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HOMEOMORPHISMS, GLOBAL UNIVALENCE AND SURJECTIVITY OF MAPS BETWEEN BANACH SPACES

by

Roy Plastock

Submitted in partial fulfillment of
the requirements for the degree of
Doctor of Philosophy
in the Belfer Graduate School of Science
Yeshiva University
New York

September 1972

The committee for this doctoral dissertation consisted of:

Melvyn S. Berger, Ph.D., Chairman

Lewis Coburn, Ph.D.

Arnold Lebow, Ph.D.

ACKNOWLEDGMENT

I acknowledge with great pleasure my indebtedness and gratitude to Professor Melvyn S. Berger for his constant flow of ideas and boundless patience.

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Introduction

In this paper we find precise conditions which insure that a map between two Banach spaces is a global homeomorphism. We then study the related problems of global univalence and surjectivity. We approach each of these problems by finding necessary conditions and then proceed to determine any additional assumptions that are needed to insure the sufficiency. Counterexamples are given whenever stated hypotheses cannot be weakened.

Preliminary analytic and topological results are developed in Chapter I.

Chapter II is concerned with the development of a general method for attacking the global homeomorphism problem. More precisely, we apply Theorem 2.2.1 to show that the global homeomorphism problem is equivalent to the more fundamental topological problem of finding precise conditions which insure that a map between two Banach spaces is a covering space map. We then solve this problem by proving Theorem 2.2.2: necessary and sufficient conditions for a map F between two Banach spaces to be a covering space map are (i) F is a local homeomorphism and (ii) F has the line lifting property [Definition 2.2.1]. We further reduce the problem by showing that a local homeomorphism has the line lifting property if and only if it satisfies a limiting condition which we designate by (C).

The remainder of the chapter is devoted to finding analytic hypotheses which insure the verification of condition (C). Among the theorems proven are the Hadamard-Lévy and Banach-Mazur theorems (see below). Many additional results are proven including a theorem on quasiconformal maps between Banach spaces. The class of quasiconformal maps was first introduced (for $\mathbb{R}^3$) by Lavrent'ev in [19]. In this paper he conjectured that every locally homeomorphic quasiconformal map of $\mathbb{R}^3$ into itself is a global homeomorphism. Using the concept of modulus [29], this conjecture has recently been verified by Zoric [31], and, in fact, shown to be true for $\mathbb{R}^N$, $N \geq 3$.

Our methods yield a more general approach to the global homeomorphism problem when contrasted to the earlier resolutions of this problem. These results ([1], [7],[8],[15],[20]) were based on ideas that can be traced back to a paper of Hadamard in 1904 [15]. In this paper he proved the following theorem: Let $F: \mathbb{R}^N \to \mathbb{R}^N \in C^1(\mathbb{R}^N)$ and suppose that its Jacobian determinant is never zero. Then if $\int_0^\infty \inf_{||x||=s} 1/||[F'(x)]^{-1}||ds=\infty$, F is a global diffeomorphism of $\mathbb{R}^N$ onto itself. The proof of this theorem was based on the use of the monodromy theorem and the fact that the hypotheses imply that the image of any line of infinite length is a (rectifiable) curve also of infinite length. Using Hadamard's ideas, Lévy [20] generalized this theorem to function spaces. The same method of proof was used by

Cacciopoli [7] to show that if F is a compact perturbation of the identity such that (i) F is a local homeomorphism and (ii) $||F(x)|| \to \infty$ as $||x|| \to \infty$, then F is a global homeomorphism. In 1934 Banach and Mazur [1] proved a very general result using the monodromy theorem: if X and Y are metric spaces, then a local homeomorphism F between X and Y is a global homeomorphism provided F is a proper map. In Theorem 1.4.7 we shall show that this result implies Cacciopoli's result.

More recent are papers of Browder [4] and John [17]. The paper of Browder is concerned with determining conditions for a local homeomorphism between general topological spaces to be a covering space map. By specializing to Banach spaces we obtain simpler and more readily applicable conditions than those in [4]. When F is a local homeomorphism between two Banach spaces, John uses a general mean value theorem due to Nevanlinna to weaken the differentiability of F and obtain an extension of the Hadamard-Lévy theorem. By using the same mean value theorem, these results also follow from our work.

In Chapter III we look at the global univalence problem from two viewpoints. First, in view of the invariance of domain theorem, we assume that a map $F: D \subseteq X \to Y$ is a local homeomorphism (X, Y Banach spaces, D open and connected). In this case we again use the theory of covering spaces to reduce the problem as was done in Chapter II. However, since

F(D) is not necessarily simply connected, Theorem 2.2.1 is not applicable and so we must revise the methods of Chapter II. The major new hypothesis added is (*): F is one-one at some point. We then show that the theory of covering spaces becomes applicable and the problem is reduced to determining when (*) holds and (**) (D,F) covers F(D). We solve (**) by introducing an appropriate modification of condition (C) of Chapter II. The new condition is designated $(\overline{C})$. We then introduce analytic and topological hypotheses which insure the verification of (*). As a consequence we prove Theorem 3.2.8, a quantitative estimate on the size of the neighborhoods involved in the inverse function theorem [see 30].

The second point of view studied involves removing the a priori condition that our maps be local homeomorphisms. For example, in Theorem 3.3.3 we use simple methods of critical point theory to show that a monotone, C^1 map F is globally univalent on D provided Ker $[F'(x)]^* = 0$ for all $x \in D$. Most previous results on global univalence ([2, pp. 133-141 and [24]) were based on the use of the Brouwer and Leray-Schauder degree. However in [12], Efimov establishes deep global univalence results for maps of $\mathbb{R}^2$ into itself using differential geometric techniques.

In Section 4 of Chapter III we use the degree theory of Leray-Schauder and the generalized degree for Fredholm maps of index zero to prove results on surjectivity of maps such as Theorem 3.4.1: if F is proper and there is a point p so that $d(F,p,B) \neq 0$ whenever $F^{-1}(p) \subset B$, then F is onto. As corollaries we derive a theorem of Cacciopoli [31b] and a theorem of Nijenhuis and Richardson [22] which states that a C^1 map F of $\mathbb{R}^N$ into itself is surjective provided F is proper and the Jacobian determinant of F is non-negative.

Finally we seek answers to the question of the preservation of global univalence and surjectivity under uniform limits [see 8]. We assume that the convergence is normal (i.e., uniform on bounded sets) and in Theorem 3.5.2 we establish the following necessary and sufficient condition for the limit F of a normally convergent sequence of univalent maps F_n to be univalent (where F_n and F are compact perturbations of the identity): for any bounded domain D and any point a, $F_n(x) \neq a$ on D implies $F(x) \neq a$ on D. We then show that if a map F is continuously differentiable and if the Jacobian of F is non-negative, then the above condition is satisfied provided F has isolated zeroes (i.e., the solutions $F(x) = y$ are isolated for each y). From this we derive Carotheodory's theorem which states that a normally convergent sequence of univalent and analytic maps of $\mathbb{C}^N$ into itself has as a limit a map which is either univalent or has Jacobian determinant identically zero.

Next, for the question of the preservation of surjectivity in the limit, we show that if each F_n is surjective, has the line lifting property and converge normally to a map F for which F^{-1} is locally bounded, then F is surjective. From this we derive an extension of the Banach-Mazur theorem: a C^1 monotone map F of a Hilbert space H into itself is a homeomorphism provided (i) F is proper and (ii) $\mathrm{Ker}[F'(x)]^* = 0$ for all $x \in H$.

CHAPTER I

DIFFERENTIAL CALCULUS AND TOPOLOGY

1. Introduction

Chapter I will concern itself with the analytic and topological results that are necessary for the reading of this thesis. The chapter is divided into three sections: calculus of nonlinear maps, classes of nonlinear maps and topological results.

Since only the most elementary properties of linear operators are used, we shall not find it necessary to devote a separate section to them, but shall instead introduce results where they are needed.

We use the notation $\| \ \|$ to denote the norm associated with a Banach space, and if the sequence x_n and x belong to this Banach space, then $x_n \to x$ means $\|x_n - x\| \to 0$ as $n \to \infty$ unless another type of convergence is specifically described.

Throughout this thesis, all maps are continuous and X, Y will denote Banach spaces (unless otherwise mentioned).

2. Calculus of Nonlinear Maps between Banach Spaces

(For general references see [10].)

Suppose X and Y are real Banach spaces. Let $F: D \subseteq X \to Y$ be a continuous map, D open.

Definition 1.2.1 F is (Frechet) differentiable at $x_0 = 0$ if and only if there exists a linear operator (denoted by $F'(x_0)$) of X into Y so that:

$$F(x_0+h) - F(x_0) - F'(x_0)h = R(x_0;h)$$

where

$$\lim_{||h||\to 0} \frac{||R(x_0;h)||}{||h||} = 0 .$$

In this case $F'(x_0)$ is called the (first) derivative of F at x_0.

If $D \subseteq X$ is an open set and F is a map of D into Y, then F is differentiable in D if F is differentiable at every $x_0 \in D$.

F is said to be continuously differentiable in D (for short, $F \in C'(D)$) if F is differentiable in D and the map $F': x \to F'(x)$ is a continuous map of $D \to L(X,Y)$, the space of bounded linear maps of X into Y.

If the map $F': D \to L(x,y)$ is differentiable, then F is said to be twice differentiable, and the map $F'': D \to L(X,L(X,Y)) \cong L(X \times X,Y)$ is called the second derivative of F. If the map F" is also continuous, we say that $F \in C^2(D)$. Higher derivatives of F are defined inductively in the same manner.

As with differentiablity one can also generalize the notion of complex analyticity to (complex) Banach spaces, i.e., if X and Y are complex Banach spaces then:

<u>Definition 1.2.2</u> If $D \subseteq X$ is open, then $F: D \to Y$ is complex analytic in D if and only if for every $x \in D$, $F(x+ty)$ is an analytic function of the complex number t for all directions y (t sufficiently small).

It can be shown that a complex analytic map is Frechet differentiable and has derivatives of all orders. Also one has a direct analog of the Cauchy Integral Theorem for such maps [25, p. 38].

Of use to us will be the

<u>Chain Rule for Differentiation</u>. If $H = F \circ G$, where G is differentiable at a point x and F is differentiable at G(x), then H is differentiable at x and $H'(x) = F'(G(x)) \circ G'(x)$ [10, p. 148].

A map $F: D \subseteq X \to Y$ is a local homeomorphism (diffeomorphism) on D if and only if every $x \in D$ has an open neighborhood W about it so that F(W) is open and $F|_W$ is a homeomorphism (diffeomorphism of W onto F(W). The "Inverse Function Theorem" gives us sufficient conditions for a map to be a local diffeomorphism.

<u>Theorem 1.2.1</u> Let $D \subseteq X$ be open, $F: D \to Y$ continuously differentiable. If $F'(x)$ is an invertible linear map of X onto Y for every $x \in D$, then F is a local diffeomorphism on D.

Proof: [10, p. 268].

We remark that a local homeomorphism on D is also an open map, i.e., the image of an open subset of D is an open subset of Y. This follows from the definition of a local homeomorphism.

3. Classes of Nonlinear Maps

In this section we consider several classes of (nonlinear) maps between Banach spaces and some of their associated properties. Although these classes of maps which we shall subsequently define are of interest in themselves, we shall only consider those properties of these classes which are needed for the presentation of the results of Chapters II and III.

The first class of maps which we shall consider is the class of compact maps.

Definition 1.3.1 Let X and Y be Banach spaces. $F: X \to Y$ is compact if and only if it maps bounded sets of X into relatively compact sets in Y.

Since every continuous map of $\mathbb{R}^N$ into itself maps bounded sets to bounded sets, it follows that such maps are compact. Thus the class of compact maps may be considered as a direct generalization of such finite dimensional maps. In fact, many of the topological properties of maps from $\mathbb{R}^N$ into itself have direct analogs in the class of compact maps. For example the Brouwer fixed point theorem holds verbatim for compact maps as was discovered by Schauder. This and other topological properties of compact maps follows from the Leray-Schauder degree theory which we shall discuss in Section 3 [2, pp. 97-103].

We have observed that every continuous map F of $\mathbb{R}^N$ into itself is a compact map. Thus $F = I + (F-I)$ is a compact perturbation of the identity, i.e., of the form $I + C$, C compact. If furthermore F is differentiable, then $F'(x) = I + (F'(x)-I)$ and by the argument given above, $F'(x)$ is also a linear map of the form $I+C$, C a linear compact map. The next theorem shows that this is also true for differentiable maps $F: X \to X$ (X a Banach space) of the form $I + C$, i.e., that $F'(x)$ is also a linear compact perturbation of the identity.

<u>Theorem 1.3.1</u> If $F: X \to Y$ is differentiable and compact, then $F'(x)$ is also a compact map.

<u>Proof</u>: [27, p. 51].

The converse of this theorem is not, in general, true.

These last considerations and the following observation will lead us to define a more general class of maps, the class of nonlinear Fredholm maps. Recall that a linear map of X into itself of the form $I + C$, C a compact linear map, is a linear Fredholm map of index zero. Thus we shall define a nonlinear Fredholm map as a continuously differentiable map whose derivative is a linear Fredholm map.

More precisely:

<u>Definition 1.3.2</u> A linear map $L: X \rightarrow Y$ is a Fredholm map if and only if

(1) L has closed range

(2) L has finite dimensional kernel and cokernel.

The index of L (ind L) is defined as dim ker L - - dim coker L, and is a continuous function from the set of linear Fredholm maps into the integers.

If $F: X \rightarrow Y$ is continuously differentiable and if $F'(x)$ is a linear Fredholm map for each $x \in X$, then again, F is called a nonlinear Fredholm map and ind F is defined as ind $F'(x)$. ind F is well defined (i.e., independent of x) since ind $F'(x)$ is a continuous function of X into the integers and so it is constant.

Note also that if F is a map satisfying the hypotheses of Theorem 1.2.1, then F is also a Fredholm map of index zero.

We were led to the concept of a nonlinear Fredholm map from the fact that the derivative $F'(x)$ of a map $F = I+C$ is also of this form, and so $F'(x)$ is, in particular, a Fredholm map (of index zero). If we consider maps $F: X \rightarrow X^*$, X^* the dual of X, we are led, by analogous reasoning, to that class of maps whose derivatives $F'(x)$ are symmetric linear maps, i.e., $L: X \rightarrow X^*$ is symmetric if $(Lx,y) = (Ly,x)$, $\forall x,y \in X$ where the notation (Lx,y) indicates the value of the linear functional Lx at y. We shall see that this property characterizes the class of maps known as gradient or potential maps.

Let X^* denote the dual space of the Banach space X. For $y \in X^*$ we use the notation (y,x) to mean $y(x)$ for $x \in X$. If x_n is a sequence in X we say that x_n converges to x weakly if $(y,x_n-x) \to 0$, $\forall y \in X^*$.

A set $D \subset X$ is said to be weakly (sequentially) compact if every sequence of elements of D has a weakly convergent subsequence whose limit is a point in X.

The following theorem will be useful to us.

Theorem 1.3.2 A set in a reflexive Banach space X is weakly compact if and only if it is bounded.

Proof [11, Part I, p. 68].

We now define the concept of a gradient map.

Definition 1.3.3 $F: X \to X^*$ is a gradient map if and only if there is a real valued function $f: X \to \mathbb{R}$ so that

$$(1) \quad \lim_{t \to 0} \frac{f(u+tv) - f(u)}{t} = (F(u),v), \quad \forall u,v \in X.$$

We write $F(u) = \text{grad } f(u)$.

Although there are many characterizations of gradient maps [2, pp. 107-116], the following one shall be especially useful for us.

Theorem 1.3.3 Let $F: X \to X^* \in C'(X)$. F is a gradient map if and only if $F'(x)$ is a symmetric linear map for each $x \in X$.

Proof: [27, p. 56].

A class of maps that has many applications to nonlinear partial differential equations is the class of monotone maps [18].

<u>Definition 1.3.4</u> $F: X \to X^*$ is monotone if and only if $(F(x) - F(y), x-y) \geq 0$, $\forall x,y \in X$. If the inequality is strict, F is said to be strictly monotone.

<u>Theorem 1.3.4</u> Let $F: X \to X^* \in C'(X)$. Then F is monotone if and only if $(F'(x)y,y) \geq 0$, $\forall x,y \in X$. F is strictly monotone if and only if the strict inequality holds.

<u>Proof</u>:

Suppose F is monotone. From Definition 1.1.1:

$$F'(x)ty + R(x;ty) = F(x+ty) - F(x), \quad t > 0.$$

Thus

$$(F'(x)y,y) + \left(\frac{R(x;ty)}{t}, y\right) \geq 0.$$

Let $t \to 0$ and we have that $(F'(x)y,y) \geq 0$.

Conversely, suppose $(F'(x)y,y) \geq 0$. Let $x(t) = (1-t)y + tx$ and let $\ell \in X^{**}$ be the linear functional identified with $x-y \in X$ under the imbedding J of X in X^{**} given by $(J(x),w) = (w,x)$, $w \in X^*$. Then from the mean value theorem [27, p. 37] we have $(F(x)-F(y),x-y)=$ $= (F'(x(\bar{t})(x-y),x-y) \geq 0$. (Strictness follows from the strictness of the inequalities used in the above arguments.)

We say a map satisfies condition (E) if $x_n \to x$ weakly and $F(x_n) \to y$ strongly implies $F(x) = y$.

Condition (E) is satisfied by a large class of elliptic differential operators and, as the next theorem shows, by monotone maps. (Note that if F is a map between finite dimensional Banach spaces, then (E) is equivalent to the continuity of F.)

Theorem 1.3.4 Let $F: X \to X^*$ be monotone. Then F satisfies condition (E).

Proof:

Let $x_n \to x$ weakly, $F(x_n) \to y$ strongly. For any $v \in X$ we have:

$$0 \leq (F(v) - F(x_n), v-x_n) \to (F(v)-y, v-x) .$$

Thus $(F(v)-y,v-x) \geq 0$, $\forall v \in X$. Choose $v = x-\lambda z$, $\lambda > 0$. Then $\lambda(F(x-\lambda z) -\lambda y, z) \leq 0$ and so $(F(x-\lambda t)-y,z) \leq 0$, $\forall z \in X$. Letting $\lambda \to 0$, we have $(F(x)-y,z) \leq 0$, $\forall z \in X$ and so $F(x) = y$.

Definition 1.3.5 A real valued $f: X \to \mathbb{R}$ is convex if and only if $f((1-t)x + ty) \leq (1-t) f(x) + t f(y)$ $(0 \leq t \leq 1)$ $\forall x$ and $y \in X$. f is said to be strictly convex if the strict inequality holds for $0 < t < 1$.

The next theorem describes the relationship between gradient, monotone and convex maps:

Theorem 1.3.5 Let $f: X \to \mathbb{R} \in C'(X)$. Then f is (strictly) convex <=> grad f is (strictly) monotone.

Proof:

Suppose f is convex. Then

(1) $$f(y+t(x-y)) \leq (1-t)\, f(y) + t\, f(x) .$$

From Definition 1.3.3, equation (1), we have that

(2) $$\lim_{t\to 0} \frac{f(y+t(x-y)) - f(y)}{t} = (\text{grad } f(y),\ x-y) .$$

From (1) and (2) we have

(4) $$(\text{grad } f(y), x-y) \leq \lim_{t\to 0} \frac{(1-t)f(y) + tf(x) - f(y)}{t} = f(x)-f(y)$$

Similarly

(5) $$(\text{grad } f(x), y-x) \leq f(y) - f(x) .$$

Adding (4) and (5) we have $(\text{grad } f(y) - \text{grad } f(x), y-x) \geq 0$, thus grad f is monotone. (The strictness follows from the strict inequality in (1) and thus (4).)

Conversely, suppose grad f is monotone. Let $\phi(t) = f(L(t))$, where $L(t) = y + t(x-y)$. Then from the mean value theorem we have

(1) $$\phi(t) - \phi(0) = t\, \phi'(t_1) , \quad 0 < t_1 < t \quad \text{and}$$

(2) $$\phi(1) - \phi(t) = (1-t)\, \phi'(t_2), \quad t < t_2 < 1.$$

However

(3) $$\phi'(t) = (\text{grad } f(L(t)), x-y)$$
$$= (\text{grad } f(L(t)),\ L(t_1)-L(t_2))/\ t_1-t_2 .$$

Upon subtracting (2) from (1) and using (3) we have

(4) $$\frac{\phi(t) - \phi(0)}{t} - \frac{\phi(1) - \phi(t)}{1-t}$$
$$= (\text{grad } f(L(t_1))-\text{grad } f(L(t_2)),\ L(t_1)-L(t_2))/\ t_1-t_2 .$$

Since $t_1-t_2 < 0$ and grad f is monotone, the right-hand side of (4) is ≤ 0, and the left-hand side is equal to $f(y+t(x-y)) - f(y) - t(f(x)-f(y))$. Thus (4) yields $f(y+t(x-y)) \leq f(y) + t(f(x)-f(y))$ and so f is convex. (Strict convexity follows from strictness in (4).)

The next class of maps we define is the class of quasiconformal maps. This class was first introduced for the special cases of $\mathbb{R}^2$ and $\mathbb{R}^3$ by Lavrentiev [19], where he also proves global homeomorphism results for them. This class of maps proves to be a useful generalization of the concept of a conformal map.

<u>Definition 1.3.6</u> Let $F: X \to Y \in C'(X)$ and also $F'(x)$ is an invertible linear map of X onto Y. F is quasiconformal if and only if there is a number $K \geq 1$ so that $||F'(x)|| \; ||[F'(x)]^{-1}|| \leq K, \forall x \in X$.

For a very complete exposition on quasiconformal maps see [29] .

4. Topological Results

A. Covering Spaces

Suppose X and Y are connected and locally pathwise connected topological spaces, $F: X \to Y$ continuous.

Definition 1.4.1 (X,F) is a covering space of Y (and F is then called a covering map) if and only if every $y \in Y$ has an open neighborhood U about it such that $F^{-1}(U)$ is the disjoint union of open sets O_i in X, each of which is mapped homeomorphically onto U by F.

Note in particular that if (X,F) covers Y then F is a local homeomorphism.

Theorem 1.4.1 If (X,F) covers Y then F has the following properties:

(i) Unique path lifting: If $F(x_0) = y_0$ and L is a path in Y (i.e., a continuous map of [0,1] into Y) with $L(0) = y_0$, then there is a unique path P in X with $P(0) = x_0$ such that $F \circ P(t) = L(t)$.

(ii) Covering homotopy property: Suppose L_1 and L_2 are paths in Y with fixed endpoints which are homotopic (with fixed endpoints), then these paths can be lifted to paths P_1 and P_2 in X which are homotopic with fixed end points. (Recall that paths L_1 and L_2 in Y with $L_1(0) = L_2(0)$ and $L_1(1) = L_2(1)$ are homotopic with fixed endpoints if there is a continuous map $H: [0,1] \times [0,1] \to Y$ with

$H(t,0) = L_0(t)$, $H(t,1) = L_1(t)$, $H(0,u) = L_0(0)$ and $H(1,u) = L_0(1)$.)

(iii) card $F^{-1}(y)$ is the same for every $y \in Y$.

Proof: [14, p. 18].

Theorem 1.4.2 Let $F: X \longrightarrow Y$ be a local homeomorphism, and let $L(t)$, $0 \leq t \leq 1$, be a path in Y. Suppose $P_1(t)$ and $P_2(t)$ are paths in X such that $F \circ P_1 = L = F \circ P_2$. If $P_1(\bar{t}) = P_2(\bar{t})$ for some $0 \leq \bar{t} \leq 1$, then $P_1(t) = P_2(t)$, $\forall t$.

Proof:

Let $S = \{t \mid P_1(t) = P_2(t)\}$.

By hypothesis S is nonempty. Also since P_1 and P_2 are continuous, S is closed. Thus, by connectivity, it suffices to show that S is open. Let $t_1 \in S$. Since F is a local homeomorphism, $\exists$ a neighborhood O about $P_1(t_1)$ $(= P_2(t_1))$ so that F maps O homeomorphically onto F(O), an open neighborhood about $L(t_1)$. Thus $P_1(t) = P_2(t)$ for $|t-t_1|$ sufficiently small and so S is open.

Definition 1.4.2 A space Y is simply connected if and only if every closed path in Y is homotopic to a point.

Theorem 1.4.3 (Monodromy Principle). Let X be a simply connected space. Assume that we have assigned to every $p \in X$ a non-empty set E_p. Assume furthermore that we have assigned to every point (p,q) of a certain subset D of X×X a mapping ϕ_{pq} of E_p into E_q, in such a way that the following hold:

(i) D is a connected neighborhood of the diagonal (all pairs (p,p)) in $X \times X$.

(ii) Each ϕ_{pq} is a one-one map of E_p onto E_q; ϕ_{pp} is the identity map.

(iii) If ϕ_{pq}, ϕ_{qr}, ϕ_{pr} are all definied, then $\phi_{pr} = \phi_{qr} \circ \phi_{pq}$.

If (i)-(iii) hold, then there exists a map ψ which assigns to every $p \in X$ an element $\psi(p) \in E_p$ in such a way that $\psi(q) = \phi_{pq}(\psi(p))$ whenever ϕ_{pq} is defined. Moreover, if p_0 is a given point of X, ψ may be chosen in such a way that $\psi(p_0)$ is any preassigned element, say $e^0_{p_0}$, of E_{p_0}, and ψ is then uniquely determined.

Proof: See [9, pp. 46-48].

B. Degree Theory.

Suppose $D \subseteq X$ is open bounded and connected, $F: \overline{D} \to Y$ continuous. We would like to "measure" the number of solutions of $F(x) = y_0$ in D, or more precisely, we would like to find an integer valued function, called the degree of F in D (denoted $d(F,y_0,D)$) with the following properties:

(i) If $d(F,y_0,0) \neq 0$ then $F(x) = y_0$ has a solution in D.

(ii) If $F_n \to F$ uniformly in D, then $d(F_n,p,D) \to d(F,p,D)$.

(iii) $d(F,p,D) = d(F,q,D)$ whenever p and q are in the same component of $Y - F(\partial D)$.

Such functions exist and we shall describe them briefly, referring the reader to the texts cited for more detailed information.

I. <u>Brouwer degree</u> [see 2, pp. 32-54 and 25, Chapter III].

The Brouwer degree, $d(F,p,D)$, is defined for continuous functions $F: \bar{D} \subset \mathbb{R}^N \to \mathbb{R}^N$ (provided $F(x) \neq p$ for $x \in \partial D$) by the following successive steps:

(a) Let $F \in C'(\bar{D})$ and $\det F'(x) \neq 0$ for $x \in F^{-1}(p)$. We define $d(F,p,D) = \sum_{x \in F^{-1}(p)} \operatorname{sgn} \det F'(x)$. This is a finite sum since $F^{-1}(p)$ is a discrete set by Theorem 1.2.1 and thus has no limit point in the compact set $\bar{D}$.

(b) Let $F \in C'(\bar{D})$ and suppose $\det F'(x) = 0$ for some $x \in F^{-1}(p)$. In this case we apply a special case of Sard's theorem: Let $S = \{x \mid x \in D, \det F'(x) = 0\}$; then $F(S)$ has empty interior. Thus we can find a sequence $p_n \to p$ so that $\det F'(x) \neq 0$ for $x \in F^{-1}(p_n)$. By (a), $d(F,p_n,D)$ is defined and we then define $d(F,p,D) = \lim_{n\to\infty} d(F,p_n,D)$. (Of course it must be shown that the limit exists and is finite and independent of the approximating sequence.)

(c) Finally let $F \in C(\bar{D})$. By the Wierstrass approximation theorem, we can find a sequence $F_n \to F$ uniformly on $\bar{D}$ and $F_n \in C'(\bar{D})$. Then we define $d(F,p,D) = \lim_{n\to\infty} d(F_n,p,D)$. (Again it must be shown that the limit exists, is finite and independent of the approximating sequence.)

Using this definition one verifies that the properties (i)-(iii) are satisfied.

II. Leray-Schauder Degree [2, pp. 95-103, and 25].

The Leray-Schauder degree extends the Brouwer degree to maps which are compact perturbations of the identity defined on a bounded open subset D which meets every finite dimensional subspace in a bounded open set, e.g., we may choose $D = \left\{x \mid ||x|| \leq r\right\}$.

The underlying method of extending the Brouwer degree to such maps is the following approximation lemma which characterizes compact maps.

Lemma 1.4.4 F is compact on the bounded set D, if and only if for every $\varepsilon > 0$ there exists a continuous and bounded (maps bounded sets to bounded sets) map P which satisfies

(i) P has finite dimensional range

(ii) $||F(x)-P(x)|| < \varepsilon, \quad \forall x \in D.$

Proof : [27, p. 12] .

If $F = I + K$, K compact, and $F(x) \neq p$ on ∂D, then one defines $d(F,p,D) = \lim_{n\to\infty} d(F_n,p,D_n)$, where D_n is a finite dimensional open bounded set, and $F_n \to F$ uniformly on D and F_n is defined using Lemma 1.4.4. One shows that the limit exists, is finite and independent of the approximating sequence.

III. Mod 2 Degree.

(a) Finite dimensional maps [see 21, pp. 20-25].

Let $F: \overline{D} \subset \mathbb{R}^N \to \mathbb{R}^N \in C'(\overline{D})$, where D is connected, open and bounded. If $F(x) \neq p$ on ∂D we define the mod 2 degree, denoted by $d_2(F,p,D)$, as follows: if $\det F'(x) \neq 0$ for $x \in F^{-1}(p)$, then $d_2(F,p,D) =$ number $F^{-1}(p)$ (mod 2). By the same considerations used in defining the Brouwer degree, number $F^{-1}(p)$ is finite.

Following the steps used in defining the Brouwer degree, one defines $d_2(F,p,D)$ for $F \in C'(\overline{D})$ via Sard's theorem and then we finally define $d_2(F,p,D)$ for any continuous function on $\overline{D}$.

(b) Infinite dimensional maps

Suppose $F: \overline{D} \subseteq X \to Y$ is a Fredholm map of index 0, where again D is bounded, open and connected. We further suppose that F is proper on $\overline{D}$, i.e., the inverse image of a compact set of Y is a compact set of $\overline{D}$. Suppose $F(x) \neq p$ on ∂D. As before we proceed in steps.

i) Suppose $F'(x)$ is a surjective linear operator $\forall x \in F^{-1}(p)$. We then define $d_2(F,p,D) =$ number $F^{-1}(p)$ (mod 2). number $F^{-1}(p)$ is finite since $F^{-1}(p)$ is a compact set and by the closed graph theorem and Theorem 1.2.1 it is a discrete set, thus it must be finite.

ii) $F'(x)$ is not surjective for some $x \in F^{-1}(p)$. In this case we use the infinite dimensional version of Sard's theorem due to Smale [26]: If $F: D \to Y \in C^r(D)$ is a proper

Fredholm map, where $r > \max(\text{ind } F, 0)$, then the image of $S = \{x \mid F'(x) \text{ is surjective}\}$ is open and dense. Thus we can approximate p by points p_n such that $F'(x)$ is surjective for $x \in F^{-1}(p_n)$. We then define $d_2(F,p,D) = \lim_{n\to\infty} d_2(F,p_n,D)$. (One must show that this limit exists, is finite and independent of the approximating sequence.)

One can also define an oriented degree for Fredholm maps of index zero. More specifically, let D be open, bounded and connected; and $F: \bar{D} \subset X \to Y \in C^2(\bar{D})$ a Fredholm map of index zero which is proper on $\bar{D}$. Since $F'(x)$ is a linear Fredholm map of index zero, it can be written as the composition of a (linear) homeomorphism and a compact perturbation of the identity. Following Elworthy and Tromba [13], we let GL(X) denote the set of invertible linear maps of X onto itself which are compact perturabations of the identity. With the topology inherited from L(X) (the Banach space of all bounded linear operators of X into itself), GL(X) has two components if X is a real Banach space (and is connected if X is a complex Banach space). We define, for $p \notin F(\partial D)$, $d(F,p,D)$ in two steps. If $F'(x)$ is surjective for $x \in F^{-1}(p)$ then $d(F,p,D) = \sum_{x \in F^{-1}(p)} \text{sgn } F'(x)$, where $\text{sgn } F'(x)$ is +1 or -1 depending on whether the part of $F'(x)$ which is a compact perturbation of the identity lies in the identity component of GL(X) or not. As for the mod 2 degree, number $F^{-1}(p)$ is finite. One then extends to the case where

$F'(x)$ is not necessarily surjective as in the definition of the mod 2 degree by using Smale's theorem. To show that this degree satisfies properties (i)-(iii), the reader is referred to [13]. We mention that if F is of the form I+C, C compact, then this degree reduces to the Leray-Schauder degree.

The fact that GL(X) is connected when X is a complex Banach space gives us:

<u>Theorem 1.4.5</u> Let X and Y be complex Banach spaces and $F: \bar{D} \subset X \to Y$ is analytic and proper on $\bar{D}$ (D open, bounded and connected). If F is a Fredholm map of index zero, then $d(F,p,D) > 0$ whenever $p \in F(D) - F(\partial D)$.

<u>Proof</u>:

Let $x \in F^{-1}(p)$. By adding a compact linear map if necessary we may assume that $F'(x)$ is invertible. Thus by Theorem 1.2.1, we can find an open ball W of x so that $F|_W$ is a diffeomorphism of W onto the open set $F(W) \subset F(B)-F(\partial B)$. We now apply Smale's theorem to find a point $q \in F(W)$ so that $F'(z)$ is invertible $\forall\, z \in F^{-1}(q)$. Now by the preceding remark $d(F,q,B) = \sum_{z \in F^{-1}(p)} \operatorname{sgn} F'(z) > 0$. However p and q are in the same component of $F(B)-F(\partial B)$ and so $d(F,p,B) = d(F,q,B) > 0$.

If F is not analytic and $X = \mathbb{R}^N$ we do have an analog of Theorem 1.4.5.

<u>Theorem 1.4.6</u> Suppose $F: \bar{D} \subset \mathbb{R}^N \to \mathbb{R}^N$ is continuous and $F \in C'(D)$. If $\det F'(x) \geq 0$ $(\not\equiv 0)$, $\forall x \in D$, then $d(F,p,D) > 0$ whenever $p \in F(D) - F(\partial D)$.

Proof:

If $\det F'(x) \neq 0$ for $x \in F^{-1}(p)$, then from the definition, $d(F,p,D) = \sum_{x \in F^{-1}(p)} \operatorname{sgn} \det F'(x) > 0$. The general case proceeds as in Theorem 1.4.5 by using Sard's theorem.

Using the degree, one can prove the Invariance of Domain Theorem: If $F: D \subseteq X \to X$ is a compact perturbation of the identity and D is open, then if F is univalent, it is an open map on D (i.e., maps open sets of D to open sets) [see 31a].

C. Miscellaneous

We have already come across the notion of a proper map in part B. We now formally define properness and give several characterizations.

Definition 1.4.3 A continuous map F between two topological spaces X and Y is proper if and only if $F^{-1}(C)$ is a compact set in X whenever C is a compact set in Y.

Theorem 1.4.7 Let X and Y be Banach spaces, $F: X \to Y$.

(a) If X = Y and F is a compact perturbation of the identity, then if $||F(x)|| \to \infty$ as $||x|| \to \infty$, F is proper. The converse is true whenever X is finite dimensional.

(b) F is proper if and only if F is a closed map and the pre-image of any point is compact.

(c) If X is reflexive, then F is proper provided

(i) $||F(x)|| \to \infty$ as $||x|| \to \infty$

(ii) F satisfies the following stronger version of condition (E): $x_n \longrightarrow x$ weakly and $F(x_n) \longrightarrow y$ strongly implies $x_n \longrightarrow x$ strongly.

Proof:

(a) Let $C_1 = F^{-1}(C)$, C a compact subset of Y, $F = I+K$. Let the sequence $x_n \in C_1$. Then $x_n+K(x_n) = y_n \in C$ and so $\exists$ a subsequence (which we renumber) $y_n \longrightarrow y_0 \in C$. Since the sequence y_n is bounded, the coercive condition $||F(x)|| \to \infty$ as $||x|| \to \infty$ implies that the sequence x_n is also bounded. However as K is a compact map, there is a subsequence (which we again renumber) x_n so that $K(x_n) \longrightarrow y_1$. Thus since $x_n + K(x_n) \longrightarrow y_0$, we conclude that $x_n \longrightarrow y_0 - y_1 = \bar{x}$ and since C_1 is closed, $\bar{x} \in C_1$. Hence C_1 is compact.

Now suppose X is finite dimensional and F is proper. Then the pre-image of a bounded set is bounded since its closure is compact and thus the pre-image of the closure is compact and in particular bounded. However the condition that the pre-image of a bounded set is bounded is equivalent to $||F(x)|| \longrightarrow \infty$ as $||x|| \to \infty$.

(b) First suppose F is proper. Then since a point is a compact set, the pre-image of a point is compact. Now suppose C is a closed set. We wish to show F(C) is closed. To this end suppose $F(x_i) \to \bar{y}$, $x_i \in C$. Since

$S = \{\bar{y}, F(x_i)\}$, $i = 1,\ldots,$ is a compact set, $F^{-1}(S)$ is also compact. However, the sequence $x_i \in F^{-1}(S)$ and so there is a convergent subsequence $x_{n_j} \to \bar{x}$. Since the sequence $x_{n_j} \in C$ and C is closed, then $\bar{x} \in C$. By continuity $F(\bar{x}) = \bar{y}$ and so $\bar{y} \in F(C)$ and thus F(C) is a closed set.

Conversely, let $\tilde{K} = F^{-1}(K)$, $K \subset Y$ compact. Let $\tilde{K} = \bigcup_{\alpha \in J} C_\alpha$ where C_α are closed sets which have the finite intersection property (f.i.p.). We show $\bigcap_{\alpha \in J} C_\alpha \neq \emptyset$ and so $\tilde{K}$ is compact. To this end, let $\beta \subset J$ be a finite set. Thus $G_\beta = \bigcap_{\alpha \in \beta} C_\alpha$ is closed and non-empty. By hypothesis, $F(G_\beta)$ is closed and $K = \bigcup_{\beta \subset J} F(G_\beta)$ (since $\tilde{K} = \bigcup_{\beta \subset J} G_\beta$). Also $F(G_\beta)$ has the f.i.p. since if γ is any appropriate finite index, then:

$$\bigcap_{\gamma} F(G_\beta) \supset F(\bigcap_{\gamma} G_\beta) = F(G_\delta) \neq \emptyset$$

(some finite $\delta \subset J$).

Thus K compact implies $S = \bigcap_{\beta \subset J} F(G_\beta) \neq \emptyset$.
Let $y \in S$. Let $D = C \cap F^{-1}(y)$ $(\neq \emptyset)$. By hypothesis $\bigcup_{\alpha \in J} D_\alpha = F^{-1}(y)$ is compact. Thus it suffices to show that D_α has the f.i.p.; for then

$$\emptyset \neq \bigcap_{\alpha \in J} D_\alpha = (\bigcap_{\alpha \in J} C_\alpha) \cap F^{-1}(y) \text{ and so } \bigcap_{\alpha \in J} C_\alpha \neq \emptyset,$$

which was to be shown. However if $\beta \subset J$ is finite, then

$$\bigcap_{\alpha \in \beta} D_\alpha = G_\beta \cap F^{-1}(y) \neq \emptyset \text{ since } y \in S.$$

(c) Let K be a compact subset of Y. Let the sequence $x_n \in F^{-1}(K)$. Since $||F(x)|| \to \infty$ as $||x|| \to \infty$ is equivalent to saying that the inverse image of a bounded set is bounded, then $F^{-1}(K)$ is bounded. Thus by Theorem 1.3.2, there is a weakly convergent subsequence (which we again call x_n) x_n converging weakly to $\bar{x}$. However the sequence $F(x_n)$ is contained in the compact set K and so it has a strongly convergent subsequence (which we again call $F(x_n)$) so that $F(x_n) \to \bar{y}$. Thus by hypothesis, $x_n \to \bar{x}$ strongly and since $F^{-1}(K)$ is closed, $\bar{x} \in F^{-1}(K)$ and thus $F^{-1}(K)$ is compact.

We remark that if X is reflexive, condition (E) and the coerciveness condition $||F(x)|| \to \infty$ as $||x|| \to \infty$ together imply that F has closed range. Suppose $F(x_n) \to \bar{y}$. Then by the coerciveness of F, the sequence x_n is bounded. Thus by Theorem 1.3.2, the sequence x_n has a weakly convergent subsequence, $x_{n_j} \to \bar{x}$ weakly, and so by condition (E), $F(\bar{x}) = \bar{y}$ and so F has closed range.

CHAPTER II

Global Homeomorphisms

1. Introduction

Suppose we have a continuous (or possibly continuously differentiable) map F between two Banach spaces X and Y. We ask what additional assumptions must be imposed upon F to insure that it is a global homeomorphism (diffeomorphism) of X onto Y? We observe that if F is a global homeomorphism then in particular it is a local homeomorphism. Also if F is a global diffeomorphism, then $F'(x)$ is an invertible linear operator $\forall x \in X$. Since $F^{-1} \circ F(x) = x$, then by the chain rule we have that

$$[F^{-1}(F(x))]' \circ F'(x) = I_X$$

and similarly

$$F'(F^{-1}(y)) \circ [F^{-1}(y)]' = I_Y ,$$

where $F^{-1}(y) = x$. Thus $F'(x)$ is invertible.

Since these conditions are necessary conditions for our problem, we shall always assume that our maps F are either local homeomorphisms or if $F \in C'(X)$, that $F'(x)$ is invertible, and we again ask when such a map F is a global homemorphism (diffeomorphism). Using Theorem 2.2.1 we shall see that we can reduce this question to a more fundamental one, that of determining when the given local homeomorphism F is a covering space map of X onto Y [Definition 1.4.1], and in Section 2 we prove the following theorem (Theorem 2.2.2):

If $F: D \subseteq X \to Y$ is a local homeomorphism (where X and Y are Banach spaces, D open), then (D,F) is a covering space of F(D) if and only if F "lifts lines".

Section 3 is devoted to applications of Theorem 2.2.2 in the case D = X. Here we develop a systematic method for verifying the hypotheses of Theorem 2.2.2 and prove such theorems as (Banach-Mazur [1]) $F: X \to Y$ is a homeomorphism of X onto Y if and only if F is a local homeomorphism and a proper map [see Definition 1.4.3] and the following theorem due to Hadamard [15] for $X = Y = \mathbb{R}^N$, and Lévy [20] for X and Y Banach spaces: If $F: X \to Y \in C'(X)$ and $F'(x)$ is an invertible linear map for all $x \in X$ and $\int_0^\infty \inf_{||x|| \leq t} \frac{1}{||[F'(x)]^{-1}||} dt = \infty$, then F is a global diffeomorphism of X onto Y. For $X = Y = \mathbb{R}^N$, we sharpen this slightly by requiring that $\int_0^\infty \inf_{||x||=t} \frac{1}{||[F'(x)]^{-1}||} dt = \infty$.
We also prove global homeomorphism theorems for a special class of quasiconformal maps (Definition 1.3.6) (see Zoric [31]).

The background references for this chapter are: [1,4,7,8,12,15,17,19,20,31]. Further references can be found in these papers.

2. Local Homeomorphisms and Covering Spaces.

Our first step in attacking the global homeomorphism question is the following:

Theorem 2.2.1 Let X and Y be connected, locally arcwise connected spaces. Furthermore, let Y be simply connected. If (X,F) is a covering for Y, then F is a homeomorphism of X onto Y.

Proof:

Suppose $F(x_1) = F(x_2) = \bar{y}$.
Let $p(t)$ be a path in X joining x_1 to x_2.
Set $L(t) = F(p(t))$. $L(t)$ is a closed path, i.e., $L(0) = L(1) = \bar{y}$.
Since Y is simply connected, there exists a homotopy $H(t,u)$ so that $H(t,0) = L(t)$, $H(t,1) = \bar{y}$ and $H(0,u) = H(1,u)= \bar{y}$.
By the covering homotopy property (Theorem 1.4.1 (ii) there exists a homotopy $\tilde{H}(t,u)$ so that $F(\tilde{H}(t,u))=H(t,u)$, $\tilde{H}(t,0) = p(t)$, $\tilde{H}(0,u) = x_1$ and $\tilde{H}(1,u) = x_2$.
Let W_1 be a neighborhood of x_1 on which F is a homeomorphism. Thus $F(W_1)$ is a neighborhood of $\bar{y}$ and by the continuity of $H(t,u)$, there exists $\bar{u} \ni H(t,\bar{u}) \subset F(W_1)$. Let $P(t)$ be the path in W_1 with $P(0) = x_1$ and $F(P(t)) = H(t,\bar{u})$. By the unique path lifting property (Theorem 1.4.1 (i) $P(t) = \tilde{H}(t,\bar{u})$. Hence $P(0) = x_1$ and $P(1) = x_2$, which is a contradiction by our choice of W_1.

Suppose X and Y are Banach spaces, $F: X \rightarrow Y$ a local homeomorphism. In order to show that F is a global homeomorphism Theorem 2.2.1 tells us that we must show that (X,F) covers Y.

Let X and Y be Banach spaces, $D \subseteq X$ open and connected.

<u>Definition 2.2.1</u> $F: D \rightarrow Y$ lifts lines (in F(D)) if and only if for each line $L(t) = (1-t)y_1 + ty_2$ (in F(D)) and for every point $x_\alpha \in F^{-1}(y_1)$ there is a path $P_\alpha(t) \ni P_\alpha(0) = x_\alpha$ and $F(P_\alpha(t)) = L(t)$.

By Theorem 1.4.2 if F is a local homeomorphism, and F lifts lines, then the path $P_\alpha(t)$ in Definition 2.2.1 is unique. With this in mind, we prove:

<u>Theorem 2.2.2</u> Let $F: D \subseteq X \rightarrow Y$ be as above. Suppose also that: (i) F is a local homeomorphism and
(ii) F lifts lines in F(D).
Then (i) and (ii) are necessary and sufficient for (D,F) to cover F(D).

Proof:

The necessity follows from the definition of a covering space and from Theorem 1.4.1(i). To prove the sufficiency, we first observe that if $y \in F(D)$, we can find an r so that $B(y;r) = \left\{z \,\middle|\, ||z-y|| < r\right\} \subseteq F(D)$, and that any radius in B can be described by a line $L_z(t) = y + tr\,z$, $||z|| = 1$, $0 \leq t < 1$ which can be lifted. Let $x \in F^{-1}(y)$,
$\tilde{O}_x = \left\{P(t) \,\middle|\, F(P(t)) = L_z(t),\ \forall ||z|| = 1,\ 0 \leq t < 1 \text{ and } P(0)=x\right\}$.

Let $O_x = \tilde{O}_x$ considered as a point set, i.e., $O_x = \{x | x=P(\bar{t}),\ P \in \tilde{O}_x\}$ ($O_x \neq \emptyset$ by (ii)). By intuitively thinking of O_x and $B(y;r)$ as the spokes of a wheel, we shall show that these sets satisfy the conditions given in the definition of a covering space [Definition 1.4.1], i.e., we show that the O_x ($x \in F^{-1}(y)$) are disjoint, open sets mapped homeomorphically onto $B(y;r)$ by F, and $F^{-1}(B(y;r)) = \bigcup_{x \in F^{-1}(y)} O_x$.

(a) Each O_x is mapped onto $B(y;r)$ since any $\bar{y} \in B(y;r)$ lies on some radius L, hence there is a path $P(t) \in \tilde{O}_x$ and a $\bar{t}$ so that $F(P(\bar{t})) = \bar{y}$. By definition of O_x, $P(\bar{t}) \in O_x$.

(b) Each O_x is mapped homeomorphically onto $B(y;r)$. If not, let $x_1 \neq x_2 \in O_x$ and $F(x_1) = F(x_2) = \bar{y}$. By definition of O_x, x_1 and x_2 lie on paths P_1 and P_2 which are not identical, for otherwise their image would be a radius which would intersect itself. Hence $F(P_1(t))$ and $F(P_2(t))$ are distinct radii by Theorem 1.4.2. Thus $\bar{y} = y$, and so
$F(x_1) = F(P_1(t_1)) = F(P(0)) = y = F(P_2(t_2)) = F(x_2)$.
Hence $t_1 = 0$ and $t_2 = 0$ (otherwise the image of $P_i(t)$ $(i=1,2)$ would be a radius which intersects itself), and so $x_1=x_2=x$, a contradiction. The continuity of the inverse follows from the fact that $F|_{O_x}$ is a local homeomorphism and thus an open map.

(c) O_x, $x \in F^{-1}(y)$, are disjoint, for if $\bar{x} \in O_{x_1} \cap O_{x_2}$ with $x_1 \neq x_2$, then $\bar{x} = P_1(t_1) = P_2(t_2)$. The images of P_1 and P_2 under F must be the same radius, for otherwise the radii would intersect and so $F(\bar{x}) = F(x_1) = F(x_2) = y$.

By part (b), $\bar{x} = x_1 = x_2$ -- a contradiction. Thus $F(P_1(t)) = F(P_2(t)) = L(t)$, and so $L(t_1) = L(t_2)$ which implies that $t_1 = t_2$ $(= \bar{t})$. From Theorem 1.4.2, we conclude that $P_1(t) = P_2(t)$, and in particular, $x_1 = x_2$. Thus the O_x's are disjoint.

(d) Each O_x is an open set in D, for if not, then $\exists\, \bar{x} \in O_x$ and a sequence x_n so that $x_n \to \bar{x}$ and $x_n \notin O_x$. Choose a neighborhood W of $\bar{x} \ni F|_W$ is a homeomorphism and $F(W) \subseteq B(y;r)$. $\exists\, N \ni x_n \in W$ for $n \geq N$, and so $F(x_n) \in B(y;r)$ for $n \geq N$. Hence there are points $\bar{x}_n \in O_x$ with $F(\bar{x}_n) = F(x_n)$, $n = N, N+1, \ldots$. However, since F is a homeomorphism of O_x onto $B(y;r)$, then $F(\bar{x}_n) \to F(\bar{x})$ implies $\bar{x}_n \to \bar{x}$. Thus for n sufficiently large, $\bar{x}_n \in W$ and so $\bar{x}_n = x_n$, which is a contradiction since we assumed that $x_n \notin O_x$. Thus each O_x is open.

(e) $F^{-1}(B(y;r)) = \bigcup_{x \in F^{-1}(y)} O_x$.

Since $F^{-1}(B(y;r)) \subseteq \bigcup_{x \in F^{-1}(y)} O_x$, it suffices to show the opposite inclusion. So let $x \in F^{-1}(B(y;r))$. Let $L(t) = (1-t)\,F(x) + ty$, $0 \leq t \leq 1$. Then $L(t) \in B(y;r)$ and so by hypothesis there is a path $P(t)$ so that $P(0) = x$ and $F(P(t)) = L(t)$. Thus $P(1) \in F^{-1}(y)$. Let $\tilde{L}(t) = L(1-t)$ and $\tilde{P}(t) = P(1-t)$. Thus $F(\tilde{P}(t)) = \tilde{L}(t)$, $\tilde{P}(0) = P(1) \in F^{-1}(y)$ and $\tilde{P}(1) = x$. So by definition of $O_{P(1)}$, we see that $x \in O_{P(1)}$.

3. Global Homeomorphisms

In view of Theorem 2.2.1 and 2.2.2, we now proceed in developing a method (Theorem 2.3.1) for determining when a local homeomorphism lifts lines.

Again we suppose that X and Y are Banach spaces, $D \subseteq X$ is open and connected. Let $F: D \to Y$ be continuous. We introduce the following condition:

(C) Whenever $P(t)$, $0 \leq t < b$ is a path satisfying $F(P(t)) = L(t)$ for $0 \leq t < b$ (where $L(t) = (1-t)y_1 + ty_2$ is <u>any line in Y</u>), then there is a sequence $t_i \to b$ as $i \to \infty$ such that $\lim_{i \to \infty} P(t_i)$ exists and is in D.

<u>Theorem 2.3.1</u>. Let $F: D \subseteq X \to Y$ be a local homeomorphism. Then condition (C) is necessary and sufficient for F to be a homeomorphism of D onto Y.

<u>Proof</u>:

The necessity is trivial, for we let $P(b) = F^{-1}(L(b))$. For the sufficiency, we first show that F lifts lines. Let $L(t)$ be any line in $F(D)$, with $L(0) = \bar{y}$. Let $\bar{x} \in F^{-1}(\bar{y})$.

Since F is a local homeomorphism, there is an $\varepsilon > 0$ and a path $P(t)$ $(= F^{-1}(L(t)))$, $0 \leq t < \varepsilon$, such that $P(0) = \bar{x}$ and $F(P(t)) = L(t)$ for $0 \leq t < \varepsilon$. Let K (≤ 1) be the largest number for which $P(t)$ can be extended to a continuous path for $0 \leq t < K$ and satisfying $F(P(t)) = L(t)$, $0 \leq t < K$. Since F satisfies condition (C), let $z = \lim_{t_i \to K} P(t_i)$. By continuity, $F(z) = L(K)$. Let W be a neighborhood of z on which F is a homeomorphism. $\exists N \ni P(t_i) \in W$ for $i \geq N$.

Also $\exists \delta > 0$ and a path $Q(t)$ defined for $K-\delta < t < K+\delta$ so that $Q(t_M) = P(t_M)$ (where M is chosen so that $M \geq N$ and $K-\delta < t_M < K$) and $F(Q(t)) = L(t)$ for $K-\delta < t < K+\delta$.

Hence $P(t)$ can be extended to a continuous path (which we again call $P(t)$) on $0 \leq t < K+\delta$, $P(0) = \bar{x}$ and $F(P(t)) = L(t)$, $0 \leq t < K+\delta$. By the maximality of K, we conclude that $K = 1$, and hence F lifts lines.

By virtue of Theorem 2.2.2, (D,F) covers $F(D)$. We need only show that $F(D) = Y$ in order to apply Theorem 2.2.1 and thus conclude that F is a homeomorphism of D onto Y. So let $\bar{y} \in Y$. Choose $y_1 \in F(D)$ and let $L(t) = (1-t)y_1 + t\bar{y}$. If we retrace the steps of the first part of our proof, we find a path $P(t)$, $0 \leq t \leq 1$, so that $F(P(t)) = L(t)$ on $0 \leq t \leq 1$. In particular $F(P(1)) = L(1) = \bar{y}$, and so $F(D) = Y$.

From this follows a theorem due to Banach and Mazur [1] which they proved using complicated arguments based on the Monodromy theorem (Theorem 1.4.3).

Theorem 2.3.2 Let X and Y be Banach spaces, $F: X \to Y$. Then F is a homeomorphism of X onto Y if and only if F is a local homeomorphism and a proper map [Definition 1.4.3].

Proof:

The necessity is obvious for if F is a homeomorphism, then F^{-1} is continuous and thus maps compact sets into compact sets. Hence F is proper.

Suppose now that F is a local homeomorphism and F is proper. By virtue of Theorem 2.3.1, it suffices to show that F satisfies condition (C) in order to conclude that F is a homeomorphism. So suppose P(t) is defined on $0 \leq t < b$ and satisfies F(P(t) = L(t) for $0 \leq t < b$. Let $t_i \rightarrow b$. Since $S = \{L(t)\}_{0 \leq t \leq 1}$ is compact, so is $F^{-1}(S)$ and it contains the sequence $P(t_i)$. Hence there is a subsequence $t_{i_j} \rightarrow b \ni P(t_{i_j}) \rightarrow \bar{x}$, and so condition (C) is satisfied.

<u>Corollary 2.3.2</u> $F: \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a diffeomorphism if and only if $F \in C'(R^N)$ and F satisfies (i) $\det F'(x) \neq 0 \ \forall x$, and (ii) $||F(x)|| \rightarrow \infty$ as $||x|| \rightarrow \infty$.

<u>Proof</u>:

We observe that if F is a diffeomorphism, then we have already seen in Section 1 that F'(x) is invertible, and so $\det F'(x) \neq 0$. The corollary now follows from Theorem 2.3.2, and Theorem 1.4.7.

In Corollary 3.4.4, we show that if $\det F'(x) \geq 0$ and $||F(x)|| \rightarrow \infty$ as $||x|| \rightarrow \infty$, then F is onto.

<u>Corollary 2.3.3</u> Let F be a local homeomorphism of the reflexive space X into Y. If (i) $||F(x)|| \rightarrow \infty$ as $||x|| \rightarrow \infty$ and (ii) whenever $x_n \rightarrow x$ weakly, and $F(x_n) \rightarrow y$ strongly implies $x_n \rightarrow x$ strongly, then F is a homeomorphism of X onto Y.

Proof:

From Theorem 1.4.7 (c), (i) and (ii) imply that F is proper. Thus the Corollary follows from Theorem 2.3.2.

In general the coerciveness condition $||F(x)|| \to \infty$ as $||x|| \to \infty$ is not enough to insure that the map F is proper and so a direct proof as in Corollaries 2.3.2 and 2.3.3 may be unattainable. However in certain cases it is possible to show directly that a given map is a homeomorphism, i.e., showing it is one-one, onto and possesses a continuous inverse. In Chapter III, Sec. 3 we shall be able to prove, in this direct manner, the following:

Corollary 2.3.4 Let $F: X \to X^* \in C'(X)$ be a monotone Fredholm map of index zero. If (i) $\text{Ker } (F'(x))^* = 0$ and (ii) $||F(x)|| \to \infty$ as $||x|| \to \infty$, then F is a diffeomorphism of X onto X^*.

The following theorem is due to Browder [4]:

Theorem 2.3.5 $F: X \to Y$ is a homeomorphism of X onto Y if and only if F is a local homeomorphism and a closed map.

Proof:

To prove the necessity we need only notice that if F is a homeomorphism, then F^{-1} is continuous and so F maps closed sets into closed sets.

To prove the sufficiency, we need only show that F satisfies condition (C) and then apply Theorem 2.3.1 to conclude that F is a homeomorphism of X onto Y.

We now show that F satisfies condition (C). So suppose that P(t) is defined on $0 \leq t < b$ and satisfies $F(P(t)) = L(t)$ for $0 \leq t < b$. Let $S = \overline{\{P(t)\}}_{0 \leq t < b}$. By hypothesis F(S) is closed. Thus since $L(t) \in F(S)$, for all $t < b$, then by continuity $L(b) \in F(S)$. Hence $\exists\, x \in S$ so that $F(x) = L(b)$. Since $x \in S$, $\exists\, t_i$ so that $P(t_i) \to x$. Since $0 \leq t_i < b$, there exists a subsequence $t_{i_j} \to \bar{t}$. We claim $\bar{t} = b$ (and thus condition (C) is satisfied by $t_{i_j} \to b$ and $P(t_{i_j}) \to x$). However by continuity, $L(\bar{t}) = L(b)$ and so $\bar{t} = b$.

Let us remark that by Theorem 1.4.7(b) we could have deduced Theorem 2.3.2 as a corollary of Theorem 2.3.5. However we preferred to prove it directly in order to illustrate the type of arguments that one can use in verifying condition (C).

<u>Corollary 2.3.6</u> Suppose $F: X \to Y$ is a local homeomorphism. Furthermore, suppose F satisfies the following:

(1) $||F(x)|| \to \infty$ as $||x|| \to \infty$.

(2) There exists a compact operator $K: X \to Y$ such that the operator $B(x) = F(x) + K(x)$ satisfies the following condition: for any x_1 and x_2 with $||x_1||$ and $||x_2|| \leq R$ we have $||B(x_2)-B(x_1)|| \geq \phi(||x_2-x_1||;R)$; where $\phi(r;R)$ is continuous, real-valued and strictly increasing with respect to $r \geq 0$ for each $R > 0$ and $\phi(0;R) \equiv 0$.

If the above conditions are satisfied, then F is a homeomorphism of X onto Y.

Proof:

We shall show that F is a closed map and then use Theorem 2.3.5 to conclude the desired result. Let $C \subseteq X$ be closed, let the sequence $x_i \in C$ be such that $F(x_i) \to y$. By (1), the sequence x_i is bounded (by M) and since K is compact, there is a subsequence x_{n_j} such that $K(x_{n_j}) \to \bar{y}$. Hence $B(x_{n_j}) \to y + \bar{y}$. Suppose $\exists \varepsilon_0 > 0$ such that $||x_{n_j} - x_{m_j}|| \geq \varepsilon_0 > 0$ for all n,m. Then $||B(x_{n_j}) - B(x_{m_j})|| \geq \phi(||x_{n_j} - x_{m_j}||;\ 2M) \geq \phi(\varepsilon_0, 2M) > 0$, which is a contradiction since $||B(x_{n_j}) - B(x_{m_j})|| \to 0$. Hence there is a subsequence of x_{n_j} which converges to some x, and $x \in C$ since C is closed. Thus $F(x) = y$ by continuity, and so F(C) is closed, as was to be shown.

Notice that if in Corollary 2.3.6, X and Y are finite dimensional, the corollary follows from Theorems 1.4.7 and 2.3.2.

Again, let X and Y be Banach spaces. Let $B(x) > 0$ be a real valued continuous function on X. Let P(t) be a path (in X) of class C' on $0 \leq t \leq b$.

<u>Definition 2.3.2</u> The arc length of P with weight B is

$$L_0^b(P) = \int_0^b B(P(t))\,||P'(t)||\,dt.$$

<u>Definition 2.3.3</u> X is complete with respect to arc length

with weight B if and only if $L_0^b(P) < \infty \Rightarrow \lim_{t \to b} P(t)$ exists and is finite whenever $P(t)$ is a C' path on $0 \leq t < b$.

We remark that if $X = \mathbb{R}^N$, then Definition 2.3.3 is equivalent to the usual notion of $\mathbb{R}^N$ being complete with respect to the conformal metric induced by the tensor $ds^2 = [B(x)]^2\, dx^2$ (see Hartman [16]).

With $B(x)$ as above, we prove the following sufficient condition for completeness:

Theorem 2.3.7 Let $h(s) = \inf_{||x|| \leq s} B(x)$. If $\int_0^\infty h(s)\, ds = \infty$, then X is complete with respect to arc length with weight B.

Proof:

Let $P(t) \in C^1[0,b)$ and $L_0^b(P) < \infty$. Let $0 < \delta < b$. For any partition $0 = t_0 \leq t_1 \leq \ldots \leq t_N = \delta$ of $[0,\delta]$, let $t_i \leq \bar{t}_i \leq t_{i+1}$ be that point for which $\sup_{t_i \leq t \leq t_{i+1}} ||P'(t)|| = ||P'(\bar{t}_i)||$. By the mean value theorem we have:

$$L_0^\delta(P) = \int_0^\delta B(P(t))\,||P'(t)||\, dt$$

$$= \lim \sum B(P(\bar{t}_i))\,||P'(\bar{t}_i)||\,(t_{i+1}-t_i)$$

$$\geq \lim \sum B(P(\bar{t}_i))\,(||P(t_{i+1})|| - ||P(t_i)||)$$

$$= \int_0^\delta B(P(t))\, d||P(t)||,$$

this last equality following from the fact that

$\int_0^\delta B(P(t))\, d||P(t)||$ is defined since $g(t) = ||P(t)||$ is of bounded variation on $[0,\delta]$. So we have that:

$$(1) \quad \infty > \int_0^\infty B(P(t))||P'(t)||\, dt \geq \int_0^\delta B(P(t))\, d||P(t)||$$

$$\geq \int_0^\delta \inf_{||x|| \leq ||P(t)||} B(x)\, d||P(t)||$$

$$= \int_0^\delta h(||P(t)||)\, d||P(t)|| = \int_{||P(0)||}^{||P(\delta)||} h(s)\, ds .$$

By hypothesis, this implies that $\{P(t)\}_{0 \leq t < b}$ is bounded. Also $\int_0^\infty h(s)\, ds = \infty$ implies that $\sup \{s | h(s) > 0\} = \infty$, and since $h(s)$ is nonincreasing, we have that $B(x)$ is bounded from below on any bounded set. In particular, $B(P(t))$ is bounded from below by some number $\lambda > 0$, for all $0 \leq t < b$.

Let $t_i \longrightarrow b$. Then

$$(2) \quad \sum_{i=1}^n ||P(t_{i+1})-P(t_i)|| \leq \sum_{i=1}^n \sup_{t_i \leq t \leq t_{i+1}} ||P'(t)||(t_{i+1}-t_i)$$

$$\leq \int_{t_1}^{t_{n+1}} ||P'(t)||dt \leq \frac{1}{\lambda} \int_0^b B(P(t))||P'(t)||\, dt < \infty.$$

Hence $\exists \bar{x}$ so that $P(t_i) \longrightarrow \bar{x}$ as $t_i \longrightarrow b$.

Suppose $s_i \longrightarrow b$ and $P(s_i) \longrightarrow \bar{z}$. If we form the sequence $t_1, s_1, t_2, s_2, \ldots$, and call it $\bar{t}_i$, then $\bar{t}_i \longrightarrow b$ and (2) shows that $||P(s_i)-P(t_i)|| = ||P(\bar{t}_{i+1})-P(\bar{t}_i)|| < \varepsilon/3$ for large enough i. Hence for $\varepsilon > 0$,

$$||\bar{x}-\bar{z}|| \leq ||\bar{x}-P(t_i)||+||P(t_i)-P(s_i)||+||P(s_i)-\bar{z}|| < \varepsilon$$

for i sufficiently large. Thus $\lim_{t \to b} P(t)$ exists and is finite.

For $X = \mathbb{R}^N$, we have the following slightly stronger version of Theorem 2.3.7.

Theorem 2.3.8 Let $h(s) = \inf_{||x||=s} B(x)$. If $\int_0^\infty h(s)\,ds = \infty$, then $\mathbb{R}^N$ is complete with respect to arc length with weight B.

Proof:

Let $P(t) \in C^1[0,b)$ and suppose $L_0^b(P) < \infty$. Let $0 < \delta < b$. Following the proof given in Theorem 2.3.7, we have that $\infty > \int_{||P(0)||}^{||P(\delta)||} h(s)\,ds$. Hence $\{P(t)\}_{0 \le t < b}$ is bounded. Since $B(x)$ is a continuous, real valued function on $\mathbb{R}^N$, it maps bounded sets into bounded sets. Thus $B(P(t))$ is bounded from below by some positive number (since $B(x)$ is positive). Again, as in Theorem 2.3.7, we find that if $t_i \to b$, then $\sum_{i=1}^{n} ||P(t_{i+1}) - P(t_i)|| \le \frac{1}{\lambda}\int_0^b B(P(t))\,||P'(t)||\,dt < \infty$, and thus we conclude that $\lim_{t \to b} P(t)$ exists and is finite.

An immediate consequence of Theorem 2.3.7 is the following theorem due to Hadamard [15]:

Theorem 2.3.9 Let X and Y be Banach spaces, $F: X \to Y \in C'(X)$ and $F'(x)$ is invertible for all $x \in X$. If $\int_0^\infty \inf_{||x|| \le s} \frac{1}{||[F'(x)]^{-1}||}\,ds = \infty$, then F is a diffeomorphism of X onto Y.

Proof:

By Theorem 1.2.1, F is a local diffeomorphism, thus in view of Theorem 2.3.1, we need only show that F satisfies condition (C). To this end we apply Theorem 2.3.7 as follows:

Suppose $P(t)$ is defined on $0 \leq t < b$ and satisfies $F(P(t)) = L(t)$ for $0 \leq t < b$. If we look at the proof of Theorem 2.3.1, we see that it suffices to show that F satisfies condition (C) only for those paths $P(t)$ that are constructed by the method used in Theorem 2.3.1. Also, if $P(t)$ is such a path, we may assume, by the inverse function theorem, that $P(t)$ is continuously differentiable on $0 \leq t < b$. Since $F(P(t)) = L(t)$ on $0 \leq t < b$, we use the chain rule and get $F'(P(t))P'(t) = L'(t)$ $(= z)$. Thus $P'(t) = [F'(P(t))]^{-1}z$ for $0 \leq t < b$. Let $B(x) = 1/||[F'(x)]^{-1}||$.

By our hypothesis, combined with Theorem 2.3.7, we have that X is complete with respect to arc length with weight B. Also

$$L_0^b(P) = \int_0^b B(P(t)) \, ||P'(t)|| \, dt$$

$$= \int_0^b \frac{1}{||[F'(P(t))]^{-1}||} \, ||[F'(P(t))]^{-1}z|| \, dt$$

$$\leq b||z|| \quad .$$

Thus, by Definition 2.3.3, F satisfies condition (C).

Corollary 2.3.10 If $F: X \to Y \in C'(X)$, $F'(x)$ is invertible for all $x \in X$ and further there exists $M > 0$ so that for each $x \in X$, $||F'(x)z|| \geq M||z||$ for all z, then F is a diffeomorphism of X onto Y.

Proof:

$||F'(x)z|| \geq M||z||$ for all z implies that $||[F'(x)]^{-1}|| \leq \frac{1}{M}$ for every x. Hence $\int_0^\infty \inf_{||x|| \leq t} \frac{dt}{||[F'(x)]^{-1}||} \geq \int_0^\infty M \; dt = \infty$. So by Theorem 2.3.9, F is a diffeomorphism of X onto Y.

The next corollary pertains to a class of maps that are related to quasiconformal maps [see Definition 1.3.6].

Corollary 2.3.11 Suppose $F: X \to Y \in C'(x)$ and $F'(x)$ is invertible $\forall x \in X$. Also suppose that there are continuous, positive non-decreasing real valued functions $M(t)$, $\bar{M}(t)$ so that $||F'(x)|| \leq M(||x||)$ and $||[F'(x)]^{-1}|| \leq \bar{M}(||x||)$. Then if $M(t)\bar{M}(t) \leq \lambda(t)$ for all real t where $\lambda(t) > 0$ and $\int_0^\infty 1/\lambda(t)\; dt = \infty$, F is a diffeomorphism of X onto Y.

Proof:

$$||F'(0)|| \; ||[F'(y)]^{-1}|| \leq \sup_{||z|| \leq ||y||} ||F'(z)|| \cdot \sup_{||z|| \leq ||y||} ||[F'(z)]^{-1}||$$

$$\leq M(||y||)\bar{M}(||y||) \leq \lambda(||y||)$$

Thus

$$\frac{1}{||[F'(y)]^{-1}||} \geq \frac{||F'(0)||}{\lambda(||y||)} .$$

So by virtue of Theorem 2.3.9, F is a diffeomorphism of X onto Y.

Corollary 2.3.12 Let H be a Hilbert space and $F: H \to H \in C'(H)$. Furthermore suppose there is a positive real valued function $\lambda(x)$ such that $(F'(x)z,z) \geq \lambda(x)||z||^2$. Then if $\int_0^\infty \inf_{||x|| \leq t} \lambda(x)\, dt = \infty$, F is a diffeomorphism of H onto itself.

Proof:

By the Lax-Milgram Theorem, $F'(x)$ is invertible and $||[F'(x)]^{-1}|| \leq 1/\lambda(x)$. Thus F is a local diffeomorphism. Also

$$\int_0^\infty \inf_{||x|| \leq t} \frac{1}{||[F'(x)]^{-1}||}\, dt \geq \int_0^\infty \inf_{||x|| \leq t} \lambda(x)\, dt = \infty.$$

Hence Theorem 2.3.9 is applicable.

Theorem 2.3.13 Suppose $F: \mathbb{R}^N \to \mathbb{R}^N$, $F \in C'(\mathbb{R}^N)$ and also $F'(x)$ is invertible for all $x \in X$. If $\int_0^\infty \inf_{||x||=t} 1/||[F'(x)]^{-1}||\, dt = \infty$, then F is a diffeomorphism of $\mathbb{R}^N$ onto itself.

Proof:

The proof mimics that of Theorem 2.3.9, except that we use Theorem 2.3.8 in place of Theorem 2.3.7.

Lemma 2.3.14 Let $L: \mathbb{R}^N \to \mathbb{R}^N$ be an invertible linear operator. Then

$$|\det L|\ |(L^{-1}x,y)| \leq ||x||\ ||y||\ ||L||^{n-1}\ (n-1)^{-(n-1)/2}$$

for all $x,y \in \mathbb{R}^N$.

Proof:

[See 11, Part II, p. 1020.]

Corollary 2.3.15 Let $F: \mathbb{R}^N \to \mathbb{R}^N \in C'(\mathbb{R}^N)$. Suppose that (i) $|\det F'(x)| > \alpha > 0$, and (ii) $||F'(x)|| \leq M$. Then F is a diffeomorphism of $\mathbb{R}^N$ onto $\mathbb{R}^N$.

Proof:

From Lemma 2.3.14, we have that:

(1) $|\det F'(x)| \; |([F'(x)]^{-1}z,w)| \leq c(n)||z|| \; ||y|| \; ||F'(x)||^{n-1}$,

where $c(n) = (n-1)^{-(n-1)/2}$. Choose z so that $||z|| = 1$, and let $w = [F'(x)]^{-1}z$. With these choices, (1) becomes:

$$|\det F'(x)| \; ||[F'(x)]^{-1}z||^2 \leq c(n)||[F'(x)]^{-1}z|| \; ||F'(x)||^{n-1}.$$

Using hypotheses (i) and (ii) we have that $||[F'(x)]^{-1}z|| \leq c(n)M^{n-1}/\alpha$, for all $||z|| = 1$. Hence $||[F'(x)]^{-1}|| \leq c(n)M^{n-1}/\alpha$, for each x.
Thus by Corollary 2.3.10, F is a diffeomorphism of $\mathbb{R}^N$ onto $\mathbb{R}^N$.

Corollary 2.3.16 Suppose $F: \mathbb{R}^N \to \mathbb{R}^N$ is continuously differentiable. Also, suppose that (i) $|\det F'(x)| > \alpha > 0$, and (ii) F is quasiconformal, i.e., $\exists M \ni$ $||F'(x)|| \; ||[F'(x)]^{-1}|| \leq M$, for all x. Then F is a diffeomorphism of $\mathbb{R}^N$ onto $\mathbb{R}^N$.

Proof:

By Lemma 2.3.14, we have with $c(n) = (n-1)^{-(n-1)/2}$:

(1) $|\det F'(x)|\ |([F'(x)]^{-1}z,w)| \le c(n)||z||\ ||w||\ ||F'(x)||^{n-1}$

Choosing z = w and $||z|| = 1$, (1) becomes:

(2) $|\det F'(x)|\ |([F'(x)]^{-1}z,z)| \le c(n)||F'(x)||^{n-1},\ \forall ||z||=1.$

Taking $\sup_{||z||=1}$ of the left side of (2), we get:

$$|\det F'(x)|\ ||[F'(x)]^{-1}|| \le c(n)||F'(x)||^{n-1},$$

and so

(3) $|\det F'(x)|\ ||[F'(x)]^{-1}||^n \le c(n)||F'(x)||^{n-1}||[F'(x)]^{-1}||^{n-1}$

Using hypotheses (i) and (ii), we have

$$||[F'(x)]^{-1}||^n \le \frac{c(n)\ M^{n-1}}{\alpha}, \text{ and finally}$$

$$||[F'(x)]^{-1}|| \le \left[\frac{c(n)\ M^{n-1}}{\alpha}\right]^{1/n} \text{ for all } x.$$

By Corollary 2.3.10, F is a diffeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$.

Corollary 2.3.16 is true under the weaker hypothesis det $F'(x) \neq 0$, provided $N \geq 3$ [31].

In general, the hypotheses of Theorems 2.3.9 and 2.3.13 cannot be entirely omitted as the following examples show:

<u>Example 3.1</u> $F(x)=F(x_1,\ldots,x_n)= (\tan^{-1}x_1, x_2(1+x_1^2)^2, x_3,\ldots,x_n)$ is a C' map of $\mathbb{R}^n$ into $\mathbb{R}^n$ and:

$$F'(x) = F'(x_1,\ldots,x_n) = \left(\begin{array}{cc|c} \frac{1}{1+x_1^2} & 4x_1x_2(1+x_1^2) & \\ 0 & (1+x_1^2)^2 & 0 \\ \hline & & \\ & 0 & I_{n-2} \end{array}\right)$$

Since $\det F'(x) = x_1^2+1 > 1$, F satisfies the conditions of the inverse function theorem. By looking at the characteristic polynomial of $F'(x)$, we see that $\lambda = 1/(1+x_1^2)$ is an eigenvalue, and so $1/\lambda = 1+x_1^2$ is an eigenvalue of $[F'(x)]^{-1}$. Therefore $\|[F'(x)]^{-1}\| \geq 1+x_1^2$, and so we have that $\inf_{\|x\|\leq t} 1/\|[F'(x)]^{-1}\| \leq 1/(1+t^2)$. Since $\int_0^\infty dt/(1+t^2) < \infty$, F does not satisfy the hypotheses of Theorems 2.3.9 and 2.3.13 and we observe that F is one-one, but not onto.

Example 3.2 A second example of a univalent map which isn't onto is the famous example of Fatou and Bieberbach [3, p. 45]. This is an example of an analytic map F of $\mathbb{C}^2$ into itself whose Jacobian (of the map considered as mapping $\mathbb{R}^4 \to \mathbb{R}^4$) is identically equal to 1 and F is univalent, however the range of F omits a full open neighborhood of a point in $\mathbb{C}^2$.

Example 3.3 $F(x_1,x_2) = e^{x_1}(\cos x_2 , \sin x_2)$.

F is a C' map of $\mathbb{R}^2 \to \mathbb{R}^2$ which is neither one-one nor onto (it omits 0). Now:

$$F'(x_1,x_2) = e^{x_1}\begin{pmatrix} \sin x_2 & \sin x_2 \\ -\sin x_2 & \cos x_2 \end{pmatrix} .$$

We observe that $||F'(x_1,x_2)|| = e^{x_1}$, thus

$\frac{1}{||[F'(x_1,x_2)]^{-1}||} \leq e^{x_1}$. So $\inf_{||x|| \leq t} \frac{1}{||[F'(x_1,x_2)]^{-1}||} \leq e^{-t}$

and $\int_0^{\infty} e^{-t}\, dt < \infty$.

Examples 3.1 and 3.2 also show that the condition $|\det F'(x)| \geq \alpha > 0$ is not in itself sufficient to insure that F is a global homeomorphism, for if $F(x_1,x_2) = (\tan^{-1} x_1 , x_2(1+x_1^2)^2, x_3,\ldots,x_n)$ then $\det F'(x_1,x_2) > 1$, yet F is not a homemorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$. However we shall show [Corollary 3.3.5] that if F is a gradient map of $\mathbb{R}^2$ into itself, then $|\det F'(x)| > \alpha > 0$ insures that F is globally one-one.

The following theorem can be used when the integral condition of Theorem 2.3.9 fails.

<u>Theorem 2.3.17</u> Let $F: X \rightarrow Y$ be continuously differentiable and also $F'(x)$ is invertible for all $x \in X$. Suppose that

(i) $||F(x)|| \rightarrow \infty$ as $||x|| \rightarrow \infty$

(ii) $||[F'(x)]^{-1}|| \leq M(||x||)$, where $M(t)$ is a continuous positive function of $\mathbb{R} \rightarrow \mathbb{R}$.

Then F is a diffeomorphism of X onto Y.

<u>Proof</u>:

By Theorem 2.3.1, it suffices to show that F satisfies condition (C). Using the argument of the beginning of

Theorem 2.3.9, it suffices to show that condition (C) is verified for C' paths. So suppose $P(t)$ is defined on $0 \leq t < b$, is continuously differentiable and satisfies

(1) $F(P(t)) = L(t) \quad (= (1-t)y_1 + ty_2)$ for $0 \leq t < b$.

By (i), $S = F^{-1}(L(t))_{0 \leq t \leq 1}$ is a bounded set, and so $\{P(t)\}_{0 \leq t < b}$ is bounded. Since $P(t)$ is continuously differentiable, we can apply the chain rule to 1 and thus

$$F'(P(t))\, P'(t) = y_2 - y_1 = z \,, \quad 0 \leq t < b.$$

Therefore

$$P'(t) = [F'(P(t))]^{-1} z \,, \qquad 0 \leq t < b.$$

By (ii), $\exists$ C so that $\|[F'(P(t))]^{-1}\| \leq C$ for $0 \leq t < b$.

Let $t_i \to b$.

$$P(t_M) - P(t_N) = \int_{t_N}^{t_M} P'(t)\, dt = \int_{t_N}^{t_M} [F'(P(t))]^{-1} z\, dt \,.$$

So

$$\|P(t_M)-P(t_N)\| \leq \int_{t_N}^{t_M} \|[F'(P(t))]^{-1}\|\, \|z\|\, dt \leq C\|z\|\, |t_M-t_N|$$

Thus $\{P(t_i)\}$ is a Cauchy sequence, and so condition (C) is verified, as was to be shown.

Chapter III

Global Univalence and Surjectivity.

Section 1. Introduction.

If we inspect the methods that were employed in Chapter II to determine when a mapping between two Banach spaces was a homeomorphism, we see that the properties of global univalence and surjectivity were related in an intimate way, i.e., once we showed that the mapping $F: X \to Y$ was a covering space map of the simply connected set Y, then F was automatically univalent by Theorem 2.2.1. In the general case of a mapping F of some domain D into a Banach space we usually have little information about F(D). Thus the assumption that F(D) is simply connected is not viable and so Theorem 2.2.1 is not applicable. In this chapter we investigate the questions of global univalence and surjectivity independently and without any assumptions on the range of the mapping at hand.

In Sections 2 and 3 we concern ourselves with the problem of global univalence. We assume, in Section 1, that the mapping is a local homeomorphism and so the question of global univalence is equivalent to the problem of determining when a local homeomorphism is a global homeomorphism between its domain and range. We then show that the method of covering spaces is applicable to this problem and deduce such theorems as Theorem 3.2.3: If $D \subseteq X$ is open and starshaped about some point a and if $F: D \to Y$ is a local homeomorphism, then the following are necessary and sufficient for F to be a homeomorphism of D onto F(D): (i) whenever $M \subset D$ is open and starshaped about a, then $(F|M)^{-1}$ is locally finite [Def. 3.2.1] and

(ii) F is one-one on every line emanating from a. We also prove analogs of the Banach-Mazur Theorem 2.3.2 and of Theorem 2.3.9 and Corollary 2.3.11.

As mentioned, Section 3 is also concerned with the question of global univalence, however we start with a different point of view from Section 2. More specifically, in Section 3 we do not a priori assume that our maps are local homeomorphisms. Thus the methods of Section 2 cannot be applied and so we must employ specialized techniques to handle the cases which we deal with. We show, for example, in Theorem 3.3.3 that if F is monotone, continuously differentiable and $\mathrm{Ker}[F'(x)]^* = 0$, then F is globally univalent.

In Section 4 we investigate surjectivity using the topological method of degree of a mapping. We prove that if a map $F: X \to Y$ is proper and if there is some point p so that $d(F,p,B) \neq 0$ whenever $F^{-1}(p) \subset B$, then F maps X onto Y. From this we deduce as Corollary 3.4.4 a result of Nijenhuis and Richardson [22] which says that if $F: \mathbb{R}^N \to \mathbb{R}^N$ is continuously differentiable, $\det F'(x) \geq 0$ and proper, then F maps $\mathbb{R}^N$ onto $\mathbb{R}^N$.

The final question that arises in this chapter is that of the preservation of univalence and surjectivity in the limit. We give necessary and sufficient conditions for the preservation of univalence (Theorem 3.5.2) and then generalizing the Hurwicz Theorem for analytic maps, we show in Theorem 3.5.3 that if F_n converges normally to F and if $\det F'(x) \geq 0$, then F is univalent provided it has isolated zeroes and the F_n's are univalent. From this we deduce as

Corollary 3.5.5 that if F_n: $\mathbb{C}^n \to \mathbb{C}^n$ are analytic and converge normally to F, then F is either degenerate in the sense that the Jacobian of F is identically zero, or F is univalent. We also show that if F_n converge to F normally then F is surjective provided that (i) F_n is surjective and lifts lines and (ii) F^{-1} is locally bounded [Def. 3.5.2]. We then apply this theorem to the class of monotone maps and give a proof of a result of Browder's [5, p.16] which states that a monotone map F is surjective if F^{-1} is locally bounded.

Section 2. Global Univalence via Covering Spaces.

Suppose X and Y are Banach spaces and $D \subseteq X$ is an open and connected set. If F: $D \to Y$ is a continuous map, we ask what additional assumptions must be imposed upon F to guarantee that it is univalent on D. The natural starting point for this investigation is the search for necessary conditions.

Definition 3.2.1. Let F: $D \subseteq X \to Y$ be continuous. We say F^{-1} is locally finite if and only if $F^{-1}(y)$ is a finite set and every $y \in Y$ has an (relative to F(p)) open neighborhood N about it such that number $F^{-1}(x)$ = number $F^{-1}(y)$ for every $x \in N$.

So if F: $D \to Y$ is univalent, the first necessary condition that arises is that F^{-1} is locally finite, for number $F^{-1}(y) = 1$ $\forall y \in F(D)$, and 0 otherwise.

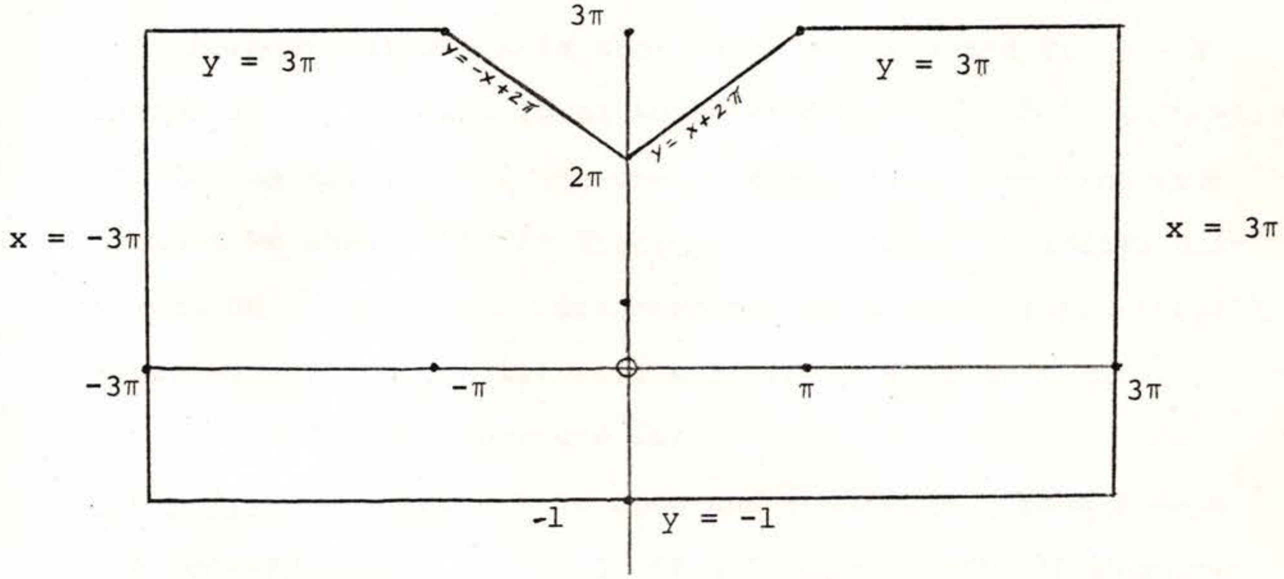
Secondly, if D is starshaped about a point a, then whenever F is univalent on D, F must certainly be univalent on every line in D whose initial point is a, i.e., if L(t) is such a line, then g(t) = F(L(t)) is one-one for every t in the interval [0,1].

Finally, suppose F is a compact perturbation of the identity. If F is univalent on D then the invariance of domain theorem tells us that F must be an open mapping, thus F is a local homeomorphism since every open set O of D is mapped homeomorphically onto F(O), which is also open. In view of this and of example 2.1 which follows, we shall (in this section) always a priori assume that F is a local homeomorphism. Thus, if F is a local homeomorphism on D, the question of univalence is equivalent to the problem of determining when F is a homeomorphism of D onto F(D). It is this last reformulation of our original problem which we investigate.

Example 2.1. Let X be a complex Banach space, F: $D \subseteq X \to X$ be analytic and of the form I + C. If $F'(z_0)$ is not invertible for some $z_0 \in D$, then there is an open neighborhood of z_0 on which F is not one-one. So in particular F is not univalent on D.

The following example shows that the assumptions F: $D \to Y$ a local homeomorphism and F one-one on every line emanating from a (where D is starshaped about a) are not sufficient for F to be a homeomorphism of D onto F(D).

Example 2.2. Let $F(x,y) = e^x(\cos y, \sin y)$ be a map of $D \to \mathbb{R}^2$ where D is defined as follows:



(D is the interior region)

D is open and starshaped about 0. One checks that F is a local homeomorphism on D, and F is one-one on every line through 0. However F is not one-one in D since, for example, $F(x,y) = e^{2\pi}(1,0)$ has two solutions in D.

We observe that F^{-1} is not locally finite, for if $-1 < y < 0$ then $F(0,y) = F(0,2\pi+y)$ and so number $F^{-1}(F(0,y)) = 2$. If $0 \leq y < 1$, then number $F^{-1}(F(0,y)) = 1$.

The next example is a local homeomorphism for which F^{-1} is locally finite on an open set D, but again is not a homeomorphism.

<u>Example 2.3.</u> Let $F(x_1,x_2) = (x_1^2-x_2^2,\ 2x_1x_2)$ be a map defined on $D' = B - \{0\}$ where B = open unit ball. Although

F^{-1} is locally finite and F is a local homeomorphism on D', F fails to be one-one since for example $F(x_1,x_2) = F(-x_1,-x_2)$. Note also that F is everywhere two to one on D'.

However, if $D \subseteq X$ is starshaped about a and $F: D \to Y$ satisfies (i) F is a local homeomorphism (ii) F^{-1} is locally finite and (iii) F is one-one on every line emanating from a, then we shall show in Theorem 3.2.3 that F is indeed univalent on D, in fact a homeomorphism of D onto F(D). Actually a weaker version of (ii) will suffice as will be seen.

Our point of departure is:

Lemma 3.2.1. Let $D \subseteq X$ be open and connected. Then F maps D homeomorphically onto F(D) if and only if (i) (D,F) covers F(D) and (ii) for some $y \in F(D)$, $F^{-1}(y)$ contains exactly one point.

Proof: If F maps D homeomorphically onto F(D) then (D,F) covers F(D) and $F^{-1}(y)$ contains exactly one point for every $y \in F(D)$.

Conversely, if (i) and (ii) are satisfied, then Theorem 1.4.1 (iii) is applicable and so for every $y \in F(D)$ $F^{-1}(y)$ contains exactly one point. Thus F maps D onto F(D) in a one-one way. Also a covering map is a local homeomorphism and so in particular is an open map. Hence F is a homeomorphism of D onto F(D).

Thus Lemma 3.2.1 shows us the direction we must follow is that of covering spaces. So we ask when a local homeomorphism $F: D \to Y$ covers F(D). Here Theorem 2.2.2 is applicable and says that necessary and sufficient conditions for (D,F) to cover F(D) is that (i) F is a local homeomorphism and (ii) F lifts any line in F(D) [recall Def. 2.2.1]. Thus we are lead, as in Section 3 of Chapter II, to introduce the following weak version of condition (C):

($\overline{C}$) whenever $P(t)$, $0 \leq t < b$ is a path satisfying $F(P(t)) = L(t)$ for $0 \leq t < b$ (where $L(t)$ is <u>any line contained in F(D))</u>, then there is a sequence $t_i \to b$ as $i \to \infty$ so that $\lim_{i\to\infty} P(t_i)$ exists and is in D.

The essential difference between (C) and ($\overline{C}$) is that for (C) we require that the hypothesis is satisfied for any line in Y, whereas in ($\overline{C}$) it is only required that the hypothesis be satisfied for any line in F(D). Thus, as the development of Chapter II indicates, condition (C) guarantees that $F(D) = Y$, which condition ($\overline{C}$) does not guarantee, as is illustrated by the map $F(x) = e^x$: $\mathbb{R}^1 \to \mathbb{R}^1$. F satisfies condition ($\overline{C}$), but not condition (C) since if $L(t) = 1-t$, then $P(t) = \ln(1-t)$ and so $P(t) \to -\infty$ as $t \to 1$.

<u>Theorem 3.2.2.</u> Let $F\colon D \subseteq X \to Y$ be a local homeomorphism. Then condition ($\overline{C}$) is necessary and sufficient for (D,F) to cover F(D).

<u>Proof:</u> If (D,F) covers F(D) and L(t) is any line in F(D), then if x is any point in $F^{-1}(L(0))$. Theorem 1.4.1 (i) says that there is a path (in D) $P(t)$, $0 \leq t \leq 1$, with $P(0) = x$ and $F(P(t) = L(t))$ for $0 \leq t \leq 1$. Thus $\lim_{t\to b} P(t)$ always exists and is in D for any b in [0,1].

Conversely, we can apply the argument of Theorem 2.3.1

to obtain the desired conclusion.

Theorem 3.2.3. Suppose $D \subseteq X$ is an open set which is starshaped about some point a. Let $F: D \to Y$ be a local homeomorphism. Then the following conditions are necessary and sufficient for F to be global homeomorphism of D onto F(D):

1) Whenever $M \subseteq D$ is open, bounded and starshaped about a, then $(F|_M)^{-1}$ is locally finite.

2) F is one-one on every line in D which emanates from a.

Proof: If F is a homeomorphism of D onto F(D), then number $(F^{-1}(y)) = 1$ for every $y \in F(D)$, and so for any set M described in (i), number $(F|_M(y))^{-1} = 1$, and thus is locally finite. Also as F is univalent, F is one-one on every line in D.

For the sufficiency it is enough to show that F is univalent on D, for then F will map D one-one onto F(D) and F^{-1} will be continuous since F is a local homeomorphism.

So suppose $F(x_1) = F(x_2)$. Then we can find a bounded, open set M containing x_1 and x_2 and so that M is starshaped about a. Since $F|_M$ is a local homeomorphism and by (i) $(F|_M)^{-1}$ is locally finite and (ii) $F|_M$ is one-one on every line emanating from a, it suffices to show that $F|_M$ maps M homeomorphically onto F(M). Applying Lemma 3.2.1, we first show that with $\overline{F} = F|_M$, $(M,\overline{F})$ covers F(M) and there is a point $y \in F(M)$ so that $\overline{F}^{-1}(y)$ contains exactly one point.

A) $(M,\overline{F})$ covers F(M).

We shall show that $\overline{F}$ satisfies condition $(\overline{C})$, and then invoke Theorem 3.2.2 to obtain the desired conclusion. So suppose L(t) is a line in F(M) and P(t) is a path defined for $0 \leq t < b$

and satisfying $\overline{F}(P(t)) = L(t)$ for $0 \leq t < b$. By hypothesis there is an open neighborhood U of L(b) so that number $\overline{F}^{-1}(x) =$ number $\overline{F}^{-1}(L(b))\ \forall x \in U$. For each $x_i\ (i=1,\ldots N) \in \overline{F}^{-1}(L(b))$ let O_i be disjoint open neighborhoods of x_i so that $\overline{F}|_{O_i}$ is a local homeomorphism. Let $S = \bigcap_{i=1}^{N} [F(O_i) \cap U]$, and $A_i = O_i \cap \overline{F}^{-1}(S)$. One can check that $\overline{F}(A_i) = S$ and A_i and S are open and $\overline{F}|_{A_i}$ is a homeomorphism. Choose $\overline{t} < b$ so that $L(\overline{t}) \in S$. In particular $L(\overline{t}) \in U$, hence each point of $\overline{F}^{-1}(L(\overline{t}))$ lies in one and only one A_i. Thus $P(\overline{t}) \in A_j$ for some j. Since $\overline{F}|_{A_j}$ is a homeomorphism onto S, there is a path Q(t) defined on $\overline{t} \leq t \leq b$ so that $Q(\overline{t}) = P(\overline{t})$ and $F(Q(t)) = L(t)$ for $\overline{t} \leq t \leq b$. However from Theorem 1.4.2 we conclude that $P(t) = Q(t)$ for $\overline{t} \leq t < b$, and so if $t_i \to b$ is any sequence, then $P(t_i) \to Q(b)$. Thus condition $(\overline{C})$ is verified.

B) There is some point $y \in F(M)$ so that number $\overline{F}^{-1}(y) = 1$. In fact $y = F(a)$, for suppose $F(x_1) = F(a)$. If we let $L(t) = (1-t)a + tx_1$, then (ii) implies that $x_1 = a$.

Related to Theorems 2.3.2 and 2.3.5 we have:

Theorem 3.2.4. Let $D \subseteq X$ be open and connected, $F: D \to Y$ a local homeomorphism. Then the following are necessary and sufficient for F to be a homeomorphism of D onto F(D):

(i) a) Either F is a relatively proper map or

b) F is a relatively closed map, and (ii) for some point $y \in F(D)$, $F^{-1}(y)$ consists of exactly one point.

Proof: If F is a homeomorphism of D onto F(D), then (i) a, b and (ii) follow immediately.

Conversely, we show first that either (i) a or (i) b implies that (D,F) covers F(D). Applying Theorem 3.2.2 it suffices to show that F satisfies condition $(\overline{C})$.

However, the arguments of Theorems 2.3.2 and 2.3.5 are applicable and so we conclude that (D,F) covers F(D). That F is a homeomorphism now follows from (ii) and Lemma 3.2.1.

As we have seen, Lemma 3.2.1 provides us with a viewpoint for attacking the problem mentioned at the beginning of this section. Its use depends on showing two things: 1) that (D,F) covers F(D) and 2) that for some $y \in F(D)$, $F^{-1}(y)$ contains exactly one point. For the former we know that it suffices to show that F satisfies condition $(\overline{C})$. To verify 2, one condition that we have already introduced is as follows: suppose D is starshaped about a, then if F is one-one on every line emanating from a, 2 is satisfied. In fact $F^{-1}(F(a))=a$ for if $F(x) = F(a)$ then, since F is one-one on the line $L(t) = (1-t)a+tx$, we conclude that $x = a$.

If D is not starshaped the hypothesis on F given above does not make sense and so other methods of verifying 2 must be looked for. One of these is topological in nature and involves the use of the degree of a map (Sect. 3B, Chapter I).

We shall assume once again that $F\colon\ D \to F(D)$ is a local homeomorphism and that D is an open and connected set. We also assume that D is bounded. This involves no loss of generality for the question of univalence because F univalent on D is equivalent to F being univalent on every bounded open subset of D (if $F(x)=F(y)$ in D we can always find a bounded open subset of D which contains x and y).

The usefulness of the degree of a map in our situation will be apparent from the following theorem of Rothe [24].

We first note that if $F = I+K$, K compact is a local homeomorphism then number $F^{-1}(p)$ is always finite provided $F(x) \neq p$ on ∂D. For if $x_n \in F^{-1}(p)$, $n=1,2,\dots$, then $x_n + K(x_n) = p$, and since D is bounded, $K(x_n)$ has a convergent subsequence $K(x_{n_j}) \to Y$. Thus x_{n_j} converges to some x and since F is a local homeomorphism on D, $x \in \partial D$ which is a contradiction since $F(x) = p$.

Let O_x denote an open ball of $x \in D$ such that $\overline{O_x}$ is contained in an open set V_x of x for which $F|_{V_x}$ is a homeomorphism.

__Theorem 3.2.5.__ Let $D \subseteq X$ be open, bounded and connected, $F\colon\ D \to X$ a local homeomorphism. If $F = I + K$ and $d(F,p,D) = \pm 1$ then $F^{-1}(p)$ contains exactly one point.

__Proof:__ Since $F^{-1}(p)$ is a finite set, we have

(1) $d(F,p,D) = \sum_{x \in F^{-1}(p)} d(F,p,O_x)$, where O_x is as above and disjoint. Thus it suffices to show that $d(F,F(P(t)),O_{P(t)}) =$ constant whenever $P(t)$ is any path in D. For if this is so, then $d(F,p,O_x) = d(F,p,O_y) = C$ for x and $y \in F^{-1}(p)$. So by (1), $\pm 1 = d(F,p,D) = C$ number $F^{-1}(p)$ and this implies that number $F^{-1}(p) = 1$.

So suppose $P(t)$, $0 \leq t \leq 1$, is any path in D. For fixed $\bar{t}$ choose $O_{P(\bar{t})}$ as above. Let t_1 be any number such that $P(t_1) \in O_{P(\bar{t})}$. By definition of $O_{P(\bar{t})}$, $d(F,F(P(t)),O_{P(\bar{t})})$ is defined and is a continuous function of $t_1 \leq t \leq \bar{t}$ (we assume, without loss of generality that $t_1 < \bar{t}$), and so it is constant. In particular

(2) $d(F,F(P(t_1),O_{P(\bar{t})}) = d(F,F(P(\bar{t})),O_{P(\bar{t})})$. Let $W_{P(t_1)}$ be an open ball about $P(t_1)$ so that $\overline{W_{P(t_1)}} \subset O_{P(t_1)} \cap O_{P(\bar{t})}$. Then

(3) $d(F,F(P(t_1),O_{P(t_1)}) = d(F,F(P(t_1),W_{P(t_1)}) =$ $d(F,F(P(t_1)),O_{P(\bar{t})})$. So 2 and 3 yield $d(F,F(P(t_1)),O_{P(t_1)}) =$ $d(F,F(P(\bar{t})),O_{P(\bar{t})})$ $\forall\, t_1$ such that $(t_1-\bar{t})$ is small enough to insure that $P(t_1) \in O_{P(\bar{t})}$. Thus $d(F,F(P(t)),O_{P(t)})$ is a continuous function of $t \in [0,1]$ into the integers, and so it is constant.

<u>Corollary 3.2.6.</u> Suppose $D \subset X$ is open, bounded and connected and $F\colon \bar{D} \to X$ is continuous and is a local homeomorphism on D. If (i) $F = I + K$, (ii) $F(D) \cap F(\partial D) = \phi$ (iii) $d(F,p,D) = \pm 1$ for some p, then F is a homeomorphism of D onto F(D).

<u>Proof:</u> By virtue of the Invariance of Domain Theorem it suffices to show that F is univalent on D. Now (ii) implies that $F(D) \subset Y - F(\partial D)$. Thus $d(F,q,D)$ is defined for every $q \in F(D)$ and is equal to some constant. However from (iii) we conclude that $d(F,q,D) = \pm 1\ \forall\, q \in F(D)$ and so we now apply Theorem 3.2.5 to obtain the desired conclusion.

We now apply our methods to derive "local" versions of Theorem 2.3.9 and Corollary 2.3.11, i.e.,:

<u>Theorem 3.2.7 [17].</u> Let $B = \{x \mid \|x-a\| < r\}$, $F\colon B \to Y \in C'(B)$ and also $F'(X)$ is an invertible linear map $\forall\, x \in B$. Then there is a domain $B_1 \subseteq B$ so that $a \in B_1$ and F maps B_1 one-one onto $D = \{y \mid \|y-F(a)\| < \int_0^r \inf_{\|x-a\| \le t} \frac{1}{\|[F'(x)]^{-1}\|}\, dt\}$. (Where this integral is defined as $\lim_{\lambda \to r} \int_0^\lambda \inf_{\|x-a\| \le t} \frac{1}{\|[F'(x)]^{-1}\|}\, dt$, $\lambda < r$).

<u>Proof:</u> Without loss of generality, let $a = 0$, $F(a) = 0$. We proceed as follows: let $L(t)=t\xi$, $0 \le t \le C < 1$, $\xi \in \partial D$, describe any segment of any radius of D. We shall show that F lifts any such L(t) to a path $P(t) \in B$, where $P(0) = 0$. Then we let

$B_1 = \{x | x=P(\bar{t}),\ 0 \le \bar{t} < 1, P(t)$ as above$\}$. Then using the argument of Theorem 2.2.2, we conclude that B_1 satisfies the conditions stated in our theorem. So suppose $P(t)$ is defined for $0 \le t \le b \le C < 1$, $P(0) = 0$ and $F(P(t)) = L(t)$, a radius of D. We shall show that condition $(\bar{C})$ is verified for such paths, and hence F lifts $L(t)$. By the usual construction of $P(t)$ (see Theorem 2.3.1) we may assume that P is continuously differentiable on $0 \le t < b$. Thus $P'(t) = [F'(P(t)]^{-1}\xi$ on $0 \le t < b$, where $\xi = L'(t)$. Also, we may suppose that $r = \lambda = \sup\{t,\ 0 \le t < r | h(t)>0\}$, where $h(t) = \inf\limits_{\|x\| \le t} \dfrac{1}{\|[F'(x)]^{-1}\|}$.
Otherwise we apply all our arguments to the set $B' = \{x | \|x\| < \lambda\}$ (D is the same since $\int_0^\lambda h(t)dt = \int_0^r h(t)dt$). Hence $\dfrac{1}{\|[F'(x)]^{-1}\|} \ge \alpha > 0$ on B. Let $t'', t' < b$, then with $P'(t) = [F'(P(t))]^{-1}\xi$ we have:

$$|t''-t'|\,\|\xi\| \ge \int_{t'}^{t''} \frac{\|P'(t)\|}{\|[F'(P(t))]^{-1}\|}\,dt \ge \alpha\|P(t'')-P(t')\|$$

Hence $\lim\limits_{t \to b} P(t)$ exists and call it $P(b)$. We now must show that $\|P(b)\| < r$. Suppose $\|P(b)\| = r$. Then from the inequality (1) of Theorem 2.3.7 (where $B(x) = \dfrac{1}{\|[F'(x)]^{-1}\|}$) we have since $\xi \in \partial D$

$$b\|\xi\| \ge \int_0^b \frac{\|P'(t)\|}{\|[F'(x)^{-1}]\|}\,dt \ge \int_0^{\|P(b)\|} h(t)dt = \int_0^r h(t)dt = \|\xi\|$$

which is a contradiction, since $b < 1$. Hence condition $(\bar{C})$ is verified, and the result follows from our introductory arguments.

The next theorem gives us more information on the size of B_1 of Theorem 3.2.7.

Theorem 3.2.8. Let $B = \{x \mid \|x-a\| < r\}$, $F: B \to Y \in C'(B)$ and $F'(x)$ is invertible $\forall x \in B$. Let $M = \sup_{\|x-a\| < r} \|F'(x)\|$ and $\overline{M} = \sup_{\|x-a\| < r} \|[F'(x)]^{-1}\|$. Then there is a domain $B_1 \subseteq B$ so that $a \in B$ and F maps B_1 one-one onto $D = \{y \mid \|F(a)-y\| < \frac{r}{\overline{M}}\}$. Furthermore, $B_1 \supseteq B(a, \frac{r}{M\overline{M}}) = \{x \mid \|x-a\| < \frac{r}{M\overline{M}}\}$.

Proof: Since $\int_0^r \inf_{\|x-a\| \le t} \frac{1}{\|[F'(x)]^{-1}\|} dt \ge \frac{r}{\overline{M}}$, we conclude from Theorem 3.2.7 the existence of a B_1 which is mapped in a one-one way onto D by F. It remains to show that $B_1 \ge B(a, \frac{r}{M\overline{M}})$. Let $L(t) = (1-t)a+tx$, $\|x-a\| \le \frac{r}{M\overline{M}}$ be any radius of $B(a, \frac{r}{M\overline{M}})$.

We show that $P(t) = F(L(t))$ is in D. This is enough, because as $P(0) = F(a)$, there is a path $\tilde{L}(t)$ in B_1 so that $\tilde{L}(0) = a$ and $F(\tilde{L}(t)) = P(t)$. From Theorem 1.4.2, $\tilde{L}(t) = L(t)$. Thus $B_1 \supseteq B(a, \frac{r}{M\overline{M}})$. So with $P(t) = F(L(t))$, then:

$$\|P(t)-F(a)\| \le \int_0^t \|F'(L(t))L'(t)\| dt$$

$$\le t\|F'(L(t))\| \; \|L'(t)\| \le t \frac{Mr}{M\overline{M}} \le \frac{r}{\overline{M}} .$$

We remark that for $X = Y = R^n$, Theorem 3.2.8 was originally due to Wazewski [30].

We remark that the proofs of Theorems 2.2.2 and 3.2.2 show us that these theorems remain valid when D is pathwise and locally pathwise connected and F(D) is locally convex. Thus Theorem 3.2.4 is also valid with the above modifications. In view of this we can now apply Theorem 3.2.4 to prove two theorems of Miranda's [31c, p. 142] which have interesting applications to nonlinear partial differential equations [see 31d, pp. 461-465].

<u>Theorem 3.2.9</u> Let F be a map between two Banach spaces X and Y. Let $Z \subseteq Y$ be closed, connected and locally convex. Suppose that:

(i) F is a local homeomorphism at each point of $F^{-1}(Z)$.

(ii) The preimage of a compact set in Z is a compact set in Y.

(iii) There is a point $\bar{z} \in Z$ which is the image of exactly one point.

Then F is a univalent map on $F^{-1}(Z)$.

<u>Proof</u>:

As $F^{-1}(Z)$ is closed, (ii) implies that F is a relatively proper map between $F^{-1}(Z)$ and Z. Thus if we show that $F^{-1}(Z)$ is pathwise and locally pathwise connected we can then apply Theorem 3.2.4 which yields the desired conclusion. Since $F|_{F^{-1}(Z)}$ is a local homeomorphism, this and the local convexity of Z gives us the local pathwise connectedness of $F^{-1}(Z)$. It now suffices to show that $F^{-1}(Z)$ is connected. Let $A \subseteq F^{-1}(Z)$ be a component. $F^{-1}(Z)$ locally connected implies that A is both open and closed (relative to $F^{-1}(Z)$). However (i) and (ii) gives us that F is both an open and closed map and so F(A) is an open and closed subset of Z. Thus the connectivity of Z implies that F(A) = Z, and in particular, that every component A contains a point of $F^{-1}(\bar{z})$. However $F^{-1}(\bar{z})$ contains exactly one point and so the disjointness of the components implies that there is only one component. Thus Z is connected.

<u>Theorem 3.2.10</u> Let X and Y be Banach spaces, $Z \subseteq Y$ closed, connected and locally convex. Suppose F: $X\times[0,1]$ is a map with the following properties:

(i) Each point $(\bar{x},\bar{t}) \in F^{-1}(Z)$ has an open neighborhood about it so that $F_{t_1} = F(x,t_1)$ is a homeomorphism on it whenever $|\bar{t}-t_1|$ is sufficiently small.

(ii) The preimage of a compact set in Z is a compact set of $X\times[0,1]$.

(iii) For some particular t_0, there is a point of Z which is the image (under F_{t_0}) of exactly one point.

Then F_t is a univalent map on $F_t^{-1}(Z)$ for each $t \in [0,1]$.

<u>Proof</u>:

Let $S_{t_0} = \{(x,t_0) \mid F(x,t_0) \in Z\}$. The hypotheses imply that the map F_{t_0} and the set S_{t_0} satisfy the conditions of Theorem 3.2.9. Thus F_{t_0} is univalent on S_{t_0}.

Let $\bar{w} \in Z$ and let $G = \{(x,t) \mid F(x,t) = \bar{w}\}$. We define the map P: $G \to [0,1]$ by $P(x,t) = t$. We shall show that P satisfies the hypotheses of Theorem 3.2.4. First, P is a local homeomorphism on G for if $(\bar{x},\bar{t}) \in G$, then by (i) we can find an open neighborhood (of $(\bar{x},\bar{t})$), $W_{(\bar{x},\bar{t})} = U\times(\bar{t}-\varepsilon,\bar{t}+\varepsilon)$ so that F_t is a homeomorphism on it for each fixed $t \in (t-\varepsilon,t+\varepsilon)$. Hence $W_{(\bar{x},\bar{t})} \cap G$ is an open neighborhood of $(\bar{x},\bar{t})$ in G on which P is a homeomorphism. Also P is a proper map for if $C \subset [0,1]$ is compact, then it is, in particular, closed. Thus $P^{-1}(C)$ is closed in G and, by (ii), it is compact since G is compact. Since F_{t_0} is univalent on S_{t_0}, there is exactly one point of G whose image under P is the point t_0. In order to apply Theorem 3.2.4, we must show that G is pathwise and locally pathwise connected. This, however, follows by the same arguments used in Theorem 3.2.9. Thus by Theorem 3.2.4, P is univalent on G and this in turn implies the desired conclusion.

Section 3. Global Univalence Once Again.

Throughout Section 3 we concern ourselves once more with the problem of global univalence and global homeomorphisms. However, we add a slight twist, whereas in Chapter II and in Section 2 of this chapter we always assumed that our maps were local homeomorphisms, in this section we shall try to remove this assumption. In other words if $D \subseteq X$ is open and connected and $F\colon D \to Y$ is continuous (or continuously differentiable), when is F globally univalent or when is it a homeomorphism of X onto Y? Since we do not a priori assume that F is a local homeomorphism, the techniques of Chapter II and of Section 2 do not, in general, work. Thus we must use specialized methods which suit the problem at hand.

Theorem 3.3.1. Let $D \subset X$ be open and convex. Let $F\colon D \to X^*$ be a gradient map, where $F = \nabla f$. If f is convex, then F is globally univalent on D if and only if f is strictly convex.

Proof: If f is strictly convex, then by Theorem 1.3.5 $(F(x)-F(y),x-y) > 0 \; \forall\, x,y \in D$. Thus F is one-one on D. Conversely, suppose f is not strictly convex. Then $\exists\, x_1 \neq x_2$ and

(1) $\bar{x} = (1-\bar{t})x_1+\bar{t}x_2$, $0 \leq \bar{t} < 1$ so that $f(\bar{x})=(1-\bar{t})f(x_1)+\bar{t}f(x_2)$.

It suffices to show that $f(\bar{x}) = f(x_i)+(F(\bar{x}),\bar{x}-x_i)$, $i=1,2$. For suppose this is so, then for any $z \in D$

$$\begin{aligned} f(z) - f(x_1) &= f(z)-f(\bar{x})+f(\bar{x})-f(x_1) \\ &\geq (F(\bar{x}),z-\bar{x}) + (F(\bar{x}),\bar{x}-x_1) \quad \text{(by convexity)} \\ &= (F(\bar{x}),z-x_1) \;. \end{aligned}$$

Let $y \in X$. Then for λ small enough, $z = x_1+\lambda y \in D$. Thus

$f(x_1+\lambda y)-f(x_1) \geq (F(\bar{x}),\lambda y)$. Hence $\frac{f(x_1+\lambda y)-f(x_1)}{\lambda} \geq (F(\bar{x}),y)$. Letting $\lambda \to 0$ we get $(F(x_1),y) \geq (F(\bar{x}),y)$ $\forall y \in X$. Thus $F(x_1) = F(\bar{x})$, a contradiction since F is one-one. We now show that $f(\bar{x}) = f(x_1)+(F(\bar{x}),\bar{x}-x_1)$, $i = 1,2$. Let $L(t) = (1-t)x_1+tx_2$, hence $\bar{x} = L(\bar{t})$. Let $g(t)=f(\bar{x})-f(L(t)) -(F(\bar{x}),\bar{x}-L(t)) \leq 0$ by convexity from (4) of Theorem 1.3.5. However, by (1), $g(0) = \bar{t}[-(f(x_1)-f(x_2))+(F(\bar{x}),x_1-x_2)]$ and $g(1) = (1-\bar{t})[f(x_1)-f(x_2)-(F(\bar{x}),x_1-x_2)]$. Thus $g(0)$ and $g(1)$ have opposite signs. However $g(t) \leq 0$ and $0 < \bar{t} < 1$ imply $g(0) = g(1) = 0$, which is what was to be shown.

The following corollary is related to Theorem 2.3.2.

Corollary 3.3.2. Let H be a Hilbert space, $F = H \to H$ be a gradient map where $F = \nabla f$. If f is convex then necessary and sufficient conditions for F to be a homeomorphism of H onto H is that (i) f be strictly convex and (ii) F is proper.

Proof: The necessity follows from Theorem 3.3.1 and the fact that a homeomorphism is a proper map. The sufficiency follows from Theorem 3.3.1, Corollary 3.5.9 (Section 5) and the continuity of the inverse comes from the fact that a proper map is a closed map.

The finite dimensional case of Corollary 3.3.2 was first proven by Rockefeller [23, p.260] using technical results in the theory of convex functions.

In the last two results F was the gradient of a convex functional and so in particular F was a monotone map. Since a strictly monotone map is always one-one, one may ask what can be said if F is just monotone?

Theorem 3.3.3. Let $F: D \to X^*$ be a monotone map, $D \subset X$ open. Suppose F is continuously differentiable and $\mathrm{Ker}[F'(x)]^* = 0$ $\forall x \in D$ (equivalently $F'(x)$ has dense range). Then F is globally univalent on D.

Proof: Suppose $F(x_1) = F(x_2) = a$. Let $g(x) = (F(x)-F(x_1), x-x_1) \geq 0$. Then $\inf_{x\in D} g(x) = 0$ is attained at $x = x_2$. Also $\nabla g(x) = [F'(x)]^*(x-x_1)+F(x)-F(x_1)$. Now the zeroes of ∇g are in one-one correspondence with the critical points of $g(x)$. In particular $\inf_{x\in D} g(x)$ is a critical value , thus $x = x_2$ is a critical point of g and so it is a zero of ∇g. Hence $0 = \nabla g(x_2) = [F'(x_2)]^*(x_2-x_1)$. By hypothesis we conclude that $x_2 = x_1$.

Recalling Theorem 2.3.5 we prove:

Corollary 3.3.4. Let $F: X \to X^*$ be a monotone map. Suppose also that F is continuously differentiable and satisfies $\mathrm{Ker}[F'(x)]^* = 0$. Then a necessary and sufficient condition for F to be a homeomorphism is that F is a closed map.

Proof: The necessity is obvious. For the proof of the sufficiency it is enough to show that F maps H one-one onto itself. Then since F is a closed map, F^{-1} is continuous. The fact that F is one-one follows from Theorem 3.3.3. The ontoness follows from a generalization, due to Browder [6], of a theorem of Pokhozhaev.

We are now in a position to prove Corollary 2.3.4 which states: Let $F: X \to X^* \in C'(X)$ be a monotone Fredholm map of index zero. If (i) $\mathrm{Ker}[F'(x)]^* = 0$ and (ii) $\|F(x)\| \to \infty$ as

$\|x\| \to \infty$, then F is a diffeomorphism of X onto X*. Since F is monotone, F satisfies condition (E) by Theorem 1.3.4. By the remark following Theorem 1.4.7, condition (E) and (ii) imply that F has closed range. Since F'(x) is a linear Fredholm map of index zero, (i) and the closed graph theorem imply that F'(x) is invertible, and so by Theorem 1.2.1 F is a local diffeomorphism. In particular, F is an open map and thus has open range. Since the range of F is both open and closed, it is all of X*. F is also univalent by Theorem 3.3.3. Since F is a local diffeomorphism, F^{-1} is differentiable and so F is a diffeomorphism of X onto X*.

Theorem 3.3.3 is a direct generalization of the following theorem of Berger [2, p.139 Theorem 4.4]:

Let F(x) be a continuously differentiable mapping of a convex open bounded set D in $\mathbb{R}^N$ into $\mathbb{R}^N$. Suppose that det F'(x) > 0 and that $\frac{F'(x)+[F'(x)]^*}{2}$ has non-negative principal minor determinants. Then F(x) is univalent in D. This theorem follows from Theorem 3.3.3 if we observe that the hypothesis on $\frac{[F'(x)]^*+F'(x)}{2} = \tilde{F}$ implies that $(\tilde{F}y,y) \geq 0$. However $(F'(x)y,y) = (\tilde{F}y,y) \geq 0$, and so F is monotone. Also det F'(x) > 0 implies Ker(F'(x))* = 0.

Corollary 3.3.5. Suppose F: $\mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable and $|\det F'(x)| > \alpha > 0$. If F is a gradient map then F is globally univalent on R^2.

Proof: Case I. det F'(x) > α > 0 .

Let F(x,y) = (U(x,y),V(x,y)). By Theorem 1.2.1, F'(x) is a symmetric matrix and so det F'(x) = $U_xV_y - V_x^2 > \alpha > 0$. Hence

$U_x \neq 0$. By continuity either $U_x > 0$ or $U_x < 0$. If $U_x > 0$, then the principal minor determinants of $F'(x)$ are positive and so by the remarks preceding the corollary on Theorem 3.3.3, F is globally one-one. If $U_x < 0$, we apply the above arguments to the map $\tilde{F}(x,y) = (-U(x,y), -V(x,y))$.

<u>Case II.</u> $\det F'(x) < -\alpha < 0$.

This is a theorem of Efimov [12].

Our next theorem is related to Corollary 3.2.6 and Theorem 3.5.3 of Section 5.

<u>Theorem 3.3.6.</u> Let $\bar{D} \subseteq \mathbb{R}^N$ be continuous on $\bar{D}$ and continuously differentiable in D and $\det F'(x) \geq 0$ ($\not\equiv 0$). Suppose that the solutions of $F(x) = p$ are isolated and $F(D) \cap F(\partial D) = \phi$. If $(F(x), x) > 0$ on ∂D then F is globally univalent in D.

<u>Proof:</u> $(x, F(x)) > 0$ implies that $d(F, 0, D) = 1$. Also $F(D) \cap F(\partial D) = \phi$ implies that $F(D) \subseteq \mathbb{R}^N - F(\partial D)$, thus $d(F, F(x), D)$ is defined and is constant. Hence $d(F, F(x), D) = 1 \ \forall x \in D$.

Suppose $F(x_1) = F(x_2) = a$, x_1 and $x_2 \in D$. Then since the solutions of $F(x) = a$ are isolated in D and since $F(D) \cap F(\partial D) = \phi$, then there are only a finite number of solutions, say $x_1, \ldots x_n \in F^{-1}(a)$. Let B_i be disjoint open neighborhoods of x_i. Then $1 = d(F, a, D) = \sum_{i=1}^{n} d(F, a, B_i)$. However from Theorem 1.4.6 $d(F, a, B_i) > 0$. Thus there can be only one solution, and so F is globally univalent on D.

4. Surjectivity

In this section we shall consider the problem of determining when a given map between Banach spaces is surjective. Throughout this section we only consider those classes of maps F between Banach spaces that are described in Chapter I, Section 3B, i.e., for which one can associate a certain integer valued function called the degree d(F,P,B) which satisfies properties (i)-(iii) of Sect. 3B.

First of all we observe that if F maps X onto Y, then $\limsup_{||x|| \to \infty} ||F(x)|| = \infty$. For if not then Range F is bounded and so F cannot be surjective. However this necessary condition is far from being sufficient as the example $F(x_1,\dots,x_n) = (x_1^2,\dots,x_n^2)$ shows. In fact F is proper since

$$||F(x)|| \longrightarrow \infty \text{ as } ||x|| \longrightarrow \infty.$$

In our search for sufficient conditions let us for the moment consider a polynomial P(x) as a function of $\mathbb{R}'$ into itself. Since every polynomial is proper, under what conditions does it map $\mathbb{R}^1$ onto $\mathbb{R}^1$? If P(x) is of odd order, then this is certainly true. Taking a closer look at our polynomial of odd order we notice that it has the property that if $|b|$ is sufficiently large, then $d(P(x),b,I) \neq 0$ whenever I is chosen so large that $P^{-1}(b) \subset I$. Surprisingly, this condition along with properness, is sufficient to guarantee surjectivity for more general mappings, i.e.:

Theorem 3.4.1 Suppose $F: X \rightarrow Y$ is proper. If $\exists\, p \in Y$ so that $d(F,p,B) \neq 0$ whenever $B \supseteq F^{-1}(p)$, then F maps X onto Y.

Proof:

Since F is proper, F has closed range. Also by property (i) of Section 3B, Chapter I, $F(X) \neq \emptyset$. Thus it suffices to show that F has open range.

So let $q \in y$ and let $L(t) = (1-t)p + tq$, $0 \leq t \leq 1$, where p is a point which satisfies $d(F,p,B) \neq 0$ when $F^{-1}(p) \subset B$. Since $L(t)$ is a compact set in Y, $F^{-1}(L(t))$ is a compact subset of X. Choose r large enough so that $F^{-1}(L(t)) \subset B(0,r) = \{x \mid ||x|| < r\}$. Hence $d(F,L(t),B)$ is defined, and is a continuous function of $t \in [0,1]$. Since d is integer valued, $d(F,L(t),B) = \text{Const.} = d(F,p,B) \neq 0$. In particular, $d(F,q,B) = d(F,p,B) \neq 0$. Also $F(\partial B)$ is a closed set since a proper map is also a closed map. Thus $Y - F(\partial B)$ is open and so it is the union of open components. Let C_q be the (open) component containing q. We show that $C_q \subset$ Range F, and so F has open range. However $d(F,y,B)$ $= d(F,q,B)$ whenever $y \in C_q$. Thus $d(F,y,B) \neq 0$ and so $C_q \subset F(B) \subset$ Range F by property (i) of Section 3B, Chapter I.

A direct analog of the case of an odd order polynomial is

Corollary 3.4.2 Let $F: X \rightarrow Y$ be an odd map, i.e., $F(-x) = -F(x)$. Suppose $d(F,0,B)$ is an odd integer whenever B is a ball about the origin. If F is proper, then F maps X onto Y.

Proof:

This follows from Theorem 3.4.1.

The example $F(x_1,\ldots,x_n) = (x_1^2,\ldots,x_n^2)$ shows that the hypothesis concerning the degree of F cannot be removed from Theorem 3.4.1. For $d(F,p,B) = 0$ whenever $F^{-1}(p) \subset B$ since F always omits negative directions. F is proper but not onto.

Corollary 3.4.3 Let X,Y be a complex Banach spaces, $F: X \to Y$ complex analytic which is also a Fredholm map of index zero. If F is proper, then F is onto.

Proof:

Since F is proper, we can define an oriented degree for F as in Section 3B, Chapter I. By Theorem 3.4.1 it suffices to show that $d(F,p,B) > 0$ whenever $F^{-1}(p) \subset B$ for any bounded domain B. However, by Theorem 1.4.5, $d(F,p,B) > 0$ whenever $p \in F(B) - F(\partial B)$.

In the same manner we give a simple proof of a result of Nijenhuis and Richardson [22].

Corollary 3.4.4 Suppose $F: \mathbb{R}^N \to \mathbb{R}^N \in C'(\mathbb{R}^N)$. If F is proper and $\det F'(x) \geq 0$ (and $\not\equiv 0$), then F maps $\mathbb{R}^N$ onto itself.

Proof:

The proof is the same as Corollary 3.4.3, except we use Theorem 1.4.6. $d(F,p,B) > 0$ whenever $p \in F(B) - F(\partial B)$.

In Section 5, Corollary 3.5.10 we shall prove a Hilbert space analog of Corollary 3.4.4, for monotone maps.

Again $F(x_1,\ldots,x_n) = (x_1^2,\ldots,x_n^2)$ serves as a counter-example for this corollary, if the condition $\det F'(x) \geq 0$ is not fulfilled. Also we cannot remove the properness condition as is seen from the map $F(x_1,x_2) = e^{x_1}(\cos x_2, \sin x_2)$. For this map $\det F'(x_1,x_2) > 0$, however F is not proper since for fixed x_1, $\lim_{x_1,x_2 \to -\infty} F(x_1,x_2) = 0$. Since $F(x_1,x_2) \neq 0$, F is not onto.

In the case where $F: ¢^n \longrightarrow ¢^n$ is analytic, we know that $\det F'(z) \geq 0$. Thus it suffices to show that an analytic map is proper in order to show that it is onto. If $n = 1$, proper <=> F being a polynomial. Thus Corollary 3.4.3 is equivalent to the Fundamental Theorem of Algebra in this case.

5. Preservation of Univalence and Surjectivity Under Normal Convergence.

Suppose F_n, $n = 1,2,\ldots$ is a sequence of univalent or surjective maps from one Banach space into another and they converge (in some sense) to a map F. Under what conditions does F inherit the properties of univalence or surjectivity from the F_N's? The Hurwitz theorem for functions of a single complex variable tells us that F is univalent provided the F_N's are analytic and univalent and converge to F uniformly on bounded sets of $\mathbb{C}^1$. One of our aims is to see how this theorem can be generalized to non-analytic functions defined on any space $\mathbb{R}^N$.

Definition 3.5.1 We say that F_n converges to F normally (or F can be normally approximated by F_n) if and only if F_n converges to F uniformly on any bounded subset.

In view of the Hurwitz theorem we shall always assume that the convergence of our sequences is taken to be normal convergence. Of course just the fact that we have normal convergence is not in itself sufficient to guarantee the univalence or surjectivity of the limit as is illustrated by the sequence $F_n(x) = \frac{1}{n}x$. This is a sequence of homeomorphisms which converges normally to $F(x) \equiv 0$ which is of course neither univalent nor surjective.

Restating our problem, we ask: If F_N converges to F normally and if the F_n's are univalent or surjective, under what conditions does F also have these properties?

To be consistent with the approach used in previous sections, we first establish a certain necessary condition. (For details in the finite dimensional case, see Cartan [8]).

Lemma 3.5.1 Let $F_j: X \to X$ be maps such that $F_j \to F$ normally in X, where F_j and F are of the form I+C. Let $p \in D$ where D is any open bounded set. If F is a local homeomorphism at p, then $\exists$ R and N so that $B_p = B(F(p);R) \subset F_n(D)$ for all $n \geq N$.

Proof:

Let $B(p;r) = B$ be such that $F|_B$ is a homeomorphism; since $F_n \to F$ uniformly on B, then $U_n = F_nF^{-1} \to I$ uniformly on F(B). Choose R so that $\tilde{B} = B(F(p);2R) \subset F(B)$. Let $B_p = B(F(p);R) \subset F(B)$. Then $\exists$N so that $||U_n(x) - x|| < R$ for all $n \geq N$ and for all $x \in \partial\tilde{B}$. Let $z \in B_p$. Then $d(U_n,z,\tilde{B}) = d(I,z,\tilde{B}) = 1$ for all $z \in B_p$. Hence U_n maps $\tilde{B}$ onto B_p for all $n \geq N$, or by definition of U_n, F_nF^{-1} maps $\tilde{B}$ onto B_p for all $n \geq N$. Thus $F_n(D) \supset F_nF^{-1}(\tilde{B}) \supset B_p = B(F(p);R)$.

From Lemma 3.5.1 we can now derive necessary and sufficient conditions for the limit (in the normal sense) of a sequence of univalent maps to be univalent.

Theorem 3.5.2 Suppose $F_j: X \to X$ form a sequence of univalent maps converging normally to a map F where F and F_j are of the form I+C. The following condition is both necessary and sufficient for F to be univalent:

(U) If D is any open bounded set in X and $a \in X$ then $F_n(x) \neq a$ on D for all n implies $F(x) \neq a$ on D.

Proof:

First suppose F is univalent. Then the invariance of domain theorem tells us that F is an open map and so in particular, F is a local homeomorphism. Thus by Lemma 3.5.1, $F_n(x) \neq a$ in D implies $F(x) \neq a$ in D for if $\exists p \in D$ so that $F(p) = a$, then we conclude that there are points $p_n \in D$ for which $F_n(p_n) = a$ for n sufficiently large -- a contradiction.

Conversely, suppose condition (U) and suppose $F(x_1) = F(x_2) = a$. If $x_1 \neq x_2$ let O_1 and O_2 be disjoint open sets about x_1 and x_2 respectively.

By (U), there is a subsequence F_{n_j}, and points $p^1_{n_j} \in O_1$ so that $F_{n_j}(p^1_{n_j}) = a$. Similarly in O_2, there is a subsequence of the F_{n_j}'s (which we renumber and call F_{n_j}) and points $p^2_{n_j} \in O_2$ for which $F_{n_j}(p^2_{n_j}) = a$. From the disjointness of O_1 and O_2 we have that $p^1_{n_j} \neq p^2_{n_j}$ which contradicts the univalence of F_{n_j}.

We now proceed to find suitable conditions which insure that condition (U) of Theorem 3.5.2 are satisfied.

Theorem 3.5.3 Let F_n be a sequence of maps of $\mathbb{E}^N$ into itself which converge normally to a map F. Suppose furthermore that F is continuously differentiable and satisfies:

(i) det $F'(x) \geq 0$ and (ii) the solutions of $F(x) = p$ are isolated for every p. Then if the F_n's are univalent, then either F is univalent or det $F'(x) \equiv 0$.

Proof:

Suppose det $F'(x) \not\equiv 0$, then it suffices, by Theorem 3.5.2, to show that condition (U) is satisfied. Thus let D be a bounded open connected set of $\mathbb{R}^N$ so that $F_n(x) \neq a$ for all $x \in D$. Suppose $\exists \bar{x} \in D$ so that $F(\bar{x}) = a$. Let $a \in B$ be an open ball in D so that $\bar{x}$ is the only solution of $F(x) = a$ in $\bar{B}$. Thus $d(F,a,B) > 0$ by Theorem 1.4.6, and since F_n converges to F uniformly on B, we have that $0 < d(F,a,B) = d(F_n,a,B)$ for n sufficiently large. Hence $F_n(x) = a$ has a solution in B (for n large) -- a contradiction.

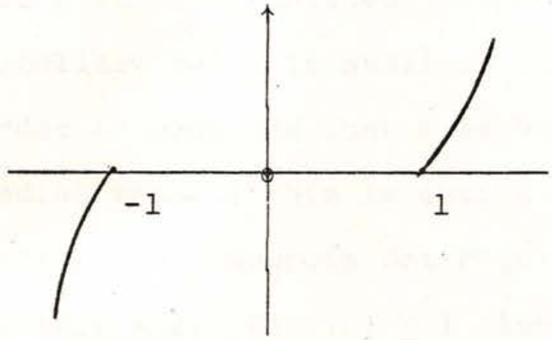
If in the proof of Theorem 3.5.3 we use Theorem 1.4.5 instead of Theorem 1.4.6 we could just as easily prove: If X is a complex space, $F_n \to F$ normally where F_n and F are complex analytic and of the form I + C, then if F has isolated zeroes, F is univalent if the F_n's are.

Corollary 3.5.4 Suppose F_n converges to F normally and $F_n: \mathbb{R}^N \to \mathbb{R}^N$ are univalent. If F is continuously differentiable and satisfies det $F'(x) > 0$ then F is univalent.

Proof:

This follows directly from Theorem 3.5.3 and the fact that F is a local homeomorphism and thus has isolated zeroes.

In general we cannot remove the condition that the solutions of $F(x) = p$ are isolated from Theorem 3.5.2 and still retain the conclusion of the theorem. This is shown by the following example:



$$F(x) = \begin{cases} (x-1)^2, & x \geq 1 \\ 0, & |x| \leq 1 \\ -(x+1)^2, & x \leq -1 \end{cases}$$

Then $F'(x) \geq 0$, F is not one-one and F can be approximated by $F_n(x) = F(x) + \frac{1}{n} x$ which converge normally to F. In fact $F_n'(x) > 0$ and $|F_n(x)| \to \infty$ as $|x| \to \infty$, thus F_n's are homeomorphisms of $\mathbb{R}'$ onto $\mathbb{R}'$.

Let $F: \mathbb{C}^N \to \mathbb{C}^N$ be analytic where $F(z_1,\ldots,z_N) = (F_1(z_1,\ldots,z_N), \ldots, F_N(z_1,\ldots,z_N))$. We introduce the following notations:

Let $F'(z) = \dfrac{\partial(F_1,\ldots,F_N)}{\partial(z_1,\ldots,z_N)}$ and

$F'(x,y)$ = Jacobian of F considered as a map of $\mathbb{R}^{2N} \to \mathbb{R}^{2N}$.

It is well known that $\det F'(x,y) = |\det F'(z)|^2$, and in particular, $\det F'(x,y) \geq 0$. With this in mind we prove:

Corollary 3.5.5 Let $F_n: \mathbb{C}^N \rightarrow \mathbb{C}^N$ be analytic and suppose F_n converges normally to a map F. If the F_n's are univalent then either F is univalent or $\det F'(x,y) \equiv 0$.

Proof:

Since $F_n \rightarrow F$ normally, then F is also analytic and so $\det F'(x,y)$ is defined. Suppose $\det F'(x,y) \not\equiv 0$, then by Corollary 3.5.4 it suffices to show that $\det F'(x,y) > 0$ in order to conclude that F is univalent. However by our preceding remarks this is equivalent to showing that $\det F'(z) \neq 0$. Suppose $\det F'(\bar{z}) = 0$. Let z_1 be such that $\det F'(z_1) \neq 0$. For $|\lambda| \leq 1$ let $g_n(\lambda) = \det F_n'((1-\lambda)\bar{z}+\lambda z_1)$, $g(\lambda) = \det F'((1-\lambda)\bar{z} + \lambda z_1)$. We observe that g_n and g are analytic functions of λ and $g_n \rightarrow g$ uniformly on $|\lambda| \leq 1$. Also since F_n is univalent, $g_n(\lambda) \neq 0$ and since $g(1) \neq 0$, $g(\lambda) \not\equiv 0$ on $|\lambda| \leq 1$. However since g is analytic, it has isolated zeroes on $|\lambda| \leq 1$ and so the argument of Theorem 3.5.4 is applicable and shows that condition (U) is satisfied, i.e., $g_n \neq 0$ implies $g \neq 0$ on $|\lambda| < 1$ -- a contradiction since $g(0) = 0$. Thus $\det F'(z) \neq 0$, and the corollary follows.

We shall now investigate the limit of a sequence of surjective maps and impose conditions which guarantee that this limit is also surjective. As before, our convergence will mean normal convergence.

For $F: X \rightarrow Y$ we have already introduced in Chapter I the following condition which is satisfied by a large class of nonlinear differential operators:

(E) If $x_n \to \bar{x}$ weakly in X and $F(x_n) \to \bar{y}$ strongly in Y, then $F(\bar{x}) = \bar{y}$.

<u>Theorem 3.5.6</u> Suppose X is reflexive and F: X $\to$ Y satisfies condition (E). Furthermore suppose that F can be normally approximated by a sequence F_j with the following properties (i) F_j is onto for each j = 1,..., (ii) $F_n(x_n) = a \Rightarrow ||x_n|| \leq M_a$ for every a, where M_a is a constant depending on a. Then F maps X onto Y.

<u>Proof</u>:

Let $\bar{y} \in Y$. By (i) and (ii) $\exists x_n$, n = 1,..., so that $F_n(x_n) = \bar{y}$ and $||x_n|| \leq M_{\bar{y}}$. Since X is reflexive a subsequence $x_{n_j} \to \bar{x}$ in the weak sense. Thus it suffices to show that $F(x_{n_j}) \to \bar{y}$ since F satisfies (E). However $||x_{n_j}|| \leq M_{\bar{y}}$ and the normal convergence of F_j to F implies that $||F(x_{n_j}) - \bar{y}|| = ||F(x_{n_j}) - F_{n_j}(x_{n_j})|| \to 0$ as $n_j \to \infty$.

<u>Definition 3.5.2</u> Let F: X $\to$ Y. Then F^{-1} is locally bounded if and only if each point y $\in$ Y has an open neighborhood N about it so that $F^{-1}(N)$ is bounded. ($F^{-1}(N)$ may be empty.)

Notice that F^{-1} locally bounded is a weaker condition than $||F(x)|| \to \infty$ as $||x|| \to \infty$, which is equivalent to F^{-1} being bounded, i.e., the inverse image of a bounded set in Y, is a bounded set in X.

<u>Theorem 3.5.7</u> Let X be a reflexive Banach space, $F: X \longrightarrow Y$ satisfying condition (E) and F^{-1} is locally bounded. Suppose $F_n: X \longrightarrow Y$, $n = 1,2,\ldots$ is a sequence of surjective maps such that F_n lifts any line in Y. If F_n converges to F normally, then F maps X onto Y.

<u>Proof</u>:

We show that every boundary point of F(X) is an interior point. Thus F(X) has empty boundary and thus F(X) = Y by connectivity. Let $\bar{y} \in \partial F(X)$. Let N be an open neighborhood of $\bar{y}$ so that $F^{-1}(N)$ is bounded. Choose r so large that $\overline{F^{-1}(N)} \subset B(0;r) = \{x \,|\, ||x|| < r\}$ and choose δ so that $B(\bar{y};4\delta) \subset N$.

Since $\bar{y} \in \partial F(X)$, $\exists y_1 \in F(X) \cap B(\bar{y};\delta)$ and $F(x_1) = y_1$ where $x_1 \in B(0;r)$. From the normal convergence $\exists$ M so that $||F_n(x)-F(x)|| < \delta$, for all $n \geq M$, and for all $x \in B(0;r)$. In particular $||F_n(x_1)|| \leq ||y_1||+\delta$, so $F_n(x_1) \in B(\bar{y},2\delta)$.

We claim that $B(\bar{y};2\delta) \subset F(X)$ and thus $\bar{y} \in \partial F(X)$ is an interior point. So let $y_0 \in B(\bar{y};2\delta)$ and let $L_n(t)=(1-t)y_0 + t\, F_n(x_1)$. Since F_n, $n = 1,2,\ldots$ lifts lines, $\exists$ paths $P_n(t)$ so that $P_n(0) = x_1$ and $F_n(P_n(t)) = L_n(t)$. Now $P_n(t) \subset B(0;r)$. If not, then $P_n(t)$ passes out of $B(0;r)$ and since $P_n(t)$ is continuous $\exists t_n \ni x_n = P_n(t_n) \in \partial(B(0;r))$. Therefore dist $(F(x_n), B(\bar{y};\ 2\delta)) \leq \delta$ and so $F(x_n) \in B(\bar{y};3\delta) \subset N$. Hence $x_n \in F^{-1}(N) \subset B(0;r)$ and in particular $x_n \notin \partial B(0;r)$, a contradiction. Thus $P_n(t) \subset B(0;r)$ and so $P_n(1) \in B(0;r)$. (Remember that $F_n(P_n(1)) = y_0$.) Let

$x_n^0 = P_n(1)$. As $x_n^0 \in B(0;r)$ for all n there is a subsequence $x_{n_j}^0 \rightarrow x_0$ weakly. However

$$||F(x_{n_j}^0)-y_0|| = ||F(x_{n_j}^0)-F_{n_j}(x_{n_j}^0)|| \rightarrow 0 \quad \text{as} \quad n_j \rightarrow \infty .$$

By condition (E) we conclude that $F(x_0) = y_0$ and so $B(\bar{y};2\delta) \subset F(X)$ as was to be shown.

Corollary 3.5.8 Suppose $F: \mathbb{R}^N \rightarrow \mathbb{R}^N$ is such that F^{-1} is locally bounded. Also suppose that $F_j: \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a sequence of maps each of which lifts any line in $\mathbb{R}^N$. If $F_j \rightarrow F$ normally, then F maps $\mathbb{R}^N$ onto $\mathbb{R}^N$.

Proof:

This follows immediately from Theorem 3.5.7 by observing that for finite dimensional spaces condition (E) reduces to the continuity of F.

Corollary 3.5.9 [5, p. 16]. Let $F: H \rightarrow H$ be monotone. If F^{-1} is locally bounded, then F is onto.

Proof:

In view of Theorem 3.5.7 and Theorem 1.3.4 it suffices to show that there is a sequence $F_n: H \rightarrow H$ such that F_n lifts lines and $F_n \rightarrow F$ normally. Let $F_n(x) = F(x) + \frac{1}{n} x$. Then $F_n \rightarrow F$ normally and we claim that F_n, $n = 1,2,...$ are homeomorphisms of H onto H. In fact, $(F_n(x)-F_n(y),x-y) = \frac{1}{n} ||x-y||^2$. Thus F_n are 1-1 and F_n maps H onto H [2, p. 105]. Thus F_n^{-1} is defined and satisfies $\frac{1}{n}||F_n^{-1}(x)-F_n^{-1}(y)||^2=||x-y||^2$. Hence F_n^{-1} is continuous and so F_n is a homeomorphism of H onto H. In particular F_n lifts any line in H.

We now have the following Hilbert space analog of Corollary 3.4.2.

Corollary 3.5.10 Suppose $F: H \to H$ is continuously differentiable and satisfies $(F'(x)y,y) \geq 0$, $\forall x,y \in H$. If $||F(x)|| \to \infty$ as $||x|| \to \infty$, then F maps H onto H.

Proof:

$(F(x)-F(y),x-y) = (F'(x(\bar{t}))(x-y),x-y) \geq 0$ by the mean value theorem, where $x(\bar{t})$ is a point on the line joining y to x. Thus F is monotone and $||F(x)|| \to \infty$ as $||x|| \to \infty$ is equivalent to $F^{-1}(N)$ is bounded whenever N is bounded. In particular F^{-1} is locally bounded. Hence F maps H onto H by Corollary 3.5.9.

One may ask if the properties of univalence and surjectivity are preserved in the limit of a sequence which converges uniformly on the whole space X. Corollary 3.5.12 shows us that in fact the homeomorphism property is preserved in this situation.

Lemma 3.5.11 Suppose $F_n: \mathbb{R}^N \to \mathbb{R}^N$ converges uniformly to F. If the F_n's are proper, so is F.

Proof:

Since F is a compact perturbation of the identity, by Theorem 1.4.7 it suffices to show that $||F(x)|| \to \infty$ as $||x|| \to \infty$. However the uniform convergence implies, for n large enough, that $||F_n(x)-F(x)|| \leq 1$ for all $x \in X$. Thus

(1) $\quad ||F_n(x)|| - 1 \leq ||F(x)||$ for all $x \in X$.

Since the F_n's are proper, $||F_n(x)|| \to \infty$ as $||x|| \to \infty$, and so from (1) the result follows.

Corollary 3.5.12 Suppose F is a local homeomorphism on $\mathbb{R}^N$. If F can be uniformly approximated by homeomorphisms, then F is also a homeomorphism of $\mathbb{R}^N$ onto itself.

Proof:

Since a homeomorphism is a proper map (Theorem 2.3.2), the corollary follows from Lemma 3.5.11 and Theorem 2.3.2.

We remark that by virtue of Theorem 1.4.7, Lemma 3.5.11 can be extended to Banach spaces if we require that F_n and F are compact perturbations of the identity and that F_n satisfies the coercive condition $||F_n(x)|| \longrightarrow \infty$ as $||x|| \to \infty$. We can then extend Corollary 3.5.12 accordingly.

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