_m}$$

$$z_m = x_m + iy_m = r_m e^{im\theta}$$

$$X = (x_1, \ldots x_k),\ Y = (Y_1, \ldots y_k),\ r = (r_1, \ldots r_m),\ \theta = (\theta_1, \ldots \theta_m) .$$

We let N = the K-tuple $(n_1, \ldots n_k)$ of non-negative integers and write

$$z^N = z_1^{n_1} z_2^{n_2}, \ldots z_k^{n_k}$$

$$|N| = \sum_{i=1}^{k} n_i .$$

For two functions f and h in S we define their inner product

$$\langle f,h \rangle = \int f\bar{h}\, dm(Z) = \underbrace{\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty}}_{2k} f\bar{h} \prod_{m=1}^{x} dx_m dy_m$$

$$\underbrace{\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty}}_{k} \underbrace{\int_{0}^{2\pi} \ldots \int_{0}^{2\pi}}_{k} f\bar{h} \prod_{m=1}^{k} r_m dr_m d\theta_m .$$

<u>Definition:</u> f is said to be <u>orthogonal</u> to g if $\langle f,g \rangle = 0$.

S is then a normed linear space, and in particular a pre-Hilbert space. We recall some basic facts about such spaces which will be used frequently throughout this chapter.

(a) <u>Schwartz's Inequality</u>. $|\langle f + h \rangle| < \|f\| \, \|h\|$, f,h in S.

(b) Minkowski's Inequality. $\|f + h\| \leq \|f\| + \|h\|$, $f,h$ in $S$.

(c) If $S$ is a Hilbert space, then a set $G \subset S$ is complete if and only if $\langle g,f\rangle = 0$ for every $g$ in $G$ implies $f \equiv 0$.

We shall make frequent use of the concept of weak convergence.

Definition: A sequence $\{f_n\}$ in a normed linear space $S$ is said to converge weakly to $f$ in $S$ if $\lim_{n\to\infty} L(f_n) = L(f)$ for every bounded linear functional $L$ on $S$.

By the Riesz Representation Theorem if $S$ is, in particular, a pre-Hilbert space the sequence $\{f_n\}$ converges to $f$ weakly if the $\lim_{n\to\infty} \langle f_n,h\rangle = \lim_{n\to\infty} \langle f,h\rangle$ for every $h$ in $\bar{S}$ where $\bar{S}$ is the Hilbert space completion of $S$.

(Remark: The space $S$ is not necessarily complete in the norm. A sufficient condition for the space $S$ to be complete is that for every compact set $Q$ there exists a constant $C(Q)$ such that

$$* \qquad \sup_{Z\in Q} |f(z)| < C(Q)\|f(Z)\| \quad \text{for every } f \text{ in } S.$$

Proof: Let $\bar{S}$ be the completion of $S$. Let $\{g_n\}$ be a Cauchy sequence in $S$. There exists $g(Z)$ in $\bar{S}$ such that $\|g_n(Z) - g(Z)\| \to 0$. Hence there exists a subsequence $\{g_{n_j}\} = \{f_n\}$ which converges to $g$ pointwise almost everywhere. But by * for every fixed compact set $Q$, given $\varepsilon > 0$, there exists $N$ such that for $m,n > N$ and $Z$ in $Q$,

$|f_n(Z) - f_m(Z)| < \sup_{Z\in Q} |f_n(Z) - f_m(Z)| < C(Q)\|f_n(Z) - f_m(Z)\| < \varepsilon$. Therefore $f_n(Z)$ converges to a function $h(Z)$ uniformly on compact sets. $h(Z) = g(Z)$ almost everywhere. Since $h(Z)$ is entire, $g(Z)$ is entire and $S = \bar{S}$, i.e., $S$ is a complete Hilbert space.

When $dm(Z)$ is absolutely continuous, i.e., $dm(Z) = K(Z)dA_Z = K(X,Y)dV$, * is true if $K(X,Y)$ is bounded away from zero on every compact set. This is true because for every $f \in S$ we have

$$f(Z) = \left(\frac{1}{2\pi}\right)^k \int_{|\xi_m - z_m| = r_m} \frac{f(\xi)}{(\xi_1 - z_1)(\xi_2 - z_2)\cdots(\xi_k - z_k)} d\xi_1, \ldots d\xi_k$$

$$|f(Z)| \leq \left(\frac{1}{\pi}\right)^k \int_{|w_1| \leq 1} \cdots \int_{|w_k| \leq 1} |f(Z+W)| \, [K(Z+W)]^{\frac{1}{2}} \, [K(Z+W)]^{-\frac{1}{2}} \, dA_W$$

$$= \left(\frac{1}{\pi}\right)^k \int_{|\xi_m - z_m| \leq 1} |f(\xi)| \, [K(\xi)]^{\frac{1}{2}} \, [K(\xi)]^{\frac{1}{2}} \, dA_\xi$$

$$< \left(\frac{1}{\pi}\right)^k \sup_{\substack{\xi \\ |\xi_m - z_m| \leq 1}} \frac{1}{[K(\xi)]^{1/2}} \int_{|\xi_m - z_m| \leq 1} |f(\xi)| \, [K(\xi)]^{\frac{1}{2}} \, dA_\xi$$

Applying Schwartz's inequality to this last integral and extending the domain of intebration to all of $C_K$ we have

$$|f(Z)| < \left(\frac{1}{\pi}\right)^{\frac{k}{2}} \sup_{\substack{\xi \\ |\xi_m - z_m| \leq 1}} \frac{1}{[K(\xi)]^{1/2}} \|f(Z)\|$$

and

$$\sup_{\substack{Z \\ Z \in Q}} |f(Z)| < \left(\frac{1}{\pi}\right)^{\frac{k}{2}} \sup_{\substack{Z \\ Z \in Q}} \sup_{\substack{\xi \\ |\xi_m - z_m| \leq 1}} \frac{1}{[K(\xi)]^{1/2}} \|f(Z)\| = C \|f(Z)\|$$

It is clear that if $K$ does not satisfy the above condition, * need not hold, as seen by the following trivial example.

$$K(x,y) = K(Z) = 1 \quad \text{when} \quad |\xi| \leq 1$$
$$= 0 \quad \text{when} \quad |\xi| > 1 .$$

The general question whether $L^2(dm)$ is complete, while not necessarily directly relevant to the considerations of this paper, does have independent interest. We do, however, use the fact that the space S of Theorem 1.4 is complete.)

One known condition that polynomials be dense in the entire functions of $L^2(dm)$, i.e., in the space S, is that the measure dm be rotation invariant, i.e., $\int_E dm(Z) = \int_{UE} dn(Z)$ for every Borel set E and every unitary transformation U. For the sake of completeness we include the proof.

<u>Theorem 1.1</u>: Let m be rotation invariant. Then polynomials are dense in the entire functions in the metric of $L^2(dm)$, i.e., in the space S as defined above.

<u>Proof</u>: Since m is rotation invariant, $dm(Z) = K(r)\prod_{m=1}^{k} r_m dr_m d\theta_m$. For every f, let $\sum_{|N|=0}^{\infty} a_N Z^N$ be the Taylor expansion of f. Clearly the monomials are orthogonal. $\int|f|^2 dm(Z) = \sum_{|N|=0}^{\infty} |a_N|^2 P(N)$, i.e., $\int|f|^2 dm$ may be expressed as a weighted sum of squares of the absolute values of the Taylor coefficients of f. It therefore follows that the Taylor expansion of f converges to f in the given metric.

We now restrict our attention to the case K = 1, i.e., we consider the problem in the complex plane. It will be shown that theorems 1.2 and 1.3 can easily be extended to $C_K$, K arbitrary. For simplicity we let $K_1(r,\theta) = K(r,\theta)r = rK(x,y)$; since there will be no possibility of confusion we drop the subscript.

Taking Theorem 1.1 as a point of departure, consider the case where $K(r,\theta)$ is not necessarily rotation invariant but for fixed r, $K(r,\theta)$ does not vary very much with $\theta$. We introduce the following notation.

$$(2) \qquad K_1(r) = \inf_\theta K(r,\theta)$$

$$0 \leq \theta < 2\pi \, .$$

$$(3) \qquad K_2(r) = \sup_\theta K(r,\theta)$$

A simple extension of Theorem 1.1 is

<u>Theorem 1.2</u>: Let $K_2(r) \leq AK_1(r)$ where A is a fixed constant. Then polynomials are dense in the space S.

<u>Proof</u>: Let $S_2$ be the space of entire functions f in E such that $\int_0^\infty \int_0^{2\pi} |f|^2 K_2(r)\,dr d\theta < \infty$. Obviously $S \subset S_2$ since for every f in S

$$\begin{aligned}\int_0^\infty \int_0^{2\pi} |f|^2 K_2(r)\,dr d\theta &\leq A \int_0^\infty \int_0^{2\pi} |f|^2 K_1(r)\,dr d\theta \\ &\leq A \int_0^\infty \int_0^{2\pi} |f|^2 K(r,\theta)\,dr d\theta \, .\end{aligned}$$

But $K_2(r)$ is rotation invariant. Hence by Theorem 1.1 f can be approximated by polynomials in $S_2$, i.e., given $\varepsilon > 0$ there exists a P such that

$$\int_0^\infty \int_0^{2\pi} |f-P|^2 K(r,\theta)\,dr d\theta < \varepsilon \, .$$

The same polynomial obviously approximates f to within $\varepsilon$ in S because

$$\int_0^\infty \int_0^{2\pi} |f-P|^2 K(r,\theta)\,drd\theta \leq \int_0^\infty \int_0^{2\pi} |f-P|^2 K_2(r)\,drd\theta < \varepsilon .$$

Before proceeding with Theorem 1.3 we give two simple examples of spaces in which we show that polynomials are dense. These will motivate both the statement and proof of Theorem 1.3. The following notation will be used in both examples and in the proof of Theorem 3.1. As above, given $K(r,\theta)$

(4) $S$ = the set of functions $f(Z)$ in $E$ such that

$$\int_0^\infty \int_0^{2\pi} |f(Z)|^2 K(r,\theta)\ drd\theta < \infty$$

We let

(5) $G$ = the set of functions $f(\lambda Z)$, $\frac{1}{2} \leq \lambda < 1$, such that $f(Z)$ belongs to $S$.

(6) $S_2$ = the set of functions $f(Z)$ in $E$ such that

$$\int_0^\infty \int_0^{2\pi} |f(Z)|^2 K_2(r)\,drd\theta < \infty$$

where $K_2(r) = \sup_\theta K(r,\theta)$ as defined in (3).

<u>Remark</u>: $G$ is clearly a convex set: Let $f(\lambda_1 z)$, $g(\lambda_2 z)$ be elements of $G$. If $\lambda_1 = \lambda_2$, $\alpha f(\lambda z) + (1-\alpha)g(\lambda z) = h(\lambda z)$ where $h(z) = \alpha f(z) + (1-\alpha)g(z)$. If $\lambda_1 < \lambda_2$, $\alpha f(\lambda_1 z) + (1-\alpha)g(\lambda_2 z) = h(\lambda_2 z)$ where $h(z) = \alpha f\left(\frac{\lambda_1}{\lambda_2} z\right) + (1-\alpha)g(z)$. If $0 < \alpha < 1$, by Minkowski's inequality $h(z)$ clearly belongs to $S$.

Example 1. Let

$$K(r,\theta) = r^n e^{-r^2} \quad \text{for} \quad 0 \le \theta < \pi$$

$$= e^{-r^2} \quad \text{for} \quad \pi \le \theta < 2\pi.$$

Note: $K_2(r) = r^n K_1(r)$ for $r \ge 1$.

We will show that G is dense in S and that $G \subset S_2$. These two facts imply that polynomials are dense in S. G is dense in S implies that for any f(z) in S, given $\varepsilon$ there exists a $\lambda$ such that $\|f(z) - f(\lambda z)\|_S < \frac{\varepsilon}{2}$. Since $K_2(r)$ is rotation invariant, $G \subset S_2$ implies that given $\varepsilon > 0$ and f(z) in S there exists a polynomial P such that

$$\|f(\lambda z) - P\|_{S_2} < \frac{\varepsilon}{2}.$$

The same polynomial approximates f(z) in S to within $\varepsilon$ because

$$\|f(z) - P(z)\| \le \|f(z) - f(\lambda z)\|_S + \|f(\lambda z) - P(z)\|_S$$

$$\le \|f(z) - f(\lambda z)\|_S + \|f(\lambda z) - P(z)\|_{S_2} < \varepsilon$$

Therefore polynomials are dense in S.

1. G is dense in S.

Since $f(\lambda z)$ in G obviously converges to f(z) in S pointwise as $\lambda \to 1$, it suffices to show that $f(\lambda z)$ is bounded in norm in S. It then follows that $f(\lambda z)$ converges weakly to f(z) in S, i.e., G is weakly dense in S. Since G is convex, G is in fact dense in S (see [4], page 207). We show that $f(\lambda z)$ is bounded in norm in S.

$$\|f(\lambda z)\|_S^2 = \int_0^\infty \int_0^\pi |f(\lambda re^{i\theta})|^2 r^n e^{-r^2} d\theta dr + \int_0^\infty \int_\pi^{2\pi} |f(\lambda re^{i\theta})|^2 e^{-r^2} d\theta dr .$$

By the obvious change of variables $\lambda r = r'$

$$\|f(\lambda z)\|_S^2 = \frac{1}{\lambda^{n+1}} \int_0^\infty \int_0^\pi |f(r'e^{i\theta})|^2 r'^n e^{-(\frac{r'}{\lambda})^2} d\theta dr'$$

$$+ \frac{1}{\lambda} \int_0^\infty \int_\pi^{2\pi} |f(r'e^{i\theta})|^2 e^{-(\frac{r'}{\lambda})^2} d\theta dr'$$

$$< 2^{n+1} \left[ \int_0^\infty \int_0^\pi |f(re^{i\theta})|^2 r^n e^{-r^2} d\theta dr + \int_0^\infty \int_\pi^{2\pi} |f(re^{i\theta})|^2 e^{-r^2} d\theta dr \right]$$

$$= 2^{n+1} \|f(z)\|_S^2 .$$

To prove that $G \subset S_2$ we need the following fact which will be proven in more generality in the proof of Theorem 3.1. Let $g(z)$ be entire. Then

$$* \qquad \int_0^1 \int_0^{2\pi} |g(z)|^2 e^{-r^2} d\theta dr \le C \int_2^3 \int_0^{2\pi} |g(z)|^2 r^n e^{-r^2} d\theta dr$$

where C is independent of $g(z)$. Assume * is proven.

2. $\underline{G \subset S_2}$

Let $f(\lambda z)$ belong to G. We wish to show that $\|f(\lambda z)\|_{S_2} < \infty$. Using * and then letting $\lambda r = r'$, we have

$$(1) \quad \|f(\lambda z)\|_{S_2}^2 = \int_0^1\int_0^{2\pi} |f(\lambda z)|^2 e^{-r^2} d\theta dr + \int_1^\infty\int_0^{2\pi} |f(\lambda z)|^2 r^n e^{-r^2} d\theta dr$$

$$< C \int_2^3\int_0^{2\pi} |f(\lambda z)|^2 r^n e^{-r^2} d\theta dr + \int_1^\infty\int_0^{2\pi} |f(\lambda z)|^2 r^n e^{-r^2} d\theta dr$$

$$< C' \int_1^\infty\int_0^{2\pi} |f(\lambda z)|^2 r^n e^{-r^2} d\theta dr = \frac{C'}{\lambda^{n+1}} \int_\lambda^\infty\int_0^{2\pi} |f(r'e^{i\theta})|^2 r'^n e^{-(\frac{r'}{\lambda})^2} d\theta dr'$$

$$< C''\left[\int_0^\infty\int_0^{\pi} |f(z)|^2 r^n e^{-r^2} d\theta dr + \int_0^\infty\int_\pi^{2\pi} |f(z)|^2 r^n e^{-(\frac{r}{\lambda})^2} d\theta dr\right].$$

By standard methods of calculus one can easily show that the function $g(r) = r^n e^{-r^2[\frac{1}{\lambda^2} - 1]} < A(\lambda)$, i.e., $r^n e^{-(\frac{r}{\lambda})^2} < A(\lambda) e^{-r^2}$ where $A(\lambda)$ is positive constant depending on $\lambda$ and finite for $\frac{1}{2} \le \lambda < 1$. From (1) we therefore have

$$\|f(\lambda z)\|_{S_2}^2 < B(\lambda)\left[\int_0^\infty\int_0^{\pi} |f(z)|^2 r^n e^{-r^2} d\theta dr + \int_0^\infty\int_\pi^{2\pi} |f(z)|^2 e^{-r^2} d\theta dr\right]$$

$$= B(\lambda) \|f(z)\|_S < \infty .$$

Hence $G \subset S_2$.

Proof of *: $g(z)$ is entire, therefore for $r \le 1$,

$$(1) \qquad g(z) = \frac{1}{2\pi i} \int_{\substack{|\xi|=t \\ 2\le t\le 3}} \frac{g(\xi)}{\xi - z} d\xi = \frac{1}{2\pi}\int_0^{2\pi} \frac{g(te^{i\phi}) te^{i\phi}}{(te^{i\phi} - z)} d\phi \qquad 2 \le t \le 3$$

$$(2) \qquad |g(z)| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{|g(te^{i\phi})|t}{|te^{i\phi} - z|}\,d\phi < \frac{1}{2\pi}\int_0^{2\pi} |g(te^{i\phi})|t\,d\phi .$$

Integrating both sides of (2) from t=2 to t=3 we get

$$(3) \qquad |g(z)| \leq \frac{1}{2\pi}\int_2^3\int_0^{2\pi} |g(te^{i\phi})|t\,d\phi\,dt .$$

By the Schwartz inequality

$$\int_2^3\int_0^{2\pi} |g(te^{i\phi})|t\,d\phi\,dt < \left[\int_2^3\int_0^{2\pi} |g(te^{i\phi})|^2 d\phi\,dt\right]^{\frac{1}{2}}\left[\int_2^3\int_0^{2\pi} t^2 d\phi\,dt\right]^{\frac{1}{2}}$$

$$= \left[\int_2^3\int_0^{2\pi} |g(te^{i\phi})|^2 d\phi\,dt\right]^{\frac{1}{2}}\sqrt{\frac{38\pi}{3}} .$$

Hence

$$(4) \qquad |g(re^{i\theta})|^2 < C\int_2^3\int_0^{2\pi} |g(te^{i\phi})|^2 d\phi\,dt .$$

If we multiply (4) by $e^{-r^2}$ and integrate $dr d\theta$ from $r = 0$ to $r = 1$, $\theta = 0$ to $\theta = 2\pi$ we have

$$(5) \qquad \int_0^1\int_0^{2\pi} |g(re^{i\theta})|^2 e^{-r^2} d\theta\,dr < C\int_2^3\int_0^{2\pi} |g(te^{i\phi})|^2 d\phi\,dt \int_0^1\int_0^{2\pi} e^{-r^2} d\theta\,dr$$

$$= C'\int_2^3\int_0^{2\pi} |g(te^{i\phi})|^2 d\phi\,dt .$$

For $2 \leq t \leq 3$, $t^n e^{-t^2} > B > 0$. Therefore

$$(6)\qquad \int_0^1\int_0^{2\pi} |g(re^{i\theta})|^2 e^{-r^2} d\theta dr < \frac{C'}{B}\int_0^3\int_0^{2\pi} |g(te^{i\phi})|^2 B d\phi dt$$

$$< C''\int_2^3\int_0^{2\pi} |g(te^{i\phi})|^2 t^n e^{-t^2} d\theta dr$$

which proves *.

<u>Example 2.</u> Let

$$K(r,\theta) = e^r e^{-r^2} \quad \text{for} \quad 0 \le \theta \le \pi$$

$$= e^{-r^2} \quad \text{for} \quad \pi < \theta < 2\pi \ .$$

<u>Note</u>: $K_2(r) = e^r e^{-r^2} = e^r K_1(r)$.

As in Example 1 we show that G is dense in S and that $G \subset S_2$.

1. <u>G is dense in S</u>

As pointed out in Example 1, it suffices to show that given f(z) in S, $f(\lambda z)$ is bounded in norm in S.

$$(1)\quad \|f(\lambda z)\|_S^2 = \int_0^\infty\int_0^\pi |f(\lambda re^{i\theta})|^2 e^r e^{-r^2} d\theta dr + \int_0^\infty\int_\pi^{2\pi} |f(\lambda re^{i\theta})|^2 e^{-r^2} d\theta dr \ .$$

Letting $\lambda r = r'$ we have

$$(2)\quad \|f(\lambda z)\|_S^2 = \frac{1}{\lambda}\left[\int_0^\infty\int_0^\pi |f(r'e^{i\theta})|^2\, e^{\frac{r'}{\lambda} - (\frac{r'}{\lambda})^2} d\theta dr'\right.$$

$$\left. + \int_0^\infty\int_\pi^{2\pi} |f(r'e^{i\theta})|^2 e^{-(\frac{r'}{\lambda})^2} d\theta dr'\right] < 2\left[\int_0^\infty\int_0^\pi |f(r'e^{i\theta})|^2 e^{\frac{r}{\lambda} - (\frac{r}{\lambda})^2} d\theta dr\right.$$

$$\left. + \int_0^\infty\int_\pi^{2\pi} |f(re^{i\theta})|^2 e^{-r^2} d\theta dr\right] .$$

By ordinary method of calculus one can easily show that for $\frac{1}{2} \leq \lambda < 1$
$e^{\frac{r}{\lambda} - \frac{r^2}{\lambda^2}} < e^{\frac{1}{4}}[e^{r-r^2}]$. We therefore have

$$(3) \quad ||f(\lambda z)||_S^2 < 2e^{\frac{1}{4}}\left[\int_0^\infty \int_0^\pi |f(z)|^2 e^{r-r^2} d\theta dr + \int_0^\infty \int_\pi^{2\pi} |f(z)|^2 e^{-r^2} d\theta dr\right]$$

$$= 2e^{\frac{1}{4}} ||f(z)||_S^2 .$$

2. $\underline{G \subset S_2}$

Let $f(\lambda z)$ belong to $G$.

$$(1) \quad ||f(\lambda z)||_{S_2}^2 = \int_0^\infty \int_0^{2\pi} |f(\lambda re^{i\theta})|^2 e^{r-r^2} d\theta dr$$

$$= \frac{1}{\lambda} \int_0^\infty \int_0^{2\pi} |f(r'e^{i\theta})|^2 e^{\frac{r'}{\lambda} - (\frac{r'}{\lambda})^2} d\theta dr' \quad .$$

As pointed out above $e^{\frac{r}{\lambda} - (\frac{r}{\lambda})^2} < e^{\frac{1}{4}}[e^{r-r^2}]$. Similarly it is easy to show that $e^{\frac{r}{\lambda} - (\frac{r}{\lambda})^2} < A(\lambda)e^{-r^2}$ where $A(\lambda)$ is some constant depending on $\lambda$ and finite for $\frac{1}{2} \leq \lambda < 1$. We therefore have from (1)

$$(2) \qquad \|f(\lambda z)\|^2_{S_2} < B(\lambda)\left[\int_0^\infty \int_0^\pi |f(z)|^2 e^{r-r^2} d\theta dr\right.$$

$$\left. + \int_0^\infty \int_\pi^{2\pi} |f(z)|^2 e^{-r^2} d\theta dr\right] = B(\lambda)\|f(z)\|_S < \infty$$

i.e., $G \subset S_2$.

Remark: Example 2 was somewhat easier to handle because for fixed $\theta$ $K(r,\theta)$ was bounded away from zero in every r-interval.

Theorem 1.3: As above, we let S = the set of f(z) in E such that $\int_0^\infty \int_0^{2\pi} |f|^2 K(r,\theta) dr d\theta < \infty$, $K_1(r) = \inf_\theta K(r,\theta)$, and $K_2 = \sup_\theta K(r,\theta)$.

Assume $K(r,\theta)$ satisfies the following conditions.

(a) $K_1(r) = e^{-P(r)}$, where P(r) is a convex function of r, for $r > r_1$.

(b) For all $\theta$, $\theta$ fixed, $K(r,\theta)$ is a decreasing function of r for $r > r_1$ and $\lim_{r\to\infty} K(r,\theta) = 0$.

(c) $K(r,\theta) = 1$, $0 \le r \le r_1$.

(d) $K(r,\theta)$ is uniformly bounded with respect to $\theta$ in every r-interval.

(e) $\ell n\ K_1(r)$ is asymptotic to $\ell n\ K_2(r)$.

Then polynomials are dense in S.

Remark: (a) and (b) are reasonable conditions on $K(r,\theta)$ to ensure that all polynomials do indeed belong to the space S. It will be clear from the proof that we may replace (a) by a somewhat weaker condition, namely that there exist constants C, C' and a convex function P(r) such that $Ce^{-P(r)} < K_1(r) < C'e^{-P(r)}$ for $r_1 < r$. (c) and (d) are needed for the method employed. We will show that in actual fact once we assume (a) the assumption that $K(r,\theta) = 1$ for $0 \le r \le r_1$ is no restriction at all. Condition (e) is our main assumption about $K(r,\theta)$. As above, we let

$S_2$ = the set of functions $f(z)$ in E such that $\int_0^\infty \int_0^{2\pi} |f(z)|^2 K_2(r)\,dr\,d\theta < \infty$.

G = the set of functions $f(\lambda z)$, $\frac{1}{2} \le \lambda < 1$ such that $f(z)$ belongs to S.

Proof: We will prove that G is dense in S and that $G \subset S_2$. That these two facts are enough to guarantee that polynomials are dense in S follows from the argument given in Example 1. The reasoning is the same and we do not repeat it here.

1. G is dense in S

As in the above examples, since $f(\lambda z)$ converges to $f(z)$ pointwise as $\lambda \to 1$, it suffices to show that for every $f(\lambda z)$ in G, $\|f(\lambda z)\|_S < C\|f(z)\|_S$. Then $f(\lambda z)$ converges weakly to $f(z)$ in the norm of S, i.e., G is weakly dense in S. Since, in addition G is convex, G is, in fact, dense in S. ([4], page 207).

$$\|f(\lambda z)\|_S^2 = \int_0^\infty \int_0^{2\pi} |f(\lambda re^{i\theta})|^2 K(r,\theta)\,d\theta\,dr\,. \tag{1}$$

Let $\lambda r = r'$. We than have

$$\|f(\lambda z)\|_S^2 = \frac{1}{\lambda}\int_0^\infty \int_0^{2\pi} |f(r'e^{i\theta})|^2 K(\tfrac{r'}{\lambda},\ \theta)\,d\theta\,dr' \tag{2}$$

$$\le 2\int_0^\infty \int_0^{2\pi} |f(re^{i\theta})|^2 K(\tfrac{r}{\lambda},\ \theta)\,d\theta\,dr\,.$$

Condition (b) says that $K(\frac{r}{\lambda},\ \theta) \le K(r,\theta)$ when $r_1 < r < \infty$. Conditions (c) and (d) imply that when $0 \le r \le r_1$, $\theta$ fixed,

$$\frac{K(\frac{r}{\lambda}, \theta)}{K(r,\theta)} \le \sup_{0\le r\le 2r_1} K(r,\theta) < M \text{ where } M \text{ is a constant independent of } \theta.$$

We therefore have from (2)

$$(3)\ \|f(\lambda z)\|^2 \le 2\left[\int_0^{r_1}\int_0^{2\pi} |f(re^{i\theta})|^2 K(\tfrac{r}{\lambda},\theta)\,d\theta dr + \int_{r_1}^{\infty}\int_0^{2\pi} |f(re^{i\theta})|^2 K(\tfrac{r}{\lambda},\theta)\,d\theta dr\right]$$

$$< 2[M\int_0^{r_1}\int_0^{2\pi} |f(re^{i\theta})|^2 K(r,\theta)\,d\theta dr + \int_{r_1}^{\infty}\int_0^{2\pi} |f(re^{i\theta})|^2 K(r,\theta)\,d\theta dr]$$

$$= C\int_0^{\infty}\int_0^{2\pi} |f(re^{i\theta})|^2 K(r,\theta)\,d\theta dr = C\|f(z)\|_S^2 .$$

2. $\underline{G \subset S_2}$

Let $f(\lambda z)$ belong to G. We wish to show that $\|f(\lambda z)\|_{S_2} < \infty$, i.e., $f(\lambda z)$ belongs to $S_2$. Conditions (c) and (d) imply that when $0 \le r \le r_1$,

$$\frac{K_2(\frac{r}{\lambda})}{K(r,\theta)} = \sup_{\theta} K(\tfrac{r}{\lambda},\theta) < \sup_{\theta}\ \sup_{0\le r\le 2r_1} K(r,\theta) < M$$

i.e.,

(1) $\quad K_2(\frac{r}{\lambda}) < MK(r,\theta)$ when $0 \le r \le r_1$, M a fixed constant independent of $\theta$.

We now wish to show

(2) $\quad K_2(\frac{r}{\lambda}) < M_1(\lambda)K(r,\theta)$ when $r_1 < r < \infty$ where $M_1(\lambda)$ depends only on $\lambda$ and is finite for $\frac{1}{2} \le \lambda < 1$.

We will show the stronger fact,

(3) $$K_2(\tfrac{r}{\lambda}) < M_1(\lambda)K_1(r) \quad \text{where} \quad r_1 < r < \infty.$$

Condition (e) says that

$$\ln K_2(r) \sim \ln K_1(r)$$

i.e.,

$$\ln K_2(r) = Q(r)\ln K_1(r) \text{ where } \lim_{r\to\infty} Q(r) = 1 .$$

Since $K_1(r) = e^{-P(r)}$, where $P(r)$ is convex, for $r > 1$ (condition (a)), we have

(4). $$K_2(r) = e^{-P(r)\ Q(r)}$$

and

(5) $$K_2(\tfrac{r}{\lambda}) = e^{-P(\frac{r}{\lambda})\ Q(\frac{r}{\lambda})}.$$

We will show that there exists $R(\lambda)$, $R(\lambda)$ a fixed constant depending on $\lambda$, such that

(6) $$P(r) - P(\tfrac{r}{\lambda})\ Q(\tfrac{r}{\lambda}) < \text{constant when } r > R(\lambda)$$

and

(7) $K_2(\frac{r}{\lambda}) < C(\lambda)K_1(r)$ when $r_1 < r \leq R(\lambda)$ where $C(\lambda)$ is finite for $0 < \lambda < 1$.

It will follow from (5) and (6) that

(8) $$K_2(\tfrac{r}{\lambda}) = e^{-P(\frac{r}{\lambda})\, Q(\frac{r}{\lambda})} < \text{constant } e^{-P(r)} = C_1K_1(r) \text{ for } r > R(\lambda)\ .$$

Combining (7) and (8) we get (3) which was the desired assertion.

$P(r)$ is given to be convex for $r_1 \leq r < \infty$. Let $A(\lambda) = \frac{r(1-\lambda)}{r-\lambda r_1}$. Since $r_1 < r$ and $\frac{1}{2} \leq \lambda < 1$, $A(\lambda) < 1$,

$$1 - A(\lambda) = \frac{\lambda(r-r_1)}{r-\lambda r_1}$$

$$r = A(\lambda)r_1 + (1 - A(\lambda))\,\frac{r}{\lambda}\ .$$

From the definition of convexity we have

(9) $$P(r) \leq A(\lambda)P(r_1) + (1 - A(\lambda))P(\tfrac{r}{\lambda})\ .$$

Recall that $\lim_{r\to\infty} Q(r) = 1$, i.e., given $\varepsilon > 0$ there exists $R(\varepsilon)$ such that $1 - \varepsilon < Q(r)$ when $r > R(\varepsilon)$. We take $\varepsilon = A(\lambda)$. Since $0 < \lambda < 1$, we have

(10) $$Q(\tfrac{r}{\lambda}) > 1 - A(\lambda) \quad \text{when} \quad r > R(A(\lambda))$$

and therefore,

(11) $$Q(\tfrac{r}{\lambda})\ P(\tfrac{r}{\lambda}) > (1 - A(\lambda))P(\tfrac{r}{\lambda}) \quad \text{when} \quad r > R(A(\lambda))\ .$$

From (9) and (11) we have

(12) $$P(r) < A(\lambda)P(r_1) + Q(\tfrac{r}{\lambda})\ P(\tfrac{r}{\lambda})$$

i.e., $P(r) - Q(\frac{r}{\lambda})\ P(\frac{r}{\lambda}) < A(\lambda)P(r_1) < P(r_1)$ when $r > R(A(\lambda))$, which is (6). Statement (8) follows quite easily. Consider $r_1 < r \leq R(A(\lambda))$. By condition (d) of the hypothesis,

$$\frac{K_2(\frac{r}{\lambda})}{K_1(r)} < \frac{\sup\limits_{r_1<r\leq 2R} K_2(r)}{\inf\limits_{r_1<r\leq R} K_1(r)} < C(\lambda)$$

i.e., $K_2(\frac{r}{\lambda}) < C(\lambda)K_1(r)$ where $C(\lambda)$ is finite when $\frac{1}{2} \leq \lambda < 1$. We have thus shown (4), namely

$$K_2(\frac{r}{\lambda}) < M_1(\lambda)K_1(r) \quad \text{when} \quad r_1 < r < \infty.$$

It is now easy to see that $\|f(\lambda z)\|_{S_2} < \infty$. From (1) and (2),

$$\|f(\lambda z)\|^2_{S_2} = \int_0^{2\pi}\int_0^{\infty} |f(\lambda r e^{i\theta})|^2 K_2(r)\,dr d\theta = \frac{1}{\lambda}\int_0^{2\pi}\int_0^{\infty} |f(re^{i\theta})|^2 K_2(\frac{r}{\lambda})\,dr d\theta$$

$$< 2\int_0^{2\pi}\int_0^{\infty} |f(re^{i\theta})|^2 K_2(\frac{r}{\lambda})\,dr d\theta$$

$$= 2[\int_0^{2\pi}\int_0^{r_1} |f(re^{i\theta})|^2 K_2(\frac{r}{\lambda})\,dr d\theta + \int_0^{2\pi}\int_{r_1}^{\infty} |f(re^{i\theta})|^2 K_2(\frac{r}{\lambda})\,dr d\theta]$$

$$< 2[M\int_0^{2\pi}\int_0^{r_1} |f(re^{i\theta})|^2 K(r,\theta)\,dr d\theta + M_1(\lambda)\int_0^{2\pi}\int_{r_1}^{\infty} |f(re^{i\theta})|^2 K(r,\theta)\,dr d\theta]$$

$$= B(\lambda)\int_0^{2\pi}\int_0^{\infty} |f(re^{i\theta})|^2 K(r,\theta)\,dr d\theta = B(\lambda)\|f(z)\|^2_S < \infty .$$

This completes the proof of Theorem 1.3.

As pointed out in the remark following the statement of Theorem 1.3, it should be clear that we need only assume that there exist constants C, C' and a convex function $P(r)$ such that $Ce^{-P(r)} < K_1(r) < C'e^{-P(r)}$ when $r > r_1$. We now show that condition (c), namely the requirement that $K(r,\theta) = 1$ for $0 \leq r \leq r_1$, is, in fact, no restriction. Suppose $K(r,\theta)$ satisfies all the conditions of Theorem 1.3 except (c). We show that we may define an equivalent measure $\tilde{K}(r,\theta)$ where

$$\begin{aligned} \tilde{K}(r,\theta) &= 1 \qquad \text{when} \quad 0 \leq r \leq r_1 \\ &= K(r,\theta) \quad \text{when} \quad r_1 < r < \infty . \end{aligned}$$

Let $\tilde{S}$ = the set of $f(z)$ in E such that $\int_0^{2\pi} \int_0^{\infty} |f|^2 \tilde{K}(r,\theta)\,dr\,d\theta < \infty$. To say the two measures are equivalent means there exist positive constants A, A' such that for every $g(z)$ in E

$$\text{(I)} \qquad A\|f\|_{\tilde{S}} < \|f\|_S < A'\|f\|_{\tilde{S}} .$$

Clearly $f(z)$ belongs to S if and only if $f(z)$ belongs to $\tilde{S}$, and polynomials are dense in S if and only if polynomials are dense in $\tilde{S}$.

We will show that if $L(r,\theta) = L_1(z)$ has the properties

(a) $L(r,\theta) \geq 0$

(b) $\int_0^{\infty} \int_0^{2\pi} L(r,\theta)\,d\theta\,dr < \infty$ ,

(c) there exists a constant $C > 0$ such that for $r > r_1$, $Ce^{-P(r)} < L(r,\theta)$ where $P(r)$ is a convex function of r,

then there exists $B > 0$ such that for every $g(z)$ in E,

$$\text{(II)} \quad \int_0^{r_1}\int_0^{2\pi} |g(z)|^2 L(r,\theta)\,d\theta dr < \text{constant} \int_{2r_1}^{3r_1}\int_0^{2\pi} |g(z)|^2 L(r,\theta)\,d\theta dr$$

Note: We can assume that $L(r,\theta) > 0$ almost everywhere for $0 \le \theta < 2\pi$, $0 < r \le r_1$. Statement (II) implies statement (I) for since both $K(r,\theta)$ and $\tilde{K}(r,\theta)$ satisfy the conditions (a), (b), and (c), we have

$$\|f\|_{\tilde{S}}^2 = \int_0^{2\pi}\int_0^{\infty} |f(z)|^2 \tilde{K}(r,\theta)\,dr d\theta$$

$$= \int_0^{2\pi}\int_0^{r_1} |f(z)|^2 \tilde{K}(r,\theta)\,dr d\theta + \int_0^{2\pi}\int_{r_1}^{\infty} |f(z)|^2 \tilde{K}(r,\theta)\,d\theta dr$$

$$< B \int_0^{2\pi}\int_{2r_1}^{3r_1} |f|^2 \tilde{K}(r,\theta)\,dr d\theta + \int_0^{2\pi}\int_{r_1}^{\infty} |f(z)|^2 \tilde{K}(r,\theta)\,dr d\theta$$

$$< B_1 \int_0^{2\pi}\int_{r_1}^{\infty} |f|^2 K(r,\theta)\,dr d\theta < B_1 \int_0^{2\pi}\int_0^{\infty} |f|^2 K(r,\theta)\,dr d\theta = B_1 \|f\|_S^2 .$$

Similarly

$$\|f\|_S^2 = \int_0^{2\pi}\int_0^{\infty} |f(z)|^2 K(r,\theta)\,dr d\theta$$

$$= \int_0^{2\pi}\int_0^{r_1} |f(z)|^2 K(r,\theta)\,dr d\theta + \int_0^{2\pi}\int_{r_1}^{\infty} |f(z)|^2 K(r,\theta)\,dr d\theta$$

$$< B' \int_0^{2\pi}\int_{2r_1}^{3r_1} |f|^2 K(r,\theta)\,dr d\theta + \int_0^{2\pi}\int_{r_1}^{\infty} |f|^2 K(r,\theta)\,dr d\theta <$$

$$< B_1' \int_0^{2\pi} \int_{r_1}^{\infty} |f|^2 \tilde{K}(r,\theta)\,dr\,d\theta < B_1' \int_0^{2\pi} \int_0^{\infty} |f|^2 \tilde{K}(r,\theta)\,dr\,d\theta = B_1' \|f(z)\|^2_{\tilde{S}}$$

Proof of II: The proof is essentially the same as that given in Example 1. Let $g(z)$ be entire. Then for $|z| \leq r_1$

(1) $$g(z) = \frac{1}{2\pi i} \int_{\substack{|\xi|=t \\ 2r_1 \leq t \leq 3r_1}} \frac{g(\xi)}{\xi - z}\,d\xi = \frac{1}{2\pi} \int_0^{2\pi} \frac{g(te^{i\phi})te^{i\phi}}{(te^{i\phi} - z)}\,d\phi \,, \quad (2r_1 \leq t \leq 3r_1) .$$

(2) $$|g(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|g(te^{i\phi})|t}{|te^{i\phi} - z|}\,d\phi < \frac{1}{2\pi} \int_0^{2\pi} |g(te^{i\phi})|t\,d\phi .$$

Integrating both sides of (2) $dt$ from $t = 2r_1$ to $t = 3r_1$ we get

(3) $$r_1 |g(z)| < \frac{1}{2\pi} \int_{2r_1}^{3r_1} \int_0^{2\pi} |g(te^{i\phi})|t\,d\phi\,dt .$$

By the Schwartz inequality

(4) $$|g(z)| < \frac{1}{2\pi r_1} \left[ \int_{2r_1}^{3r_1} \int_0^{2\pi} |g(te^{i\phi})|^2 d\phi\,dt \right]^{\frac{1}{2}} \left[ \int_{2r_1}^{3r_1} \int_0^{2\pi} t^2 d\phi\,dt \right]^{\frac{1}{2}} .$$

Squaring both sides of (4) and letting $z = re^{i\theta}$ we have

(5) $$|g(re^{i\phi})|^2 < \text{constant} \left[ \int_{2r_1}^{3r_1} \int_0^{2\pi} |g(te^{i\phi})|^2 d\phi\,dt \right]$$

If we multiply both sides of the inequality in (5) by $L(r,\theta)$ and integrate $dr\,d\theta$ from $r = 0$ to $r = 1$, $\theta = 0$ to $\theta = 2\pi$, we have

$$(6)\qquad \int_0^{r_1}\int_0^{2\pi} |g(re^{i\phi})|^2 L(r,\theta)\,d\theta dr$$

$$< \text{constant} \int_{2r_1}^{3r_1}\int_0^{2\pi} |g(te^{i\phi})|^2 d\phi dt \int_0^{r_1}\int_0^{2\pi} L(r,\theta)\,d\theta dr\ .$$

Since $L(r,\theta)>0$ a.a., and $\int_0^{\infty}\int_0^{2\pi} L(r,\theta)\,drd\theta < \infty$, $0 < \int_0^{r_1}\int_0^{2\pi} L(r,\theta)\,d\theta dr < \infty$ and therefore

$$(7)\qquad \int_0^{r_1}\int_0^{2\pi} |g(re^{i\theta})|^2 L(r,\theta)\,d\theta dr < \text{constant} \int_{2r_1}^{3r_1}\int_0^{2\pi} |g(te^{i\phi})|^2 d\phi dt\ .$$

We are given that for $r > r_1$, $L(r,\theta) > Ce^{-P(r)}$ for some positive constant C and some convex function $P(r)$. Since $P(r)$ is convex, $P(r)$ is continuous and therefore in the interval $2r_1 \le t \le 3r_1$ $P(t) < \alpha < \infty$, i.e., $e^{-P(t)} > e^{-\alpha} = B > 0$. Consequently from (7) we have

$$(8)\qquad \int_0^{r_1}\int_0^{2\pi} |g(re^{i\theta})|^2 L(r,\theta)\,d\theta dr < \frac{\text{constant}}{B} \int_{2r_1}^{3r_1}\int_0^{2\pi} |g(te^{i\phi})|^2 B\,dtd\phi$$

$$< \text{constant} \int_{2r_1}^{3r_1}\int_0^{2\pi} |g(te^{i\phi})|^2 e^{-P(t)}\,d\phi dt$$

$$< \text{constant} \int_{2r_1}^{3r_1}\int_0^{2\pi} |g(te^{i\phi})|^2 L(t,\phi)\,d\phi dt\ .$$

This completes the proof of statement II.

We now show that Theorem 1.3 can easily be extended to the case $k \neq 1$. Recalling the standard notation introduced in the beginning of this chapter we let

$$Z = (z_1, \dots z_n) = (x_1 y_1, x_2 y_2, \dots x_k, y_k) \text{ where } z_m = x_m + iy_m$$

$$X = (x_1, \dots x_k), \quad Y = (y_1, \dots y_k)$$

$$\langle Z \cdot Z \rangle = \|Z\|^2 = r^2 = (|z|^2 + |z_2|^2 + \dots + |z_k|^2)$$

$$dV = \prod_{m=1}^{k} dx_m dy_m$$

Given $dm(Z) = K(x_1, \dots x_k, y_1, \dots y_k)dV$, let S be the set of functions in $E_K$ (the set of entire functions in $C_K$) such that

$$\|f(z)\|^2 = \int |f(z)|^2 K(X,Y)\,dV < \infty \; . \tag{1}$$

Here the integration is over all of $C_K$ considered as a 2k-dimensional real Euclidean space. We introduce the variables $\xi_m = x_m/r$, $\eta_m = y_m/r$. Let $\Theta = (\xi_1\eta_1, \xi_2\eta_2, \dots \xi_k\eta_k)$. It is clear that $\|\Theta\| = 1$. Z can then be written as $r\Theta$. In (1) we make the change of variables $x_m = r\xi_m$, $y_m = r\eta_m$. $\eta_k^2 = 1 - (\xi_1^2 + \xi_2^2 + \dots + \xi_k^2)$ and the Jacobean $J = \frac{\partial(X,Y)}{\partial(r,\Theta)} = r^{2k-1}g(\Theta)$. (For a complete discussion of this transformation see [3], chapter IV.) For f(Z) in S

$$\|f(Z)\|^2 = \underbrace{\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty}}_{2K\text{-fold}} |f(Z)|^2 K(X,Y)\,dV = \int_{0}^{\infty} \int_{\|\Theta\|=1} |f(r,\Theta)|^2 K(r,\Theta)\,|J|\,d\Theta dr$$

$$= \int_{0}^{\infty} \int_{\|\Theta\|=1} |f(r,\Theta)|^2 K_1(r,\Theta)\,d\Theta dr \; .$$

Theorem 1.3 then states the corresponding result in $C_K$ with the understanding that $\theta = \theta$ as defined above. Because of the parametrization the proof for arbitrary K is identical to the proof of Theorem 1.3 with the obvious modifications, and we do not include it.

Before we proceed to Theorem 1.4 which gives another class of measures $\{dm(z)\}$ for which polynomials are dense in the entire functions in the metric of $L^2(dm(z))$, we give an example of a space in which polynomials are not dense. The space we give is one to which not all exponentials belong. Given a measure dm(z) having the property that all of the exponentials belong to $L^2(dm)$, we might ask whether this property is sufficient for polynomials to be dense in $L^2(dm)$. We shall later show by example that this is not the case. In Theorem 1.4 we consider spaces having this property but with added restrictions. In our example we use the known fact that $e^{ix}$ cannot be approximated in $L^2(0,\infty)$ with the weight $e^{-x^\alpha}$, $0 < \alpha < \frac{1}{2}$. For the sake of completeness we include the proof of this fact.

Notation: 1) By $L^2[0,\infty)$ we mean all measurable functions f(x) on $[0,\infty)$ such that $\int_0^\infty |f(x)|^2 dx < \infty$.

2) By $L^2(e^{-x^\alpha})$ we mean all measurable functions f(x) on $[0,\infty)$ such that $\int_0^\infty |f(x)|^2 e^{-x^\alpha})dx < \infty$. As usual we say a set is complete in $L^2[0,\infty)$ or $L^2(e^{-x^\alpha})$ if the set is complete in the given $L^2$ metric. We shall need the folling lemma.

Lemma 1: Let H(x) be a continuous strictly positive function on $[0,\infty)$ satisfying

(i) $$\lim_{x\to\infty} \frac{x^n}{H(x)} = 0 \quad n = 0,1,2,\ldots$$

(ii) $$\int_1^\infty \frac{\ln H(x)}{x^{3/2}}\,dx < \infty .$$

Then the sequence $\{\frac{x^n}{H(x)}\}$ is not complete in $L^2[0,\infty)$.

Proof: ([ 6 ], page 40.)

Note: In particular $\{x^n e^{-x^\alpha/2}\}$, $0 < \alpha < \frac{1}{2}$, is not complete in $L^2[0,\infty)$.

Proposition: The function $e^{ix}$ cannot be approximated by polynomials in $L^2(e^{-x^\alpha})$ when $0 < \alpha < \frac{1}{2}$.

Proof: We first show that polynomials are not dense in $L^2(e^{-x^\alpha})$ when $0 < \alpha < \frac{1}{2}$. We then show that the set $\{e^{i\lambda x}\}$, $\frac{1}{2} \le \lambda < 1$, is complete in $L^2(e^{-x^\alpha})$, and finally that if $e^{ix}$ could be approximated by polynomials in $L^2(e^{-x^\alpha})$, for every $\lambda$, $e^{i\lambda x}$ could be approximated by polynomials.

I. Polynomials are not dense in $L^2(e^{-x^\alpha})$ for $0 < \alpha < \frac{1}{2}$: Let $g(x)$ be any function in $L^2[0,\infty)$. Define $f(x) = g(x)e^{x^\alpha/2}$.

$\int_0^\infty |f(x)|^2 e^{-x^\alpha} dx = \int_0^\infty |g(x)|^2 dx < \infty$. Hence $f(x)$ belongs to $L^2(e^{-x^\alpha})$. Now suppose polynomials were dense in $L^2(e^{-x^\alpha})$, $0 < \alpha < \frac{1}{2}$. Then for every $\varepsilon > 0$ there exists $P(x)$ such that $\int_0^\infty |f(x) - P(x)|^2 e^{-x^\alpha} dx < \varepsilon$. But then

$$\int_0^\infty |g(x) - P(x)\, e^{-x^\alpha/2}|^2\, dx = \int_0^\infty |f(x) - P(x)|^2\, e^{-x^\alpha} dx < \varepsilon ,$$

i.e., the set $\{x^n e^{-x^\alpha/2}\}$ is complete in $L^2[0,\infty)$ which contradicts the lemma.

II. The set $\{e^{i\lambda x}\}$, $\frac{1}{2} \le \lambda < 1$, is complete in $L^2(e^{-x^\alpha})$: It suffices to show that for every $f(x)$ in $L^2(e^{-x^\alpha})$, if $\int_0^\infty f(x)e^{i\lambda x}\, e^{-x^\alpha} dx = 0$ for all $\lambda$ $\frac{1}{2} \le \lambda < 1$, then $f(x) \equiv 0$. Let $F(x) = \int_0^\infty f(x)e^{izx}\, e^{-x^\alpha}$. Assume $F(z) = 0$ for $z$ real, $\frac{1}{2} \le z < 1$. We assert that $F(z)$ is analytic in the upper half plane $\operatorname{Im} z \ge 0$. Since an analytic function cannot vanish on a segment unless it is identically zer0, $F(z) \equiv 0$. ([10], page 88.) This obviously implies that $f(x) \equiv 0$. $F(z)$ is analytic for $\operatorname{Im} z \ge 0$ for by the Schwartz inequality,

$$\int_0^\infty \max_z |f(x)e^{izx}e^{-x^\alpha}|dx = \int_0^\infty \max_z |f(x)|e^{-x^\alpha}e^{-(\operatorname{Im} z)x}\, dx$$

$$< \int_0^\infty |f(x)|e^{-x^\alpha}dx < \left[\int_0^\infty |f(x)|^2 e^{-x^\alpha} dx\right]^{\frac{1}{2}} \left[\int_0^\infty e^{-x^\alpha}dx\right]^{\frac{1}{2}} < \infty .$$

Therefore $\int_0^\infty f(x)e^{izx}\, e^{-x^\alpha} dx$ is uniformly convergent in the upper half plane which implies that $F(z)$ is analytic ([10], page 95). We have shown that $\{e^{i\lambda x}\}$, $\frac{1}{2} \le \lambda < 1$, is complete in $L^2(e^{-x^\alpha})$.

III. $e^{ix}$ cannot be approximated by polynomials in $L^2(e^{-x^\alpha})$: Assume the converse, i.e., for every $\varepsilon > 0$ there exists a polynomial $P(x)$ such that

$$\int_0^\infty |e^{ix} - P(x)|^2\, e^{-x^\alpha} dx < \varepsilon .$$

For fixed $\lambda$, $\frac{1}{2} \le \lambda < 1$,

$$\int_0^\infty |e^{i\lambda x} - P(\lambda x)|^2 e^{-x^\alpha} dx < \int_0^\infty |e^{i\lambda x} - P(\lambda x)|^2 e^{-(\lambda x)^\alpha} dx$$

$$= \frac{1}{\lambda} \int_0^\infty |e^{ix} - P(x)|^2 e^{-x^\alpha} dx < 2\varepsilon \ ,$$

We have shown that the assumption that polynomials approximate $e^{ix}$ in $L^2(e^{-x^\alpha})$ implies that polynomials approximate $e^{i\lambda x}$, $\frac{1}{2} \le \lambda < 1$, in $L^2(e^{-x^\alpha})$. But since $\{e^{i\lambda x}\}$ is complete in $L^2(e^{-x^\alpha})$ it would follow that polynomials are dense in $L^2(e^{-x^\alpha})$ which contradicts I.

We now give an example of a space S in which polynomials are not dense. We divide the complex plane into two sets $R_1$ and $R_2$, where

$$R_1 = \{z|\ \mathrm{Re}\ z \le 0 \ \text{ or } \ \mathrm{Re}\ z > 0 \text{ and } |\mathrm{Im}\ z| > 1\}$$

$$R_2 = \{z|\ \mathrm{Re}\ z > 0 \ \text{ and } \ |\mathrm{Im}\ z| \le 1\} \ .$$

Let $z = x + iy$. We define $dm(z)$

$$dm(z) = e^{-|z|^{1+\delta}} dxdy \ (\delta > 0) \quad \text{for } z \text{ in } R_1$$

$$= e^{-|z|^\alpha} dxdy \left(0 < \alpha < \frac{1}{2}\right) \text{for } z \text{ in } R_2 \ .$$

Let S be the set of entire functions $f(z)$ in C such that

$$\|f\|^2 = \int |f(z)|^2 dm(z) < \infty \ .$$

The function $e^{ix}$ is easily seen to be in S.

$$\|e^{iz}\|_S^2 = \int |e^{iz}|^2 dm(z) < \int_{R_1 \cup R_2} e^{-2y}\, e^{-(x^2+y^2)^{\frac{1+\delta}{2}}} dxdy + \int_{R_2} e^{-2y}\, e^{-(x^2+y^2)^{\frac{\alpha}{2}}} dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2y}\, e^{-(x^2+y^2)^{\frac{1+\delta}{2}}} dxdy + \int_{-1}^{1}\int_{0}^{\infty} e^{-2y}\, e^{-(x^2+y^2)^{\frac{\alpha}{2}}} dxdy .$$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2y}\, e^{-(x^2+y^2)^{\frac{1+\delta}{2}}} dxdy$ is obviously convergent since $\delta > 0$.

$\int_{-1}^{1}\int_{0}^{\infty} e^{-2y}\, e^{-(x^2+y^2)^{\frac{\alpha}{2}}} dxdy < C \int_0^{\infty} e^{-x^{\alpha}} dx < \infty$ . Thus $\|e^{iz}\|_S^2 < \infty$, i.e. $e^{iz}$ belongs to S. We will prove

** There exists a constant K such that for every $f(z)$ in S

$\int_0^{\infty} |f(x)|^2\, e^{-x^{\alpha}} dx < K\|f(z)\|_S^2$ .

** implies that in particular the function $e^{iz}$ cannot be approximated by polynomials in S. For suppose the converse, i.e., given $\varepsilon > 0$ there exists a polynomial $P(z)$ such that $\|e^{iz} - P(z)\|_S^2 < K\varepsilon$ . By ** $\int_0^{\infty} |e^{ix} - P(x)|^2\, e^{-x^{\alpha}} dx \leq K\|e^{iz} - P(z)\|^2 < \varepsilon$ , i.e., $e^{ix}$ can be approximated by polynomials in $L^2(e^{-x^{\alpha}})$, $0 < \alpha < \frac{1}{2}$ , which contradicts the previous proposition.

<u>Proof of **</u>: Let $f(z)$ be any function in S. Then for any point u, $u \geq 0$,

$$(1) \quad f^2(u) = \frac{1}{2\pi i} \int_{|\xi-u|=r} \frac{f^2(\xi)}{\xi-u}\, d\xi = \frac{1}{2\pi} \int_0^{2\pi} f^2(u + re^{i\theta})\, d\theta \text{ where } 0 < r \leq 1$$

$$(2) \qquad |f^2(u)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(u+re^{i\theta})|^2 d\theta \quad (0 < r \le 1)\ .$$

Multiplying both sides of (2) by $r\,e^{-u^\alpha}$ and then integrating drdu from $r = 0$ to $r = 1$ and from $u = 0$ to $u = \infty$, we get

$$(3) \qquad \int_0^\infty |f(u)|^2 e^{-u^\alpha} du \le \frac{1}{\pi}\int_0^\infty \int_0^{2\pi} \int_0^1 |f(u+re^{i\theta})|^2\, e^{-u^\alpha} r\,dr\,d\theta\,du\ .$$

We make the following change of variables in the right hand integral of (3).

$$x = u + r\cos\theta$$
$$y = r\sin\theta$$
$$v = u$$
$$d\theta\,dr\,du = \frac{1}{r}\,dv\,dy\,dx$$

and

$$(4) \qquad \int_0^\infty |f(u)|^2 e^{-u^\alpha} du \le \frac{1}{\pi}\int_{-1}^{\infty} \int_{-1}^{1} |f(x+iy)|^2 \int_{\max\{0,x-1\}}^{x+1} e^{-v^\alpha} dv\,dy\,dx$$

$$= \frac{1}{\pi}\left[\int_{-1}^{2}\int_{-1}^{1} |f(x+iy)|^2 \int_{\max\{0,x-1\}}^{x+1} e^{-v^\alpha} dv\,dy\,dx + \int_2^\infty \int_{-1}^{1} |f(x+iy)|^2 \int_{x-1}^{x+1} e^{-v^\alpha} dv\,dx\,dy\right]$$

Let $z = x + iy$ and note

a) for $-1 \le x < 2$, $|y| < 1$, $e^{-v^\alpha} < 1$, and $|z| < 3$

b) for $2 \le x < \infty$, $|y| < 1$, $e^{-v^\alpha} <$ constant $e^{-|z|^\alpha}$ .

b) is true because for $2 \le x < \infty$ and $|y| < 1$

(i) $v \ge x - 1 = |x| + 1 - 2 \ge |x| + |y| - 2 \ge |x+iy| - 2 = |z| - 2 \ge 0$

and

(ii) $$|z|^{\alpha} = (|z| - 2 + 2)^{\alpha} \le (|z| - 2)^{\alpha} + 2^{\alpha} .$$

Combining (i) and (ii) we have

$$e^{-v^{\alpha}} < e^{-(|z|-2)^{\alpha}} < e^{-|z|^{\alpha}-2^{\alpha}} = \text{constant } e^{-|z|^{\alpha}}$$

which is b). From (4) we have

(5) $$\int_0^{\infty} |f(u)|^2 e^{-u^{\alpha}} du < \frac{2}{\pi}\Bigg[\int_{-1}^{0}\int_{-1}^{1} |f(z)|^2 e^{-|z|^{1+\delta}} e^{|z|^{1+\delta}} dydx$$

$$+ \int_0^2 \int_{-1}^{1} |f(z)|^2 e^{-|z|^{\alpha}} e^{|z|^{\alpha}} dydx + C\int_2^{\infty}\int_{-1}^{1} |f(z)|^2 e^{-|z|^{\alpha}}\Bigg]$$

$$< K\Bigg[\int_{-1}^{0}\int_{-1}^{1} |f(z)|^2 e^{-|z|^{1+\delta}} dydx + \int_0^{\infty}\int_{-1}^{1} |f(z)|^2 e^{-|z|^{\alpha}} dydx\Bigg]$$

$$< K\Bigg[\int_{-\infty}^{0}\int_{-\infty}^{\infty} |f(z)|^2 e^{-|z|^{1+\delta}} dydx + \int_0^{\infty}\int_{|y|>1} |f(z)|^2 e^{-|z|^{1+\delta}} dydx$$

$$+ \int_0^{\infty}\int_{-1}^{1} |f(z)|^2 e^{-|z|^{\alpha}} dydx\Bigg]$$

$$= K\Big[\int_{R_1} |f(z)|^2 dm(z) + \int_{R_2} |f(z)|^2 dm(z)\Big] = K\|f(z)\|^2 .$$

This completes the proof of ** and, as shown above, it therefore follows that $e^{iz}$ cannot be approximated in the space S.

Remark: The above case was obviously constructed in such a way so as to make use of the fact that $e^{iz}$ is in it, and $e^{ix}$ cannot be approximated by polynomials in $L^2(e^{-x^\alpha})$, $0 < \alpha < \frac{1}{2}$. If we vary the conditions on $dm(z)$ slightly, e.g., $\alpha = \frac{1}{2}$ or $\delta = 0$ (in which case $e^{iz}$ does not belong to $L^2(dm)$), the situation becomes entirely different, and whether polynomials are dense in $L^2(dm)$ remains an open problem.

In Theorem 1.4 we confine our attention to the case $k = 1$, i.e., the complex plane. We let $z = x + iy$ $dm(z) = k(x,y)dxdy$.

Theorem 1.4: Let $k(x,y) = e^{-\phi(x,y)}$ where $\phi(x,y) = \phi(z)$ is a positive convex function of $z$, $\phi(0) = 0$. Let S be the space of all entire functions $b(z)$ such that $\|b(z)\|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |b(z)|^2 k(x,y)\,dxdy < \infty$, and suppose $e^{tz}$ belongs to S for all complex t. Then the exponentials are complete in S.

Proof: S is easily seen to be a Hilbert space. Therefore to prove that the exponentials are complete, it is sufficient to show

$$(1)\qquad \int\int b(z)e^{t\bar{z}}\, e^{-\phi(x,y)}dxdy = 0 \quad \text{for all complex } t \ (b(z) \text{ in } S) \text{ implies } b(z) \equiv 0.$$

The proof will be given as a series of lemmas. We begin by introducing notation which will be used throughout the series.

Notation

1. $C(x,y) = b(z)e^{-\phi(x,y)}$.

Recalling the definition of the Fourier transform in the plane we define

2. $\hat{d}(\alpha,\beta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d(x,y)e^{i\alpha x+i\beta y}dxdy$, $\alpha,\beta$ complex.

In particular,

$$3. \quad \hat{c}(\alpha,\beta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(x,y)e^{i\alpha x+i\beta y}dxdy.$$

Since $\phi(x,y)$ is such that all exponentials belong to the space, $\hat{c}(\alpha,\beta)$ is meaningful and an analytic function of the two complex variables $\alpha,\beta$ ([ 8], page 13).

$$4. \quad \hat{h}(\alpha,\beta) = \frac{\hat{c}(\alpha,\beta)}{\alpha - i\beta} .$$

5. $\phi(x,y)$ is convex. We may therefore define the conjugate function $\psi(\tau,\tau') = \max\limits_{\substack{x,y \\ -\infty<x<\infty \\ -\infty<y<\infty}} \{\tau x + \tau' y - \phi(x,y)\}$ where $\phi$ and $\psi$ have a reciprocal relationship, i.e., $\psi$ is convex and $\phi(x,y) = \max\limits_{\substack{\tau,\tau' \\ -\infty<\tau<\infty \\ -\infty<\tau' <\infty}} \{\tau x + \tau' y - \psi(\tau,\tau')$ [5].

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} b(z)e^{t\bar{z}}\ e^{-\phi(x,y)}dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(x,y)e^{t(x-iy)}dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(x,y)e^{i(-it)x+i(-t)y}dxdy = \hat{c}(-it,-t) .$$

Assume $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} b(z)^{t\bar{z}}\ e^{-\phi(x,y)}dxdy = c(-it,-t) = 0$ for all $t$ complex, i.e., $\hat{c}(\alpha,\beta) = 0$ when $\alpha = i\beta$. It then follows that $\hat{h}(\alpha,\beta)$ is analytic in $\alpha$ and $\beta$. We will show $b(z) \equiv 0$. Let $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$. $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$ real.

<u>Lemma 1</u>: Let $0 < k < \infty$. Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{d}(\alpha,\beta)|^2 d\alpha_1 d\beta_1 < M\, e^{\psi\left(\frac{-2\alpha_2}{k},\, \frac{-2\beta_2}{k}\right)} \quad \text{if and only if}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |d(x,y)|^2\, e^{\phi(kx,ky)}\, dxdy < \infty$$

where M is some positive constant independent of $\alpha_2$ and $\beta_2$.

<u>Proof:</u> (a) Assume $\iint |d(xy)|^2 e^{\phi(kx,ky)} dxdy < \infty$. By Parseval's identity

$$(1) \quad \iint |\hat{d}(\alpha,\beta)|^2 d\alpha_1 d\beta_1 = \iint |d(x,y) e^{-\alpha_2 x - \beta_2 y}|^2 dxdy$$

$$= \iint |d(x,y)|^2\, e^{-2\alpha_2 x - 2\beta_2 y} dxdy\ .$$

Letting $x = \frac{x'}{k}$, $y = \frac{y'}{k}$, we have

$$(2) \quad \iint |d(x,y)|^2\, e^{-2\alpha_2 x - 2\beta_2 y} dxdy = \frac{1}{k^2}\iint |d(\tfrac{x'}{k}, \tfrac{y'}{k})|^2\, e^{\frac{-2\alpha_2 x'}{k} \frac{-2\beta_2 y'}{k}} dx'dy'$$

$$= \frac{1}{k^2}\iint |d(\tfrac{x'}{k}, \tfrac{y'}{k})|^2\, e^{\phi(x',y')}\, e^{\frac{-2\alpha_2 x'}{k} \frac{-2\beta_2 y'}{k} - \phi(x',y')} dx'dy'$$

$$\leq \frac{1}{k^2}\iint |d(\tfrac{x'}{k}, \tfrac{y'}{k})|^2\, e^{\phi(x',y')}\, e^{\max_{x'y'}\left\{\frac{-2\alpha_2 x'}{k} \frac{-2\beta_2 y'}{k} - \phi(x',y')\right\}} dx'dy'$$

$$= \frac{1}{k^2}\iint |d(\tfrac{x'}{k}, \tfrac{y'}{k})|^2\, e^{\phi(x',y')}\, e^{\psi\left(\frac{-2\alpha_2}{k},\, \frac{-2\beta_2}{k}\right)} dx'dy'\ .$$

Letting $x = \frac{x'}{k}$, $y = \frac{y'}{k}$, we have from (1) and (2)

(3) $$\iint |\hat{d}(\alpha,\beta)|^2 d\alpha_1 d\beta_1 \le e^{\psi(\frac{-2\alpha_2}{k},\ \frac{-2\beta_2}{k})} \iint |d(x,y)|^2\, e^{\phi(kx,ky)} dxdy$$

$$= M\, e^{\psi(\frac{-2\alpha_2}{k},\ \frac{2\beta_2}{k})}$$

(b) Assume $\iint |\hat{d}(\alpha,\beta)|^2 d\alpha_1 d\beta_1 < M\, e^{\psi(\frac{-2\alpha_2}{k},\ \frac{-2\beta_2}{k})}$. By Parseval's identity

(1) $$\iint |d(x,y)|^2\, e^{-2\alpha_2 x - 2\beta_2 y} dxdy = \iint |\hat{d}(\alpha,\beta)|^2 d\alpha_1 d\beta_1 < M\, e^{\psi(\frac{-2\alpha_2}{k},\ \frac{-2\beta_2}{k})}.$$

Hence

(2) $$\iint |d(x,y)|^2\, e^{-2\alpha_2 x\ -2\beta_2 y\ -\psi(\frac{-2\alpha_2}{k},\ \frac{-2\beta_2}{k})} dxdy < M .$$

Letting $x = \frac{x'}{k}$, $y = \frac{y'}{k}$ in (2), we have

(3) $$\frac{1}{k^2} \iint |d(\frac{x'}{k}, \frac{y'}{k})|^2\, e^{\frac{-2\alpha_2 x'}{k}\ \frac{-2\beta_2 y'}{k}\ -\psi(\frac{-2\alpha_2}{k},\ \frac{-2\beta_2}{k})} dx'dy' < M .$$

Recall that $\phi(x',y') = \max\limits_{\alpha_2,\beta_2} \left\{ \frac{-2\alpha_2 x'}{k}\ \frac{-2\beta_2 y'}{k}\ -\psi(\frac{-2\alpha_2}{k},\ \frac{-2\beta_2}{k}) \right\}$. But M is independent of $\alpha_2$ and $\beta_2$. Therefore (3) holds for that $\alpha_2$, $\beta_2$ at which the maximum occurs, i.e.,

(4) $$\frac{1}{k^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |d(\frac{x'}{k}, \frac{y'}{k})|^2\, e^{\phi(x',y')}\ dx'dy' < M .$$

Letting $x = \frac{x'}{k}$, $y = \frac{y'}{k}$, in (4) we get the desired result, i.e.,

(5) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |d(x,y)|^2 \, e^{\phi(kx,ky)} \, dxdy < M < \infty .$$

Lemma 2: Let c and c' be fixed real constants. Let $\lambda$ be any real number $0 < \lambda < 1$. Then there exist M and $M_1$ such that

a) $$\iint |\hat{c}(\alpha_1 + ic, \beta_1 + ic')|^2 d\alpha_1 d\beta_1 < M\, e^{\psi(-2c,-2c')}$$

b) $$\iint |\hat{h}(\alpha_1 + ic, \beta_1 + ic')|^2 d\alpha_1 d\beta_1 < M_1 e^{\psi(-\frac{2c}{\lambda}, -\frac{2c'}{\lambda})} .$$

Proof of a): Since b(z) is in S,

$$\iint |c(x,y)|^2 \, e^{\phi(x,y)} dxdy = \iint |b(z)|^2 \, e^{-\phi(x,y)} dxdy = \|b(z)\|^2 < \infty .$$

Therefore, by Lemma 1

$$\iint |\hat{c}(\alpha_1 + ic, \beta_1 + ic')|^2 d\alpha_1 d\beta_1 < M\, e^{\psi(-2c,-2c')} .$$

Proof of b): We first show

(A) $$|\hat{c}(\alpha_1 + i\alpha_2, \beta_1 + i\beta_2)|^2 < \text{constant } e^{\max \psi(-2(\alpha_2 \pm 1), -2(\beta_2 \pm 1))}$$

where the max is taken over the four possibilities of sign. Since $(\hat{c}(\alpha,\beta))^2$ is an analytic function of $\alpha$ and $\beta$ we have

(1) $$(\hat{c}(\alpha,\beta))^2 = \frac{1}{\pi^2}\int_0^1\int_0^1\int_0^{2\pi}\int_0^{2\pi} (\hat{c}(\alpha + r_1 e^{i\theta_1}, \beta + r_2 e^{i\theta_2}))\, d\theta_1 d\theta_2 dr_1 dr_2$$

and therefore

$$(2)\quad |\hat{c}(\alpha,\beta)|^2 \le \frac{1}{\pi^2}\int_0^1\int_0^1\int_0^{2\pi}\int_0^{2\pi} |\hat{c}(\alpha+r_1e^{i\theta_1},\ \beta+r_2e^{i\theta_2})|^2 r_1r_2d\theta_1d\theta_2dr_1dr_2$$

Letting $z_1 = x_1 + iy_1 = r_1e^{i\theta_1}$, $z_2 = x_2 + iy_2 = r_2e^{i\theta_2}$ we have

$$(3)\quad |\hat{c}(\alpha,\beta)|^2 \le \frac{1}{\pi^2}\int_{-1}^1 dy_1\int_{-1}^1 dy_2\int_{-1}^1\int_{-1}^1 |\hat{c}[\alpha_1+x_1+i(\alpha_2+y_1),\beta_1+x_2+i(\beta_2+y_2)]|^2dx_1dx_2$$

$$< \frac{1}{\pi^2}\int_{-1}^1 dy_1\int_{-1}^1 dy_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{c}[\alpha_1+x_1+i(\alpha_2+y_1),\beta_1+x_2+i(\beta_2+y_2)]|^2dx_1dx_2$$

Letting $\alpha_1 + x_1 = \alpha_1'$, $\beta_1 + x_2 = \beta_1'$ in the last integral of (3), we have from part a) of Lemma 2

$$(4)\quad |\hat{c}(\alpha,\beta)|^2 \le \frac{1}{\pi^2}\int_{-1}^1 dy_1\int_{-1}^1 dy_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{c}[\alpha_1'+i(\alpha_2+y_1),\ \beta_1'+i(\beta_2+y_2)]|^2d\alpha_1'd\beta_1'$$

$$< \frac{M}{\pi^2}\int_{-1}^1\int_{-1}^1 e^{\psi(-2(\alpha_2+y_1),-2(\beta_2+y_2))}dy_1dy_2\ .$$

$$< \frac{4M}{\pi^2}\max_{\substack{-1\le y_1\le 1\\ -1\le y_2\le 1}} e^{\psi(-2(\alpha_2+y_1),-2(\beta_2+y_2))}$$

However since $\psi$ is convex, $e^{\psi}$ is convex and

$$\max_{\substack{-1\le y_1\le 1\\ -1\le y_2\le 1}} e^{\psi(-2(\alpha_1+y_1),-2(\beta_2+y_2)]} = \max_{\pm} e^{\psi(-2(\alpha_2\pm 1),-2(\beta_2\pm 1))}$$

and thus for fixed $\alpha,\beta$

$$|\hat{c}(\alpha_1+i\alpha_2,\ \beta_1+i\beta_2)|^2 < \frac{4M}{\pi^2}\, e^{\max\limits_{\pm} \psi(-2\alpha_2 \pm 2, -2\beta_2 \pm 2)}$$

which was the assertion of (A).

We now prove the second assertion of Lemma 2, namely

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{h}(\alpha_1+ic,\ \beta_1+ic')|^2 d\alpha_1 d\beta_1 < M_1\, e^{\psi(-\frac{2c}{\lambda}, \frac{2c'}{\lambda})}$$

where $\lambda$ is any fixed number, $0 < \lambda < 1$.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{h}(\alpha_1+ic,\ \beta_1 + ic')|^2 d\alpha_1 d\beta_1$$

$$= \int_{|\beta_1-c|>1}\int_{-\infty}^{\infty} |\hat{h}(\alpha_1+ic,\beta_1+ic')|^2 d\alpha_1 d\beta_1 + \int_{|\beta_1-c|<1}\int_{|\alpha_1+c'|>1} |\hat{h}(\alpha_1+ic,\beta_1+ic')|^2 d\alpha_1 d\beta_1$$

$$+ \int_{c-1}^{c+1}\int_{-c'-1}^{-c'+1} |\hat{h}(\alpha_1+ic,\ \beta_1+ic')|^2 d\alpha_1 d\beta_1 .$$

When $|\beta_1-c| > 1$ or $|\alpha_1+c'| > 1$, $|\hat{h}(\alpha_1+ic,\ \beta_1+ic')| < |\hat{c}(\alpha_1+ic,\ \beta_1+ic')|$.

Hence

$$(2)\quad \int_{|\beta_1-c|>1}\int_{-\infty}^{\infty} |\hat{h}(\alpha_1+ic,\beta_1+ic')|^2 d\alpha_1 d\beta_1 + \int_{|\beta_1-c|<1}\int_{|\alpha_1+c'|>1} |\hat{h}(\alpha_1+ic,\beta_1+ic')|^2 d\alpha_1 d\beta_1$$

$$< \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{c}(\alpha_1+ic,\ \beta_1+ic')|^2 d\alpha_1 d\beta_1 < 2M\, e^{\psi(-2c,-2c')} \quad \text{(by Lemma 2, part a).}$$

Moreover for fixed $\alpha_1$, $|\alpha_1+c'| \le 1$, and fixed $\beta_1$, $|\beta_1-c| \le 1$,

$\hat{h}(\alpha_1+iz_1, \beta_1+iz_2)$ is an entire function of $z_1$ and $z_2$. In particular, it is analytic in the polydisc $|z_1-c| \le 2$, $|z_2-c'| \le 5$. Hence by the maximum modulus principle, for some $\theta_1$, $\theta_2$ depending respectively on $\alpha_1$ and $\alpha_2$

$$|\hat{h}(\alpha_1+ic, \beta_1+ic')| \leq |\hat{h}(\alpha_1+i(c+2e^{i\theta_1}), \beta_1+i(c'+5e^{i\theta_2})|$$

Since

$$|\alpha-i\beta| = |\alpha_1+c'+5e^{i\theta_2}+i(c-\beta_1+2e^{i\theta_1})| \ge \left| |\alpha_1+c'+5e^{i\theta_2}| - |c-\beta_1+2e^{i\theta_1}| \right| \ge |4-3| = 1,$$

we have

$$|\hat{h}(\alpha_1+ic, \beta_1+ic'| \leq |\hat{c}(\alpha_1+i(c+2e^{i\theta_1}), \beta_1+i(c'+5e^{i\theta_2})|$$

and therefore by (A)

$$|\hat{h}(\alpha_1+ic,\beta_1+ic')|^2 < \frac{4M}{\pi^2} e^{\max\limits_{\pm} \psi(-2(c+2\cos\theta_1 \pm 1),-2(c'+5\cos\theta_2 \pm 1))}$$

$$\le \text{(since } \psi \text{ is convex) constant } e^{\max\limits_{\pm} \psi(-2(c\pm 3),\, -2(c'\pm 6))}.$$

Now consider the last integral in (1). By the above reasoning

$$(3) \qquad \int_{c-1}^{c+1} \int_{-c'-1}^{-c'+1} |\hat{h}(\alpha_1+ic, \beta_1+ic')|^2 d\alpha_1 d\beta_1 < \frac{16M}{\pi^2} e^{\max\limits_{\pm} \psi(-2(c\pm 3),-2(c'\pm 6))}.$$

It follows from (2) and (3) that

$$(4)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{h}(\alpha_1+ic,\ \beta_1+ic')|^2 d\alpha_1 d\beta_1$$

$$< M_1[e^{\psi(-2c,-2c')} + e^{\psi(-2c+6,-2c'+12)} + e^{\psi(-2c-6,-2c'+12)}$$

$$+ e^{\psi(-2c+6,-2c'-12)} + e^{\psi(-2c-6,-2c'-12)}] .$$

Since $\psi$ is a convex function, we have for $0 < \lambda < 1$

$$(5)\qquad \psi(-2c\pm 6,\ -2c'\pm 12) \leq \lambda\psi(-\frac{2c}{\lambda},\ -\frac{2c'}{\lambda}) + (1-\lambda)\psi(\pm\frac{6}{1-\lambda},\ \pm\frac{12}{1-\lambda}) .$$

We note that $\psi(\pm\frac{6}{1-\lambda},\ \pm\frac{12}{1-\lambda})$ is finite for $0 < \lambda < 1$. Moreover, since every exponential belongs to S,

$$(6)\quad \psi(-2c,\ -2c') < \psi(-\frac{2c}{\lambda},\ -\frac{2c'}{\lambda}) .$$

Combining (5) and (6) we have from (4)

$$(7)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{h}(\alpha_1+ic,\ \beta_1+ic')|^2 d\alpha_1 d\beta_1 < M_2\, e^{\psi(-\frac{2c}{\lambda},\ -\frac{2c'}{\lambda})} .$$

This completes the proof of Lemma 2.

<u>Lemma 3</u>: $c(x,y) = i\,\frac{dh(x,y)}{dz}$, $\left(\frac{dh}{dz} = \frac{dh}{dx} - i\,\frac{dh}{dy}\right)$.

<u>Proof</u>: By definition

$$(1)\quad \hat{\frac{dh}{dz}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dh}{dz}(x,y)\,e^{i\alpha x+i\beta y}dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{dh}{dx} - i\,\frac{dh}{dy}\right) e^{i\alpha x+i\beta y}\,dxdy\,.$$

We integrate (1) by parts letting "u" $= e^{i\alpha x+i\beta y}$, "dv" $= \frac{dh}{dz}$. We show that the boundary terms vanish. "uv" $= e^{i\alpha x+i\beta y}\,h(xy)$. By Fubini's Theorem we need only show

$$(2)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|e^{i\alpha x+i\beta y}\,h(x,y)\right| dxdy < \infty\,.$$

By Lemma 1 and Lemma 2, for $\frac{1}{2} \le \lambda < 1$, $\iint |h(x,y)|^2\, e^{\phi(\lambda x,\lambda y)}dxdy < \infty$. Since exponentials are in the space S

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |e^{i\alpha x+i\beta y}|^2 e^{-\phi(\lambda x,\lambda y)}\,dxdy = \frac{1}{\lambda^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|e^{\frac{i\alpha x}{\lambda}+\frac{i\beta y}{\lambda}}\right|^2 e^{-\phi(x,y)}dxdy < \infty$$

We therefore have by the Schwartz inequality

$$\iint \left|e^{i\alpha x+i\beta y}h(x,y)\right|dxdy = \iint \left|e^{i\alpha x+i\beta y}\, e^{\frac{-\phi(\lambda x,\lambda y)}{2}}\, h(x,y)\, e^{\frac{\phi(\lambda x,\lambda y)}{2}}\right|dxdy$$

$$\le \left[\iint |e^{2i\alpha x+2i\beta y}|\; e^{-\phi(\lambda x,\lambda y)}\,dxdy\right]^{\frac{1}{2}} \left[\iint |h(x,y)|^2\, e^{\phi(\lambda x,\lambda y)}dxdy\right] < \infty\,.$$

Hence

$$\hat{\frac{dh}{dz}} = -i(\alpha-i\beta)\iint h(x,y)^{\,i\alpha x+i\beta y}dxdy = -i(\alpha-i\beta)\hat{h}(\alpha,\beta) = -i\hat{c}(\alpha,\beta)$$

and

$$c(x,y) = i\,\frac{dh}{dz}\,(x,y)$$

Lemma 4: Let $g(z)$ be any function in S. Then for $\lambda$, $\frac{1}{2} \leq \lambda < 1$,
$\iint b(z)\bar{g}(\lambda z)e^{-\phi(x,y)}dxdy = 0$.

Proof: By Lemma 4,

$$(1)\quad \iint b(z)\bar{g}(\lambda z)^{-\phi(x,y)}dxdy = \iint c(x,y)\bar{g}(\lambda z)dxdy = i\iint [\tfrac{dh}{dz}]\;\bar{g}(\lambda z)dxdy$$

We integrate (1) by parts letting "u" = $\bar{g}(\lambda z)$, "dv" = $\frac{dh}{dz}$. It is easy to see that "u v" = $\bar{g}(\lambda z)h(x,y)$ vanishes on the boundary. For, as in Lemma 3, we need only show

$$(2)\quad \iiint |h(x,y)\bar{g}(\lambda z)|dxdy < \infty$$

$$(3)\quad \iiint |h(x,y)\bar{g}(\lambda z)|dxdy = \iiint \left|h(x,y)\; e^{\frac{\phi(\lambda x,\lambda y)}{2}}\;\bar{g}(\lambda z)e^{\frac{-\phi(\lambda x,\lambda y)}{2}}\right|dxdy\ .$$

Applying the Schwartz inequality to the left hand side of (3) we have

$$(4)\quad \iiint |h(x,y)\bar{g}(\lambda z)|dxdy$$

$$\leq [\iiint |h(x,y)|^2\; e^{\phi(\lambda x,\lambda y)}dxdy]^{\frac{1}{2}}\;[\iiint |\bar{g}(\lambda z)|^2\; e^{-\phi(\lambda x,\lambda y)}dxdy]^{\frac{1}{2}}\ .$$

As shown in Lemma 2, $\iint |h(x,y)|^2\; e^{\phi(\lambda x,\lambda y)}dxdy < \infty$. Letting $\lambda z = z'$

$$\iint |\bar{g}(\lambda z)|^2 \, e^{-\phi(\lambda x, \lambda y)} dxdy = \frac{1}{\lambda^2} \iint |g(z)|^2 \, e^{-\phi(x,y)} dxdy < 4\|g(z)\|^2 < \infty \, ,$$

because g(z) is in S. We have shown that (2) is true and therefore the boundary terms vanish. Moreover, since g is analytic, $\frac{d\bar{g}}{dz} = 0$ and therefore

$$(5) \qquad \iint b(z)\bar{g}(\lambda z) \, e^{-\phi(x,y)} dxdy = -i \iint h(x,y) \, \frac{d\bar{g}}{dz} \, (\lambda z) dxdy = 0$$

Lemma 5: Let $\frac{1}{2} \le \lambda < 1$. Let G = the set of functions $f(\lambda z)$ such that $f(z)$ belongs to S. Then G is weakly dense in S. In particular, $f(\lambda z)$ converges weakly to $f(z)$ as $\lambda \to 1$.

Proof: As shown in the proof of Theorem 1.3 since, for any $f(z)$, $f(\lambda z)$ converges to $f(z)$ pointwise, it suffices to show that $f(\lambda z)$ is bounded in norm in S, i.e., $\|f(\lambda z)\| < M \|f(z)\|$ where M is independent of $\lambda$. Let $\lambda z = z'$. Since $\phi(z)$ is an increasing function of $|z|$ on every half ray, $\phi(\frac{x}{\lambda}, \frac{y}{\lambda}) > \phi(x,y)$ and therefore

$$\|f(\lambda z)\|^2 = \iint |f(\lambda z)|^2 \, e^{-\phi(x,y)} dxdy = \frac{1}{\lambda^2} \iint |f(z')|^2 \, e^{-\phi(\frac{x'}{\lambda}, \frac{y'}{\lambda})} dx'dy'$$

$$< 4 \iint |f(z)|^2 \, e^{-\phi(x,y)} dxdy = 4\|f(z)\|_S^2 \quad .$$

Theorem 1.4 is now obvious. Let $g(z) = b(z)$ in Lemma 4. By Lemma 5

$$\|b(z)\|^2 = \lim_{\lambda \to 1} \int b(z)\bar{b}(\lambda z) \, e^{-\phi(x,y)} dxdy = 0 \quad .$$

Therefore $b(z) \equiv 0$. This completes the proof of Theorem 1.4.

Corollary: Polynomials are dense in the space S as defined in Theorem 1.4.

Proof: We prove that the exponentials can be approximated by polynomials. Since the exponentials are complete in S this gives the desired result. In particular, we show that the Taylor expansion of $e^{az}$ converges to $e^{az}$ in norm for all a, a complex.

$$e^{az} = \sum_{k=0}^{n} \frac{(az)^n}{n!} + \frac{1}{n!}\int_0^1 (az)^{n+1} e^{azt}(1-t)^n dt \quad ([1], \text{ page } 76).$$

Therefore

$$(1) \qquad \left|e^{az} - \sum_{k=0}^{n} \frac{(az)^n}{n!}\right| = \frac{1}{n!}\left|\int_0^1 (az)^{n+1} e^{azt}(1-t)^n dt\right|$$

$$\leq \frac{1}{n!}\int_0^1 |az|^{n+1} e^{\text{Re } azt} dt = \frac{|az|^{n+1}}{n!}\int_0^1 e^{t \text{ Re } az} dt$$

$$< \frac{|az|^{n+1}}{n!}\left[\int_0^1 e^{2\text{Re } azt} dt\right]^{\frac{1}{2}} = \frac{|az|^{n+1}}{n!}\left[\frac{e^{2\text{Re } az} - 1}{2\text{Re } az}\right]^{\frac{1}{2}}$$

$$< \frac{|az|^{n+1}}{n!}\left[1 + |e^{2az}|\right]^{\frac{1}{2}}.$$

Furthermore

$$(2) \qquad \left(\frac{|az|^{n+1}}{n!}\right)^2 < \sum_{k=1}^{4} \left|e^{4(i)^k az}\right|^2$$

One verifies (2) quite easily. $|az| \leq |\text{Re } az| + |\text{Im } az| \leq 2\max(|\text{Re } az|, |\text{Im } az|)$.

When $\max(|\text{Re } az|, |\text{Im } az|) = |\text{Re } az|$

$$\frac{|az|^{n+1}}{n!} \le \frac{(2\text{Re } az)^{n+1}}{n!} = (2\text{Re } az)\frac{(2\text{Re } az)^n}{n!} < e^{4\text{Re } az} = |e^{4az}|$$

(Re az > 0)

$$\frac{|az|^{n+1}}{n!} \le \frac{(-2\text{Re } az)^{n+1}}{n!} = (-2\text{Re } az)\frac{(-2\text{Re } az)^n}{n!} < e^{-4\text{Re } az} = |e^{-4az}|$$

(Re az > 0)

Similarly, when $\max(|\text{Re } az|, |\text{Im } az|) = |\text{Im } az|$

$$\frac{|az|^{n+1}}{n!} < |e^{-4iaz}| \ , \ (\text{Im } az > 0)$$

$$\frac{|az|^{n+1}}{n!} < |e^{4iaz}| \ , \ (\text{Im } az < 0) \ .$$

Combining (1) and (2) we have

$$\left|e^{az} - \sum_{k=0}^{n} \frac{(az)^n}{n!}\right|^2 < \sum_{k=1}^{4}\left|e^{4(i)^k az}\right|^2 + \sum_{k=1}^{4}\left|e^{(4(i)^k+1)az}\right|^2 = g(z) \quad .$$

Since by assumption every exponential is in S, $\int g(z)e^{-\phi(z)}dxdy < \infty$, and therefore by the Lebesgue dominated convergence theorem

$$\lim_{n \to \infty} \left\|e^{az} - \sum_{k=0}^{n} \frac{(az)^n}{n!}\right\|_S = 0 \ .$$

We now give an example of a Hilbert space S to which all the exponentials belong and yet in which the exponentials are not complete. We divide the complex plane into four regions. As usual $z = x + iy$. We let

$$R_1 = \{z \mid |z| > 3\ ,\ x > 0\ ,\ |xy| \le 1\}\ .$$

$$R_2 = \{z \mid |z| > 3\ ,\ x > 0\ ,\ y > 0\ ,\ |x^2-y^2| \le 2\}$$

$$R_3 = \{z \mid |z| \le 3\}$$

$$R_4 = C - (R_1 \cup R_2 \cup R_3)\ .$$

We let $dm(z) = k(x,y)dxdy$ where

$$k(x,y) = \begin{cases} e^{-2x^2}x^{12} & (z \in R_1) \\ e^{-(x+y)^2}(x+y)^{12} & (z \in R_2) \\ 1 & (z \in R_3) \\ e^{-3|z|^8} & (z \in R_4) \end{cases}$$

As above we let S be the space of entire functions $f(z)$ such that $\|f(z)\|_S^2 = \int |f(z)|^2 dm(z) < \infty$. S is easily seen to be a Hilbert space and clearly $\int |e^{az}|^2\, dm(z) < \infty$ for all complex $a$, i.e., every exponential belongs to S. We first define the space

$S_1$ = the set of entire functions $f(z)$ such that

$$\|f(z)\|^2_{S_1} = \int_0^\infty (|f(x)|^2 + |f(\xi x)|^2)\,dm_1(x) < \infty ,$$

$$\text{where } dm_1(x) = \frac{e^{-2x^2} x^{11}}{1-e^{-x^8}}\,dx \quad \text{and} \quad \xi = e^{\frac{i\pi}{4}} .$$

It is clear that $e^{az} \varepsilon\, S_1$ for all complex numbers $a$. The inner product of two functions $g$, $h$ in $S_1$ is, as usual, defined as

$$\langle f,g\rangle = \int_0^\infty [f(x)\overline{g(x)} + f(\xi x)\overline{g(\xi x)}]\,dm_1(x)$$

$S_1$ is then a pre-Hilbert space. We will show that the exponentials are not complete in $S_1$ by exhibiting a function $F(z) \not\equiv 0$ such that

(i) $F(z)$ belongs to $S_1$

and

(ii) $F(z)$ is orthogonal to every exponential.

$$\text{Let } F(z) = \frac{e^{(1-i)z^2}(1-e^{-z^8})}{z^8} .$$

(i) $F(z)$ belongs to $S_1$:

$$\|F(z)\|^2_{S_1} = 2\int_0^\infty e^{2x^2}\frac{\left(1-e^{-x^8}\right)^2}{x^{16}}\,\frac{e^{-2x^2} x^{11}}{1-e^{-x^8}}\,dx$$

$$= 2\int_0^\infty \frac{\left(1-e^{-x^8}\right)}{x^5}\,dx < \infty$$

and therefore $F(z) \varepsilon S_1$.

(ii) $\langle e^{\tau z}, F(z)\rangle_{S_1} = 0$:

We wish to show

(1) $$\int_0^\infty (e^{\tau x}\,\overline{F(x)} + e^{\tau\xi x}\,\overline{F(\xi x)})\,dm_1(x) = 0 \text{ for all complex } \tau, \text{ i.e.,}$$

(2) $$\int_0^\infty e^{\tau x}\, e^{-(1-i)x^2} x^3 + e^{\tau\xi x}\, e^{-(1+i)x^2}\, x^3 dx = 0 .$$

Now $g(z) = e^{\tau z}\, e^{-(1-i)z^2}\, z^3$ is analytic and therefore since

$$\lim_{R\to\infty} \int_0^{\frac{\pi}{4}} R|g(Re^{i\theta})|\,d\theta = 0$$

$$\int_0^\infty g(x)\,dx = \xi \int_0^\infty g(\xi x)\,dx .$$

Hence (2) holds. Therefore $\langle e^{\tau z}, F(z)\rangle_{S_1} = 0$ for all $\tau$ complex, so the exponentials are not complete in $S_1$. As in our previous example (page 28) we will now show

(iii) $F(z)$ is in $S$

and that for any $f(z)$ in $S$

(iv) $\|f(z)\|_{S_1} < \text{constant}\, \|f(z)\|_S$ .

And again, as in our previous example, (iv) implies that, in particular,

$F(z)$ cannot be approximated by linear combinations of exponentials in S.

(iii) $F(z)$ is in S:

$$\int_{R_1} |F(z)|^2 dm(z) \leq 4 \int_2^\infty \int_{-\frac{1}{x}}^{\frac{1}{x}} \frac{e^{2(x^2-y^2)+4xy}}{|z|^{16}} e^{-2x^2} x^{12} dx < K \int_2^\infty \frac{1}{x^5} < \infty$$

Similarly letting $z = \xi\eta$, $\eta = t + iw$,

$$\int_{R_2} |F(z)|^2 dm(z) \leq 4 \cdot 2^6 \int_2^\infty \int_{-\frac{1}{t}}^{\frac{1}{t}} \frac{e^{2(t^2-w^2)-4tw}}{|\eta|^{16}} e^{-2t^2} t^{12} dt < K \int_2^\infty \frac{1}{t^5} < \infty.$$

Obviously $\int_{R_4 \cup R_1} |F(z)|^2 dm(z) < \infty$ and therefore $F(z)$ is in S.

Proof of (iv):

Given $f(z)$ entire we show

$$\text{(A)} \quad \int_0^\infty |f(u)|^2 \frac{e^{-2u^2} u^{11}}{1-e^{-u^8}} du < M \left[ \int_{|z|=4} |f(z)|^2 dxdy + \int_{R_1} |f(z)|^2 dm(z) \right]$$

and similarly

$$\text{(B)} \quad \int_0^\infty |f(\xi u)|^2 \frac{e^{-2u^2} u^{11}}{1-e^{-u^8}} du < M \left[ \int_{|z|=4} |f(z)|^2 dxdy + \int_{R_2} |f(z)|^2 dm(z) \right] .$$

Since $f(z)$ is entire

$$\text{(1)} \qquad f(u) = \frac{1}{2\pi i} \int_0^{2\pi} f(u + re^{i\theta}) d\theta \qquad \left(0 < r \leq \frac{1}{2u}\right)$$

and hence

$$(2) \qquad |f(u)|^2 \le \frac{4u^2}{\pi} \int_0^{2\pi} \int_0^{\frac{1}{2u}} |f(u + re^{i\theta})|^2 r\,dr\,d\theta \,.$$

Multiplying both sides of (2) by $\dfrac{e^{-2u^2} u^{11}}{1-e^{-u^8}}$ and integrating from $u = 3$ to $u = \infty$, we have

$$(3) \qquad \int_3^\infty \frac{|f(u)|^2 e^{-2u^2} u^{11}}{1-e^{-u^8}}\,du \le \frac{4}{\pi} \int_3^\infty \int_0^{2\pi} \int_0^{\frac{1}{2u}} \frac{|f(u+re^{i\theta})|^2 u^{13}\, e^{-2u^2}}{1-e^{-u^8}}\, r\,dr\,d\theta\,du$$

$$< \frac{4e}{\pi(e-1)} \int_3^\infty \int_0^{2\pi} \int_0^{\frac{1}{2u}} |f(u + re^{i\theta})|^2\, u^{13} e^{-2u^2} r\,dr\,d\theta\,du \,.$$

We make the following change of variables in the last integral of (3).

$$u + r\cos\theta = x$$

$$r \sin\theta = y$$

$$u = v \,.$$

We then have

$$(4) \qquad \int_3^\infty \frac{|f(u)|^2 e^{-2u^2} u^{11}}{1-e^{-u^8}}\,du < \frac{4e}{\pi(e-1)} \int_{3-\frac{1}{6}}^\infty dx \int_{-\frac{1}{2(x-1)}}^{\frac{1}{2(x-1)}} |f(x+iy)|^2 dy \int_{x-\frac{1}{2(x-1)}}^{x+\frac{1}{2(x-1)}} e^{-2v^2} v^{13} dv.$$

Since

$$v < x + \frac{1}{2(x-1)} < x+1 < 2x \quad ,$$

$$-x^2 < -(x - \frac{1}{2(x-1)})^2 < -x^2+2$$

and

$$\frac{x}{x-1} < 2 \text{ when } x \geq 2$$

we have

$$(5) \quad \int_3^\infty \frac{|f(u)|^2 e^{-2u^2} u^{11}}{1-e^{-u^8}} du < \frac{2^{15} e^5}{\pi(e-1)} \int_{3-\frac{1}{6}}^\infty \int_{-\frac{1}{2(x-1)}}^{\frac{1}{2(x-1)}} |f(x+iy)|^2 e^{-2x^2} \frac{x^{13}}{x-1} dydx$$

$$< \frac{2^{16} e^5}{\pi(e-1)} \int_{3-\frac{1}{6}}^\infty dx \int_{-\frac{1}{2(x-1)}}^{\frac{1}{2(x-1)}} |f(x+iy)|^2 e^{-2x^2} x^{12} dydx$$

$$< \frac{2^{16} e^5 3^{12}}{\pi(e-1)} [\int_{3-\frac{1}{6}}^3 \int_{-\frac{1}{x}}^{\frac{1}{x}} |f(x+iy)|^2 dydx + \iint_{R_1} |f(x+iy)|^2 dydx]$$

$$(6) \quad < \text{constant } [\iint_{|z|=4} |f(x+iy)|^2 dydx + \iint_{R_1} |f(x+iy)|^2 dydx] .$$

For $0 \leq u \leq 3$, we have $0 \leq \frac{u^{11} e^{-2u^2}}{(1-e^{-u^8})} <$ constant and therefore,

$$(7) \quad \int_0^3 |f(u)|^2 \frac{u^{11} e^{-2u^2}}{1-e^{-u^8}} du < \text{constant} \int_0^3 |f(u)|^2 du$$

$$< \text{constant} \int_0^3 \int_0^1 \int_0^{2\pi} |f(u+re^{i\theta})|^2 r dr d\theta du < \text{constant} \int_{|z|\leq 4} |f(z)|^2 dxdy .$$

Combining (6) and (7) we have (A).

In exactly the same manner we have, letting $\eta = t + iw$,

$$(8) \qquad \int_0^\infty |f(\xi u)|^2 \frac{u^{11} e^{-2u^2}}{1-e^{-u^8}}\,du < M \Big[ \int_{|\eta|\le 4} |f(\xi\eta))|^2 dt dw$$

$$+ \iint_{\substack{|\eta|\ge 3\\ |tw|\le 1\\ t>0}} |f(e^{\frac{\pi i}{4}}(\eta))|^2 e^{-2t^2} t^{12} dt\Big]$$

Letting $z = e^{\frac{\pi i}{4}} \eta$ we have from (8)

$$(9) \qquad \int_0^\infty |f(\xi u)|^2 \frac{u^{11} e^{-2u^2}}{1-e^{-u^8}}\,du$$

$$< M \Big[ \int_{|z|\le 4} |f(z)|^2 dx dy + \int_{R_2} |f(z)|^2 \frac{e^{-(x+y)^2}(x+y)^{12}}{2^6}\, dx dy\Big]$$

which is (B).

Moreover, for $3 \leqslant |z| \leqslant 4$, $dm(z) > M_1 > 0$. Therefore from (A) and (B) we have

$$(1)\quad \|f(z)\|_{S_1} < 2M\Big[\int\limits_{|z|\le 3} |f(z)|^2 dxdy + \frac{1}{M_1}\int\limits_{3\le|z|\le 4} |f(z)|^2 M_1 dxdy + \int\limits_{R_1\cup R_2} |f(z)|^2 dm(z)\Big]$$

$$< 2M\Big[\int\limits_{R_3} |f(z)|^2 dm(z) + \frac{1}{M_1}\int\limits_{3\le|z|\le 4} |f(z)|^2 dm(z) + \int\limits_{R_1\cup R_2} |f(z)|^2 dm(z)\Big]$$

$$< \text{constant } \|f(z)\|_S^2 .$$

This completes the proof of (iv), and therefore the exponentials are not complete in S.

Remark: It easily follows that polynomials are not dense in S.

Theorem 1.4 can be generalized to the space of entire functions in $L^p(\phi(z))$, $1 \le p \le \infty$. The proof is essentially that of B. A. Taylor [12]. We have managed to simplify it somewhat.

Theorem 1.5: Let $\phi(xy) = \phi(z)$ be a positive convex function of $z$, $\phi(0) = 0$. Let $S^p$, $1 \le p < \infty$, be the space of entire functions $f(z)$ such that

$$\|f\|^p = \int |f(z)|^p e^{-\phi(z)} dxdy < \infty .$$

Let $S^\infty$ be the space of entire functions such that $f(z)e^{-\phi(z)} \to 0$ $|z| \to \infty$. As customary we define

$$\|f(z)\|_{S^\infty} = \sup\{|f(z)|e^{-\phi(z)}\}$$

where the sup is taken over the complex plane. Assume that every exponential belongs to the space $S^p$, $1 \le p \le \infty$. Then polynomials are dense in $S^p$, $1 \le p \le \infty$.

Proof:

Lemma 1: Let $f(z)$ belong to $S^p$. Then the sequence $\{f_\lambda(z)\} = \{f(\lambda z)\}$, $\frac{1}{2} \le \lambda < 1$, converges in norm to $f(z)$ as $\lambda \to 1$.

Proof: (a) Let $1 \le p < \infty$. We use the fact that if a sequence $\{f_n\} \to f$ pointwise, and if $\|f_n\|_{S^p} \to \|f\|_{S^p}$ then in fact $\|f_n - f\|_{S^p} \to 0$ (see [4], page 209). $f(\lambda z)$ obviously converges to $f(z)$ pointwise as $\lambda \to 1$. Moreover,

$$\|f(\lambda z)\|_{S^p}^p = \int |f(\lambda z)|^p e^{-\phi(z)} dxdy = \frac{1}{\lambda^2} \int |f(z)|^p e^{-\phi(\frac{z}{\lambda})} dxdy .$$

For fixed $z$, $\phi(\frac{z}{\lambda})$ decreases monotonically to $\phi(z)$ as $\lambda \to 1$. Therefore $-\phi(\frac{z}{\lambda})$ increases monotonically to $-\phi(z)$ and by the theorem of Beppo Levi,

$$\lim_{\lambda \to 1} \|f(\lambda z)\|_{S^p}^p = \int |f(z)|^p e^{-\phi(z)} dxdy = \|f(z)\|_{S^p}^p ,$$

i.e., $\|f(\lambda z)\|_{S^p}^p$ converges to $\|f(z)\|_{S^p}^p$.

(b) Let $p = \infty$. If $f(z)$ is in $S$, by definition $f(z)e^{-\phi(z)} \to 0$ as $|z| \to \infty$, i.e., given $\varepsilon > 0$, there exists $N$ such that for $|z| > N$

$$|f(z)|\, e^{-\phi(z)} < \frac{\varepsilon}{2} .$$

Let $|z| > 2N$. Then $|\lambda z| > N$ and

$$|f(\lambda z)|\, e^{-\phi(\lambda z)} < \frac{\varepsilon}{2} ,$$

$$\|f(\lambda z) - f(z)\|_{S^\infty} = \sup\{|f(\lambda z) - f(z)|\, e^{-\phi(z)}\}$$

$$= \max\left( \sup_{|z|\le 2N} \{|f(\lambda z) - f(z)|e^{-\phi(z)}\},\ \sup_{|z|>2N} \{|f(\lambda z) - f(z)|e^{-\phi(z)}\}\right).$$

For $|z| > 2N$ we have, since $-\phi(z) < -\phi(\lambda z)$,

$$|f(\lambda z) - f(z)|e^{-\phi(z)} < |f(\lambda z)|e^{-\phi(\lambda z)} + |f(z)|e^{-\phi(z)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

On the other hand, since $f(z)$ is uniformly continuous on $|z| \le 2N$, for $\lambda > \lambda'$, $|f(\lambda z) - f(z)| < \frac{\varepsilon}{M}$ where $M = \sup_{|z|\le 2N} \phi(z)$.

Hence $\lim_{\lambda\to 1} \|f(\lambda z) - f(z)\|_{S^\infty} = 0.$

Lemma 2: Let $\frac{1}{2} \le k < k' < k'' < 1$. Then

(a) $\sup_{|u|\le 1} \phi(kz + u) < \phi(k'z) + \text{constant}$,

(b) there exists $M$ such that for $|z| > M$

$$\phi(k''z) - \phi(k'z) > (k'' - k')|z| .$$

Proof: (a) Since $\phi(z)$ is convex

$$\phi(kz + u) = \phi\left(\frac{k}{k'} (k'z) + \left(1 - \frac{k}{k'}\right)\left(\frac{k'}{k'-k} u\right)\right)$$

$$\le \frac{k}{k'} \phi(k'z) + \left(\frac{k'-k}{k'}\right) \phi\left(\frac{k'}{k'-k} u\right) < \phi(k'z) + \phi\left(\frac{k'}{k'-k} u\right) .$$

Hence, $\sup_{|u|\le 1} \phi(kz + u) < \phi(k'z) + \sup_{|u|\le 1} \phi\left(\frac{k'}{k'-k} u\right) < \phi(k'z) + \text{constant}.$

(b) Let $z = t\xi$, $t \ge 0$, $|\xi| = 1$. Define $h(t) = \phi(t\xi) = \phi(z)$. For fixed $\xi$ $h(t)$ is obviously a convex function of the real variable $t$. From the definition of convexity, it can be easily verified that

$h(t) \geq h(t_o) + L_{t_o}(t-t_o)$ where $L_{t_o} = \sup_{t<t_o} \dfrac{h(t_o) - h(t)}{t_o - t}$. $L_t$ is a non-decreasing function of t since for $t < t_o < t_1$ we have $t_o = \alpha t + (1-\alpha)t_1$ and by convexity $h(t_o) \leq \alpha h(t) + (1-\alpha)h(t_1)$. Therefore

$$L_{t_o} = \sup_{t<t_o} \frac{h(t_o) - h(t)}{t_o - t} \leq \sup_{t<t_1} \frac{h(t_o) - h(t)}{t_o - t}$$

$$\leq \sup_{t<t_1} \frac{(\alpha-1)h(t) + (1-\alpha)h(t_1)}{t_o - t} = \sup_{t<t_1} \frac{(1-\alpha)[h(t_1) - h(t)]}{t_o - t}$$

$$= \sup_{t<t_1} \frac{[h(t_1) - h(t)]}{t_1 - t} = L_{t_1} .$$

Moreover, since every exponential belongs to the space $S^p$, $\lim_{t\to\infty} \dfrac{\phi(t\xi)}{|t\xi|} = \lim_{t\to\infty} \dfrac{h(t)}{t} = \infty$ and hence $\lim_{t\to\infty} L_t = \infty$. In particular, there exists t' such that for $t > t'$

$$\frac{h(k''t) - h(k't)}{(k'' - k')t} > 1 .$$

For every $\xi$, $|\xi| = 1$, let $t_o(\xi) = \min_t\{t\}$ such that $\dfrac{h(k''t) - h(k't)}{(k'' - k')t} > 1$.

We will show that the function $t_o(\xi)$ is bounded from above. This will give the desired result, for suppose $t_o(\xi) \leq M$. Then for all $\xi$, $|\xi| = 1$, $t \geq M$ implies

$$\frac{h(k''t\xi) - h(k't\xi)}{(k'' - k')t} > 1 .$$

Since $|z| = |t\xi| = t$, we will then have for $|z| \geq M$

$$\frac{\phi(k''z) - \phi(k'z)}{(k'' - k')|z|} > 1 .$$

We will show that $t_o(\xi)$ is upper semi-continuous. Since $\{\xi \mid |\xi| = 1\}$ is compact, this will imply that $t_o(\xi)$ is bounded from above. Fix $\xi_o$, and let $\varepsilon > 0$ be given. Choose $t_1$ such that $t_o(\xi_o) < t_1 < t_o(\xi_o) + \varepsilon$. Since $t_1 > t_o(\xi_o)$ we have

$$\frac{\phi(k''t_1\xi_o) - \phi(k't_1\xi_o)}{(k'' - k')t_1} \geq 1 + \delta, \quad \delta > 0 .$$

$\phi(z)$ is convex in z, hence continuous. Therefore one can find $\delta'$ such that

$$\frac{\phi(k''t_1\xi) - \phi(k't_1\xi)}{(k'' - k')t_1} \geq 1 + \frac{\delta}{2} \quad \text{when } |\xi - \xi_o| < \delta' .$$

Hence for $|\xi - \xi_o| < \delta'$, $t_o(\xi) \leq t_1 < t_o(\xi_o) + \varepsilon$, i.e., $t_o(\xi)$ is upper semi-continuous. This completes the proof of Lemma 2.

We now prove the assertion of the theorem. By Lemma 1, it suffices to show that given $f(z)$ in $S^p$, $f(\lambda z)$, $\frac{1}{2} \leq \lambda < 1$, can be approximated by polynomials in $S^p$.

(a) Let $1 \leq p < \infty$. Let $\lambda$ be fixed. Choose constants a, b, c such that $\lambda < a < b < c < 1$. Let H = the space of entire functions $g(z)$ such that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |g(z)|^2 e^{-\frac{2}{p}\phi(bz)} dxdy < \infty$. We first show that $f(\lambda z)$ belongs to H. By Theorem 1.4 $f(\lambda z)$ can then be approximated by polynomials in H. We then show that the same sequence of polynomials which approximates $f(\lambda z)$ in H, approximates $f(\lambda z)$ in $S^p$.

$$(1) \qquad f(\lambda z) = \frac{1}{\pi}\int_0^1 \int_0^{2\pi} f(\lambda z + re^{i\theta})\, rdrd\theta$$

hence

$$(2)\qquad |f(\lambda z)| \le \frac{1}{\pi}\int_0^1\int_0^{2\pi} |f(\lambda z + re^{i\theta})|\,rdrd\theta$$

$$= \frac{1}{\pi}\int_0^1\int_0^{2\pi} |f(\lambda z + re^{i\theta})|\,e^{-\frac{1}{p}\phi(\lambda z+re^{i\theta})}\,e^{\frac{1}{p}\phi(\lambda z+re^{i\theta})}\,rdrd\theta$$

$$\le e^{\frac{1}{p}\sup_{|u|\le 1}\phi(\lambda z+u)}\;\frac{1}{\pi}\int_{|u|\le 1} |f(\lambda z + u)|\,e^{-\frac{1}{p}\phi(\lambda z+u)}\,dA_u\,.$$

Applying Hölder's inequality to this last integral and then extending the domain of integration to the entire plane, we have from (2)

$$(3)\qquad |f(\lambda z)| \le K\,e^{\frac{1}{p}\sup_{|u|\le 1}\phi(\lambda z+u)}\,||f(z)||_{S^p}\,.$$

Using the estimate from (3) and Lemma 2, we have

$$(4)\qquad ||f(\lambda z)||_H^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(\lambda z)|^2\,e^{-\frac{2}{p}\phi(bz)}\,dA_z$$

$$\le K||f(z)||_{S^p}^2 \iint e^{\frac{2}{p}\sup_{|u|\le 1}\phi(\lambda z+u)}\,e^{-\frac{2}{p}\phi(bz)}\,dA_z$$

$$< K_1||f(z)||_{S^p}^2 \iint e^{\frac{2}{p}[\phi(az)-\phi(bz)]}\,dA_z$$

$$(5)\qquad = K_1||f(z)||_{S^p}^2 \iint_{|z|\le M} e^{-\frac{2}{p}[\phi(bz)-\phi(az)]}\,dA_z + \iint_{|z|>M} e^{-\frac{2}{p}[\phi(bz)-\phi(az)]}\,dA_z$$

The first integral of (5) is finite by the M. L. formula. On the other hand by Lemma 2, M can be chosen such that for $|z| > M$

$$\phi(bz) - \phi(az) > (b-z)|z| .$$

Hence

$$\iint_{|z|>M} e^{-\frac{2}{p}[\phi(bz)-\phi(az)]} dA_z < \iint_{|z|>M} e^{-\frac{2}{p}(b-a)|z|} dA_z < \infty .$$

We have shown that given $f(z)$ in $S^p$, $1 \le p < \infty$, $f(\lambda z)$, $\frac{1}{2} \le \lambda < 1$ is in the space H. Let $P_n(z)$ be the sequence of polynomials which converges to $f(\lambda z)$ in H. These polynomials also converge to $f(\lambda z)$ in $S^p$. For as above

$$f(\lambda z) - P_n(z) = \frac{1}{\pi}\int_0^1\int_0^{2\pi} f(\lambda(z + re^{i\theta})) - P_n(z + re^{i\theta})rdrd\theta ,$$

hence

$$(2) \quad |f(\lambda z) - P_n(z)|$$

$$\le \frac{1}{\pi}\int_0^1\int_0^{2\pi} |f(\lambda(z+re^{i\theta})) - P_n(z+re^{i\theta})| e^{-\frac{1}{p}\phi(b(z+re^{i\theta}))} e^{\frac{1}{p}\phi(b(z+re^{i\theta})} rdrd\theta$$

$$< e^{\frac{1}{p}\sup_{|u|\le 1}\phi(bz+u)} \frac{1}{\pi}\int_0^1\int_0^{2\pi} |f(\lambda(z+re^{i\theta})) - P_n(z+re^{i\theta})| e^{-\frac{1}{p}\phi(b(z+re^{i\theta}))} rdrd\theta.$$

Applying the Schwartz inequality to this last integral and then extending domain of integration to the entire plane we have from (2)

$$(3) \qquad |f(\lambda z) - P_n(z)| < K\, e^{\frac{1}{p} \sup\limits_{|u|\le 1} \phi(bz+u)} \|f(\lambda z) - P_n(z)\|_H .$$

Using this estimate and Lemma 2 we have

$$\|f(\lambda z) - P_n(z)\|^p_{S^p} < K_1 \|f(\lambda z) - P_n(z)\|^p_H \int e^{\phi(cz)-\phi(z)} dA_z$$

$$< K_2 \|f(\lambda z) - P_n(z)\|^p_H .$$

But $\lim\limits_{n\to\infty} \|f(\lambda z) - P_n(z)\|_H = 0$ and therefore $\lim\limits_{n\to\infty} \|f(\lambda z) - P_n(z)\|_{S^p} = 0.$

(b) Let $p = \infty$.

The proof for $p = \infty$ is essentially the same. As above, let $\lambda$ be fixed. We choose constants a, b, c such that $\lambda < a < b < c < 1$. Let H = the space of entire functions g(z) such that $\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} |g(z)|^2 e^{-2\phi(bz)} dxdy < \infty$. We have

$$(1) \qquad |f(\lambda z)| \le \frac{1}{\pi} \int\limits_{|u|\le 1} |f(\lambda z + u)| e^{-\phi(\lambda z+u)} e^{\phi(\lambda z+u)} dA_u$$

$$\le e^{\sup\limits_{|u|<1} \phi(\lambda z+u)} \sup_{|u|\le 1} |f(\lambda z + u)| e^{-\phi(\lambda z+u)} < \|f(z)\|_{S^\infty} e^{\sup\limits_{|u|\le 1} \phi(\lambda z+u)} .$$

Using this estimate and Lemma 2, we have

$$(2) \qquad \|f(\lambda z)\|^2_H = \iint |f(\lambda z)|^2 e^{-2\phi(bz)} dA_z < \text{constant} \iint e^{-2[\phi(bz)-\phi(az)]} dA_z$$

$$= \text{constant} \left[ \iint\limits_{|z|\le M} e^{-2[\phi(bz)-\phi(az)]} dA_z + \iint\limits_{|z|>M} e^{-2(b-a)|z|} dA_z \right] < \infty$$

Hence $f(\lambda z)$ belongs to H. By Theorem 1.4, $f(\lambda z)$ can be approximated by polynomials in H. Let $\{P_n(z)\}$ be the sequence of polynomials which approximate $f(\lambda z)$ in H. Then $\{P_n(z)\}$ converges to $f(\lambda z)$ in $S^\infty$ as well. For as above we have the estimate

$$|f(\lambda z) - P_n(z)| < K e^{\sup\limits_{|u|\leq 1} \phi(bz+u)} \|f(\lambda z) - P_n(z)\|_H$$

and therefore by Lemma 2

$$\|f(\lambda z) - P_n(z)\|_{S^\infty} = \sup |f(\lambda z) - P_n(z)| e^{-\phi(z)}$$

$$< K_1 \|f(\lambda z) - P_n(z)\|_H \sup e^{-[\phi(z)-\phi(cz)]} < K_2 \|f(\lambda z) - P_n(z)\|_H ,$$

and as before since $\lim\limits_{n\to\infty} \|f(\lambda z) - P_n(z)\|_H = 0$, we have $\lim\limits_{n\to\infty} \|f(\lambda z) - P_n(z)\|_{S^\infty} = 0$.

We have shown that given $f(z)$ in $S^p$, $1 \leq p \leq \infty$, $f(\lambda z)$, $\frac{1}{2} \leq \lambda < 1$, can be approximated by polynomials in $S^p$. As remarked above, since by Lemma 1, $f(\lambda z)$ converges to $f(z)$ in $S^p$ as $\lambda \to 1$, this completes the proof of Theorem 1.5.

Remark: Theorem 1.4 was recently proven by B. A. Taylor in [9] with slightly different conditions on $\phi(z)$, e.g., he does not assume every exponential belongs to the space S. The theorem is given there for the much more general n-variable case. Our proof, discovered independently, is simpler and more direct for the single-variable situation, but unfortunately it does not generalize.

## 2. On the Existence of Solutions to the Equation $\bar{P}(D)P(Z)f(Z) = 0$

For convenience we begin this chapter by restating notation previously introduced. We let $Z = (z_1, \ldots z_k)$ be a point in Euclidean K-space. $N$ = the K-tuple $(n_1, \ldots n_k)$ of non-negative integers. We write

$$Z^N = z_1^{n_1} z_2^{n_2}, \ldots z_k^{n_k}$$

and

$$|N| = \sum_{i=1}^{k} n_i \, .$$

$P(Z) = P(z_1, \ldots z_k)$ will always denote a polynomial $\sum_N a_N Z^N$ $(0 \leq |N| \leq \ell)$ and $\bar{P}(Z)$ the polynomial obtained from $P(Z)$ by replacing each coefficient by its complex conjugate. By $\bar{P}(D) = \bar{P}(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots \frac{\partial}{\partial z_k})$ we shall mean the differential operator obtained from $\bar{P}(Z)$ by replacing each $z_i$ by $\frac{\partial}{\partial z_i}$. We recall some basic facts about the Fischer space $\mathfrak{F}_Z$ of entire functions mentioned in our introduction.

$\mathfrak{F}_Z$ = the Hilbert space of entire functions $f(z)$ such that

$$\|f(Z)\|^2_{\mathfrak{F}_Z} = \frac{1}{\pi^K} \int |f(Z)|^2 e^{-|Z|^2} dA_Z < \infty \, .$$

As usual, the inner product of two functions $f$, $h$ in $\mathfrak{F}_Z$ is defined by

$$\langle f,h \rangle = \frac{1}{\pi^K} \int f(Z)\overline{g(Z)} e^{-|Z|^2} dA_Z$$

We let

$$K = \{\phi(Z) \,|\, \phi(Z) = O[e^{\frac{|Z|^2}{2} - A|Z|}]\} \, .$$

It is clear that $K \subset \mathfrak{F}_Z$ and that if $\phi(Z)$ is in K then every exponential times $\phi(Z)$ is also in $\mathfrak{F}_Z$. It can easily be verified that the operators "multiplication by P" and $\bar{P}(D) = \bar{P}[\frac{\partial}{\partial z_1}, \ldots \frac{\partial}{\partial z_k}]$ are formally adjoint. Thus, given P(Z) and functions $\phi(Z)$, h(Z) such that $\phi(Z)$ is in K, h(Z) is in $\mathfrak{F}_Z$ and $\bar{P}(D)h(Z)$ is in $\mathfrak{F}_Z$, then

(1) $$\langle P(Z)\phi(Z), h(Z)\rangle = \langle \phi(Z), \bar{P}(D)h(Z)\rangle .$$

D. J. Newman and H. S. Shapiro have shown [7] that in fact these operators are truly adjoint, i.e., if $P(Z)F(Z) \in \mathfrak{F}_Z$ and $h(Z) \in \mathfrak{F}_Z$, F(Z) entire, then

(2) $$\langle P(Z)F(Z), h(Z)\rangle = \langle F(Z), \bar{P}(D)h(Z)\rangle .$$

It is clear that (2) implies that there is no non-trivial solution within the Fischer space to $\bar{P}(D)g(Z) = 0$ where $g(Z) = P(Z)f(Z)$, f(Z) entire. For from (2) we have

$$\|P(Z)f(Z)\|^2_{\mathfrak{F}_Z} = \langle P(Z)f(Z), P(Z)f(Z)\rangle$$

$$= \langle f(Z), \bar{P}(D)P(Z)f(Z)\rangle = \langle f(Z), 0\rangle = 0 .$$

Hence $\|g(Z)\| = 0$ and $g(Z) \equiv 0$.

We now ask whether there are any non-trivial entire solutions to

$$\bar{P}(D)P(Z)f(Z) = 0 ,$$

where we no longer require that $P(Z)\cdot f(Z)$ belong to $\mathcal{F}_Z$. For $K = 1$ the result is known. For the sake of completeness we include the proof.

<u>Theorem 2.1</u>: Let $K = 1$, and assume that $f(Z)$ is entire. Then $\bar{P}(D)P(Z)f(Z) = 0$ implies $f \equiv 0$.

<u>Proof</u>: Let $P(z) = \sum_{k=0}^{N} \bar{b}_k z^k$, $b_N = 1$. Let $g(z) = P(z)f(z)$. By definition $\bar{P}(D)g(z) = 0$ implies that $g(z)$ is an entire solution to the ordinary differential equation

$$b_0 g(z) + b_1 g'(z) + \dots + g^{(N)}(z) = 0 \,. \tag{1}$$

We show that an entire solution to such an equation can grow at most exponentially. We let

$$y_0 = g(z),\ y_1 = g'(z), \dots\ y_N = g^{(N)}(z)$$

$$Y = (y_0, y_1, \dots\ y_{N-1})$$

$$\frac{dY}{dz} = (y_1, y_2, \dots\ y_N) = (y_1, y_2, \dots\ y_{N-1}, -b_0 y_0 - b_1 y_1, \dots -b_{N-1} y_{N-1})$$

$A =$ the $N \times N$ matrix $(a_{ij})$ where

$$\text{for } 1 \le i < N,\ 1 \le j \le N,\ a_{ij} = \begin{cases} 0 \text{ when } j \ne i+1 \\ 1 \text{ when } j = i+1 \end{cases}$$

and for $1 \le j \le N$, $a_{Nj} = -b_{j-1}$ .

(1) can then be expressed as

$$\text{(2)} \qquad \frac{dY}{dz} = AY$$

where multiplication is ordinary matrix multiplication. One further defines

$$\|Y\| = \sum_{i=0}^{N-1} |y_i|, \quad \left\|\frac{dY}{dz}\right\| = \sum_{i=1}^{N} |y_i|, \quad \|A\| = \sum |a_{ij}| .$$

It is clear that $\left\|\frac{dY}{dz}\right\| = \|A\|\,\|Y\|$ and since A is a constant matrix we have

$$\text{(3)} \qquad \left\|\frac{dY}{dz}\right\| = M\|Y\| .$$

Letting $z = re^{i\theta}$, $\frac{dy_i}{dr} = \frac{dy_i}{dz} e^{i\theta}$ and $\left\|\frac{dY}{dr}\right\| = \left\|\frac{dY}{dz}\right\|$ . Since $\left|\frac{d}{dr}\|Y\|\right| \le \left\|\frac{dY}{dr}\right\|$ (see [2], pg. 18) we have from (3)

$$\text{(4)} \qquad \left|\frac{d}{dr}\|Y\|\right| < M\|Y\| .$$

Let $z_0$ be any point for which $g(z) = y_0 \neq 0$. Then

$$\text{(5)} \qquad \ln \frac{\|Y(z)\|}{\|Y(z_0)\|} \le \left|\ln \frac{\|Y(z)\|}{\|Y(z_0)\|}\right| = \left|\int_{|z_0|}^{|z|} \frac{\frac{d}{dr}\|Y\|}{\|Y\|} dr\right|$$

$$\le \int_{|z_0|}^{|z|} \frac{\left|\frac{d}{dr}\|Y\|\right|}{\|Y\|} dr < M \int_{|z_0|}^{|z|} dr = M(|z| - |z_0|) .$$

Hence $\|Y(z)\| <$ constant $e^{M|z|}$, so in particular, $|y_0| = |g(z)| <$ $<$ constant $e^{M|z|}$. It then follows that $g(z)$ is in $\mathfrak{F}_z$ and since $g(z) = P(z)f(z)$, $g(z) \equiv 0$.

We generalize the above theorem.

Theorem 2.2: Let $k = 1$, assume that $f(z)$ is entire. Then $\sum_{i=1}^{n} \bar{P}_i(D)P_i(z)f(z) = 0$ implies $f(z) \equiv 0$.

Proof: As above we show that $|f(z)| <$ constant $e^{M|z|}$. This implies that $f(z) \in K$, and $P_i(z)f(z) \in \mathfrak{F}_z$. Therefore,

$$\sum_{i=1}^{n} \|P_i f\|^2 = \sum_{i=1}^{n} \langle P_i(z)f(z), P_i(z)f(z)\rangle = \sum_{i=1}^{n} \langle f(z)\bar{P}_i(D), P_i(z)f(z)\rangle$$

$$= \langle f(z), \sum_{i=1}^{n} \bar{P}_i(D)P_i(z)f(z)\rangle = 0 \; .$$

Hence $\sum_{i=1}^{n} \|P_i f\|_{\mathfrak{F}_z} = 0$ and $f \equiv 0$. Let $N_i$ be the degree of $P_i(z)$, $1 \leq i \leq N$, and suppose that $N = N_n > N_i$ for $i < n$. Assume

$$(1) \qquad \sum_{i=1}^{n} \overline{P_i(D)}P_i(z)f(z) = \sum_{i=1}^{n} [\sum_{k=0}^{N_i} b_{k_i} D^k(P_i(z)f(z))] = 0 \; .$$

Consider any term $b_{k_i} D^k(P_i(z)f(z))$. By Leibnitz's rule we have

$$b_{k_i} D^k(P_i(z)f(z)) = b_{k_i} [P_i(z)f^{(k)}(z) + kP_i'(z)f^{(K-1)}(z) + \ldots + P_i^{(k)}(z)f(z)] \; .$$

Since $N > N_i$, $i < n$, the only term involving the $N^{th}$ derivative of $f(z)$ will be

$$b_N D^N[P_n(z)f(z)] = b_N[P_n(z)f^{(N)}(z) + NP_n'(z)f^{(N-1)}(z) + \ldots + N!\ f(z)] .$$

Thus

$$(2) \qquad \sum_{i=1}^{n} \bar{P}_i(D)P_i(z)f(z) = b_N P_n(z)f^N(z)$$

$$+ A_{N-1}(z)f^{N-1}(z) + \ldots + A_o f(z) = 0 ,$$

where $A_p$, $0 \le p \le N-1$ is a polynomial of degree at most p. Choose R such that for $|z| > R$, $P_n(z) \neq 0$. For such z, we have

$$b_N P_n(z)[f^{(N)}(z) + B_{N-1}(z)f^{(N-1)}(z) + \ldots + B_o f(z)] = 0$$

where $B_p(z) = \dfrac{A_p(z)}{b_N P_n(z)}$ and $|B_p(z)|$ is bounded. Hence for $|z| > R$ we have

$$(3) \qquad [f^{(N)}(z) + B_{N-1}(z)f^{(N-1)}(z) + \ldots + B_o f(z)] = 0.$$

Recalling the notation from Theorem 2.1, (3) may be written as $\frac{dY}{dz} = AY$ where A now is the matrix $(a_{ij})$ such that for $1 \le i < N$, $1 \le j \le N$, $a_{ij} = \begin{cases} 0 \text{ when } j \neq i+1 \\ 1 \text{ when } j = i+1 \end{cases}$ and for $1 \le j \le N$, $a_{Nj} = -B_{j-1}(z)$. Thus for $|z| > R$, $\|A\| = \sum |a_{ij}| < M$ and exactly as in the proof of Theorem 2.1,

$$\left|\frac{d}{dr}\|Y\|\right| < \left\|\frac{dY}{dr}\right\| = \left\|\frac{dY}{dz}\right\| \ , \ (z = re^{i\theta}) \ ,$$

and

$$\|Y\| = \sum_{i=0}^{N-1} |y_i| = \sum_{i=0}^{N-1} |g^i(z)| < \text{constant } e^{M|z|} \text{ for } |z| > R.$$

Thus in particular, $|y_o| = |g(z)| <$ constant $e^{M|z|}$ for $|z| > R$. For $|z| \leq R$, $|g(z)|$ is obviously bounded because of continuity.

<u>Note:</u> It should be clear that the assumption $N_n > N_i$, $i < n$, was unnecessary. Its only purpose was to simplify our notation.

We now consider in more general terms the statement

$$\text{(A)} \qquad \bar{P}(D)P(Z)g(Z) = 0 \Rightarrow g(Z) \equiv 0$$

where $P(Z)$ is again any polynomial. We have shown that for $k = 1$, (A) is true if we ask that $g(Z)$ be entire. One might ask whether (A) is still true if we ask only that $P(Z)g(Z)$ be entire. Clearly this is false. Let $k = 1$, $P(z) = 1 - z$, $g(z) = \frac{e^z}{1 - z}$. $P(z)g(z) = e^z$ and $\bar{P}(D)P(z)g(z) = 0$. We now ask whether (A) is true when $P(z)$ is any formal power series. As seen by the above example, this is clearly false. In that case the power series $g(z)$ has in fact a positive radius of convergence. We show, however, that for a certain class of polynomials (A) is true for any formal power series $g(Z)$. This, of course, implies that for polynomials in this class no non-trivial entire solution $g(Z)$ exists which satisfies $\bar{P}(D)P(Z)g(Z) = 0$.

<u>Theorem 2.3:</u> Let $P(Z) = \sum b_j Z^{N_j}$, $N_j = (n_{j1}, \dots n_{jk})$. Suppose there are positive constants $a_1, a_2, \dots a_k$, $M$, such that $\sum_{i=1}^{k} a_i n_{ji} = M$ for all $j$. Then if $g(Z)$ is any formal power series

$$\bar{P}(D)P(Z)g(Z) = 0 \Rightarrow g(Z) \equiv 0 .$$

Proof: $g(Z)$ may be written as $\sum_{m=0}^{\infty} Q_m(Z)$ where $Q_m(Z)$ is the sum of monomials $\sum d_i Z^{L_{i'}}$, $L_{i'} = (\ell_1, \ldots \ell_k)$, such that $\sum_{i=1}^{k} a_i \ell_i = m$. Consider any monomial, constant $Z^{L_{i'}+N_j}$ of $P(Z)Q_m(Z)$. Either $\bar{P}(D)Z^{L_{i'}+N_j} = 0$ or $\bar{P}(D)Z^{L_{i'}+N_j} = \sum_{j'} C_{j'}[Z^{L_{i'}+N_j-N_{j'}}]$. Since $\sum_{i=1}^{k} a_i(\ell_i + n_{ji} - n_{j'i})$ $= \sum_{i=1}^{k} a_i \ell_i = m$, there can be no cancellation of terms between $\bar{P}(Z)P(Z)Q_m(Z)$ and $\bar{P}(D)P(Z)Q_{m'}(Z)$, $m \neq m'$. Now let n be arbitrary and assume

$$\bar{P}(D)P(Z)g(Z) = \bar{P}(D)P(Z) \sum_{m=0}^{n} Q_m(Z) + \bar{P}(D)P(Z) \sum_{m=n+1}^{\infty} Q_m(Z) = 0.$$

Each of the above terms must vanish separately. Hence

$$\bar{P}(D)P(Z) \sum_{m=0}^{n} Q_m(Z) = 0.$$

But since the $a_i$ are positive, $\sum_{m=0}^{n} Q_m(Z)$ is a polynomial and therefore in the Fischer space. Therefore, as in Theorem 2.1, $\sum_{m=0}^{n} Q_m(Z) \equiv 0$. Since n was arbitrary, $g(Z) \equiv 0$.

Remark: The above class of polynomials clearly includes any homogeneous polynomial. In the case $k = 2$, the condition on $P(z) = \sum_{k,\ell} z_1^k z_2^\ell$ has a simple geometric interpretation. It simply means that the points $(k,\ell)$ lie on a line of negative slope. Theorem 2.3 can easily be generalized.

Theorem 2.4: Let $P_r(Z) = \sum b_j^r Z^{N_j^r}$ and assume there exist positive constants $a_1, \ldots a_k$, $M_r$ such that $\sum_{i=1}^{k} a_i n_{ji}^r = M_r$ for all j. Let $g(Z)$ be any formal power series. Then

$$\sum_{r=1}^{n} \bar{P}_r(D)P_r(Z)g(Z) = 0 \text{ implies } g(Z) \equiv 0.$$

Proof: The proof is almost identical to that of Theorem 2.3. We let $g(Z) = \sum_{m=0}^{\infty} Q_m(Z)$ where $Q_m(Z)$ is the sum of monomials constant $\sum d_{i'} Z^{L_{i'}}$, $L_i' = (\ell_1, \ldots \ell_k)$ such that $\sum_{i=1}^{k} a_i \ell_i = m$. As above, we note that there can be no cancellation of terms between $\sum_{r=1}^{n} \bar{P}_r(D)P_r(Z)Q_m(Z)$ and $\sum_{r=1}^{n} \bar{P}_r(D)P_r(Z)Q_{m'}(Z)$, $m \neq m'$. Let $n'$ be arbitrary and assume

$$\sum_{r=1}^{n} \overline{P_r(D)} P_r(Z)g(Z)$$

$$= \sum_{r=1}^{n} \overline{P_r(D)} P_r(Z) \sum_{m=0}^{n'} Q_m(Z) + \sum_{r=1}^{n} \overline{P_r(D)} P_r(Z) \sum_{m=n'+1}^{\infty} Q_m(Z) = 0 .$$

Hence the above two terms must vanish separately, and, in particular, $\sum_{r=1}^{n} \bar{P}_r(D)P_r(Z) \sum_{m=0}^{n'} Q_m(Z) = 0$. But since the $a_i$ are positive, $\sum_{m=0}^{n'} Q_m(Z) = Q(Z)$ is a polynomial and therefore in the Fischer space. We then have

$$\sum_{r=1}^{n} \|P_r(Z)Q(Z)\|^2_{\mathcal{F}_Z} = \sum_{r=1}^{n} \langle P_r(Z)Q(Z), P_r(Z)Q(Z)\rangle$$

$$= \sum_{r=1}^{n} \langle Q(Z), \bar{P}_r(D)P_r(Z)Q(Z)\rangle = \langle Q(Z), \sum_{r=1}^{n} \overline{P_r(D)} P_r(Z)Q(Z)\rangle = \langle Q(Z), 0\rangle = 0.$$

Hence $\|P_r(Z)Q(Z)\|^2_{\mathcal{F}_Z} = 0$, $1 \le r \le n$, and $Q(Z) \equiv 0$. Since $n'$ was arbitrary $g(Z) \equiv 0$. This completes the proof of Theorem 2.4.

We conjecture that when $P(Z)$ does not satisfy the condition of Theorem 2.3, then there always exists a non-trivial formal power series $g(Z)$ satisfying $\bar{P}(D)P(Z)g(Z) = 0$. While for some individual cases this is easy to see, we have not been able to prove this in general.

Finally we remark that given a specific polynomial one can sometimes show that no non-trivial entire function $g(Z)$ exists which satisfies $\bar{P}(D)P(Z)g(Z) = 0$. For example, let $k = 2$, $P(Z) = z_1^2 + z_1 + z_2$. We use the shift rule

$$\bar{P}(D_{z_1} D_{z_2})(e^{\alpha z_1 + \beta z_2} h(z_1, z_2)) = e^{\alpha z_1 + \beta z_2} \bar{P}(D_{z_1} + \alpha, D_{z_2} + \beta) h(z_1, z_2) .$$

Now suppose $\bar{P}(D)P(Z)g(Z) = 0$ where $g(Z)$ is entire, $g(Z) \not\equiv 0$. Let

$$f(z_1, z_2) = \frac{g(z_1, z_2)}{e^{-\frac{1}{2} z_1 + \frac{1}{4} z_2}} .$$ Then

$$\bar{P}(D)P(Z)g(Z) = \bar{P}(D)P(Z)\, e^{-\frac{1}{2} z_1 + \frac{1}{4} z_2} f(z_1, z_2)$$

$$= e^{-\frac{1}{2} z_1 + \frac{1}{4} z_2} \left[\frac{d^2}{dz_1^2} + \frac{d}{dz_2}\right](z_1^2 + z_1 + z_2) f(z_1, z_2) = 0,$$

and therefore

$$\left[\frac{d^2}{dz_1^2} + \frac{d}{dz_1}\right](z_1^2 + z_1 + z_2) f(z_1, z_2) = 0 .$$

Let
$$w_1 = z_1 + \frac{1}{2}$$
$$w_2 = z_2 - \frac{1}{4}.$$

Then

$$\left[\frac{d^2}{dz_1^2} + \frac{d}{dz_2}\right](z_1^2 + z_1 + z_2)f(z_1,z_2)$$

$$= \left[\frac{d^2}{dw_1^2} + \frac{d}{dw_2}\right](w_1^2 + w_2)f(w_1 - \frac{1}{2}, w_2 + \frac{1}{4}) = 0 .$$

Let $f(w_1,w_2) = \sum_{m,n} C_{mn} w_1^m w_2^n$

$$f(w_1 - \frac{1}{2}, w_2 + \frac{1}{4}) = \sum_{m,n} C_{m,n}(w_1 - \frac{1}{2})^m (w_2 + \frac{1}{4})^n .$$

We note that $w_1^2 + w_2 = P_1(w_1,w_2)$ satisfies the condition of Theorem 2.3, and, as in the proof of that theorem, it follows that

$$\sum_{m,n} C_{m,n}(w_1 - \frac{1}{2})^m(w_2 + \frac{1}{4})^n \equiv 0,$$

i.e., $f(w_1,w_2)$ cannot be expanded in a power series about the point $(\frac{1}{2}, -\frac{1}{4})$. Hence $f(w_1,w_2)$ cannot be entire. Since

$f(w_1,w_2) = \dfrac{g(w_1,w_2)}{e^{-\frac{1}{2}w_1 + \frac{1}{4}w_2}}$, $g(w_1,w_2)$ cannot be entire. Clearly this example is very special. It is conjectured that for any polynomial $P(Z)$, if $g(Z)$ is entire and satisfies $\overline{P}(D)P(Z)g(Z) = 0$, then $g(Z) \equiv 0$.

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