ERGODIC THEORY AND INFINITE SYSTEMS

by

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1. Introduction

Statistical mechanics is concerned with the description and justification of the thermodynamic behavior of macroscopic physical systems on the basis of their underlying structure: The systems are composed of a very large number of identical subsystems (particles) and evolve (classically) as Hamiltonian dynamical systems [14, 35]. The Hamiltonian typically consists of two parts: H_o, the kinetic energy, which gives rise to free motion; and a potential energy term, V, which is typically the sum of pair interactions over all pairs of particles. By thermodynamic behavior we mean, typically, that states of isolated systems approach equilibrium states (as time approaches infinity) which consist of one or more macroscopically homogeneous phases and which are characterized by a small number of parameters and thermodynamic functions (energy, temperature, pressure, etc.) obeying the laws of thermodynamics. The approach to equilibrium may be characterized by kinetic and transport equations. The difficulty involved in the justification of thermodynamic behavior can be appreciated if one considers that thermodynamic behavior is clearly irreversible whereas the underlying Hamiltonian dynamics is completely reversible, and that the systems are so complex that an exact (pointwise) dynamical description is a practical impossibility.

In attempting to solve these problems it is natural to look for general, abstract features, common to all realistic systems, which provide a framework for attacking these problems. Typical of such a formal approach is the consideration of infinite systems, C^* algebras [9], and ergodic theory.

One of the earliest of formal results within the compass of ergodic theory is Liouville's theorem [14]. The phase space Γ of a Hamiltonian system is the set of all possible microscopic states, each of which is determined by 2dN variables: $q_1, q_2, \ldots q_{dN}$, the configurational coordinates, and $p_1, p_2, \ldots, p_{dN}$, the canonical momenta. (d is the dimension of the space in which a single particle is located and N is the number of particles in the system under consideration.) Thus Γ can be identified with a subset of $\mathbb{R}^{2dN}$. In a <u>Hamiltonian dynamical system</u> the dynamics is induced by differential equations of the form

$$dq_i/dt = \partial H/\partial p_i \ , \quad dp_i/dt = -\partial H/\partial q_i ,$$

where $H = H(q_i, p_i)$ is a function on Γ (called the Hamiltonian of the system; see previous description.) A natural measure (Liouville measure) on Γ is Lebesgue measure ($= dq_1 \ldots dq_{dN} \ldots dp_{dN}$). Liouville's theorem asserts that for a Hamiltonian system this measure is invariant under the time evolution; i.e., any measurable subset

A of Γ is mapped via the time evolution T_t to a new set T_tA of the same Liouville measure. This result, powerful in its own right, puts us squarely within the context of ergodic theory, which deals typically with the quadruple $(X,\Sigma,\mu,\{T_t\})$; here (X,Σ,μ) is a(probability) measure space and $\{T_t\}$ is a measurable flow on (X,Σ,μ), i.e., a one parameter group of measure preserving transformations for which $\mathbb{R}\otimes X \to X$ by $(t,x) \mapsto T_t x$ is measurable in the product measure on $\mathbb{R}\otimes X$ (Lebesgue measure $\otimes\ \mu$). One also considers the case of the discrete dynamical system for which t assumes values in $\mathbb{Z}$, the group of integers[1]; i.e., the dynamics is generated by a single automorphism T [2, 17].

2. Ergodicity

One of the simplest and most important of facts about Hamiltonian systems is that the Hamiltonian (the energy) is a constant of the motion: $H(x) = H(T_t x)$, $x \in \Gamma$, $t \in \mathbb{R}$. It is thus natural to take as our space X not Γ but rather $\Gamma_E = \{x \in \Gamma: H(x)=E\}$, the energy surface at energy E, since such surfaces are invariant

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1. Since the definitions which we shall give are essentially the same for flows as for discrete dynamical systems, we will usually give the definitions using the notation appropriate for discrete dynamical systems; the corresponding definitions for flows can be obtained by replacing $n \in \mathbb{Z}$ by $t \in \mathbb{R}$ and by replacing $1/N \sum_{n=o}^{N-1}$ by $1/T \int_o^T$ (and vice versa).

under T_t. It is not difficult to see that $d\mu_E = (d\sigma/|\text{grad } H|) \times 1/\text{Normalization}^2$, the normalized "projection" of the Liouville measure onto Γ_E (the microcanonical measure or ensemble) is invariant under T_t. Using μ_E one can compute averages of phase space functions by $\langle f\rangle_E = \int d\mu_E\, f$. In statistical mechanics one identifies $\langle f\rangle_E$ with the value of the quantity f in the equilibrium state characterized by the energy E (and the volume V and particle number N implicit in the foregoing discussion.) If such an identification can be justified, part of the problem of the justification of thermodynamic behavior will be solved; the equilibrium values of physical quantities would be determined by the microcanonical ensemble, which depends upon only a small number of parameters.

The problem of the justification of this use of the microcanonical ensemble has two aspects:

i) Why is μ_E superior to other measures on Γ_E (i.e., to $f\mu_E$, f a positive function on Γ_E with μ_E - integral unity?

ii) Why should a microcanonical (or any other) average of a quantity represent the value of an equilibrium measurement of that quantity? It is often argued, in answer to ii), that in the

2. The Normalization is chosen in such a way that $\mu_E(X) = \int_X d\mu_E = 1$, so that we have a probability measure.

thermodynamic limit (i.e., as $N \to \infty$, $V \to \infty$ in such a way that $N/V \to \rho$ (density) and $E/V \to \epsilon$ (energy density)) the microcanonical measures approach delta functions with respect to the functions of physical significance (the sum functions [35].) Though this is a fact of great importance, it is not, in view of i), a completely satisfactory solution: Why must sets of small microcanonical measure actually be of small probability?[3]

The traditional justification lies in the hypothetical equality of time average and phase averages; i.e., $\langle f \rangle_E = \lim_{T \to \infty} (1/T) \int_0^T f(T_t x)dt$. It is often asserted that since measurements are not instantaneous but rather take place in a time span which is large relative to typical microscopic times, the time average of a quantity should be identified with its equilibrium value [45]. This explanation is unsatisfactory in that the measurement times are in fact small relative to the time intervals necessary for the attainment of a time average (i.e., recurrence times or even relaxation times.) We can argue, however, as follows: Systems which behave thermodynamically will spend an overwhelming majority of their time "in

3. It does, however, seem plausible-and will in fact prove necessary to make the probablistic assumption - that the Lebesgue measure is special at least to the extent that sets of microcanonical measure zero do in fact have probability zero.

equilibrium"; hence, we can identify the time average of a quantity with its equilibrium value; if we then have equality between time averages and microcanonical averages, the latter are validated (and selected.)[4] [35].

The problem thus becomes one of justifying the replacement of time averages by phase averages. Significant progress in this direction was made by Birkhoff, who showed that for abstract dynamical systems [2]

a) time averages exist a.e. (almost everywhere):

$$f^+(x) = \lim_{N \to \infty} 1/N \sum_{n=o}^{N-1} f(T^n x) \qquad \text{a.e.}$$

b) $f^+(x)$ is integrable and $\int d\mu\, f^+(x) = \int d\mu\, f(x)$

c) $f^+(x)$ is invariant a.e.:

$$f^+(Tx) = f^+(x) \quad \text{a.e.}.$$

A dynamical system (X, Σ, μ, T) is said to be ergodic if the only sets $A \in \Sigma$ invariant under T (i.e., $TA = A$) have $\mu(A) = 0$ or $\mu(A) = 1$. It is easily seen that ergodicity is equivalent to the requirement that invariant measurable functions be constant a.e. [2]. Thus for ergodic systems Birkhoff's theorem implies that

4. This answer presupposes the attainment of a satisfactory account of approach to equilibrium.

$f^+(x) = \int d\mu \, f(x)$, i.e., that time averages equal phase averages almost everywhere[5]. It follows from the above that an ergodic measure is a measure which is the unique invariant member of the family of measures absolutely continuous with respect to it. We thus have in another form a validation of ergodic microcanonical ensembles (essentially equivalent to the one given previously). It has proven difficult, however, to establish the ergodicity of the energy surfaces of specific realistic Hamiltonian systems. In fact, much of the progress which has been made has consisted in the establishing of stronger ergodic theoretic properties, of which ergodicity is a consequence. We will next turn to these stronger properties, alluding to other formulations and implications of ergodicity when appropriate.

3. Mixing

An important ergodic theoretic property, introduced by Hopf [47] in 1932, is mixing; intuitively, a system is mixing if any subset becomes uniformly distributed over the phase space under the action of the time evolution as t approaches infinity. Formally a dynamical system (X, Σ, μ, T) is mixing if

$$\lim_{n \to \infty} \mu(T^n A \cap B) = \mu(A)\,\mu(B) \tag{1}$$

for all $A, B \in \Sigma$.

5. We are here using our assumption concerning sets of measure zero.

Equivalently, a system is mixing if and only if

$$\lim_{n \to \infty} \int d\mu\, f(T^n x)\, g(x) = (\int d\mu\, f)\,(\int d\mu\, g) \qquad (2)$$

for all $f, g \in L^2(\mu)$.

Thus mixing implies the decay of correlations.[6] Furthermore, it is not difficult to see that for ρ a positive function of unit integral and g a bounded measurable function, we have as a consequence of mixing that

$$\lim_{t \to \infty} \int d\mu\, \rho(T^{-t} x)\, g(x) = \int d\mu\, g(x) \qquad (3)$$

Since $\rho(T^{-t} x)\,\mu$ represents the time evolution of the measure determined by the density ρ, we see that if a system is mixing, "reasonable" (i.e., absolutely continuous) nonequilibrium states weakly approach the equilibrium measure (in the sense that averages approach the equilibrium average.) Thus mixing illustrates the possibility of a deterministic reversible dynamics in which can be found irreversible behavior.

If in (1), (2), and (3) we replace convergence by Cesaro convergence (i.e., $\alpha_n \to \beta$ by $1/N \sum_{n=1}^{N-1} \alpha_n \to \beta$), we obtain conditions equivalent to ergodicity. Thus, whereas mixing can be interpreted (at least for finite systems) as approach to equilibrium or decay

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6. Mixing also has many other implications related to the decay of correlations [25].

of correlation functions, ergodicity can be interpreted as time averaged approach or decay.[7] Needless to say mixing implies ergodicity.

4. Isomorphism and invariants

Two systems (X, Σ, μ, T) and (X', Σ', μ', T') are isomorphic if they are the same from the standpoint of their ergodic theoretic structure, i.e., if there exists a one to one mapping φ from X onto X' such that both φ and φ^{-1} are measure preserving and such that $T'\varphi(x) = \varphi(Tx)$, $x \in X$. Since in ergodic theory one adopts the point of view that sets of measure zero are of no consequence, one normally employs the concept of isomorphism (mod 0) rather than isomorphism. (X, Σ, μ, T) and (X', Σ', μ', T') are isomorphic (mod 0) if there exist invariant subsets $\tilde{X}$ and $\tilde{X}'$ of X and X', respectively, whose complements are of zero measure and such that $(\tilde{X}, \tilde{\Sigma}, \tilde{\mu}, \tilde{T})$ is isomorphic to $(\tilde{X}', \tilde{\Sigma}', \tilde{\mu}', \tilde{T}')$[8]. In general we will say that (X, Σ, μ, T) has a property (mod 0) if a system (X', Σ', μ', T') obtained from (X, Σ, μ, T) by removal of a set of measure zero has the property.[9]

Invariants of abstract dynamical systems are properties which are shared by all systems isomorphic to each other. Hence

7. For infinite systems the situation is more complicated, as we shall see.

8. Here $\tilde{\Sigma}$ denotes Σ restricted to $\tilde{X}$, etc..

9. We will very often delete the expression "(mod 0)" from "isomorphism (mod 0)", as well as from similar expressions. All expressions of isomorphism are to be so understood.

they serve to classify dynamical systems. It is clear that both mixing and ergodicity are invariants. Properties of Hamiltonian systems which are not invariants are obviously those which cannot be encompassed within the abstract framework of ergodic theory.

5. Spectral invariants

An important class of invariants is composed of the spectral invariants. These are the unitary invariants of the unitary operator U_T on $L^2(\mu)$ induced by T via

$$U_T f = f \circ T, \; f \in L^2(\mu) \qquad [2]$$

For example, the spectrum of U_T is a spectral invariant. U_T has a simple eigenvalue 1 if and only if T is ergodic.[10] If the spectrum of U_T on the orthogonal complement in $L^2(\mu)$ of the constants is absolutely continuous with respect to Lebesgue measure, the system is mixing, while the continuity of the spectrum of U_T there is equivalent to weak mixing [2, 17]. The absolute continuity of the spectrum (apart from the eigenvalue 1) of unitary operators of the form U_T (induced unitaries) is equivalent to their having (homogeneous) Lebesgue spectrum, a necessary and sufficient condition for which is that there exist an orthonormal basis (of the

10. We will often say that T has a certain property rather than saying that (X, Σ, μ, T) has that property.

orthogonal complement of the constants) of functions f_j^i ($j \in \mathbb{Z}$, $i = 1, 2, \ldots, I$; I is the multiplicity of the Lebesque spectrum) for which we have $U_T f_j^i = f_{j+1}^i$.

For ergodic discrete dynamical systems with discrete spectrum, the spectral invariants (the eigenvalues and their multiplicites) form a complete set of invariants: two such systems are isomorphic [11] if they have the same spectral invariants [17]. We shall see that in general the spectral invariants are not complete.

6. K-systems

We now come to the more recent ergodic theoretic concepts, which illustrate the manner in which determinism on the one hand, and instability, indeterminism, and intrinsic statistics, on the other, can appear as different aspects of the same underlying structures. The first of these are the K-systems (or flows), which were introduced by Kolmogorov [19], and are a generalization of the Anosov flows or C-systems [2] (about which we shall have nothing further to say). Heuristically, these are systems which possess sufficient instability to render "practical" measurements completely nondeterministic, in a sense which we shall later elucidate.

Before we proceed to a formal description, it will be convenient to comment briefly on continuous Lebesgue spaces [37].

11. Strictly speaking, conjugate [17,4]

These are measure spaces isomorphic to the unit interval with Lebesgue measure.[12] Restriction to Lebesgue spaces avoids pathological situations and leads to harmony between the point set and the measure algebraic points of view [17, 37].[13] Furthermore, such a restriction is not really very stringent since, in fact, most spaces encountered in practice are Lebesgue [37, 2]. Henceforth, all measure space to which we refer will be assumed to be continuous Lebesgue spaces, unless we explicitly indicate the contrary.

An important fact about Lebesgue spaces is that they admit of a natural correspondence between sub - σ - algebras (mod 0) [33] and an important class of partitions, the measurable partitions (mod 0) [37]. A partition of a space X is a family of disjoint subsets of X (the elements or fibers of the partition) whose union is X. It is natural to consider only partitions whose elements are measurable. However, if the partition is uncountable, the measurability of each of its elements does not preclude

12. A general Lebesgue space is a probability space composed of a part isomorphic to a subinterval of the unit interval (with Lebesque measure) and a part consisting of a finite or countable number of atoms.

13. E.g., conjugacy and isomorphism are equivalent for Lebesque spaces.

the possibility that the partition is, in a significant sense, unmeasurable, since typical elements of the partiton may be of measure zero. A measurable partition ζ [2, 37] of a Lebesque space X can be <u>generated</u> by a countable family $\{\Gamma_i\}_{i \in Z}$ of measurable subsets of X (we write $\zeta = \zeta(\{\Gamma_i\})$) in the sense that two points of X are in the same element of ζ if, and only if, for every $i \in Z$ they are either both in Γ_i or both in the complement of Γ_i. By means of such families one can establish a one to one correspondence between measurable partitions and sub - σ - algebras. It is the measurable partitions which possess a "canonical system of measure", admitting a generalization of iterated integrals [37, 12].

We conclude the discussion of Lebesgue spaces with two important theorems [37]:

1) A countable family $\{\Gamma_i\}$ of measurable subsets generates the full σ - algebra Σ (mod 0) if, and only if, it separates the points of X (mod 0)[14].

2) A factor space of a Lebesgue space with respect to a measurable partition is a Lebesgue space.[15]

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14. In the sense that for any pair of points in X we can find a member of the family containing one of the points but not the other.

15. The <u>factor space</u> of the measure space (X, Σ, μ) with respect to the partition ζ is the space whose elements are the fibers of ζ, with measure induced by μ.

If ζ_1 and ζ_2 are partitions we write $\zeta_1 \leq \zeta_2$ and say that ζ_2 is finer than ζ_1 (ζ_1 is coarser than ζ_2) if the elements of ζ_1 are unions of elements of ζ_2. If $\{\zeta_\alpha\}$ is any family of measurable partitions (mod 0), we denote by $\bigvee_\alpha \zeta_\alpha$ the coarsest measurable partition (mod 0) finer (mod 0) than all the ζ_α, and by $\bigwedge_\alpha \zeta_\alpha$, the finest measurable partition (mod 0) coarser (mod 0) than all the ζ_α. $\zeta_1 \vee \zeta_2$ is the partition whose elements are the intersections of the elements of ζ_1 and ζ_2. For P a countable partition and T an automorphism of a measure space X, we will also denote by $\bigvee_{i=-\infty}^{\infty} T^i P$ the σ-algebra generated by the sets of the family of partitions $\{T^j P\}_{j \in \mathbb{Z}}$: We will say that P is a generator for T if $\bigvee_{i=-\infty}^{\infty} T^i P$ is the full σ-algebra, Σ.

We can map the dynamical system (X, Σ, μ, T) onto a process (with the shift on doubly infinite sequences as the automorphism) determined by P (the (P,T) - process) by mapping each point $x \in X$ onto the doubly infinite sequence of labels of elements of P whose j<u>th</u> member is the label of the element of P containing $T^j x$ (the P-name of x) and equipping the sequences with the (stationary) measure induced by μ. If P is a generator for T, (X, Σ, μ, T) is, in fact, isomorphic to the (P,T)-process [32].

Due to the coarseness of realistic measurements of physical systems and other practical limitations, we can associate with such a measurement a finite partition P of the phase space of the system (representing the set of distinguishable outcomes.) If we subject the system to "constant" observation, the best we could hope to accomplish would be to perform a sequence of such measurements separated by time intervals of some nonvanishing length τ. Thus a (P, T_τ)-process can be regarded as a mathematical model of the realistic observation of a physical system. In the case of a K-system, as we shall see, such a process must be nondeterministic in the sense that the present is not uniquely determined by the entire past.

Formally, a dynamical system (X, Σ, μ, T) is said to be a K-system (and T a K-automorphism) if there exists a measurable partition ζ (a K-partition) such that

1) $T\zeta \geq \zeta$ (mod 0);

2) $\bigvee_n T^n \zeta = \varepsilon$(mod 0), where ε is the partition of X into its points;

3) $\bigwedge_n T^n \zeta = \nu$(mod 0), where ν is the trivial partition of X whose sole element is X itself. (For the definition of a K-flow, see Chapter IV, section 5.) Geometrically, this definition indicates a sense in which K-systems are unstable: the fibers of ζ, which as time evolves in one direction contract to single points, in the other direction expand to "fill the entire space".

There are many equivalent formulations of the concept of a K-system. We here give two other useful formulations [43]:

a) A system (X, Σ, μ, T) is a K-system if, and only if, for every finite partition P and every subset $A \in \Sigma$ we have

$$\lim_{n \to \infty} \sup_{C \in \bigvee_{j=-\infty}^{-n} T^j P} |\mu(A \cap C) - \mu(A)\,\mu(C)| = 0.$$

b) (X, Σ, μ, T) is a K-system if, and only if, all finite partitions P have trivial tails (i.e., $\bigcap_{n=0}^{\infty} \bigvee_{j=-\infty}^{-n} T^j P$ contains only sets of measure zero or measure one.)

It follows from b) that K-systems are completely nondeterministic: The "remote past" of all processes determined by a nontrivial finite partition P of a K-system contains no information, implying, in particular, that such processes are nondeterministic (i.e., $P \not\leq \bigvee_{j=1}^{\infty} T^j P$). a) implies that K-systems are mixing. Not only is the K-system property stronger than mixing, but K-systems, in fact, have homogeneous Lebesgue spectrum of (countably) infinite multiplicity [2]. Thus, since, as we shall see, not all K-systems are isomorphic, the spectral invariants do not form a complete set of invariants.

An example of a K-system, of which we shall later make much use, is the baker's transformation [2], $(B, \Sigma_o, \mu_o, T_o)$. (B, Σ_o, μ_o) is the unit square, $\{(x,y) \in \mathbb{R}^2 : 0 \leq x,y < 1\}$, with Lebesgue measure and

$$T_o(x,y) = \begin{cases} (2x, \tfrac{1}{2}y) & \text{if } x < \tfrac{1}{2} \\ (2x-1, \tfrac{1}{2}y + \tfrac{1}{2}) & \text{if } x \geq \tfrac{1}{2} \end{cases} .$$

It is not difficult to see that γ_o, the partition of B into vertical lines, is a K-partition for the baker's transformation.

An important class of K-systems consists of the Bernoulli shifts, which we denote by $B(p_o, p_1, \ldots, p_{n-1})$ ($p_i > 0$, $\Sigma p_i = 1$). The measure space of $B(p_o, \ldots, p_{n-1})$ is the measure theoretic product of a doubly infinite sequence of copies of the space $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with measure given by the probability vector $(p_o, \ldots, p_{n-1})$. The automorphism S of $B(p_o, \ldots, p_{n-1})$ is the shift on doubly infinite sequences:

$$(S\xi)_j = \xi_{j+1} \quad , \quad \xi = (\ldots, \xi_{-1}, \xi_o, \xi_1, \ldots), \; \xi_i \in Z_n, \; i \in Z.$$

The partition corresponding to the σ-algebra generated by the variables ξ_j, $j \geq 0$ is a K-partition, as follows from the zero-one law for tail events [10].

The baker's transformation is isomorphic to the Bernoulli shift $B(\frac{1}{2}, \frac{1}{2})$ [2]; the isomorphism is realized by the mapping

$$\varphi: \; \mathbb{Z}_n^{\mathbb{Z}} \to B$$

$$\xi \mapsto \varphi(\xi) = (x,y) = (.\,\xi_o\, \xi_1\, \xi_2 \ldots, .\xi_{-1}\, \xi_{-2} \ldots).$$

In the above we have expressed x and $y \in [0,1)$ in binary notation. Until recently it was believed that every K-system is isomorphic to some Bernoulli shift. However, Ornstein [40] has found an uncountable family of K-systems which are not isomorphic to Bernoulli shifts.

Since two Bernoulli shifts cannot be distinguished by any of the invariants we have so far discussed, it was wondered for a long time whether all Bernoulli shifts might not be isomorphic. The question was answered negatively with the introduction by Kolmogorov [19] of a new metric invariant: the entropy.

7. Entropy [4]

The entropy can be regarded as a measure of the extent to which a process or a system is nondeterministic. We will define it in stages.

The entropy of a countable partition $P = \{P_i\}$, defined by $H(P) = - \sum_i \mu(P_i) \log \mu(P_i)$, is a measure of the information contained in P (or of the average "uncertainty" removable by a determination of which element of P contains the state of our system.) A key fact about the entropy of a partition is that it is of the form $\Sigma\, \eta(\mu(P_i))$, with η strictly concave in the unit interval.

The conditional entropy of the partition $P = \{P_i\}$ given the partition $Q = \{Q_j\}$ is defined by

$$H(P\|Q) = \sum_j \mu(Q_j)\, H(P|Q_j) = -\sum_j \mu(Q_j) \sum_i \mu(P_i|Q_j) \log \mu(P_i|Q_j),$$

with $\mu(P_i|Q_j) = \mu(P_i \cap Q_j)/\mu(Q_j)$.

It is a measure of the information contained in P above and beyond the information already contained in Q. Some important relations involving conditional entropy are the following:

1) $H(P \vee Q \| R) = H(P \| R) + H(Q \| P \vee R)$,

2) $H(P \| R) \leq H(Q \| R)$ if $P \leq Q$,

3) $H(P \| Q) \geq H(P \| R)$ if $Q \leq R$,

and, in particular,

4) $0 \leq H(P \| Q) \leq H(P)$, with equality attained on the left if, and only if, $P \leq Q$, and on the right if, and only if, P and Q are independent (i.e., $\mu(P_i \cap Q_j) = \mu(P_i)\, \mu(Q_j)$ for all i,j).[16]

The <u>entropy of a partition</u> $P = \{P_i\}$ <u>relative to an automorphism</u> T[17] is given by

$$h(P,T) = \lim_{n\to\infty} 1/n \; H\Big(\bigvee_{j=0}^{n-1} T^{j}P\Big)$$

$$\Big(= \lim_{n\to\infty} 1/n \; \big(H(P) + H(P \| T^{-1}P) + \cdots + H(P \| \bigvee_{j=1}^{n-1} T^{-j}P)\big)$$

$$= \lim_{n\to\infty} H(P \| \bigvee_{j=1}^{n-1} T^{-j}P) = H(P \| \bigvee_{j=1}^{\infty} T^{-j}P)\Big).$$

It is a measure of the asymptotic rate at which the (P,T) process produces information. It follows from 4) that $h(P,T) > 0$ if, and only if, the (P,T)-process is nondeterministic. If $h(P,T) > 0$ for

16. The above results can be extended to embrace general measurable partitions as well as countable ones [33].

17. The entropy of the (P,T)-process

every nontrivial partition P, the automorphism T is said to have completely positive entropy. A theorem of Rohlin and Sinai says that T is a K-automorphism if, and only if, T has completely positive entropy [43]. Thus K-systems are precisely those systems which are completely nondeterministic.

Finally, the entropy of an automorphism T is defined by $h(T) = \sup_{P \text{ finite}} h(P,T)$. (The supremum could in fact be taken over all partitions of finite entropy [33].)[18] h(T) is clearly an invariant. Furthermore, by virtue of a theorem [4] of Kolmogorov and Sinai which says that if P is a generator for T $h(T) = h(P,T)$, the entropy can be easily computed for many systems. In particular, the partition P_o determined by the coordinate ξ_o of a Bernoulli shift is clearly a generator for S. In addition, it has the property that the sequence P_o, SP_o, $S^2 P_o, \ldots$ forms an independent sequence of partitions [39] (thus P_o is said to be an independent generator), so that, as follows from 1) and 4), $h(P_o,S) = H(P_o)$. The entropy of $B(p_o,p_1,\ldots,p_i,\ldots)$ is thus given by $-\sum_i p_i \log p_i$. It is trivial that two Bernoulli shifts with different entropies cannot be isomorphic; but whether all Bernoulli shifts with the same entropy (e.g., $B(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$ and

18. The entropy of a flow $\{T_t\}$ is defined as the entropy of T_1. By a "formula of Abramov" [5] $h(T_t) = |t|\, h(T_1)$.

B(½, 1/8, 1/8, 1/8, 1/8)) are isomorphic proved to be a difficult problem.

8. Bernoulli systems and Ornstein's theorems

It is convenient to extend the notion of Bernoulli Shift to that of the generalized Bernoulli Shift; a generalized Bernoulli Shift is constructed as is a Bernoulli shift except that the probability space of which we form a doubly infinite product can be taken to be any Lebesque space rather than only a discrete space. In investigating the question of isomorphism between Bernoulli Shifts it proved useful to characterize systems isomorphic to $B(\ldots p_i \ldots)$ as systems which possess an independent generator $P = \{P_i\}$ for which $\mu(P_i) = p_i$ for all i (with an analagous result for a generalized Bernoulli shift.) The (generalized) Bernoulli shifts are clearly systems with the strongest possible stochastic properties: if P is an independent generator then the (P,T)-process is completely random.

Ornstein's main result concerning (generalized) Bernoulli shifts is the following: Two Bernoulli shifts with the same entropy (which may be infinite) are isomorphic [26, 27]. Thus the entropy is a complete invariant for Bernoulli shifts.

The flow $\{T_t\}$ is said to be a Bernoulli flow if T_1 is a Bernoulli shift. Ornstein [29,30] has shown that Bernoulli flows exist, that two Bernoulli flows of finite entropy are isomorphic (except possibly for a change in the scale of time which may be necessary to insure that the flows have equal entropy), and that two Bernoulli flows of infinite entropy are isomorphic. Since there exists a simple standard Bernoulli flow $\{S_t\}$ (a certain flow built under a function) such that for each t, S_t can be shown to be Bernoulli [29], it follows from Ornstein's theorem on Bernoulli flows that if $\{T_t\}$ is a Bernoulli flow, T_τ is a Bernoulli shift for any $\tau \in \mathbb{R}$. Thus to show that a flow $\{T_t\}$ is Bernoulli, it suffices to show that for some t_o, T_{t_o} is a Bernoulli shift.

If P is an independent generator for T, the (P,T)-process is obviously isomorphic to the process representing the behavior of a fair roulette wheel. Ornstein has shown that if P is any finite partition and T a Bernoulli shift, the (P,T)-process (a B-process) can be approximated arbitrarily well by finite codings of a roulette wheel, or by a multistep mixing Markov process[19] [32].

Sinai has shown that the time evolution of the microcanonical ensemble of the hard sphere gas in a box is a K-flow [41]. Gallavotti and Ornstein [31] have augmented Sinai's argument to show that this system is, in fact, a Bernoulli flow. We thus

19. Intuitively, two processes are close if one of the processes can be obtained by infrequent modification of the other process.

have an example of a mechanical system which is a moderately accurate model of a realistic physical system and which has a representation as, and for which certain complete measurements may form, a totally random process[20]. Furthermore, by virtue of a theorem of Sinai asserting that for an ergodic automorphism T with $h(T) \geq -\Sigma\, p_i \log p_i$ ($p_i \geq 0$, $\Sigma p_j = 1$), there exists a partition $P = \{P_i\}$ for which $\mu(P_i) = p_i$ for all i , and such that the $T^j P$ are independent [43], typical (i.e., ergodic with nonzero entropy) mechanical (Hamiltonian) systems are homomorphic to totally random processes.

We conclude by observing that any mechanical realization of a Bernoulli flow provides and "upper bound" on the extent to which the formal ergodic theoretic structure to which we have referred can account for "good thermodynamic behavior", since any two Bernoulli flows are formally identical (unless one has finite entropy and the other infinite entropy.)

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20. The measurements to which we refer are complete in the sense that if they are performed periodically throughout all of time, the state of the system can be completely determined.

II. Ergodic Properties Of Simple Model System With Collisions+

1. Introduction

We are interested in the ergodic properties of dilute gas systems. These may be thought of as Hamiltonian dynamical systems in which the particles move freely except during binary 'collisions'. In a collision the velocities of the colliding particles undergo a transformation with 'good' mixing properties (c.f. Sinai's study of the billiard problem [41]). To gain an understanding of such systems we have studied the following simple discrete time model: The system consists of a single particle with coordinate $\underline{r} = (x,y)$ in a two dimensional torus with sides of length (L_x, L_y), and 'velocity' $\underline{v} = (v_x, v_y)$, in the unit square, $v_x \in [0,1)$, $v_y \in [0,1)$. The phase space Γ is thus a direct product of the torus and the unit square. The transformation T which takes the system from a dynamical state $(\underline{r}, \underline{v})$ at 'time' j to a new dynamical state $T(\underline{r}, \underline{v})$ at time j + 1 may be pictured as resulting from the particle moving freely during the unit time interval between j and j + 1 and then undergoing a 'collision' in which its velocity changes according to the baker's transformation, i.e.

+Most of this chapter has been taken from [15].

$$T(\underline{r}, \underline{v}) = (\underline{r} + \underline{v}, B\underline{v}),$$

with

$$B(v_x, v_y) = \begin{cases} (2v_x, \tfrac{1}{2} v_y), & 0 \leq v_x < \tfrac{1}{2} \\ (2v_x - 1, \tfrac{1}{2} v_y + \tfrac{1}{2}), & \tfrac{1}{2} \leq v_x < 1 . \end{cases}$$

The normalized Lebesgue measure $d\mu = dxdydv_x dv_y / L_x L_y = d\underline{r}\, d\underline{v} / L_x L_y$ in Γ is left invariant by T. We call U_T the unitary transformation induced by T on $L^2(d\mu)$, $U_T \emptyset = \emptyset \circ T$. Our interest lies then in the ergodic properties of T and in the spectrum of U_T.

We note first that the transformation B on the velocities is, when taken by itself as a transformation of the unit square with measure $d\underline{v}$, well known to be isomorphic to a Bernoulli shift. It has therefore got very good mixing properties. The isomorphism is obtained by setting

$$v_x = \sum_{j=1}^{\infty} 2^{-j} \xi_j , \quad v_y = \sum_{j=1}^{\infty} 2^{-j} \xi_{1-j} ,$$

with the ξ_j independent random variables taking the values 0 and 1 each with probability $\frac{1}{2}$. We then have

$$(B\underline{v})_x = \sum_{j=1}^{\infty} 2^{-j} \xi_{j+1} = 2v_x - \xi_1 ,$$

$$(B\underline{v})_y = \sum_{j=1}^{\infty} 2^{-j} \xi_{2-j} = \tfrac{1}{2} v_y + \tfrac{1}{2} \xi_1 .$$

2. Ergodic properties.

The ergodic properties of our system which combines B with free motion turn out to depend on whether L_x^{-1} and L_y^{-1} satisfy the independence condition (I),

$$n_x L_x^{-1} + n_y L_y^{-1} \notin \mathbb{Z} \text{ for } n_x \text{ and } n_y \text{ integers unless } n_x = n_y = 0$$

Theorem 1: When (I) holds the spectrum of U_T, on the complement of the one-dimensional subspace generated by the constants, is absolutely continuous with respect to Lebesgue measure and has infinite multiplicity.

It follows from Theorem 1 that when (I) holds the dynamical system (Γ, T, μ) is at least mixing. We do not know at present whether it is also a Bernoulli shift or at least a K-system.

Theorem 2: When (I) does not hold the system (Γ, T, μ) is not ergodic.

The proof of Theorem 1 has two parts: a general characterization of unitary operators with Lebesgue spectrum and a set of estimates.

Lemma: Let U be a unitary operatory on a Hilbert space $\mathcal{H}$, with spectral representation $U = \int_0^{2\pi} e^{i\theta} \underline{P}(d\theta)$. Assume that there exists a total* set of vectors $\{\phi_i\}$ such that $\sum_{n=1}^{\infty} |(U^n\phi_i|\phi_i)| < \infty$ for all i. Then the spectral measure $\underline{P}(d\theta)$ is absolutely continuous with respect to Lebesgue measure, i.e., if E is a Borel set of Lebesque measure 0, then $\underline{P}(E) = 0$.

<u>Proof</u>: We have

$$(U^n \phi_i | \phi_i) = \int e^{in\theta} (\underline{P}(d\theta) \phi_i | \phi_i), \text{ i.e., the function}$$

$$n \mapsto (U^n\phi_i|\phi_i)$$

is the Fourier transform of the measure $(\underline{P}(d\theta)\phi_i|\phi_i)$. On the other hand, $\sum_n |(U^n\phi_i|\phi_i)| < \infty$, so we can compute its inverse Fourier transform in the elementary way. By the uniqueness of the Fourier transform, we get:

$$(\underline{P}(d\theta)\phi_i|\phi_i) = \frac{d\theta}{2\pi} \cdot \sum_{n=-\infty}^{\infty} e^{-in\theta} (U^n \phi_i|\phi_i) \quad ,$$

so the numerical measure $(\underline{P}(d\theta)\phi_i|\phi_i)$ is absolutely continuous with respect to Lebesque measure. If E is a Borel set of Lebesque measure 0,

$$\|\underline{P}(E) \phi_i\|^2 = (\underline{P}(E) \phi_i|\phi_i) = 0, \text{ so } \underline{P}(E)\phi_i = 0 \text{ for all } \phi_i.$$

But the vectors $\{\phi_i\}$ form a total set, so $\underline{P}(E) = 0$ as desired.

Now the estimates: Let $\chi(1) = 1$; $\chi(0) = -1$. For each finite subset X of Z, we define

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* A set of vectors is said to be <u>total</u> if the finite linear span of this set of vectors is dense.

$$\chi_X(\underline{v}) = \prod_{j\in X} \chi(\xi_j).$$

The χ_X form an orthonormal basis for $L^2(d\underline{v})$. Similarly the functions $\exp(i\underline{k}\cdot\underline{r})$, $\underline{k}=(k_x,k_y)$, $k_x = 2\pi n_x/L_x$, $k_y = 2\pi n_y/L_y$, n_x and n_y integers , form an orthogonal basis for $L^2(d\underline{r})$. Thus, the functions $\emptyset_{X,\underline{k}} = \exp(i\underline{k}\cdot\underline{r})\,\chi_X(\underline{v})$ form a orthonormal basis for $L^2(d\mu)$. We will prove that $\sum_{n=1}^{\infty} |(U_T^n \emptyset_{X_1,\underline{k}_1} | \emptyset_{X_2,\underline{k}_2})| < \infty$ unless $\underline{k}_1 = \underline{k}_2 = 0$; $X_1 = X_2 = 0$.

By straightforward computation,

$$U_T^n \emptyset_{X_1,\underline{k}_1} = \emptyset_{X_1+n}(\underline{v}) \exp(i\underline{k}\cdot\underline{r}) \exp(i\underline{k}\cdot(\underline{v} + B\underline{v} + \ldots + B^{n-1}\underline{v})).$$

Thus

$$\int d\underline{r}\, (U_T^n \emptyset_{X_1,\underline{k}_1})\, \overline{\emptyset_{X_2,\underline{k}_2}} = 0 \text{ unless } \underline{k}_1 = \underline{k}_2 \ (=\underline{k}).$$

so we assume $\underline{k}_1 = \underline{k}_2 = \underline{k}$. Also,

$$\int d\underline{v} (U_T^n \emptyset_{X_1,0}) \emptyset_{X_2,0} = 0 \text{ unless } X_2 = X_2 + n,$$

so the result is trivally true for $\underline{k} = 0$. We therefore assume $\underline{k} \neq 0$.

Now

$$(L_xL_y)^{-1}\int d\underline{r}d\underline{v}\left(U_T^n\phi_{X_1,\underline{k}}\right)\bar{\phi}_{X_2,\underline{k}}=\int d\underline{v}\chi_{X_1}(B^n\underline{v})\chi_{X_2}(\underline{v})\exp(i\underline{k}(\underline{v}+B\underline{v}+\ldots B^{n-1}\underline{v})),$$

$$(B^j\underline{v})_x=\sum_{i=1}^{\infty}\xi_{j+1}2^{-i}\quad\sum_{j=0}^{n-1}(B^j\underline{v})_x=\sum_{j=0}^{n-1}\sum_{i=1}^{\infty}\xi_{j+1}2^{-i}=\sum_{\ell=1}^{\infty}\xi_\ell\sum_{i=1\vee(\ell-n+1)}^{\ell}2^{-i}$$

$=\sum_{\ell=1}^{\infty}\xi_\ell\alpha_\ell^n$ (where this equation defines α_ℓ^n).

$$(B^j\underline{v})_y=\sum_{i=1}^{\infty}2^{-i}\xi_{j+1-i},\quad\sum_{j=0}^{n-1}(B^j\underline{v})_y=\sum_{j=0}^{n-1}\sum_{i=1}^{\infty}2^{-i}\xi_{j+1-i}$$

$$=\sum_{\ell=-\infty}^{n-1}\xi_\ell\sum_{i=1\vee(-\ell+1)}^{n-\ell}2^{-i}=\sum_{\ell=-\infty}^{\infty}\xi_\ell\beta_\ell^n\,.$$

Now let $\ell_2=1\vee\max\{X_2\}$, $\ell_1=\inf\{X_1\}\wedge 0$.

Then

$$U_T^n\phi_{X_1,\underline{k}}\cdot\bar{\phi}_{X_2,\underline{k}}=\prod_{\ell=\ell_2+1}^{n+\ell_1-1}e^{i(\alpha_\ell^n k_X+\beta_\ell^n k_y)\xi_\ell}\times[\text{fn of the }\xi_\ell\text{'s for }\ell\notin(\ell_2,n+\ell_1)]$$

By independence, the integral of the product on the right is the product of the integrals, and the unspecified function of the ξ_ℓ's, $\ell\notin(\ell_2,n+\ell_1)$ is no greater than one in absolute value, so

$$(L_xL_y)^{-1}\left|\int d\underline{v}d\underline{r}\,U_T^n\phi_{X_1,\underline{k}}\cdot\bar{\phi}_{X_2,\underline{k}}\right|\le\prod_{\ell=\ell_2+1}^{n+\ell_1-1}\left|\tfrac{1}{2}\left[\exp(i\alpha_\ell^n k_x+i\beta_\ell^n k_y)+1\right]\right|.$$

For ℓ's within the limits of the product, we have

$$\alpha_\ell^n = \sum_{i=1}^{\ell} 2^{-i} = 1 - 2^{-\ell}$$

$$\beta_\ell^n = \sum_{i=1}^{n-\ell} 2^{-i} = 1 - 2^{-(n-\ell)} \quad .$$

Thus, for most of the terms in the product, $\alpha_\ell^n \approx \beta_\ell^n \approx 1$, and the number of terms is n - const. for large n. In particular, if we put

$\gamma = \frac{1}{2}\left|\exp\left[i(k_x + k_y)\right]+1\right| < 1$ (by our fundamental assumption),

$\left|(U_T^n \emptyset_{X_1,\underline{k}} \mid \emptyset_{X_2,\underline{k}})\right| < \gamma^{n/2}$ for all sufficiently large n, and we have

$$\sum_{n=1}^{\infty} \left|(U_T^n \emptyset_{X_1,k} \mid \emptyset_{X_2,k})\right| < \infty$$

as desired.

The fact that the mulitplicity is infinite is trivial. We have $L^2(d\underline{v}) \subseteq L^2(d\underline{r}d\underline{v})$, and we already know that the spectrum of U_T restricted to $L^2(d\underline{v})$ has infinite multiplicity.

To obtain a proof of Theorem 2, we note that ergodicity is equivalent to

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \int d\mu \, (U_T^n \emptyset)\overline{\Psi} = \left(\int d\mu \emptyset\right) \left(\int d\mu \overline{\Psi}\right), \ \emptyset, \ \Psi \in L^2(d\mu).$$

For $\emptyset$ or Ψ orthogonal to the constants we must then have Cesaro convergence to zero when the system is ergodic. We prove that the system is nonergodic by finding $\emptyset$ and Ψ orthogonal to the constants such that the above integral converges (strictly) to a non zero number.

Let n_x, n_y be such that $n_x/L_x + n_y/L_y \in Z$ and n_x and n_y are not both 0, and let $k_x = 2\pi\, n_x/L_x$, $k_y = 2\pi\, n_y/L_y$. We set $\emptyset = \Psi = \emptyset_{0,\underline{k}}$ and compute as before the relevant integrals:

$$I_n = \int d\mu\; U_T^n \emptyset_{0,\underline{k}} \cdot \bar{\emptyset}_{0,\underline{k}} = \int d\underline{v}\, \exp\left[i\underline{k} \cdot \left(\sum_{j=0}^{n-1} B^j\, \underline{v} \right) \right]$$

$$= \int d\underline{v} \prod_{\ell=\infty}^{\infty} \exp\left[i(k_x\, \alpha_\ell^n + k_y\, \beta_\ell^n)\, \xi_\ell \right]$$

$$= \prod_{\ell=-\infty}^{\infty} \tfrac{1}{2} \left[1 + \exp\,(i\, \alpha_\ell^n\, k_x + i\, \beta_\ell^n\, k_y) \right].$$

Here $\alpha_\ell^n = \sum_{i=1\vee(\ell-n+1)}^{\ell} 2^{-i} = 2^{-\ell} \sum_{m=0}^{(n-1)\wedge(\ell-1)} 2^m = 2^{-\ell}(2^{n\wedge\ell}-1)$ for $\ell > 0$ and vanishes for $\ell \leq 0$, and $\beta_\ell^n = \sum_{i=1\vee(-\ell+1)}^{n-\ell} 2^{-i} = 2^{\ell-1} \sum_{m=0\vee\ell}^{n-1} 2^{-m} = 2^{0\wedge\ell} - 2^{\ell-n}$

for $\ell < n$ and vanishes for $\ell \geq n$.

We thus have found that

$$I_n = \prod_{\ell=-\infty}^{0} \tfrac{1}{2} \left\{ 1 + \exp\left[i\,(2^{\ell} - 2^{\ell-n})\, k_y \right] \right\}$$

$$\times \prod_{\ell=1}^{n-1} \tfrac{1}{2} \left\{ 1 + \exp i \left[(1-2^{-\ell})\, k_x + (1-2^{-(n-\ell)})\, k_y \right] \right\}$$

$$\times \prod_{\ell-n}^{\infty} \tfrac{1}{2} \left\{ 1 + \exp\left[i\, k_x (2^{-(\ell-n)} - 2^{-\ell}) \right] \right\}$$

$$= F_n^1(\underline{k})\, F_n^2(\underline{k})\, F_n^3(\underline{k})$$

with $F_n^1(\underline{k}) = F_n^1(k_y) = \prod_{m=0}^{\infty} \tfrac{1}{2} \left\{ 1 + \exp\left[i\, k_y\, (2^{-m} - 2^{-(m+n)}) \right] \right\}$

$$F_n^3(\underline{k}) = F_n^3(k_x) = F_n^1(k_x)$$

$$F_n^2(\underline{k}) = \prod_{\ell=1}^{n-1} \tfrac{1}{2} \left(1 + \exp\left[i(1-2^{-\ell})\, k_x + i(1-2^{-(n-\ell)})k_y\right]\right)$$

Since $k_x + k_y \in 2\pi\,\mathbb{Z}$, we have

$$F_n^2(\underline{k}) = \prod_{\ell=1}^{n-1} \tfrac{1}{2} \left\{1 + \exp\left[-i\,(k_x\, 2^{-\ell} + k_y\, 2^{-(n-\ell)})\right]\right\}.$$

We now complete the proof by showing that (for $k_x + k_y \in 2\pi\,\mathbb{Z}$)

$$\lim_{n\to\infty} F_n^i(\underline{k}) = \alpha^i \neq 0 \;,\; i = 1,2,3.$$

We do this by showing that the $\log F_n^i(\underline{k})$ converge to a finite limit.

We first show that

$$\sum_{m=0}^{\infty} \left|\log \tfrac{1}{2}\, (1 + \exp ik/2^m)\right| < \infty.$$

We have $\log \frac{1}{2}\,(1 + \exp i\, k/2^m) = \log \frac{1}{2}\,\left|1+\exp ik/2^m\right| + i\emptyset(i+\exp ik/2^m)$ where $\emptyset(z) = \theta$ for $z = |z|\, e^{i\theta}$. Now $\log \frac{1}{2}\,\left|1+\exp ik/2^m\right| = \frac{1}{2} \log \frac{1}{2}(1+\cos k/2^m)$ $> \log \cos k/2^m > \frac{1}{2} \log\,(1-\frac{1}{2}k^2/4^m) > -\frac{1}{2}\, k^2/4^m$, and $\emptyset\,(1+\exp ik/2^m)$ $> \emptyset\,(\exp ik/2^m) = k/2^m$ for m sufficiently large. Since $\Sigma(\frac{1}{2})^m < \infty$ and $\Sigma(\frac{1}{4})^m < \infty$ we have the desired result (assuming k is such that the first few terms are well behaved).

We now consider $\log F_n^1(\underline{k}) = \sum_{m=0}^{\infty} \log \frac{1}{2}\,(1+\exp ik_y(2^{-m}-2^{-(m+n)}))$. We show that $\lim_{n\to\infty} \log F_n^1(\underline{k})$ exists and is finite. For m sufficiently large we have

$$\left|\log \tfrac{1}{2} \left\{1+\exp[ik_y(2^{-m}-2^{-(m+n)})]\right\}\right| < \left|\log \tfrac{1}{2} (1+\exp ik_y/2^m)\right|$$

and we have just shown that $\Sigma\left|\log \tfrac{1}{2} (1+\exp ik_y/2^m)\right| < \infty$. Thus $\lim_{n \to \infty} \sum_{m=0}^{\infty} \log \tfrac{1}{2} \{1+\exp[ik_y(2^{-m}-2^{-(m+n)})]\} = \sum_{m=0}^{\infty} \lim_{n \to \infty} \log\{\tfrac{1}{2} \times (1+\exp[ik_y(2^{-m}-2^{-(m+n)})])\} = \sum_{m=0}^{\infty} \log \tfrac{1}{2} (1+\exp ik_y/2^m) = \alpha$ and $|\alpha|<\infty$.

We have thus shown that F_n^1 and F_n^3 have the desired properties.

We now consider

$$\log F_n^2(\underline{k}) = \sum_{m=1}^{n-1} \log \tfrac{1}{2} \{1+\exp[-i\,(k_x/2^m + k_y/2^{n-m})]\}$$

$$= \sum_{m=1}^{n-1} Amn = \sum_{m=1}^{[(n-1)/2]} Amn + \sum_{m=[n/2]+1}^{n-1} Amn + Cn \quad ,$$

$$\text{where } Cn = \begin{cases} 0 & , n \text{ odd} \\ \log A_{n/2,n}, & n \text{ even} \end{cases} .$$

$$\log F_n^2(\underline{k}) = G_n(k_x,k_y) + G_n(k_y,k_x) + C_n \quad ,$$

where $G_n(k_x,k_y) =$

$$\sum_{m=1}^{[(n-1)/2]} \log \tfrac{1}{2} \left\{1+\exp[-i\,(k_x/2^m + k_y/2^{n-m})]\right\}.$$

Since $C_n \to 0$, we conclude the proof by showing that G_n converges to a finite limit.

$$G_n(k_x,k_y) = \sum_{m=1}^{\infty} B_{mn},$$

$$B_{mn} = \begin{cases} A_{mn}, & m \leq [(n-1)/2] \\ 0, & m > [(n-1)/2] \end{cases}$$

and

$$|B_{mn}| < |\log \tfrac{1}{2} \{1 + \exp[-i\,(k_x + k_y)/2^m]\}| = D_m,$$

for m sufficiently large.

We have shown that $\Sigma\, Dm < \infty$. Thus $\lim_{n\to\infty} G_n(k_x,k_y) = \sum_{m=1}^{\infty} \lim_{n\to\infty} Bmn = \sum_{m=1}^{\infty} \log \tfrac{1}{2}(1 + \exp ik_x/2^m) < \infty$.

(If k_x and k_y are such that some of the terms at the beginning of the series which we discussed are singular, one easily removes the difficulty by an appropriate change in the functions $\emptyset$ and Ψ introduced at the beginning of the proof of theorem 2.

We also note that for the case where L_x/L_y is rational we can find explicitly a nonconstant function f which is left invariant by U_T. From the fact that $U_B\,(v_x + 2\,v_y) = 2v_x + v_y$, it follows that $f(x-y-v_x-2v_y)$ is invariant if f is doubly periodic with periods L_x and L_y, so that we can construct an infinite family of orthonormal invariant functions f_n : $f_n = \exp\{(i\,2\pi n/L)\,(x-y-v_x-2v_y)\}$ with $L_x/r = L_y/s = L$, r and s integers.)

III Infinite Systems

1. Importance

A key feature of the systems which are treated in statistical mechanics is that they consist of a very large number of subsystems; it is only in such a limit that one expects thermodynamic behavior to be exhibted. Rather than taking limits it is natural to employ infinite systems ab initio , in the hope that, in exhibiting "exact" thermodynamic behavior, the intricacies unrelated to thermodynamic behavior which are associated with a large but finite number of degrees of freedom might be avoided. Moreover, new and powerful modes of description and mathematical tools are suggested by a consideration of infinite systems. For example, the translation invariance of particle interactions, unencumbered by the walls between which finite systems evolve, implies the possibility of a translation invariant description of infinite systems, which corresponds to the homogeneity of actual physical systems and is a powerful tool. Thus the study of the ergodic properties of the (statistical) states of infinite systems under translations is suggested.

2. Measures

In view of the above remarks, and the discussion in Chapter I, it is natural to consider the ergodic properties of the time evolution of the equilibrium states of infinite systems of particles. In this regard two problems immediately arise:

1) To what extent can the time evolution determined by

Hamilton's equations for finite systems be generalized to an infinite number of degrees of freedom?

2) What, if anything, is the analogue of the finite system microcanonical ensemble?

We shall discuss the latter question in this section.

We must first describe the phase space of an infinite system of particles moving in a (physical) space of dimension ν. We take as our phase space Γ the set of infinite locally finite configurations in $\mathbb{R}^\nu \otimes \mathbb{R}^\nu$, i.e., an element of Γ is a subset x of $\mathbb{R}^\nu \otimes \mathbb{R}^\nu$ for which the cardinality of $(V \otimes \mathbb{R}^\nu) \cap x$ is finite for bounded $V \subset \mathbb{R}^\nu$ [1]; we thus consider only states for which bounded regions of space contain only a finite number of particles.

It is clear that the microcanonical ensemble cannot be directly transported to an infinite system; for one thing there is no analogue of the energy surface: essentially all configurations have infinite energy. However, corresponding to given values ϵ and ρ of energy per unit volume and density (or, equivalently, $\beta = 1/kT$ and z of inverse temperature and activity) one can define an <u>equilibrium state</u> $\mathcal{U}_{\beta,z}$ as an infinite volume limit (in a suitable topology) of a sequence $\left\{\mathcal{U}_{\beta,z}^{V_i}\right\}$ of grand canonical ensembles at inverse termperature β and activity z in a finite

1. For many systems it is necessary to define an infinite locally finite configuration as a locally finite multiplicity function on $\mathbb{R}^\nu \otimes \mathbb{R}^\nu$ (which gives the number of particles at each point of $\mathbb{R}^\nu \otimes \mathbb{R}^\nu$); this definition will not be needed for the systems which we shall consider, except in section 3.

volume V_i determined by, say, a pair potential $\Phi(q_i - q_j)$ and suitable boundary conditions (corresponding, e.g., to a configuration of particles outside of V_i) [6, 36, 24]. For a suitable interaction Φ (e.g., for superstable [46] Φ) one can obtain in this way a probability measure on the quasi-local σ-algebra over $\mathbb{R}^\nu \otimes \mathbb{R}^\nu$, which is generated by the symmetric Lebesque measurable functions on configurations in $V \otimes \mathbb{R}^\nu$, where $V \subset \mathbb{R}^\nu$ ranges over all bounded Lebesque measurable sets. This measure is uniquely specified by its restrictions to finite volumes V; these can be described by a system of (symmetric) density distributions $\{u_V^n (q_1, p_1; q_2, p_2; \ldots; q_n, p_n)\}_{n \in \mathbb{Z}^+}$ on $\sum_{n=0}^{\infty} (V \otimes \mathbb{R}^\nu)^n$.

The equilibrium state at given β, z need not be unique (even given the particle interaction); the limit may depend, e.g., upon the sequence of boundary conditions. One can prove uniqueness for a dilute gas [36], but Dobrushin [7] has found examples of lattice gases for which inequivalent translation invariant equilibrium states exist for some values of β and z. A unique equilibrium state, which, of course, must be translation invariant, is, by virtue of Doob's martingale theorem[2], a K-system under translations[3], hence has good cluster properties [24, 6], and presumably represents a pure thermodynamic phase [35, 36]. More generally, unique translation invariant equilibrium

2. See Chapter IV, section 5.
3. For ergodic properties under several automorphisms, see Chapter V.

states or extremal translation invariant equilibrium states, which, of course, are translation ergodic, represent pure thermodynamic phases, while a non-ergodic translation invariant equilibrium state represents a mixture of coexisting phases (which are represented by its ergodic components. The extremal invariant components of an invariant equilibrium state are equilibrium states [36].)

If the infinite system is composed of noninteracting particles, the description of equilibrium states is greatly simplified; since in this case boundary conditions are of no consequence, there being no interaction with the "boundary", the unique limit of grand canonical ensembles is trivial to take. As we shall be dealing primarily with systems of this type, we shall soon describe their equilibrium states in a concise manner.

3. Time evolution

One can formally write down the infinite system of equations governing the motion of a system of infinitely many particles:

$$dq_i/dt = p_i; \; dp_i/dt = \sum_{j \neq i} F(q_i - q_j), \text{ with } F(q) = - \text{grad } \Phi(q).$$

However, for many configurations these equations may not make sense; some terms may diverge. For many more configurations the equations, though initially meaningful, induce a motion which after a finite time leads the system to a catastrophic configuration, in which the equations of motion are no longer meaningful. The problem, then, is to show that the equations admit of "unique" globally defined solutions for sufficiently many initial configurations to permit significant discussion of the time evolution of (equilibrium) states. It has been solved by O. Lanford [20, 21, 23], who has shown, in particular, the following:

1) For one-dimensional systems with suitable potential Φ, the set of initial configurations which do not admit of globally defined solutions satisfying a "regularity" condition has measure zero with respect to equilibrium states characterized by a suitable potential and activity z. [21]

2) For a ν-dimensional system (ν arbitrary) with rather unrestrictive conditions on the potential Φ, the set of initial configurations which do not admit of globally defined solutions, satisfying a regularity condition much more complicated than the one in 1), has zero measure with respect to any equilibrium state for the potential Φ. [23]

3) The regularity conditions admit of at most one solution with a given initial configuration. [20, 23]

4) The equilibrium states for the potential Φ are invariant under the time evolution induced by the regular solutions of the equations determined by Φ (for suitable Φ and z.) [21, 23]

Once again, as the systems with which we shall be concerned are of noninteracting particles, they will not be subject to the above difficulties; for these systems the time evolution can be trivially obtained from the time evolution of a single particle, and the equilibrium states will be trivially invariant under this evolution.

4. Physical interpretation and significance of ergodic properties of infinite systems.

Insofar as the ergodic properties under space translations are concerned, having already referred to ergodicity and K-mixing, we will merely remark that a state of a lattice system which is Bernoulli under translations can be globally approximated by a state induced by a finite range interaction[4].

Concerning the ergodic properties under time evolution, we observe that our previous assumption concerning sets of zero measure is not valid for infinite volume equilibrium states; in fact, inasmuch as an equilibrium state corresponding to a pure thermodynamic phase is ergodic (under translations), equilibrium states representing different phases are mutually singular [4]. Hence we cannot in general ascribe probability zero to sets of measure zero with respect to an equilibrium state, without a dynamical justification. Accordingly, the justification for the use of ergodic ensembles given in Chapter I cannot be applied to ergodic infinite volume equilibrium states.

4. See chapter I, section 8.

We have seen that mixing implies the approach to equilibrium of nonequilibrium states absolutely continuous with respect to the equilibrium state. In view of the preceding paragraph the restriction to absolutely continuous nonequilibrium states is severe for infinite systems. In fact, no spatially homogeneous nonequilibrium state can be absolutely continuous with respect to an equilibrium state representing a pure phase. In view of the quasilocal structure of equilibrium measures, a measure absolutely continuous relative to an equilibrium state represents a "local perturbation" of that state. Hence, for infinite volume equilibrium states mixing implies return to equilibrium, but not approach to equilibrium.

It is also of much less significance for an infinite system to be a K-system. Unlike the situation for finite systems, the requirement that no finite partition approximate the system sufficiently well to be deterministic is hardly a restriction at all; one cannot really expect to approximate an infinite system equipped with a quasi-local σ-algebra by a finite "coarse graining".

Consider, for example, the infinite ideal gas, an equilibrium state $\mu^I_{\beta,\rho}$ of which can be characterized by saying that the positions of the particles are Poisson distributed in $\mathbb{R}^\nu$ with density ρ and the velocities of the particles, which are independent of each other and of the positions, have identical Maxwellian distributions corresponding to the inverse temperature β.[5]

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5. μ_β is Maxwellian with inverse temperature β if $d\mu_\beta = \sqrt{\beta/2\pi}$ $\exp(-\frac{1}{2}\beta\vec{v}^{\,2})\, d\vec{v}$ (taking the mass to be unity).

$\mathcal{U}^I_{\beta,\rho}$ is invariant under the ideal gas time evolution T_t (induced by the evolution $\{q(t) = q + vt, v(t) = v\}$ for a single particle), and we will soon show that the time evolution of $\mathcal{U}^I_{\beta,\rho}$ is a Bernoulli flow. We observe however, that a probability measure $\mathcal{U}_{\nu,\rho}$, constructed like $\mathcal{U}^I_{\beta,\rho}$ except that the velocities of the particles are given a distribution ν, is also invariant under T_t; hence $\mathcal{U}_{\nu,\rho}$, which is singular with respect to $\mathcal{U}_{\beta,\rho}$ if $\mu_\beta \neq \nu$, does not approach any equilibrium state $\mathcal{U}^I_{\beta',\rho}$ (unless $\mu_{\beta'} = \nu$).

5. The Poisson construction

For $\mathcal{U}$ an equilibrium measure of an infinite system of noninteracting particles we define for every bounded Lebesque measurable set $A \subset \mathbb{R}^\nu \otimes \mathbb{R}^\nu$ a random variable $N(A)$ equal to the number of particles with coordinates in A. Let $\mathcal{U}_o(A) = \int d\mathcal{U}\, N(A)$. Since the particles are noninteracting, $N(A)$ is independent of $N(B)$ for A and B disjoint. Thus $N(A)$ has Poisson distribution with mean $\mathcal{U}_o(A)$. Furthermore the dynamics T_t of the infinite system may be represented by the equation

$$T_t x = T_{o,t} x \quad , \qquad x \in \Gamma$$

(where we regard x as a set on the right.) Here $\{T_{o,t}\}$ is a $\mathcal{U}_o$-flow on $\mathbb{R}^\nu \otimes \mathbb{R}^\nu$.

Now let (X, Σ, μ, T) be an automorphism of any σ-finite non atomic measure space. Let X_∞ be the set of countable subsets of X and for any $A \in \Sigma$ let $N(A)$ be the function on X_∞ such that for $x \in X_\infty$, $[N(A)](x)$ = cardinality of $A \cap x$. Let Σ_∞ be the σ-algebra generated by the "random variables" of the form $N(A)$, $A \in \Sigma$, and let μ_∞ be the measure on Σ_∞ for which the $N(A)$ define a Poisson distribution of points in X with density given by μ (i.e., $\mu_\infty \{x \in X_\infty \mid [N(A)](x) = m\} = \exp(-\mu(A))\,\mu(A)^m/m!$). Define an automorphism T_∞ of μ_∞ by

$$T_\infty x = Tx$$

for $X \supset x \in X_\infty$. We will say that $(X_\infty, \Sigma_\infty, \mu_\infty, T_\infty)$, the <u>Poisson system built over</u> (X, Σ, μ, T), is obtained from (X, Σ, μ, T) by a <u>Poisson construction</u>. The system $(\mathcal{U}, T_t)$ is clearly the Poisson system built over the one-particle system $(\mathbb{R}^\nu \otimes \mathbb{R}^\nu, \mathcal{U}_0, T_{0,t})$, so that we have a convenient description of an infinite system of noninteracting particles.

6. <u>The Fock space representation of the induced unitaries of a Poisson system</u>[6].

In Chapter V we will have occasion to investigate the properties of the induced unitary operator U_{T_∞} of a Poisson system $(X_\infty, \Sigma_\infty, \mu_\infty, T_\infty)$ built over (X, Σ, μ, T); hence we need a convenient representation of the action of U_{T_∞} on $L^2(\mu_\infty)$, which we now provide.

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6. I am indebted to Oscar Lanford for the material of this section.

We denote $L^2(\mu)$ by H and write H_n for $H^{\otimes n}_{symm}$, n = 0,1,2,3,4,....[7] We identify H_n with the space of all symmetric square integrable functions on $(X^n, \mu^{\otimes n})$. We will show in particular that $L^2(\mu_\infty)$ may be identified with the boson Fock space built over H ($\equiv \bigoplus_{n=0}^{\infty} H_n$) in such a way that U_{T_∞} is identified with $\bigoplus_{n=0}^{\infty} U_T \otimes \ldots \otimes U_T$ (for all automorphisms T), which follows from the

Theorem: There exists a sequence of unitary mappings

$$\hat{\Sigma}_n : H_n \to L^2(\mu_\infty) \quad , \quad n = 1,2,3,\ldots$$

such that

1. $\hat{\Sigma}_n (U_T \otimes U_T \ldots \otimes U_T f) = U_{T_\infty} \hat{\Sigma}_n f$ for all $f \in H_n$ (and all automorphisms T).

2. $\hat{\Sigma}_n H_n$ is orthogonal to the constants and to $\hat{\Sigma}_m H_m$ for all $m \neq n$.

3. $L^2(\mu_\infty) = \mathbb{C} \cdot 1 \oplus \bigoplus_{m=1}^{\infty} \hat{\Sigma}_m H_m$

4. If Λ is a subset of X with finite measure, and if $f(x_1,\ldots,x_n) \in H_n$ is zero a.e. outside of Λ, then $\hat{\Sigma}_n f$ is measurable in Λ[8].

Proof: We proceed as follows: We first assume $\mu(X) = \infty$; we then prove analogues of 1. and 2. for dense subsets of the H_n and extend to all the H_n, obtaining 3. and 4. in the process. We then use 4. to remove the restriction $\mu(X) = \infty$.

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7. $H_o = \mathbb{C} \cdot 1$ and $H_1 = H$.

8. $g \in L^2(\mu_\infty)$ is measurable in Λ if $g(x \cap \Lambda) = g(x)$ for all $x \in X_\infty$.

Let $\widetilde{H}$ denote the set of all square integrable functions on (X, μ) with support in some Λ with $\mu(\Lambda) < \infty$ (i.e., $f(x) = 0$ a.e. outside of Λ) and with $\int f\, d\mu = 0$. $\widetilde{H}$ is dense in H in view of our assumption that $\mu(X) = \infty$. Let $\widetilde{H}_n$ denote the n-fold algebraic tensor product of $\widetilde{H}$ with itself regarded as a linear subset of H_n. Now $\widetilde{H}_n$ is dense in H_n and for $f_n(x_1, \ldots, x_n) \in \widetilde{H}_n$, $\int \mu(dx_i)\, f_n(x_1, \ldots, x_n) \equiv 0$ for all i. Define Σ_n on H_n by $(\Sigma_n f)(\{x_i\}) = \sqrt{n!} \sum_{i_1 < \ldots < i_n} f(x_{i_1}, \ldots, x_{i_n})$ (a function on X_∞). A straightforward computation indicates that for each n, Σ_n is a unitary mapping of $\widetilde{H}_n$ into $L^2(\mu_\infty)$ such that $\Sigma_n \widetilde{H}_n$ is orthogonal to the constants and to $\Sigma_m \widetilde{H}_m$ for $m \neq n$. We then define $\hat{\Sigma}_n$ through extension by continuity, immediately obtaining 1. and 2. of the theorem. 4. is valid for $f \in \widetilde{H}_n$; to establish it for all of H_n we compute $\hat{\Sigma}_n f$ for $f \in H_n$ vanishing outside of Λ. If, for example, $f \in H$ we find a sequence $f^n \in \widetilde{H}$ converging to f in $L^2(\mu)$; then $\hat{\Sigma}_1 f = \lim_{n \to \infty} \Sigma_1 f^n$. We may constuct the f^n by forming a sequence M_n of subsets of X with $\mu(M_n) \to \infty$ and put

$$f^n(x) = f(x) - (\varphi_{M_n}(x)/\mu(M_n)) \int f\, d\mu.^{9}$$

The latter term clearly converges to zero in $L^2(\mu)$. Also $\Sigma_1 f^n = \Sigma_1 f - \int f\, d\mu\, N(M_n)/\mu(M_n)$. Since $N(M_n)/\mu(M_n)$ converges to the constant function 1 (in $L^2(\mu_\infty)$), $\hat{\Sigma}_1 f = \Sigma_1 f - \int f\, d\mu$.

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9. φ_M is the characteristic function of $M \subset X$.

Proceeding in a similar manner, we may express $\hat{\Sigma} f_n$, for $f_n \in H_n$ vanishing outside of some Λ as a linear combination of the form $\hat{\Sigma}_n f_n = \sum_{j=0}^{n} c_j \Sigma_j f_j$, where $f_j(x_1,\ldots,x_j) = \int \mu(dx_{j+1})\ldots \mu(dx_n) f_n(x_1,\ldots,x_n)$, displaying $\hat{\Sigma}_n f_n$ as a function measurable in Λ. We see also that the finite linear span of functions of the form $\hat{\Sigma}_n f_n$ contains all functions of the form $\Sigma_n f_n$. Observing that functions of the form $\exp(i\theta N(\Lambda)) = \sum_n (i\theta N(\Lambda))^n/n!$ are in the closed linear span of functions of the form $\Sigma_n f_n$, we establish 3.

Finally, if $\mu(X)$ is finite, we replace (X,μ) by $(X \cup X', \mu \oplus \mu')$, where (X', μ') is an infinite measure space. This replaces (X_∞, μ_∞) by $(X_\infty \otimes X'_\infty, \mu_\infty \otimes \mu'_\infty)$. Since 4. implies that if $f \in H_n(X)$, $\hat{\Sigma}_n f \in L^2(\mu_\infty)$, the proof is complete.

7. The Bernoulli construction and the ideal gas

There is a simple method which can often be employed to show that a Poisson system is Bernoulli. The idea is constained in the following

Proposition: Let $\{C_i\}_{i \in \mathbb{Z}}$ be a measurable partition of the space X of the system (X, Σ, μ, T) (μ may be an infinite measure) such that $T C_i = C_{i+1}$ for all $i \in \mathbb{Z}$. For any set $A \subset X$ let $\Sigma_\infty(A)$ denote the local σ-algebra on A (i.e., the sub-σ-algebra of Σ_∞ generated by Σ_∞-measurable functions measurable in A, or equivalently by the functions of the form $N(D)$ with $D \in \Sigma$, $D \subset A$).

Then $\Sigma_\infty (C_o)$ is an independent generator for the system $(X_\infty, \Sigma_\infty, \mu_\infty, T_\infty)$.

<u>Proof</u>. Observe that

$$T_\infty \Sigma_\infty (C_i) = \Sigma_\infty (T\, C_i) = \Sigma_\infty (C_{i+1});$$

hence the $T_\infty{}^j \Sigma_\infty(C_o)$ form an independent sequence of sub-σ-algebras, since disjoint regions of X are independent under the Poisson construction. The $T_\infty{}^j \Sigma_\infty (C_o)$ also generate all of Σ_∞ because

$$\bigcup_{i \in \mathbb{Z}} C_i = X.$$

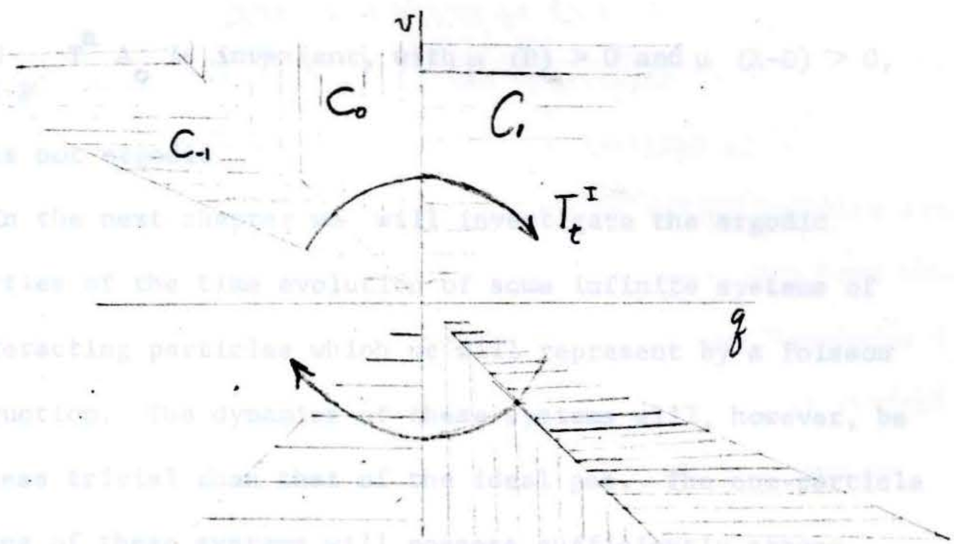
The time evolution of the infinite volume equilibrium states $\mathcal{U}^I_{\beta,\rho}$ of the ideal gas can be obtained by a Poisson construction from the system $(X^I, \mu^I, T_t{}^I)$. Here $X^I = \mathbb{R}^\nu \otimes \mathbb{R}^\nu$, $\mu_I = \rho\, \mu_L \otimes \mu_\beta$ (μ_L denotes Lebesgue measure on $\mathbb{R}$, while μ_β denotes a Maxwellian distribution with parameter β) and $T_t{}^I(q,v) = (q + vt, v)$, $(q,v) \in \mathbb{R}^\nu \otimes \mathbb{R}^\nu$. One easily verifies that $\mu_\infty{}^I = \mathcal{U}^I_{\beta,\rho}$, so that $(X_\infty{}^I, \mu_\infty{}^I, T_{t,\infty}{}^I)$ does in fact represent the time evolution of an infinite volume equilibrium state of the ideal gas.

Now for many systems, and in particular for the ideal gas, there is a natural way of obtaining a Bernoulli construction. For the ideal gas we set

$$C_n{}^I = \{x \in X^I \mid \|T_t{}^I x\| \text{ achieves its (strict) minimum for } -n \leq t < -n + 1\}$$ [10]

10. By $\|\cdot\|$ we mean Euclidean distance to the origin.

Thus $C_o^{\ I}$ is the set of one particle ideal gas initial configurations for which the nearest approach to the origin occurs between times 0 and 1; we clearly have $T_1^{\ I} C_j^{\ I} = C_{j+1}^{\ I}$, and $\bigcup_{j \in \mathbb{Z}} C_j^{\ I} = X^I$. The time evolution of the infinite ideal gas thus forms a Bernoulli flow. For $\nu = 1$ we have the following "picture" of the Bernoulli construction:



For some systems there will be no unique time of nearest approach to the origin; in such a case it may still be possible to perform a Bernoulli construction, based, for example, on the last time of nearest approach. In the next chapter we will encounter such systems, but we will also encounter systems for which no Bernoulli construction is possible at all. Indeed, as $T_t^{\ I}$ does not have very good ergodic properties, the situation encountered with the ideal gas suggests that the possibility of performing a Bernoulli construction on (X, μ, T) is, loosely, inversely proportional to the degree to which the nontrivial automorphism T possesses good ergodic properties, and we do, in fact, have the following

<u>Proposition</u>: If T is ergodic[11] no Bernoulli construction is possible.

<u>Proof</u>: Let $\{C_i\}$ be a Bernoulli construction. We can decompose C_o into disjoint sets of nonzero measure: $C_o = A_o \cup B_o$. Then $D = \bigcup_{n \in \mathbb{Z}} T^n A_o$ is invariant, with $\mu(D) > 0$ and $\mu(X-D) > 0$, so T is not ergodic.

In the next chapter we will investigate the ergodic properties of the time evolution of some infinite systems of noninteracting particles which we will represent by a Poisson construction. The dynamics of these systems will, however, be much less trivial than that of the ideal gas. The one-particle dynamics of these systems will possess sufficiently strong ergodic properties to render a Bernoulli construction impossible and to guarantee the existence of "global" K-partitions.

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11. In the sense that for an invariant set A either $\mu(A) = 0$ or $\mu(X-A) = 0$.

IV. Ergodic Properties of an Infinite System of Particles Independently Moving in a Periodic Field

1. Introduction

While some results have been obtained concerning the ergodic properties of interesting finite systems [2,41], very little is known concerning the ergodic properties of nontrivial systems with an infinite number of degrees of freedom, which are of great interest for statistical mechanics. De Pazzis [34] and Sinai [44,42] have investigated the ergodic properties of an infinite system of hard rods moving in one dimension and an infinite ideal gas in an arbitrary number of dimensions. Though they have shown these systems to have very good ergodic properties (K-systems or Bernoulli), the physical interpretation of the result is trivial: "local disturbances stream off to infinity where they are no longer visible" [22].

We investigate here the ergodic properties of an infinite system with non-trivial "collisions", i.e., the transformation which occurs during a collision possesses itself good mixing properties. Except for these collisions the particles move freely. It seems reasonable to expect that the ergodic properties of our system will be at least as good as those of the systems considered by Sinai. We must be cautious, however, since the physical explanation of the ergodic properties of those systems may not be valid here. It will be seen, in fact, that the underlying mathematical structures (partitions) which determine the ergodic properties of the respective systems are of a very different nature.

2. General description of a one-dimensional model system

We investigate first the ergodic properties of an infinite system of non-interacting particles moving freely in one dimension except for "collisions". A periodic array of barriers is the agency responsible for the "collisions"; when a particle reaches a barrier, it is equally likely that it will be either reflected or transmitted. Since we wish to study a dynamical system, we attach to each particle internal parameters whose sole function is to determine whether the particle, upon reaching a barrier, will be reflected or transmitted.

Since the particles are to be non-interacting, it will suffice to describe the dynamics of a single particle; it is clear from the previous paragraph that this will be determined once we have specified the behavior and effects of the internal space of a particle. Now it is clear from the above description of the role of the internal parameters and requirements of spatial symmetry that the sole effect of the spatial variables (position and velocity) upon the internal dynamics can be assumed to be the determination of the times at which the internal parameters undergo a transformation; this transformation will occur when the particle is in a given position relative to the barrier from which it is immediately departing. We choose the convention that the transformation occur immediately after a particle leaves a barrier. Furthermore, it is natural to choose as our internal dynamical system one which, though among the simplest of dynamical systems, has ergodic properties of the strongest kind (Bernoulli): the Bernoulli shift on an alphabet of 2 letters each with weight $\frac{1}{2}$, $B(\frac{1}{2},\frac{1}{2})$, which is equivalent to the baker's transformation[1]. It is also natural from the standpoint of the theory of Bernoulli shifts to require that the spatial dependence upon the internal space should be measurable with respect to the partition which determines the entire ergodic structure of the internal dynamics, the independent generator [39]. This is the 2-element partition of the baker's square into a left side and a right side (of the same area). The dynamics can therefore be described as follows: a particle moves freely until it comes to a barrier; if its internal parameters lie in the left side of the baker's square the particle is reflected; otherwise it is transmitted; in either case the internal parameter subsequently undergoes a baker's transformation, and the particle moves on freely until it reaches another barrier.

It is not difficult to see that the above description, obtained on the basis of requirements of simplicity and naturalness, is actually a description of the only internal dynamics which is consistent with the role we assigned to the internal space: that it provide a deterministic foundation for a Markov process. An essential feature of the spatial process we wish to consider is the independence of what happens at a given barrier from the past spatial history of the particle.

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1 See Chapter I, section 6.

From a purely dynamical point of view nothing would be gained by our considering an infinite system of particles, since the particles are non-interacting; we are considering an infinite system because we are interested in ergodic properties. Thus we must specify an invariant probability measure on the phase space (in order to obtain a dynamical system in the sense of ergodic theory). Of course, such a measure must also be natural from the standpoint of statistical mechanics, e.g., in some sense a limit of grand canonical measures on finite systems. The only natural candidate consistent with the above and with the remarks in the previous paragraph is the following: the (unlabelled) particles are distributed along the line with a Poisson distribution of density ρ ; the internal and velocity spaces associated with a particle at a given position are independent of the configuration (positions) of the particles, of the spaces associated with other particles, and of each other.

We note that whereas it is only in the infinite case that the ideal gas becomes ergodically interesting, our system, since it has a non-trivial dynamics, is ergodically interesting even for a single particle. Thus, before considering the ergodic effects of taking the infinite limit of finite systems it is reasonable to investigate what ergodic properties are present before taking the limit.

3. Ergodic properties of one-particle system

Let the barriers be situated at integral positions. Choose the unit of time so that the absolute value of the velocity of the particle is unity. (The speed of the particle is a constant of the motion.) The only modification of the description in the preceding section which we must make is that we must take for our external space $\mathbb{R}/n\mathbb{Z}$, the real line modulo some integer n, instead of $\mathbb{R}$. This is necessary because we wish to have a normalized spatially homogeneous invariant measure.

We thus have the following dynamical system, $\tau_n = (X_n, \Sigma_n, \mu_n, \{S_{n,t}\})$: The phase space $X_n = \mathbb{R}/n\mathbb{Z} \otimes \{1,-1\} \otimes B$, where B is the baker's square. The σ-algebra $\Sigma_n = \Sigma_{L_n} \otimes P\{1,-1\} \otimes \Sigma_B$ where Σ_{L_n} is the Σ-algebra of Lebesgue sets of the real line modulo n, $P\{1,-1\}$ is the power set of $\{1,-1\}$ (regarded as a σ-algebra), and Σ_B is the σ-algebra of Lebesgue sets of the baker's square. The measure $\mu_n = \mu_{L_n} \otimes \mu_2 \otimes \mu_B$, where μ_{L_n} is the normalized Lebesgue measure on Σ_{L_n}, μ_2 assigns mass $\frac{1}{2}$ to the points of $\{1,-1\}$ and μ_B is the normalized Lebesgue measure on the baker's square. $\{S_{n,t}\}$ is a measurable flow on X_n such that for $t < 1$ we have

$$S_{n,t}(x, i, \xi) = S_{n,t}u = \begin{cases} (x + i\,t, i, \xi) \text{ if } \mathbb{Z} \cap [(x, x + i\,t) \\ \quad \cup (x + i\,t, x)] = \emptyset, \text{ and} \\ (m + \tilde{\xi}_o\, i\,(t - |m-x|), \tilde{\xi}_o\, i, T\,\xi \\ \quad \text{if } \mathbb{Z} \cap [(x, x+it) \cup (x + it, x)] = m. \end{cases}$$

Here $x \in \mathbb{R}/n\mathbb{Z}$, $i \in \{1,-1\}$, $\xi \in B$, $u \in X_n$, T is the baker's transformation, and $\tilde{\xi}_k = 2\xi_k - 1 = \pm 1$, where ξ_k is the kth coordinate of the Bernoulli representation of ξ[2].

One easily checks that the above does in fact describe a dynamical system, i.e., for example, that μ_n is invariant under $\{S_{n,t}\}$. However, it is not difficult to see that this dynamical system is not mixing; in fact, $S_{n,1}$ is not even ergodic. If $A = \bigcup_{k=0}^{n-1} (k+\frac{1}{4}, k+3/4)$ then $A \otimes \{1,-1\} \otimes B$ is a subset of X_n, invariant under $S_{n,1}$, with measure $\frac{1}{2}$. More generally, all periodic functions of $\mathbb{R}/n\mathbb{Z}$ symmetric about the point $x = \frac{1}{2}$ are invariant under $S_{n,1}$.

The failure of τ_n to possess strong mixing properties is not very surprising; the breakdown occurs in precisely that "part" of τ_n which is in no way affected by the good mixing properties which we built into the collisions. To be more precise, let us define a bijection α from X_n to $X_n' = \{0,\ldots,n-1\} \otimes [0,1) \otimes \{1,-1\} \otimes B$ as follows: Let m be the "first" integral first coordinate of $S_{n,t}\, u$, $t \leq 0$, $u \in X_n$. Let t_o be the largest value of $t \leq 0$ for which $S_{n,t}\, u$ has first coordinate m. Then $\alpha(x,i,\xi) = (m, |t_o|, i, \xi)$. Thus α, regarded as a mapping defined on the configurational part of X_n, can be thought of as a transformation from the position coordinate x to coordinates (m_o, δ) which describe the location of the barrier

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2 See Chapter I, section 6

from which the particle is departing and the distance of the paeticle from this barrier, respectively. α determines, in an obvious manner, a dynamical system τ'_n which is isomorphic to τ_n. Letting $\{S'_{n,t}\}$ be the image of $\{S_{n,t}\}$ under α, we have

$$S'_{n,t}(m, \delta, i, \xi) = \begin{cases} (m, \delta+t, i, \xi) \text{ if } 0 \leq t < 1-\delta \\ (m+i, (\delta+t) \bmod 1, \tilde{\xi}_o\, i, T\xi) \\ \qquad \text{if } 1 \geq t > 1-\delta \,. \end{cases}$$

We thus see that $\{S'_{n,t}\}$ acts in a trivial way upon $[0,1)$ (the second term in the product defining X'_n). Indeed τ'_n can be factored into a skew product with a rotation for its first component:

$$S'_{n,t}\ (\delta,\omega) = ((\delta+t) \bmod 1,\ \varphi_{n,t}^{\ \delta}\ \omega),\quad \delta \in [0,1)\ ,$$

$$\omega \in \{0,\ldots, n-1\} \otimes \{-1,1\} \otimes B$$

in an obvious manner.

Note that although $\varphi_{n,t}^{\ \delta}$ does not form a one parameter group, its value changes only when $t = k-\delta$, $k \in \mathbb{Z}$ and $\varphi_{n,k}^{\ \delta} = (\varphi_{n,1}^{\ \delta})^k = (\varphi_{n,1})^k$, since $\varphi_{n,1}^{\ \delta}$ is independent of δ. Thus τ_n "factors" into the product of a rotation and an essentially discrete (space and time) dynamical system $\bar{\tau}'_n$ in which all of the ergodic activity occurs. We investigate such a system in the next section.

4. Ergodic properties of discrete one-particle system

The discrete dynamical system $\bar{\tau}'_n$, alluded to at the end of the previous section, can be described as follows:

$$\bar{\tau}'_n = (\bar{X}'_n, \bar{\Sigma}'_n, \bar{\mu}'_n, \bar{S}'_n)$$

$$\bar{X}'_n = \mathbb{Z}_n \otimes \{-1,1\} \otimes B \quad (\mathbb{Z}_n \text{ is the set of integers mod n})$$

$$\bar{S}'_n (m, i, \xi) = (m + i, \tilde{\xi}_o i, T\xi)$$

and $\bar{\Sigma}'_n$ and $\bar{\mu}'_n$ are obvious.

Since we are now dealing with a discrete system, the velocities are somewhat unnatural. Therefore, instead of investigating $\bar{\tau}'_n$ we will investigate a simpler system $\bar{\tau}_n$ which has the same ergodic properties as $\bar{\tau}'_n$. $\bar{\tau}_n$ is obtained from $\bar{\tau}'_n$ essentially by dropping the velocity part of phase space and making the appropriate modification of the dynamics. It is, in fact, isomorphic to $\bar{\tau}'_n$.

We thus investigate $\bar{\tau}_n = (\bar{X}_n, \bar{\Sigma}_n, \bar{\mu}_n, \bar{S}_n)$, where $\bar{X}_n =$ $B \otimes Z_n$, $\bar{\Sigma}_n = \tilde{\Sigma}_n \otimes \Sigma_B$, $\bar{\mu}_n = \tilde{\mu}_n \otimes \mu_B$, and for $x \in \bar{X}_n$, $\bar{S}_n x = \bar{S}_n(\xi,k) = (T\xi, \tilde{\varphi}_{n,\xi}(k)) =$ $(T\xi, k + \xi_o)$. Here $\tilde{\tau}_n = (Z_n, \tilde{\Sigma}_n, \tilde{\mu}_n, \tilde{\varphi}_{n,\xi})$ is a unit translation on the integers mod n with the discrete σ-algebra and with equal weight assigned to each integer mod n.

$\bar{\tau}_n$ is thus a skew product of a Bernoulli shift with a rotation valued function which is measurable with respect to an independent generator [39]. It is known [39,1] that such a system is Bernoulli if it is mixing. We will here prove as a special case the following:

Theorem 4.1: $\bar{\tau}_n$ is Bernoulli if and only if n is odd. (For n even $\bar{\tau}_n$ fails to be mixing.)

Proof: Let M_n be the Markov shift on Z_n with transition probabilities $\pi_{nm} = \frac{1}{2}(\delta_{n,m+1} + \delta_{n,m-1})$ and stationary distribution $p_k = 1/n$ (random walk). Since a mixing Markov shift is Bernoulli [39], the theorem follows from 2 lemmas:

Lemma 4.2: $\bar{\tau}_n$ is isomorphic to M_n.

Lemma 4.3: M_n is mixing if and only if n is odd.

Proof of Lemma 4.2: A Markov shift is mixing if and only if the nth order transition probabilities π^n_{jk} approach (in the limit $n \to \infty$) the stationary distribution p_k. For if $C = \{C_\ell\}$ is the natural Markov generator [39] for the shift S and if $A \in \bigvee_{i=m_1}^{n_1} S^i C$ and $B \in \bigvee_{i=m_2}^{n_2} S^i C$ (i.e.,

$$A = S^{m_1} C_{i_{m_1}} \cap S^{m_1+1} C_{i_{m_1+1}} \ldots \cap S^{n_1} C_{i_{n_1}}$$

$$B = S^{m_2} C_{j_{m_2}} \cap \ldots \cap S^{n_2} C_{j_{n_2}}),$$

we have for n sufficiently large (denoting the measure on M_n by μ),

$$\mu(S^n A \cap B) = \mu(S^{m_1+n} C_{i_{m_1}} \cap \ldots \cap S^{n_2} C_{j_{n_2}})$$

$$= p_{j_{m_2}} \pi_{j_{m_2} j_{m_2+1}} \cdots \pi_{j_{n_2-1}, j_{n_2}} \pi^{n+m_1-n_2}_{j_{n_2} i_{m_1}} \pi_{i_{m_1} i_{m_1+1}} \cdots \pi_{i_{n_1-1}, i_{n_1}}$$

$$= (\pi^{n+m_1-n_2}_{j_{n_2} i_{m_1}} / p_{i_{m_1}}) (p_{j_{m_2}} \pi_{j_{m_2} j_{m_2+1}} \cdots \pi_{j_{n_2-1} j_{n_2}})$$

$$\times (p_{i_{m_1}} \pi_{i_{m_1} i_{m_1+1}} \cdots \pi_{i_{n_1-1} j_{n_1}})$$

$$= (\pi^{n+m_1-n_2}_{j_{n_2} i_{m_1}} / p_{i_{m_1}}) \mu(A) \mu(B) \xrightarrow[n \to \infty]{} \mu(A) \mu(B)$$

(for all such A,B) if and only if

$$\lim_{n \to \infty} \pi^n_{jk} = p_k .$$

For a Markov chain on a finite state space the above equation is valid for precisely those chains which are irreucible and aperiodic[3]. (The chain is irreucible if every state has a non-vanishing probability of being reached from any other state. The chain is aperiodic if every state has period 1. If ν is the largest possible integer such that π^n_{jj} is nonvanishing only for n an integral multiple of ν, the state j is said to have period ν.)

It is clear that M_n is irreducible for all n. For n even all states have period 2, since the states can be partitioned into an "even" class and an "odd" class in such a way that (one step) transitions always involve a change in class. For all n we have

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3. See ref. [10], p.393.

$\pi^2_{jj} \geq 0$, so $\nu \leq 2$. Since by jumping to the right on each transition the system will eventually return to its initial state, we have $\nu = 1$ for n odd. Thus for n odd M_n is aperiodic and the lemma is established.

Proof of Lemma 4.1: One easily checks that

$$P_n = \{ C_k^{[n]} \} = \{\{(\xi,j) \mid \xi \in B,\ j = k \in \mathbb{Z}_n\}\}$$

is a Markov generator for $\bar{S}_n$ having the same conditional probabilities as M_n. (The isomorphism (mod 0), α, determined by P_n is easily seen to map every point $x \in \bar{X}_n$ into its trajectory $\{\eta_k\} \in \mathbb{Z}_n^{\mathbb{Z}}$. Equipped with the measure induced by α, $\mathbb{Z}_n^{\mathbb{Z}}$ becomes the measure space of M_n and the image of $\bar{S}_n$ under α is clearly the shift on trajectories).

In the next section we will have occasion to use a general criterion for determining whether a countable family Γ of measurable subsets of a Lebesgue space [37] (X, Σ, μ) generates Σ. A necessary and sufficient condition for Γ to generate (mod 0) is that the decomposition $\zeta(\Gamma)$ determined by Γ be the decomposition of X into points (mod 0); i.e., that there exist a set A of full measure such that for any x, $y \in A$ there exists $\Gamma_n \in \Gamma$ for which $x \in \Gamma_n$, $y \notin \Gamma_n$ or $y \in \Gamma_n$, $x \notin \Gamma_n$ [37]. For the case that Γ is generated from a (finite) partition P by the transformation T this condition reduces to the requirement that the mapping from points to trajectories determined by (P,T) be injective (mod 0). It is trivial to check that for the system $\bar{\tau}_n$, (P_n, S_n) satisfies this condition (everywhere).

5. Ergodic properties of infinite discrete system

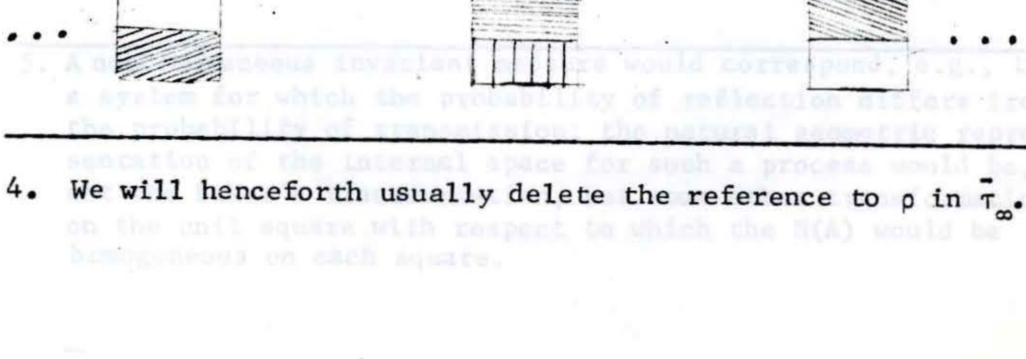
We have now descended as far as we will go in the direction of simplification, and we will now begin an ascent to the system with which we are primarily concerned. We will first investigate a system $\bar{\tau}_\infty$ which is essentially the thermodynamic limit for the model of the previous section. (Since the particles are non-interacting in all of the models which we consider, there is nothing to be gained by considering a system with several particles.) We expect the infinite system to have "strong" ergodic properties, having found finite systems for which this is the case and remembering that the thermodynamic limits of some trivial systems (i.e., the ideal gas) possess these properties.

As the dynamical system $\bar{\tau}_\infty(\rho) = (\bar{X}_\infty, \bar{\Sigma}_\infty, \bar{\mu}_\infty(\rho), \bar{S}_\infty)$ is considerably more complex than those considered so far, we will discuss its components more carefully than we have discussed the components of the models considered previously.

Recalling that (B, Σ_B, μ_B) ($= \delta$) denotes the measure space of the baker's transformation, we let $(\bar{X}_\infty, \bar{\Sigma}_\infty, \bar{\mu}_\infty(\rho)) = \bigotimes_{i=-\infty}^{\infty} \bigoplus_{n=0}^{\infty} e^{-\rho} (\rho^n/n!) \; \delta_i^n$. Here $\delta_i^n = \delta_{symm}^{\otimes n}$ denotes the symmetrized measure theoretic product of δ with itself n times. We thus have at each site a sequence of spaces with the nth member of the sequence representing a situation in which n particles are present at the site. We weigh these spaces according to a Poisson distribution of mean ρ and take the (measure theoretic) union. We then take the product over all lattice sites. (We will soon describe $\bar{S}_\infty$; its definition should, however, be obvious.)

Now it is not difficult to see that $\bar{\tau}_\infty(\rho)$[4] can be identified with the Poisson system built over the generalized Baker's transformation $(\bar{B}_\infty, \Sigma_{\bar{B}_\infty}, \rho\mu_{\bar{B}_\infty}, \bar{T}_\infty)$: $(\bar{B}_\infty, \Sigma_{\bar{B}_\infty}, \mu_{\bar{B}_\infty}) = \oplus_{i\in \mathbb{Z}} \delta_i$ (and $\bar{B}_\infty = \oplus_{i\in \mathbb{Z}} B_i$) where $\delta_i = \delta$ (and $B_i = B$) for all $i \in \mathbb{Z}$; $\bar{T}_\infty$ has a simple geometric representation: Recall that the baker's transformation can be described geometrically as a two step process:

Now if we perform the baker's transformation independently on the doubly infinite array of baker's squares and follow it with the simultaneous translation of the top half of each resulting square one unit to the right and the bottom half one unit to the left, we obtain $\bar{T}_\infty$. We thus have the following "picture" of $\bar{T}_\infty$:



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4. We will henceforth usually delete the reference to ρ in $\bar{\tau}_\infty$.

This auxiliary system is, of course, simply a one particle component of $\bar{T}_\infty$.

Using the independence of the lattice sites (there is no interaction), the homogeneity of the baker's square (i.e., that if A, $D \subset B_i$ with $\mu_B(A) = \mu_B(D)$, then N(A) and N(D) are identically distributed)[5], and the area preserving nature of the auxiliary dynamics $(\bar{B}_\infty, \bar{T}_\infty)$, we can show that $\bar{\mu}_\infty$ is the only "reasonable" invariant probability measure on $(\bar{X}_\infty, \bar{\Sigma}_\infty)$. (Note that it is not immediately obvious that the number of particles at a given site must have a Poisson distribution, although we certainly expect this to be the case).

Theorem 5.1: $\bar{\mu}_\infty$ is the unique $\bar{S}_\infty$ - invariant probability measure on $\bar{\Sigma}_\infty$ for which we have:

a) The $\bar{\Sigma}_\infty(B_i)$, i = 0, 1, -1, ..., form an independent sequence of Σ-algebras.

b) The distribution of N(A), $A \subset B_i$, $i \in \mathbb{Z}$, is determined by $\mu_{\bar{B}_\infty}(A)$ (and i) and satisfies $\langle N(A) \rangle = \int d\bar{\mu}_\infty(x)\ [N(A)](x) < \infty$.

5. A nonhomogeneous invariant measure would correspond, e.g., to a system for which the probability of reflection differs from the probability of transmission; the natural geometric representation of the internal space for such a process would be, not the baker's transformation, but some other transformation on the unit square with respect to which the N(A) would be homogeneous on each square.

Proof: We note that

$$[U^{-1}_{\bar{S}_\infty} N(D)](x) = [N(D)](\bar{S}_\infty^{-1}x) = [N(\bar{T}_\infty D)](x)$$

and that

$$\bar{S}_\infty(\{x \in \bar{X}_\infty \mid [N(D)](x) = m\}) = \{\bar{S}_\infty x \mid [N(D)](x) = m\}$$

$$= \{x \mid [N(D)](\bar{S}_\infty^{-1}x) = m\} = \{x \mid [N(\bar{T}_\infty D)](x) = m\}.$$

It now follows from the area ($\mu_{\bar{B}_\infty}$) preserving property of $\bar{T}_\infty$ that any measure for which the joint distribution of any finite sequence of random variables of the form $\{N(D_i)\}$, where $\{D_i\}, i \in I$ (a finite index set), is a sequence of pairwise disjoint subsets of $\bar{B}_\infty$, depends only on the sequence of areas $\{\mu_{\bar{B}_\infty}(D_i)\}$ is invariant under $\bar{S}_\infty$. $\bar{\mu}_\infty$ is thus invariant (since $\{N(D)\}$ is Poisson with constant density).

To prove uniqueness it is sufficient to show that $\bar{\Sigma}_\infty(A)$ is independent of $\bar{\Sigma}_\infty(C)$ when $A \cap C = \emptyset$. We can assume that A and C are both subsets of B_o. If A_o and C_o are distinct atoms of $\bigvee_{k=-n}^{n} T^k P$ ($P = \{P_i\}$, $P_i = \{\xi \in B \mid \xi_o = i\}$), there exists $j \in [-n, n+1]$ for which $\bar{T}_\infty^{-j} A_o \subset B_\ell$ and $\bar{T}_\infty^{-j} C_o \subset B_m$, $\ell \neq m$. Thus $\bar{\Sigma}_\infty(\bar{T}_\infty^{-j} A_o) = \bar{S}_\infty^{-j} \bar{\Sigma}_\infty(A_o)$ is independent of $\bar{S}_\infty^{-j} \bar{\Sigma}_\infty(C_o) = \bar{\Sigma}_\infty(\bar{T}_\infty^{-j} C_o)$ so that, by invariance, $\bar{\Sigma}_\infty(A_o$ is independent of $\bar{\Sigma}_\infty(C_o)$. By an induction on n one verifies that for any N, and for A' and B' disjoint unions of atoms of $\bigvee_{-N}^{N} T^j P$, $\bar{\Sigma}_\infty(A')$ is independent of $\bar{\Sigma}_\infty(B')$. Because $\langle N(D)\rangle = \langle N(E)\rangle$ for $\mu_{\bar{B}_\infty}(D) = \mu_{\bar{B}_\infty}(E)$, $N(D) = 0$, a.e., if $\mu_{\bar{B}_\infty}(D) = 0$. Thus, since P is a generator for T, $\bar{\Sigma}_\infty(A)$ must be independent of $\bar{\Sigma}_\infty(B)$ for $A \cap B = \emptyset$. Since $\langle N(B_i)\rangle$ clearly equals $\langle N(B_j)\rangle$, we are done.

Using methods similar to those used above, we prove the following

<u>Theorem 5.2:</u> $\bar{T}_\infty$ is mixing.

<u>Proof:</u> Let $\alpha = \{a_j\}_{j \in J}$ and $\beta = \{\beta_k\}_{k \in K}$ be finite families of disjoint subsets of $\bar{B}_\infty$ such that $\cup\alpha$ and $\cup\beta$ are contained in $\bigcup_{i=-N}^{N} B_i$ for some N. Let any set $\alpha_k \in \alpha$ an atom of $\bigvee_{j=-M}^{M} T^j P^i$ for some M and $-N \le i \le N$, where P^i is the partition of B_i corresponding to the partition P of B. Let

$$X_\alpha^{\{n_j\}_{j \in J}} = \{x \in \bar{X}_\infty \mid N(\alpha_j) = n_j \text{ for all } j \in J\}$$

and

$$X_\beta^{\{m_k\}_{k \in K}} = \{x \in \bar{X}_\infty \mid N(\beta_k) = m_k \text{ for all } k \in K\}$$

We have $\bar{S}_\infty X_\alpha^{\{n_j\}} = X_{\bar{T}_\infty \alpha}^{\{n_j\}}$ Also $\bar{T}_\infty^M \alpha$ is a family which is independent of the "future". Thus $\bar{T}_\infty^{M+m}$, m = 1,2,... induces a random walk on a point uniformly distributed over an element of α. We can, therefore, use the central limit theorem to find an $\bar{N}$ so large that

$$\mu_{\bar{B}_\infty}(\cup \bar{T}_\infty^{\bar{N}} \alpha \cap \bigcup_{i=-N}^{N} B_i) < \epsilon .$$

We can now use the independence of N(A) and N(C) for $A \cap C = \emptyset$ to conclude that

$$\lim_{n\to\infty} \bar{\mu}_\infty (\bar{S}_\infty^n X_\alpha^{\{n_j\}} \cap X_\beta^{\{m_k\}}) =$$

$$= \lim_{n\to\infty} \bar{\mu}_\infty (X_{\bar{T}_\infty^{n}\alpha}^{\{n_j\}} \cap X_\beta^{\{m_k\}}) = \bar{\mu}_\infty (X_\alpha^{\{n_j\}}) \cap \bar{\mu}_\infty (X_\beta^{\{m_k\}}).$$

Since finite unions of sets of the form $X_\alpha^{\{n_j\}}$ are dense in $\bar{\Sigma}_\infty$, we conclude that $\bar{\tau}_\infty$ is mixing.

We will now show that $\bar{\tau}_\infty$ is a K-system. We first review the definition. A continuous Lebesgue space (X, Σ, μ) equipped with an invertible measure preserving transformation S is said to be a <u>K-system</u> if there exists a measurable partition ζ_o (a K-partition) such that [37, 33, 19, 38]

1) $S^n \zeta_o = \zeta_n \geq \zeta_o$ (mod 0) for $n \geq 0$;

2) $\bigvee_n \zeta_n = \epsilon$ (mod o), where ϵ is the partition of X into its elements;

3) $\bigwedge_n \zeta_n = \nu$ (mod 0), where ν is the trivial partition of X whose sole element is X itself.

If $\{S^t\}$ is a measurable flow on (X, Σ, μ) and if in the above definition we replace "n" by "t", we obtain the definition of a <u>K-flow</u>.

We have already shown that $\bar{\tau}_n$ is Bernoulli (which implies that it is a K-system) for n odd. Let Σ'_n be the σ-algebra generated by the $\bar{S}_n^k P_n$, $k \leq 0$, where P_n was defined in section 4, and let $\zeta(\Sigma'_n)$ be the partition determined by the family of sets of the form $\bar{S}_n^k P_{n,i}$, $k \leq 0$, $P_{n,i} \in P_n$ [37]. It is easy to see that $\zeta(\Sigma_n')$ satisfies 1), 2), and 3). 1) is trivial, 2) is equivalent to the fact that the $\bar{S}_n^k P_n$, $k \in \mathbb{Z}$ generate (see final paragraph of section 4), and 3) follows from the fact that for n odd the transition probabilities approach the values of the stationary distribution. (That the partitions which we encounter are measurable and that the spaces are Lebesgue are easily verifiable in each case [6]). $\zeta(\Sigma_n')$ can be described as that partition for which $x \sim x'$ (i.e., $x = (\xi^x, m_x)$ and $x' = (\xi^{x'}, m_{x'})$ belong to the same element of $\zeta(\Sigma_n'))$ when $m_x = m_{x'}$, and $\xi_j^x = \xi_j^{x'}$ for $j \geq 0$.

We now introduce some notation for partitions of $\bar{X}_\infty$. Let $\bar{\gamma}$ be a partition of $\bar{B}_\infty$. We denote by $\zeta[\bar{\gamma}]$ the partition of $\bar{X}_\infty$ generated by functions of the form N(D), $D \in \Sigma(\bar{\gamma}) \subset \Sigma_{\bar{B}_\infty}$ [37]. Let $\hat{\gamma}$ denote the partition for which $B_i \in \Sigma(\hat{\gamma})$ for all i and which when restricted to each B_i is "identical" to γ (a partition of B). We write $\zeta[\gamma]$ for $\zeta[\hat{\gamma}]$. We denote by γ_o the partition of B into vertical line segments (i.e., $\xi \sim \xi'$ when $\xi_j = \xi_j'$ for $j \geq 0$). We recall that γ_o is a K-partition for (B, Σ_B, μ_B, T). We denote by ν_B the trivial partition of B.

Two possibilities for a K-partition for $\bar{T}_\infty$ now suggest themselves:

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6. See ref. [37], pp. 24, 37.

1) $\zeta[\gamma_o]$, corresponding directly to $\zeta(\Sigma_n')$ in an obvious way, and

2) $\bar{\zeta} = \bigvee_{k \leq 0} \bar{S}_\infty^k \zeta[\nu_B]$, constructed from $\zeta[\nu_N]$, which corresponds to P_n, in a manner analagous to the construction of $\zeta(\Sigma_n')$ from P_n; by $\bigvee_{k \leq 0} \bar{S}_\infty^k \zeta[\nu_B]$ we mean the partition determined by the σ-algebra generated from the partitions $\bar{S}_\infty^k \zeta[\nu_B]$, $k \leq 0$ [33]. We will denote by $\zeta[\varphi]$ a partition constructed in such a way from a partition φ. We note that $\zeta[\gamma_o] \geq \zeta\overline{[\nu_B]}$. This follows from the fact that if x and y are in the same element of $\zeta[\gamma_o]$, they have the same future spatial trajectories and hence belong to the same element of $\zeta\overline{[\nu_B]}$.

We will see shortly that $\zeta[\gamma_o]$ is, in fact, a K-partition for $\bar{\tau}_\infty$; $\bar{\zeta}$, however, satisfies 1) and 3) but fails to satisfy 2). It satisfies 1) essentially by construction. That $\bar{\zeta}$ satisfies 3) is an immediate consequence of the fact that $\zeta[\gamma_o]$, a K-partition, satisfies 3). $\bar{\zeta}$ fails to satisfy 2) because, with probability 1, any $x \in \bar{X}_\infty$ is such that 2 points, η and η', in some square B_i are occupied; if we exchange the "future" coordinates η_j and η_j', $j \geq 0$, of η and η' we obtain, with probability 1, a new element $x' \in \bar{X}_\infty$; by construction x and x' have identical external trajectories and hence along to the same element of $\bar{\zeta}$; hence $\bar{\zeta} \neq \epsilon$ (mod o). A similar argument indicates that any partition of the form $\zeta\left[\bigvee_{j=-N}^{N} T^j P\right]$ fails to satisfy 2).

Theorem 5.3: $\bar{\tau}_\infty$ is a K-system.

Proof: We will show that $\zeta[\gamma_o]$ satisfies 1), 2), and 3). We observe that a) $\bar{S}_\infty^{n} \zeta[\gamma] = \zeta[\bar{T}_\infty^{n}\gamma]$, $n \in \mathbb{Z}$

and

b) $\bar{S}_\infty^{-n} \zeta[\gamma] = \zeta[T^n\gamma]$, $n \in \mathbb{Z}^+$, γ a partition of B,

and $\gamma \geq P$.

1) is an immediate consequence of b) and the corresponding property of γ_o. Similarly, 2) follows from b) and the fact that $\bigvee_n T^n\gamma_o = \epsilon_B$ (mod o): for $x \neq y \in \bar{X}_\infty$, there exists an N such that $[N(A)](x) \neq [N(A)](y)$, $A \in \bigvee_{j=-N}^{N} T^j P^i$ for some $i \in \mathbb{Z}$. Thus x is separated from y by $\zeta[T^N\gamma_o]$ and hence by $\bigvee_n S_\infty^{-n}\zeta[\gamma_o]$. Hence $\bigvee_n \bar{S}_\infty^{n}\zeta[\gamma_o] = \epsilon (= \bigvee_n \zeta[T^n\gamma_o] = \zeta[\bigvee_n T^n\gamma_o] = \zeta[\epsilon_B])$ (mod 0).

We now give a (somewhat) heuristic argument for 3). Let $\sigma^n = \Sigma(\bar{S}_\infty^{-n}\zeta[\gamma_o])$ and let $\bar{\sigma} = \bigcap_n \sigma^n$. To establish 3) we must show that if $A \in \sigma$ we have $\bar{\mu}_\infty(A) = 0$ or $\bar{\mu}_\infty(A) = 1$. Let F_n be the σ-algebra generated by $\{N(D): D \subset \bigcup_{i=-n}^{n} B_i\}$. Let $F = \bigcup_{n>0} F_n$. We would like to show that if $C \in F$ and if $A \in \sigma$ then $\bar{\mu}_\infty(A \cap C) = \bar{\mu}_\infty(A)\,\bar{\mu}_\infty(C)$. For this the theorem would easily follow, because for any $A \in \sigma$ (recalling that $\bar{\Sigma}_\infty = \Sigma(F)$) we can find a sequence $\{A_n\}$ for which $A_n \in F$ for all n and $\lim_{n\to\infty}\bar{\mu}_\infty(A_n \,\Delta\, A) = 0$. We then would have

$$\bar{\mu}_\infty(A) = \bar{\mu}_\infty(A \cap A) = \lim_{n\to\infty} \bar{\mu}_\infty(A \cap A_n) =$$

$$= \lim_{n\to\infty} \bar{\mu}_\infty(A)\,\bar{\mu}_\infty(A_n) = [\bar{\mu}_\infty(A)]^2,$$ so that we would have

$\bar{\mu}_\infty(A) = 0$ or $\bar{\mu}_\infty(A) = 1$.

We now use a) to obtain the structure of the σ^n. Recall that $\hat{\gamma}_o$ partitions $\bar{B}_\infty$ into "vertical" lines. Hence $\bar{T}_\infty^{-n}\ \hat{\gamma}_o$ partitions $\bar{B}_\infty$ into unions of 2^n vertical lines in such a way that the image under $\bar{T}_\infty^{-n}$ of a line in B_o is a set of lines scattered among the B_i with a random walk distribution (i.e., the number of lines in B_j is $p_j^n\, 2^n$, with p_j^n the n-step $0 \to j$ random walk transition probability). We can thus use the central limit theorem to find an $\bar{N}$ such that for any $A \in \Sigma(\bar{T}_\infty^{-\bar{N}}\ \hat{\gamma}_o)$ we have

$$\mu_{\bar{B}_\infty}(A \cap \bigcup_{i=-M}^{M} B_i) < \bar{\epsilon}\ \mu_{\bar{B}_\infty}(A)$$

(given $\bar{\epsilon}$ and M). Thus given any $\beta \in F$ and any $\epsilon > 0$, we might expect that there would exist an N such that for any $\alpha \in \sigma^N = \Sigma(\zeta[\bar{T}_\infty^{-N}\ \hat{\gamma}_o])$ we would have

$$|\bar{\mu}_\infty(\alpha \cap \beta) - \bar{\mu}_\infty(\alpha)\ \bar{\mu}_\infty(\beta)| < \epsilon\ . \qquad *$$

Thus for $\alpha \in \sigma$ we would have $\bar{\mu}_\infty(\alpha \cap \beta) = \bar{\mu}_\infty(\alpha)\bar{\mu}_\infty(\beta)$ for any $\beta \in F$, and the proof would be complete.

The difficulty in the above argument lies in showing that * is valid uniformly as α ranges over σ^n. We bypass this difficulty by using Doob's martingale theorem [8] to directly establish 3). We need the corollary of Doob's theorem which asserts that for a decreasing sequence of σ-algebras, $\Sigma_n \downarrow \Sigma_o$, and a measurable set A we have

$$\lim_{n\to\infty} \mu(A\|\Sigma_n)\ (x) = \mu(A\|\Sigma_o),$$ a.e., where $\mu(\cdot\|\cdot)$ denotes

conditional measure (with respect to an arbitrary σ-algebra)[7].

We want to verify that $\sigma_n \downarrow \nu$. Since $\mu(A\|\Sigma) = \mu(A)$ a.e. if and only if $\Sigma = \nu$ (mod o) we must show that for $A \in \bar{\Sigma}_\infty$ we have

$$\lim_{n\to\infty} \bar{\mu}_\infty(A\|\sigma^n) = \bar{\mu}_\infty(A) \qquad \text{a.e.} \qquad **$$

But, by virtue of the remarks at the beginning of the paragraph before the preceding one, it is not difficult to see that ** is satisfied by A of the form $X_\alpha^{\{n_j\}_{j\in J}}$ (see proof of Theorem 5.2) and hence by all $A \in \bar{\Sigma}_\infty$, so the proof is complete.

6. <u>Ergodic properties of infinite continuous system</u>

The continuous case can be treated in essentially the same way as the model of the previous section. We will therefore limit ourselves to a few remarks, omitting details.

In the previous section we indicated how the system $\bar{\tau}_\infty=(\bar{X}_\infty, \bar{\mu}_\infty(\rho), \bar{S}_\infty)$ can be obtained by a Poisson construction from the (non-normalizable, one-particle) system $(\bar{B}_\infty, \rho\mu_{\bar{B}_\infty}, \bar{T}_\infty)$. The auxiliary space $\bar{B}_\infty$ can be regarded as a product of the baker's square with the discrete space Z. The continuous models $\tau_{\nu,\rho} = (X_\infty, \Sigma_\infty, \mu_{\nu,\rho}, \{S_t\})$, where ν is an even probability measure on ℝ, absolutely continuous at the origin, are flows which can be obtained by a Poisson construction from an anxiliary system

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7. See ref. [4], Chapter 3.

$(B_\infty, \mu_{B_\infty}(\rho,\nu), \{T_t\})$; $B_\infty = B \otimes \mathbb{R}^2$ and $d\mu_{B_\infty} = d\mu_B \otimes \rho dq \otimes d\nu$. We have chosen ν to be absolutely continuous at the origin so that the probability of finding a particle at rest in any given (finite) interval will vanish.

As already suggested by our notation the only "physically reasonable" invariant probability measures on Σ_∞ are of the form $\mu_{\nu,\rho}$, ν an even probability measure on $\mathbb{R}$. Letting $\hat{\beta}$ denote a Maxwellian measure on $\mathbb{R}$ with inverse temperature β (i.e., $\hat{\beta}(A) = \sqrt{\beta/2\pi}\int_A e^{-\frac{1}{2}\beta v^2} dv$, $A \subset \mathbb{R}$, taking the mass to be unity), we obtain a family of "states" $\{\mu_{\hat{\beta},\rho}\}$ natural from the standpoint of statistical mechanics (since they are infinite volume limits of grand canonical ensembles). The presence here of more general invariant measures corresponding to different velocity distributions is due to the fact that the velocities play a trivial role in the "collisions".

The partitions of X_∞ which correspond respectively to the partitions $\zeta[\gamma_0]$ and $\zeta\,\overline{[\nu_B]}$ of $\bar{X}_\infty$ coincide. Two points x and $x' \in X_\infty$ belong to the same element of this partition if they differ at most by values of some "past" Bernoulli coordinates. In essentially the same way as for its counterpart $\zeta[\gamma_0]$, this partition is seen to satisfy the conditions by virtue of which it is a K-partition. We thus have

<u>Theorem 6</u>: $\tau_{\nu,\rho}$ is a K-flow if ν is absolutely continuous at the origin.

7. A general theorem

In this section we will establish a theorem relating the ergodic properties of a (base) system (X, μ, T) to those of $(X_\infty, \mu_\infty, T_\infty)$, the Poisson system built over (X, μ, T). The theorem will concern (base) systems which share with $(\bar{B}_\infty, \mu_{\bar{B}_\infty}, \bar{T}_\infty)$ certain key features. In particular we observe that the group $\mathbb{Z}$ of integers, acting in the natural way upon $\bar{B}_\infty$, preserves $\mu_{\bar{B}_\infty}$ and commutes with $\bar{T}_\infty$. We can thus "reduce" $(\bar{B}_\infty, \mu_{\bar{B}_\infty}, \bar{T}_\infty)$ to a set $\bar{B}_n = \bigcup_{i=-n}^{n} B_i$ by replacing $\bar{B}_\infty = B \otimes \mathbb{Z}$ by $B \otimes \mathbb{Z}_{2n+1}$ ($\mathbb{Z}_n$ denoting the integers considered modulo n); we obtain in this way (after normalizing the induced measure) the systems $\bar{\tau}_{2n+1}$, which we have shown to be K-systems (in fact, Bernoulli.).

Let X have a representation as $\mathbb{R}^2$ with μ defined on Lebesque sets. (We make this assumption for the sake of convenience of expression; the appropriate generalizations of the definition we give should be clear; in particular we could take (X, μ) to be the product of $(\mathbb{R}^2, \mu_{\mathbb{R}^2})$ with any probability space and proceed in the obvious manner.) Let T be an automorphism of (X, μ) and let the representation be such that there exist $a, b \in \mathbb{R}$ for which $G_{(a,b)}$, the group generated by $(x, y) \mapsto (x+a,y)$ and $(x, y) \mapsto (x,y+b)$, preserves μ and commutes with T. Let $R_o = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x < a, 0 \leq y < b\}$ and let us call the translates of R_o by the elements of $G_{(a,b)}$ basic rectangles.

Let us call rectangles which are unions of basic rectangles <u>compound rectangles</u>. For any compound rectangle R let τ_R be the dynamical system obtained from (X, μ, T) by replaxing X with X modulo $G_{(a',b')}$, where a' and b' are the lengths of the sides of R. We will say that a sequence R_i of rectangles <u>converges to infinity</u> if the sequence of lengths of the smallest side converges to infinity. (X, μ, T) will be said to be of <u>periodic K-type</u> if τ_{R_o} has finite entropy[8], $T(R_o)$ is bounded, and

(K) there exists a sequence R_i of compound rectangles converging to infinity such that each of the systems τ_{R_i} is a K-system.

(X, μ, T) will be said to be of <u>periodic M type</u> if

(M) there exists a sequence R_i of compound rectangles converging to infinity such that each of the systems τ_{R_i} is mixing.

We can now state

<u>Theorem 7</u>: If (X, μ, T) is of periodic K-type (M-type), then $(X_\infty, \mu_\infty, T_\infty)$ is a K-system (mixing).

<u>Proof:</u> It follows from (M) that for bounded measurable subsets A and B of $\mathbb{R}^2$,

$$\lim_{n\to\infty} \mu(T^n B \cap A) = 0.$$

The mixing assertion then follows from an argument similar to the one given in the proof of Theorem 5.2.

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8. See ref. [4], Chapter 2. All classical systems have finite entropy, by virtue of Kouchnirenko's Theorem [2].

Let ζ_∞ be the partition of X_∞ according to the number of particles in each of the fibers of a partiton ζ of X. We have seen that $(\hat{\gamma}_o)_\infty$ is a K-partition for $\bar{\tau}_\infty$, where $\hat{\gamma}_o$ is the partition of $\bar{B}_\infty$ into "vertical" line segments. Letting P_∞ be the partition of $\bar{B}_\infty$ whose elements are the B_i $(i = 0, 1, -1,\ldots)$, we recall that $\bar{B}_\infty$ can be identified with the set of possible P_∞ - names (what we have previously called "spatial" trajectories)[39], and that $\hat{\gamma}_o$ can be identified with the partition of $\bar{B}_\infty$ according to "future" P_∞ - names $(\zeta(\bigvee_{j=0}^{\infty} \bar{T}_\infty^{-j} P_\infty))$. We further recall that a key element in the proof of Theorem 5.3 was the observation that by virtue of the central limit theorem the fibers of $\hat{\gamma}_o$ <u>expand toward infinity</u>; i.e., the fiber of $\bar{T}_\infty^{-n}\hat{\gamma}_o$ containing a (fixed) point $x \in \bar{B}_\infty$ increases (monotonically) with n in such a way that the fraction of the fiber intersecting any fixed bounded region $A \subset \bar{B}_\infty$ approaches zero.

For the problem at hand we proceed similarly. We let Q_o be the partition of X into basic rectangles and let Q be a finite partition of R_o which is a generator for τ_{R_o}. (Since τ_{R_o} has finite entropy, Krieger's theorem guarantees the existence of such a partition.[9])

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9. See ref. [43], Thm. 9.7, p. 56.

We then obtain Q_∞ by forming the "product" of Q_o and Q: the atoms of Q_∞ are obtained by decomposing each atom of Q_o according to Q. Finally we let our base - K - partition ζ be the partition of X according to future Q_∞-names. Now the proof that ζ_∞ is a K-partition for $(X_\infty, \mu_\infty, T_\infty)$ is similar to the proof of Theorem 5.3. We need mention only that since, by virtue of (K) and the boundedness of $T(R_o)$, the restriction of ζ to any of the rectangles R_i is a K-partition for τ_{R_i}, finite partitions of K systems having trivial tails[10], the martingale convergence theorem applied to the τ_{R_i} implies that the fibers of ζ (within a compound rectangle) expand toward infinity,[11] permitting us to infer that the analogue of ** (see proof of Thm. 5.3) is valid for $(X_\infty, \mu_\infty, T_\infty)$.

We conclude by using Theorem 8 to show that a (certain kind of) Lorentz gas [11] forms a K-system. Sinai has shown that (apart from possible pathological situations) the motion of a particle in a two dimensional rectangle, with periodic boundary conditions, containing convex barriers from which the particle, which otherwise moves freely, undergoes elastic collisons induces a K-flow on the unit tangent bundle of the rectangle [41]. Thus the dynamical system representing the motion of a particle in a two dimensional (nonpathological) periodic array of circular barriers (at unit velocity) is

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10. See ref. [43], Thm. 7.9, p. 38 .

11. To define precisely the concept of uniform expansion toward infinity, we use the canonical systems of measures possessed by the $T^{-n}\zeta$ [37].

of periodic K-type, so that the system representing an infinite gas of such particles, all moving with the same speed, (with a grand canonical configurational measure on bounded regions) is a K-system for any discrete time evolution. Though the thermodynamic limit of the grand canonical ensemble (Maxwellian velocity distribution) is not built over a system of periodic K-type (since the speed of a particle is a constant of the motion), we can still use an argument similar to the one given above to conclude that it, too, is a K-system; we choose as our base - K-partition ζ the union of the partitions $\zeta(s)$ $(s \in \mathbb{R}^+)$, the base- K-partitions on the surfaces of constant speed, and use the fact that such surfaces "support" systems of periodic-K-type. (Here we are ignoring the technical problem of showing that the partitions $\zeta(s)$ can be chosen in such a way that their union is a measurable partition. We also observe that although we have shown that our Lorentz gas is a K-system under any discrete time evolution, we have not shown it to be a K-flow, though an approach similar to the above could probably be exploited to establish this result as well).

8. <u>Remarks</u>:

We observe that though the infinite ideal gas and $\tau_{\nu,\rho}$ are both K-systems, there is certainly a sense in which the "mixing" which occurs in $\tau_{\nu,\rho}$ is of a less trivial nature than that which occurs in the ideal gas. (Recall that we have shown that certain finite submodels of $\bar{\tau}_\infty$ are Bernoulli). This difference is perhaps reflected in differences in the K-partitions for the respective systems. Two points, x and x', in the phase space of the ideal gas belong to the same element of its K-partition if the points $\tilde{x}$ and $\tilde{x}'$ obtained from them by deleting all particles outside some region of the auxiliary space coincide, suggesting, perhaps, the "nonlocal" nature of the dissipation of disturbances. Two points, y and y', of X_∞ belong to the same element of its K-partition if the points $\tilde{y}$ and $\tilde{y}'$ (belonging to some new space) obtained from them by factoring out some of the structure of the space B_∞ coincide, suggesting, perhaps, a "local" mechanism for the dissipation of disturbances.

We conclude by showing that our obtaining a K-partition of a very different nature from that of the ideal gas was unavoidable. Having denoted the one-particle ideal gas space by (X^I, T_t^I), we have seen in Chapter III, section 7 that we can easily construct a partition $F = \{F_i\}$ of X^I such that $T_1^I F_j = F_{j+1}$, and that the existence of such a partition of the auxiliary space of an infinite (Poisson) system implies that the system is isomorphic to a Bernoulli shift. We will show that though the systems we have considered may be Bernoulli, they are not of the above type.

<u>Theorem 8.1</u>: Let T be an automorphism of the measure space (X, μ). If there exists a set A of finite positive measure almost all points of which return to it infinitely often, then we cannot partition X into $\{C_i\}$ in such a way that $TC_j = C_{j+1}$.

<u>Proof:</u> Assume we have such an A and $\{C_i\}$ for which $\mu(A \cap C_o) \geq 0$. Let R_n denote the set of elements $x \in A$ for which $T^n x \in A$. Then the $T^n(A \cap C_o \cap R_n) \subset A \cap C_n$ are disjoint so that

$$\sum_n \mu(T^n(A \cap C_o \cap R_n)) = \sum_n \mu(A \cap C_o \cap R_n) < \mu(A).$$

But

$$\sum_n \mu(A \cap C_o \cap R_n) = \int_X d\mu \sum_n \chi[A \cap C_o \cap R_n](z) = \int_{A \cap C_o} d\mu\, R(z) = \infty,$$

where $R(z)$ is the number of integers n for which $T^n z \in A$, and $\chi[D]$ is the characteristic function of $D \subset X$.

Since in a symmetric random walk of dimension <3 a particle will with probability one return to its original position infinitely often [18], the above theorem applies to the auxiliary space of the models we have considered. (For $\bar{\tau}_\infty$ we can set $A = B_o$).

We observe that all that is required for the above argument is that the measure of the subset of A whose points return to A infinitely often be nonvanishing. If this is not the case we will say that A is nonrecurrent. Strengthened in this way, the theorem admits of a partial inverse.

<u>Theorem 8.2:</u> Let μ be σ-finite. If all sets of finite measure are noncurrent there exists a partition $\{C_i\}$ of X for which $TC_k = C_{k+1}$. (Hence the Poisson system built upon (X, μ, T) is isomorphic to a Bernoulli shift).

<u>Proof</u>: We have $X = \bigcup_n D_n$, with $D_n \subset D_{n+1}$ and $\mu(D_n) < \infty$ for all n, for some sequence $\{D_n\}_{n \geq 1}$. Let E_n be the set of points which will eventually be in D_n. Let $F_1 = E_1$ and $F_n = E_n - E_{n-1}$, $n > 1$. Let f be the measurable function from X to $\mathbb{Z}$ such that for $x \in F_k$, f(x) is the largest integer n for which $T^n(x) \in D_k$. f is defined almost everywhere, and $f(Tx) = f(x) - 1$. We therefore obtain a partition $\{C_i\}$ satisfying $TC_i = C_{i+1}$ by setting $C_j = \{x \mid f(x) = -j\}$.

Since all the states of a random walk in more than 2 dimensions are transient[12], the above theorem is easily seen to apply to the (auxiliary) space representing such a random walk. Furthermore, the analog of the K-partition of $\bar{\tau}_\infty$ is easily seen to be a K-partition for the Poisson system built over a random walk in any finite number of dimensions (with an infinite stationary measure). Thus a random walk in more than 2 dimensions provides a basis for a system in which a K-partition of the ideal gas type and a K-partition of the $\bar{\tau}_\infty$ type are present simultaneously. (The preceding remarks apply as well to the higher dimensional generalizations of the continuous systems $\tau_{\nu,\rho}$; the two dimensional generalization of the periodic field of barriers could be taken, say, to be a square grid from which particles are either reflected or transmitted according to the same rules as in the one-dimensional case.)

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12. See ref. [10], pp. 360-361.

V. Generalized Dynamical Systems and the Space-Time Ergodic Properties of Infinite Systems of Particles

1. Motivation

As we have seen, an infinite system, such as the infinite ideal gas, may possess the strongest possible ergodic properties without exhibiting good thermodynamic behavior. Thus, the ergodic theoretic concepts introduced so far cannot adequately account for such behavior. In fact, the situation is somewhat worse. We have found several examples, among them the infinite ideal gas, of infinite systems of particles, physically quite distinct, whose time evolutions form Bernoulli flows, and it is to be expected that infinite systems of interacting particles exhibiting better thermodynamic behavior also form Bernoulli flows. Hence, by virtue of Ornstein's theorem, these systems are indistinguishable from the standpoint of the $\{X, \Sigma, \mu, T\}$ framework[1]. Thus, as well as new ergodic theoretic concepts, we need an expanded abstract framework to support these concepts.

Fortunately, there is a rather prominent additional element

1. Infinite systems typically have infinite entropy; e.g., an independent generator obtained by a Bernoulli construction from a continuous measure space will be nonatomic; thus, all flows which we have so far shown to be Bernoulli have infinite entropy.

of structure common to infinite systems of interest in statistical mechanics: invariance under space translations. The dynamics as well as the equilibrium states of infinite systems of particles are normally required to be translation invariant. Thus, the measure spaces of these systems possess, in a natural way, a larger invariance group than considered so far: the abelian group generated by both space and time translations. We have mentioned that the ergodic properties under space translations alone of the equilibrium states of these systems have, in fact, already been subject to investigation. It thus appears natural to extend our abstract framework by replacing the flow $\{T_t\}$ in the quadruple $(X, \Sigma, \mu, \{T_t\})$ by the larger abelian group G generated by space translations and time evolution. Generalizations of the ergodic theoretic concepts and results for a group generated by a single automorphism to a group generated by several commuting automorphisms have, in fact, already been obtained[2] [18,5]. We shall see that the ergodic properties of infinite systems relative to this framework provide a much sharper tool of investigation than the ergodic properties with respect to space translations or time evolution separately.

The extension of our framework to the larger group G has as

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2. We shall review some of them in the next section.

an immediate consequence that the implications of Ornstein's theorem no longer afford us significant difficulty; though Ornstein's theorem presumably extends to a generalized dynamical system [18] ((X, Σ, μ, G), G a group of automorphisms generated by several commuting transformations) it should be much more difficult for infinite systems to be Bernoulli under the space-time translation group[3]. Furthermore, the argument given in Chapter III, section 4 to the effect that we should normally expect infinite systems to be K-systems does not generalize to the extended framework; though it is not plausible that infinite systems should be "approximable" by a finite partition, bounded regions may very well be so "approximable"; from such a finite-partition "approximation", using space translations, we may obtain a "global approximation".

It seems reasonable to expect that good mixing properties under the space-time translation group might require more than a "purely nonlocal dissipation of disturbances". We will see, in fact, that the inclusion of space translations in the automorphism subgroup allows us to control effects due to the infinite extension of our systems; for example, we shall see that though possessing

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3. See the next section.

infinite time evolution entropies, the infinite systems which we have considered have physically significant space-time entropies.

2. Properties of generalized dynamical systems[4]

We consider a (generalized) abstract dynamical system (X, Σ, μ, G), where G is generated by n commuting automorphisms of (X, Σ, μ). To simplify the notation we will explicitly treat only the case $n = 2$. Let (S,T) be a pair of commuting automorphisms which generates the group G. Every such pair determines a homomorphism from the group $\mathbb{Z}^2$ to G, permitting the representation of the elements of G by the points of $\mathbb{Z}^2$. Though some properties will be formulated in terms of the pair (S,T), they will, in fact, depend only upon G, unless we explicitly indicate otherwise. However, we intend for the definitions we shall give to apply to dynamical systems of the form $(X, \Sigma, \mu, \{S,T\})$, where S and T are commuting automorphisms possibly satisfying some relation such as $S = T$. We will say that a sequence ρ_n of parallelograms in $\mathbb{Z}^2$ approaches infinity if the smallest of its dimensions (orthogonal distance between parallel sides) approaches infinity. We will denote by $N(\rho)$ the number of points in the parallelogram ρ. For any measurable partition P and $g \in G$, we let $P_g = \bigvee_{j=-\infty}^{\infty} g^j P$, $P_g^- = \bigvee_{j=-\infty}^{-1} g^j P$, and $P_G = \bigvee_{g \in G} gP$. We denote by $\ominus$ the orthogonal complement in

4. As a general reference for much of the material of this section, see Conze [5].

$L^2(\mu)$ of the constants. The generalization of the properties which we shall describe to the case in which the group G is generated by two flows $\{S_t\}$ and $\{T_t\}$ parallels the corresponding generalization from properties of a discrete dynamical system to properties of a flow.

a) ergodicity: (X, Σ, μ, G) is ergodic if all $A \in \Sigma$ invariant under G[5] are such that $\mu(A) = 0$ or $\mu(A) = 1$. As in the case of a one parameter group, we have that if (X, Σ, μ, G) is ergodic, and only then,

$$\lim_{n \to \infty} 1/N(\rho_n) \sum_{g \in \rho_n} f(gx) = \lim_{n \to \infty} 1/N(\rho_n) \sum_{(k,\ell) \in \rho_n} f(S^k T^\ell x)$$

$$= \int d\mu \, f \quad , \text{a.e.},$$

for ρ_n a sequence of parallelograms approaching infinity, $f \in L^1(\mu)$, and $x \in X$. Ergodicity with respect to G is clearly a weaker property than, say, ergodicity with respect to T. It is the only such property which we shall encounter.

b) mixing: (X, Σ, μ, G) is mixing if

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5. I.e., satisfying the equation $g A = A$ for all $g \in G$

$$\lim_{g \to \infty} \mu(gA \cap B) = \mu(A)\, \mu(B)$$

for all A, B $\in \Sigma$.[6] Since convergence to infinity in $\mathbb{Z}^2$ is invariant under automorphisms of $\mathbb{Z}^2$, the definition of mixing does not depend on the choice of generators S and T.

c) countable Lebesgue spectrum: (X, Σ, μ, G) has <u>Lebesgue spectrum of countable multiplicity</u> if there exists a family $\left\{f^i_{(j,k)}\right\}^{i \in \mathbb{Z}}_{(j,k)\in \mathbb{Z}^2}$ of functions forming an orthonormal basis of Θ and satisfying

$$U_{S^n T^m} f^i_{(j,k)} = U^n_S U^m_T f^i_{(j,k)} = f^i_{(j+n,k+m)}$$

for all $(j,k) \in \mathbb{Z}^2$, $(n,m) \in \mathbb{Z}^2$, and $i \in \mathbb{Z}$. Just as in the case of a one parameter group, a system with Lebesgue spectrum is mixing.

d) entropy [5]: The entropy of a group G is defined in a manner completely analagous to the definition of the entropy of an automorphism T. We need mention only that the entropy of a (countable) measurable partition P relative to the group G is defined by

$$h(P,G) = \lim_{n \to \infty} 1/N(\rho_n)\, H(\bigvee_{g \in \rho_n} g P),$$

6. By $g \to \infty$ we mean in the sense of the natural locally compact topology on $\mathbb{Z}^2$. The generalization to an arbitrary locally compact topological group G is immediate.

where $\{P_n\}$ is a sequence of parallelograms approaching infinity. The limit is independent of the particular sequence of parallelograms, and, consequently, $h(P,G)$ is independent of the choice of generators T and S of G. In much the same way as in the case of a single automorphism, one verifies that if $H(P) < \infty$,

$$h(P,G) = H(P \| P_S^- \vee (P_S)_T^-).$$

We will call $P_S^- \vee (P_S)_T^-$ the past of P relative to (S,T), and denote it by P_G^-.[7] We also note that if Q is a generator for G of finite entropy (i.e., $Q_G = \epsilon$ (mod 0), and $H(Q) < \infty$), we have for the entropy of G

$$h(G) \ (= \sup_{P \text{ finite}} h(P,G) = \sup_{H(P) < \infty} H(P,G)) = h(Q,G).$$

Finally, we will say that (X, Σ, μ, G) has <u>completely positive entropy</u> if $h(P,G) > 0$ for all nontrivial partitions P. If (X, Σ, μ, G) has completely positive entropy, it is mixing.

e) K-systems [5]:

We define the K-system property for an ordered pair of

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7. There are, of course, seven other possible choices of a "past" of P which we could insert in the above relation in place of P_G^- without altering its validity.

commuting automorphisms (S,T) rather than for the group G which they generate. Insofar as space translations and the time evolution play rather different roles in statistical mechanics, this development is quite appropriate. The key to the generalization is the extension of the natural ordering on $\mathbb{Z}$, on the structure of which the notion of K-system for an automorphism T is implicitly based, to an ordering on $\mathbb{Z}^2$. We write $(n,m) \leq (p,q)$ if $m < q$, or if $m = q$ and $n \leq p$. We will say that $(X, \Sigma, \mu, (S, T))$ is a K-system if there exists a measurable partition ζ such that

1) ζ is increasing:

$$S^n T^m \zeta \leq S^p T^q \zeta \pmod 0 \quad \text{if } (n,m) \leq (p,q),$$

2) $\bigvee_{(n,m)\in \mathbb{Z}^2} S^n T^m \zeta = \epsilon \pmod 0$,

3) $\bigwedge_m S^{-m} \zeta = T^{-1} \zeta_S \pmod 0$,

4) $\bigwedge_n T^{-n} \zeta_S = \nu \pmod 0$.

Note that if $(X, \Sigma, \mu, (S,T))$ is a K-system, (X, Σ, μ, T) possess an S-invariant K-partition, namely ζ_S. If (S,T) forms a K-system, the group G generated by S and T has competely positive entropy and, by essentially the same argument as for a single automorphism T, countable Lebesgue spectrum.

f) Bernoulli systems:

(X, Σ, μ, G) is a Bernoulli system if there exists a measurable partition P which is a generator for G such that $\{g P\}_{g \in G}$

is an independent family of partitions. If (X, Σ, μ, G) is Bernoulli, then $(X, \Sigma, \mu, (S,T))$ is a K-system for any pair of generators (S,T); if P is an independent generator for G, $\zeta = \bigvee_{(n,m)\leq(0,0)} S^n T^m P$ is a K-partition for (S,T). Ornstein's theorems can be extended to this generalized framework [18].

g) formula of Abramov:

If G is generated by $\{S_t\}, \{T_t\}, \ldots, \{R_t\}$, n commuting groups depending continuously on a real parameter, a generalization of the formula of Abramov ($h(T_t) = |t|\, h(T_1)$) can be proven [5]. Let Γ be the subgroup of G generated by $S_1, T_1, \ldots, R_1$. If we regard G as a real vector space with basis $S_1, T_1, \ldots, R_1$, we can operate on Γ by a real $n \times n$ matrix M to obtain a subgroup Γ_M. Then $h(\Gamma_M) = |\det(M)|\, h(\Gamma)$.

3. <u>Invariance of space-time ergodic properties under Galilean transformations</u>

For the most part we will be working from now on with dynamical systems (X, Σ, μ, G) where (X, Σ, μ) is an equilibrium measure for a one dimensional infinite system of particles and G is the group generated by S_1, the unit space translation, and T_1, the unit time evolution. In this framework we will consider only Galilean transformations determined by integral velocities. Most of the results generalize to arbitrary Galilean and Lorentz transformations in the case where G is generated by the complete group of space and time translations.

Accordingly, let (X, Σ, μ) be a translation invariant equilibrium state of a one dimensional (for notational convenience) system of infinitely many particles. Let T_t denote its time evolution and S_q the spatial translations. We can describe a trajectory induced by T_t by specifying a family $\{q_i(t)\}_{t\in R}$ of functions representing the time evolution of the positions of the individual particles, labeled arbitrarily. A Galilean transformation G_v at velocity v transforms a trajectory $\{q_i(t)\}$ into a trajectory $\{q_i(t) - vt\} = G_v\{q_i(t)\}$. G_v, of course, also transforms the velocities according to $G_v\{v_i(t)\} = \{v_i(t)-v\}$. Thus, in an obvious manner, G_v induces a mapping from the system $(X, \Sigma, \mu, (S_q, T_t))$ to the system $G_v(X, \Sigma, \mu, \{S_q, T_t\}) = (X', \Sigma', \mu', \{S'_q, T'_t\})$. It should be clear that from the standpoint of our abstract framework we can identify $(X', \Sigma', \mu', \{S'_q, T'_t\})$ with $(X, \Sigma, \mu, \{S_q, T_t S_{vt}\})$, so that the effect of G_v can be regarded as the replacement of the pair (S_q, T_t) by the pair $(S_q, T_t S_{vt})$, or, in the discrete case, assuming v to be an integer, (S_1, T_1) by $(S_1, T_1 S_1^v)$.[8,9] Consequently, those properties which depend upon

8. The effect of a Lorentz transformation would be to replace S_q by some $S_{\alpha q} T_{\beta q}$, $\alpha, \beta \in R$, since under a Lorentz transformation both the space and the time axes become obliquely oriented with respect to the original axes [3].
9. Henceforth, we will write (S,T) for (S_1, T_1), etc..

only the group G are invariant under Galilean transformations; S and T S^v generate the same group as S and T. Furthermore, the concept of mixing for the pair (S,T), which depends upon the notion of convergence to infinity in $\mathbb{Z}^2$, is invariant under G_v, inasmuch as the automorphism of $\mathbb{Z}^2$ induced by the replacement $(S,T) \mapsto (S, T S^v)$ leaves such convergence invariant. Finally, the concept of K-system, which depends upon the ordered pair (S,T) and in particular upon the ordering of $\mathbb{Z}^2$ which the pair induces, is invariant under G_v, since $(S, T S^v)$ induces the same ordering on $\mathbb{Z}^2$ as does (S, T).

4. Space-time ergodic properties of the ideal gas

We proceed to the investigation of the space-time ergodic properties of the Poisson systems considered previously. We will work with one-dimensional systems; the results and arguments can easily be adapted to several spatial dimensions. Our investigation will provide a precise formulation of the heuristic remarks in Chapter IV, section 8 concerning distinctions between the ideal gas and, say, the system $\tau_{\beta,\rho}$. In this section we will show that though the ideal gas has countable Lebesgue spectrum even in the space-time framework, it is not a K-system for the pair (S, T).

We will first exhibit a concrete example of two systems identical from the standpoint of the framework of the time evolution considered by itself which are distinguishable from the standpoint of the space-time framework. A system identical to the infinite ideal gas except that, instead of a Maxwellian velocity distribution, all particles move with unit velocity (to the right) is clearly a Bernoulli flow under the time evolution and hence isomorphic to the ideal gas (with a Maxwellian velocity distribution). However, since the time evolution and the space translations act identically on the phase space of this sytem, it is not "jointly" mixing; in fact, $S^n T^n$ is the identify transformation though $(n,n) \to \infty$ in $\mathbb{Z}^2$. Recalling that we have described the infinite ideal gas as a Poisson system $(X^I_\infty, \mu^I_\infty, T^I_{t,\infty})$, with $X^I = \mathbb{R} \otimes \mathbb{R}$, etc. (see Chapter III, section 7), we will denote by S^I_x the spatial translation on X^I (i.e., $S^I_\alpha (q,v) = (q - \alpha, v)$, $(q,v) \in \mathbb{R} \otimes \mathbb{R}$) and by $\{S^I_{x,\infty}\}$ the flow on $(X^I_\infty, \mu^I_\infty)$ induced by $\{S^I_x\}$. That $(X^I_\infty, \mu^I_\infty, \{S^I_{x,\infty}, T^I_{t,\infty}\})$ is not isomorphic to a gas in which all particles move at constant unit velocity is a consequence of the following simple

<u>Theorem 4.1:</u> $(X^I_\infty, \mu^I_\infty, \{S^I_{x,\infty}, T^I_{t,\infty}\})$ is mixing.

<u>Proof</u>: The theorem follows[10] from the observation that if A and B are bounded subsets of $\mathbb{R}^2$,

$$\lim_{(r,s) \to \infty} \mu^I (S^I_r T^I_s A \cap B) = 0.$$

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10. See proof of Thm. IV. 5.2.

The space-time ergodic properties of the ideal gas are, in fact, somewhat stronger than mixing:

<u>Theorem 4.2:</u> $(X^I_\infty, \mu^I_\infty, G^I_\infty)$[11] has countable Lebesgue spectrum.

<u>Proof</u>: In view of the Fock space representation of the induced unitaries on $L^2(\mu^I_\infty)$, obtained in Chapter III, section 6, it suffices to show that $(X^I, \Sigma^I, \mu^I, G^I)$ has Lebesgue spectrum. We let $U_\alpha = U_{S^I_\alpha}$ and $V_\beta = U_{T^I_\beta}$. Then for $f(q,v) \in L^2(\mu^I)$ $(= L^2(\mathbb{R}^2, e^{-v^2} dqdv)$, with β and ρ adjusted to obtain a "simplified" measure), we have

$$U_\alpha V_\beta f(q,v) = f(q - \alpha + v\beta, v).$$

Let $\emptyset: L^2(\mathbb{R}^2, e^{-v^2} dqdv) \to L^2(\mathbb{R}^2, e^{-v^2} dkdv), f(q,v) \mapsto \tilde{f}(k,v)$, the q-Fourier transform of f.

Let $\tilde{U}_\alpha$, $\tilde{V}_\beta$ represent the Fourier transforms of U_α and V_β, respectively.

Then for $\tilde{f} \in L^2(\mathbb{R}^2, e^{-v^2} dkdv)$, $\tilde{U}_\alpha \tilde{V}_\beta \tilde{f}(k,v) = e^{-i(k\alpha - kv\beta)} \tilde{f}(k,v)$.

Let $\Psi: L^2(\mathbb{R}^2, e^{-v^2} dkdv) \to L^2(\mathbb{R}^2, (e^{-w^2/k^2}/|k|)dkdw)$ be the isomorphism induced by $\Psi: \mathbb{R}^2 \to \mathbb{R}^2, (k,v) \mapsto \Psi(k,v) = (k,kv) = (k,w)$.

Letting $\bar{U}_\alpha$ and $\bar{V}_\beta$ be the images, respectively, of $\tilde{U}_\alpha$ and $\tilde{V}_\beta$ under Ψ, we have for $\bar{f} \in L^2(\mathbb{R}^2, e^{-w^2/k^2}/|k|\ dkdw)$

11. G^I_∞ is, of course, the group of space-time translation of X^I_∞, induced by the group G^I of space-time translations of X^I.

$\bar{U}_\alpha \bar{V}_\beta \bar{f}(k,w) = e^{-i(k\alpha - w\beta)} \bar{f}(k,w)$. We thus have a representation of $U_\alpha V_\beta$ as the operator of multiplication by $e^{-i(k\alpha - w\beta)}$ on $L^2(\mathbb{R}^2, d\mu(k,w))$ with μ a measure on $\mathbb{R}^2$ equivalent to the the Lebesgue measure, establishing the theorem.

Fortunately, countable Lebesgue spectrum is the strongest space-time ergodic property which we shall find that the ideal gas possesses. In Chapter IV, section 8 we alluded to the non-local nature of the dissipation of disturbances of the ideal gas, as a symptom of which we might regard the manifestly non-translation-invariant nature of its K-partition. Since a space-time ((S,T)) K-system must possess, in particular, a translation invariant K-partition for the time evolution T, and since it appears implausible that the time evolution of the ideal gas should possess such a K-partition, we expect it to fail to be a K-system for (S,T). Rather than verifying that no such partition exists, we will show that the ideal gas is not a space-time K-system by establishing that the space-time entropy of the ideal gas is zero. Since K-systems have completely positive entropy, this will imply the desired result.

Theorem 4.3: $h(G_\infty^I) = 0$, so that $(X_\infty^I, \mu_\infty^I, (S_\infty^I, T_\infty^I))$ is not a K-system.

Proof: We will compute $h(G_\infty^I)$ by finding a partition P^I of finite entropy such that $P^{I}{}^{-}_{G_\infty^I} = \epsilon$ (mod 0). Then, since P^I is a generator for G^I_∞, we will have

$$h(G^I_\infty) = h(P^I, G^I_\infty) = H(P^I \,\|\, P^{I-}_{G^I_\infty}) = 0.$$

We choose for P^I the partition whose atoms are of the form $P_{\{n;(m_1,k_1),\ldots,(m_i,k_i),\ldots,(m_n,k_n)\}} = \{x \in X^I_\infty : N([0,1) \otimes \mathbb{R}) = n$ and for $(q_1,v_1),\ldots,(q_n,v_n)$, $q_i < q_j$ for $i < j$, the coordinates of the particles in $[0,1) \otimes \mathbb{R}$,[12] we have $q_i + v_i \in [m_i, m_i + 1)$ and the particle of $T^I_\infty x$ with coordinates $(q_i + v_i, v_i)$ has index k_i in $[m_i, m_i+1)$, for all $i = 1, 2, \ldots, n.\}$

The theorem now follows from two lemmas:

<u>Lemma 4.4</u>: $P^{I-}_{G^I_\infty} = \epsilon$ (mod 0).

<u>Lemma 4.5</u>: $H(P^I) < \infty$.

<u>Proof of Lemma 4.4</u>:

It suffices to show that $(P^I)^-_{S^I_\infty T^I_\infty} = \epsilon$ (mod 0), which we will do by showing that $(P^I)^-_{S^I_\infty T^I_\infty}$ contains sufficient information to determine (mod 0) all the coordinates $\{(q_i, v_i)\}$ of the particles of a point $x \in X^I_\infty$, and hence x itself. We first observe that

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12. We will say that the particle of $x \in X^I_\infty$ with coordinates (q_i, v_i) has <u>index</u> i in $[0,1)$ (at $t = 0$). If the particle of $T^I_\infty x$ with coordinates $(q_i + v_i, v_i)$ has index k_i in $[m_i, m_i+1)$, we will often say that the particle of x with coordinates (q_i, v_i) has index k_i in $[m_i, m_i+1)$ at $t = 1$.

$P^{I}_{S^{I}_{\infty}}$ determines the number of particles in each unit cell $[i, i+1) \otimes \mathbb{R}$ of $\mathbb{R}^2$ and, in addition, determines the immediate future of each particle to the extent of requiring, for example, that the particle which at $t = 0$ has index j in $[k,k+1)$ will at $t = 1$ have index ℓ in $[m, m+1)$. Similarly, $(T^{I}_{\infty})^{-1} \; P^{I}_{S^{I}_{\infty}}$ provides analagous information for times $t = 1$ and $t = 2$, and the index information contained in P^{I} enables us to unambiguously trace every particle from $t = 0$ to $t = 2$, with respect to the partition of X^{I} into unit cells, using the information contained in $P^{I}_{S^{I}_{\infty}} \vee (T^{I}_{\infty})^{-1} \; P^{I}_{S^{I}_{\infty}}$. Proceeding in this way, we see that the knowledge of the atom of $(P^{I}_{S^{I}_{\infty}})^{-}_{T^{I}_{\infty}}$ containing a point $\mathbf{x} \in X^{I}_{\infty}$ determines the trajectory of each particle in $\mathbf{x}$, with respect to the unit cells, from times $t = 0$ to $t = \infty$, so that the velocities of all the particles are uniquely determined. The Jacobi theorem for the irrational rotation of the circle implies that positions of particles with irrational velocity are also determined by $(P^{I}_{S^{I}_{\infty}})^{-}_{T^{I}_{\infty}}$, since for v irrational the sequence $q, q+v, q+2v, \ldots$ is dense in $\mathbb{R}$ (mod 1) with the standard topology. Finally, since the Maxwellian distribution μ_{β} assigns measure zero to the set of rational velocities, we have

$$(P^{I}_{S^{I}_{\infty}})^{-}_{T^{I}_{\infty}} = \epsilon \pmod 0.$$

Proof of Lemma 4.5:

The finiteness of $H(P^I)$ follows from elementary estimates, using the observations:

a) For measurable partitions α, β, and γ,
$H(\alpha \vee \beta \,\|\, \gamma) = H(\alpha \| \gamma) + H(\beta \| \alpha \vee \gamma) \leq H(\alpha \| \gamma) + H(\beta \| \gamma)$.

b) All moments of a Poisson distribution are finite.

c) All moments of a Gaussian distribution are finite.

d) For $\sum_{i=1}^{\infty} p_i = 1$ and $p_j \geq 0$ for all j,
$\sum_{n=1}^{\infty} p_n \log n \leq \log (\Sigma n p_n) = \log \langle n \rangle$, since $\log t$ is concave in $(0,\infty)$.

e) $H(P) \leq \log k$ for P a partition with k atoms.

We estimate $H(P^I)$ by writing $P^I = P_1 \vee P_2 \vee P^I$, where P_1 is the partition of X_∞^I according to the number of particles in $[0,1) \otimes \mathbb{R}$, and P_2 is the refinement of P_1 according to the cell membership at $t = 1$ of the indexed particles in $[0,1) \otimes \mathbb{R}$. We then have

$$H(P^I) = H(P_1) + H(P_2 \| P_1) + H(P^I \| P_1 \vee P_2).$$

Now $H(P_1) = \widetilde{H}(\rho) = -\sum_{n=0}^{\infty} (e^{-\rho} \rho^n/n!) \log (e^{-\rho} \rho^n/n!)$; $\widetilde{H}(0) = 0$,[13] and $\widetilde{H}(t)$ is continuous for $t \in \mathbb{R}^+$.

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13. $= \lim_{t \to 0} \widetilde{H}(t)$

$$H(P_2 \parallel P_1) = \sum_{n=0}^{\infty} (e^{-\rho} \rho^n/n!)\, H(P_2 \mid \alpha_n)$$

$$\leq \sum_{n=0}^{\infty} (e^{-\rho} \rho^n/n!)\, n\, C_\beta < \infty ,$$

where $\alpha_n = \{x \in X_\infty^I \mid N([0,1) \otimes \mathbb{R}) = n\}$ and C_β is a uniform (in i and n) bound on the entropy of the partition of α_n (normalized to unit total measure) according to the cell membership of the particle of index i in $[0,1)$ at $t = 0$. Let γ be a typical atom of $P_1 \vee P_2$ and let $P_{\gamma,j}$ be the partition of γ (normalized) according to the index at $t = 1$ of the particle with index j at $t = 0$. We have

$$H(P^I \parallel P_1 \vee P_2) = \sum_{\gamma \in P_1 \vee P_2} \mu_\infty^I(\gamma)\, H(P^I \mid \gamma)$$

$$\leq \sum_{\gamma \in P_1 \vee P_2} \mu_\infty^I(\gamma) \sum_{j=1}^{n(\gamma)} H(P_{\gamma,j}) ,$$

where $n(\gamma)$ = the value of $N([0,1) \otimes \mathbb{R})$ characteristic of γ. Let $\widetilde{P}_{\gamma,j}$ be the partition of γ according to N_j , the number of particles at time $t = 1$ in the cell containing the particle which at $t = 0$ has index j in $[0,1)$. Then

$$H(P_{\gamma,j}) \leq H(\widetilde{P}_{\gamma,j} \vee P_{\gamma,j}) \leq H(\widetilde{P}_{\gamma,j}) + H(P_{\gamma,j} \parallel \widetilde{P}_{\gamma,j}).$$

Now, using e) and d),

$$H(P_{\gamma,j} \parallel \widetilde{P}_{\gamma,j}) \leq \sum_{k=1}^{\infty} \widetilde{p}_k \log k$$

$$\leq \log \Sigma\, k\, \widetilde{p}_k \leq \log(\rho + n(\gamma)),$$

where $\widetilde{p}_k$ is the probability, given γ, that $N_j = k$. Also, $\widetilde{P}_{\gamma,j}$ is the partition of γ according to the value of $N(A_{\gamma,j})$, with $A_{\gamma,j} = \{x \in X^I \mid T^I x \in [n_j(\gamma), n_j(\gamma) + 1) \otimes \mathbb{R}\} - [0,1) \otimes \mathbb{R}$; $n_j(\gamma)$ = the left coordinate of the cell containing at $t = 1$ the particle with index j in $[0,1)$ at $t = 0$, characteristic of γ. Thus, $H(\widetilde{P}_{\gamma,j}) = \widetilde{H}(\mu^I(A_{\gamma,j}))$, with $\mu^I(A_{\gamma,j}) < \rho$, so it appears evident that $\widetilde{H}(\mu^I(A_{\gamma,j})) \leq \widetilde{H}(\rho)$. In any case, since $\widetilde{H}(0) = 0$ and $\widetilde{H}(t)$ is continuous for $t \in [0, \infty)$, we have $\widetilde{H}(\mu^I(A_{\gamma,j})) \leq \widetilde{H}(\widetilde{\rho})$ for some $\widetilde{\rho} \in (0,\rho]$ not depending upon γ and j. Thus

$$H(P^I \parallel P_1 \vee P_2) \leq \sum_n n\, \mu_\infty^I(\alpha_n)(\widetilde{H}(\widetilde{\rho}) + \log(n + \rho))$$

$$= \sum_n (e^{-\rho} \rho^n / n!)\,(n\, \widetilde{H}(\widetilde{\rho}) + n \log(n + \rho)) < \infty,$$

completing the proof.

The method of proof of Theorem 4.3 is an extension to an infinite system of particles of the method of proof for the vanishing of the entropy of a finite ideal gas [33]. We also remark that a similar method, using, in particular, a partition analagous to P^I, can be used to show that an infinite one-dimensional system of hard rods has vanishing space-time entropy [M. Aizenmann, private communication]. Finally, we observe that though the above argument works only for a velocity distribution assigning zero measure to rational velocities, the theorem is valid for an arbitrary velocity distribution, since we can always change the time scale in such a way that the argument is applicable and then apply the formula of Abramov[14] to obtain

$$h(G) = \tau_o \; h(G_{\tau_o}) = 0,$$

where G_{τ_o} is the group generated by unit space and time translations corresponding to a change of time scale by the appropriate factor τ_o.

5. Space-time ergodic properties of some Poisson systems built over systems of periodic type

a) space-time K-systems

Having shown that the ideal gas is not a K-system for (S,T), we formally distinguish it from systems such as $\bar{\tau}_\infty$ and $\tau_{\nu,\rho}$ by establishing that such systems are, in fact, space-time K-systems. That we have found K-partitions for these systems which are trans-

14. See section 2g of the present chapter.

lation (S_∞) invariant[15] strongly suggests that this is the case.

We will deal in detail with Poisson systems built over a system of periodic-K-type with one spatial dimension, using the notation and terminology established previously (for systems of two spatial dimensions)[16]. The remarks made there also apply here. We will denote by S a generator of $G_{(a,b)}$, i.e., a periodic translation, and by S_∞ the automorphism of (X_∞, μ_∞) induced S. We will prove the following:

<u>Theorem 5.1</u>: If (X, Σ, μ, T) is of periodic-K-type, then $(X_\infty, \Sigma_\infty, \mu_\infty, (S_\infty, T_\infty))$ is a K-system.

<u>Proof:</u> If we can express ζ_∞ in the form ξ_{S_∞}, where ξ satisfies 1) and 3) of the definition of an (S,T) K-system,[17] we will be done, since 2) and 4) follow from the K-properties of ζ_∞. We obtain such a ξ by setting $\xi = T_\infty^{-1} \zeta_\infty \vee \zeta_\infty^{+}$, where ζ_∞^{+} is the restriction of ζ_∞ to $\mathbb{R}^+$ (the nonnegative spatial axis), i.e.,

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15. See Chapter IV, sections 5 and 6.
16. See Chapter IV, section 7.
17. See Chapter V, section 2e.

ζ_∞^+ is the partition associated with the σ-algebra $\hat{\zeta}_\infty^+ = \hat{\zeta}_\infty \cap \Sigma_\infty(\mathbb{R}^+)$[18].

It is obvious that $\zeta_\infty = \xi_{S_\infty}$ and that ξ is increasing. The theorem thus follows from

<u>Lemma 5.2</u>: $\bigwedge_n S_\infty^{-n} \xi = T_\infty^{-1} \zeta_\infty$.

<u>Proof</u>: The lemma follows, using Doob's martingale theorem[19], from the fact that for all $A \in \Sigma_\infty$,

$$\mu_\infty(A \| \bigwedge_n S_\infty^{-n} \xi) = \lim_{n \to \infty} \mu_\infty(A \| S_\infty^{-n} \xi) = \mu_\infty(A \| T_\infty^{-1} \zeta_\infty), \quad \text{a.e..}$$

It suffices to establish that the above equality is valid for any A a member of some $\Sigma_\infty(K)$, with K any bounded region of X. But for A of this form we can find an N such that for $n \geq N$,

$$\mu_\infty(A \| S_\infty^{-n} \xi) = \mu_\infty(A \| T_\infty^{-1} \zeta_\infty), \text{ a.e.};$$

we merely pick N so large that $\alpha \cap K \cap [N, \infty) = \emptyset$, for all $\alpha \in T^{-1} \zeta$.

The preceding argument can be applied, essentially without modification, to generalizations of systems of periodic-K-type

18. For any measurable partition α, we denote by $\hat{\alpha}$ the Σ-algebra associated with α.

19. See Chapter IV, section 5.

such as $\tau_{\nu,\rho}$ and the periodic Lorentz gas. We thus have

Corollary 5.3: $\tau_{\nu,\rho}$ (if ν is absolutely continuous at the origin) and the periodic Lorentz gas (as well as $\bar{\tau}_\infty$) form space-time K-systems.

b) space-time entropy

We will now investigate the space-time entropy of infinite systems of noninteracting particles. The proof of the vanishing of the space-time entropy of the infinite ideal gas suggests - and can be generalized to show - that the space-time entropy of any infinite system whose finite volume one-particle components have vanishing time entropy vanishes as well. We will prove a stronger result. A natural quantity to consider for infinite systems is the time entropy per unit volume. It would be nice if the space-time entropy of these systems could be so interpreted; we will show that for all translation invariant (infinite) systems of the type which we have so far considered, this is, indeed, the case.

We first define the notion of the time entropy per unit volume of a Poisson system of periodic type, i.e., the Poisson system built over a system with the periodic structure described in Chapter IV, section 7. Recall that we have denoted by τ_R the restriction of the periodic system (X, Σ, μ, T) to the compound rectangle R. We will denote by $(\tau_R)_\infty$ the Poisson system built over $\mu(R)\ \tau_R$[20]. We define the T_∞-entropy per unit volume by

$$h_\infty(T_\infty) = \lim_{R \to \infty} (1/\|R\|)\ h((\tau_R)_\infty)^{21} ,$$

where $\|R\|$ is the Lebesgue measure of R.

We will say that $\tau = (X, \Sigma, \mu, T)$ is of <u>periodic</u>-<u>bounded</u>-<u>type</u> if it has periodic structure and is such that $T(R_o)$ is bounded for R_o a basic rectangle. We will first prove

<u>Theorem 5.4</u>: If τ is of periodic-bounded-type,

$$h_\infty(T_\infty) = h(G_\infty) = \mu(R_o)\ h(\tau_R) = (1/\|R\|)\ h((\tau_R)_\infty).$$

<u>Proof</u>: For P a partition of R_o of finite entropy, let us denote by Q_o P the "product" of Q_o and P[22]. Let

$$h(P,T) = \lim_{n\to\infty} (1/n) H(\bigvee_{j=0}^{n-1} T^j Q_o P \mid R_o)$$

$$= H(T Q_o P|R_o \| \bigvee_{j=0}^{\infty} T^{-j} Q_o P \mid R_o) \quad ,$$

---

20. For $\lambda > 0$ we denote by $\lambda(\tilde{X}, \tilde{\Sigma}, \tilde{\mu}, \tilde{T})$ the system $(\tilde{X}, \tilde{\Sigma}, \lambda\tilde{\mu}, \tilde{T})$; recall that in obtaining τ_R we had normalized the restriction of μ to R.

21. For $\tau' = (X', \Sigma', \mu', T')$ we often write $h(\tau')$ instead of $h(T')$. In this section R will always denote a compound rectangle.

22. See Chapter IV, section 7.

where we are using the following notation: For any partition $\tilde{P}$ of X we denote by $\tilde{P} \mid R$ the partition induced by $\tilde{P}$ on R (with normalized measure on R induced by μ).

Let $h(T) = \sup h(P,T)$, with the supremum taken over finite (measurable) partitions of R_o. We will use the fact that in the same way as for a single automorphism [33], if $\{P_n\}$ is an increasing sequence of finite entropy partitions and $\vee P_n = \epsilon$ (mod 0) (or even if $\vee P_{n_G} = \epsilon$ (mod 0)), then $h(G) = \lim_{n\to\infty} h(P_n,G)$, for any (normalized) dynamical system (X, Σ, μ, G). A completely analagous result holds for the h(T) which we introduced at the beginning of this paragraph.

Theorem 5.4 easily follows from 3 lemmas:

Lemma 5.5: $h(T) = \lim_{R\to\infty} h(\tau_R) = h(\tau_R)$.

Lemma 5.6: $h(G_\infty) = \mu(R_o)\, h(T) = \rho h(T)$.

Lemma 5.7: $h((\tau_R)_\infty) = \mu(R)\; h(\tau_R) = \rho\|R\|\; h(\tau_R)$.

Proof of Lemma 5.5: The first equality follows from the observation that for R such that $R_o \subset R$ and $T(R_o) \subset R$ we have $h(\tau_R) = h(T)$; indeed, $h(Q_o P \mid R, T_R) = h(P,T)$ for any partition of P of R_o. (T_R is the automorphism of the system τ_R.)

The second equality holds because for any finite partition P of R_o we have

$$h((Q_o P \vee T Q_o P) \mid R, T_R) = h((Q_o P \vee T Q_o P \mid R_o, T).\text{[23]}$$

23. $(Q_o P \vee T Q_o P) \mid R$ is finite since $T(R_o)$ is bounded.

Proof of Lemma 5.6: We compute $h(G_\infty)$ in a manner analagous to the method used for the computation of $h(G_\infty^I)$. For P a finite partition of R_o, let P_∞ be the partition of X_∞ constructed from $Q_o P$ in a manner analagous to the way in which P^I was constructed from the partition of X^I into unit cells: Let $P = \{P_i\}$, $(i=1,\ldots,k)$ and label the atoms of $Q_o P$ using the ordered pairs (n,i) $(n \in \mathbb{Z}, i = 1,\ldots,k)$. (n,i) is the label of the "copy" of P_i in $R_n = S^n R_o$. We order the labels lexicographically (i.e., $(n,i) \leq (m,j)$ if $n < m$ or if $n = m$ and $i \leq j$). Using this labeling we form the future Q_o P-names of elements $x \in X$ and order them lexicographically using the lexicographical ordering of labels. We index the particles in an atom of Q_o P according to this ordering. P_∞ is then the partition of X_∞ according to the number of particles in each of the atoms of P, the element of Q_o P containing, at $t = -1$, each of these indexed particles, and the index at $t = -1$ in their respective atoms of Q_o P of each of these $(t = 0)$ indexed particles. One easily verifies that, like P^I, P_∞ has finite entropy.

Using the remark preceding Lemma 5.5 we obtain

$$h(G_\infty) = \sup_P h(P_\infty, G_\infty) ,$$

P a finite partition of R_o, since we can easily construct an increasing sequence of partitions P_n of R_o for which $\bigvee_n (P_n)_\infty = \epsilon$ (mod 0).

But $h(P_\infty, G_\infty) = H(P_\infty \| P_{\infty \bar{G}_\infty}) = \sum_{n=0}^{\infty} (e^{-\rho} \rho^n/n!)\, nh(P,T) = \rho h(P,T)$, since the index information at $t = -1$ is determined by the information in $P_{\infty \bar{S}_\infty}$, particles coming from R_{-n}, $n > 0$, at $t = 0$ automatically having lower $t = -1$ indices than particles from R_o, which in turn have lower $t = -1$ indices than particles from R_m, $m > 0$. Taking the sup over P now leads to the desired result.

Proof of Lemma 5.7: We have

$$h((\tau_R)_\infty) = \sum_{n=0}^{\infty} (e^{-\mu(R)} (\mu(R))^n/n!)\, h(T_{R\ \text{symm}}^{\otimes n})$$

$$= \sum_{n=0}^{\infty} (e^{-\mu(R)} (\mu(R))^n/n!)\, h(T_R^{\otimes n})$$

$$= \sum_{n=0}^{\infty} (e^{-\mu(R)} (\mu(R))^n/n!)\, nh(T_R) = \mu(R)\, h(T_R).$$

The first equality follows from the fact that the entropy of a direct sum is the average of the entropies[24]. The second equality

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24. $(1/n)H(\bigvee_{j=0}^{n-1} T^j(P \vee \gamma)) = (1/n)H((\bigvee_{j=0}^{n-1} T^j P) \vee \gamma)$

$$= 1/n\, H(\gamma) + (1/n)H(\bigvee_{j=0}^{n-1} T^j P \,\|\, \gamma)$$

$$\underset{n \to \infty}{\to} \sum_k \mu(\gamma_k)\, h(P|\gamma_k, T_{\gamma_k}),$$

where T is an automorphism of a probability space with (invariant) components γ_k, $\gamma = \{\gamma_k\}$ is the partition into components, $P|\gamma_k$ is the restriction of P to γ_k, and T_{γ_k} is is the restriction of T to γ_k.

equality follows in a manner similar to the proof of Lemma 5.6.

$$(h(P^{\otimes n}, T^{\otimes n}) = h(P_{\text{symm-indexed}}, T^{\otimes n}_{\text{symm}}),$$

where T is an automorphism of a probability space (X, μ) and $P^{\otimes n}_{\text{symm-indexed}}$ is the partition of $X^{\otimes n}_{\text{symm}}$ according to membership in the atoms of the partition P of the (future P-name) indexed particles.) The third equality follows from the well known fact that the entropy of a direct product is the sum of the entropies (since $H(P \otimes P_2) = H(P_1) + H(P_2)$).

We have thus shown that our expectations are satisfied for Poisson systems of periodic-bounded-type: the space-time entropy equals the time entropy per unit volume (in any compound rectangle R) which in turn equals the time entropy of a single particle moving in any compound rectangle times the average number of particles per unit volume. Thus we can easily compute the space-time entropy of $\bar{\tau}_\infty$. We have $h(G_{\bar{\tau}_\infty}) = \rho h(\bar{\tau}_n) = \rho \log 2$, as we expect.

We would like the above results to be valid for the generalizations of systems of periodic-bounded-type such as $\tau_{\nu,\rho}$ and the periodic Lorentz gas. For these systems $T(R_o)$ is not bounded because R_o contains particles with arbitrarily high velocity. However, the speed of a particle is a constant of the motion. Accordingly, we define a system $\tau = (X, \Sigma, \mu, T)$ to be of <u>periodic-σ-bounded-type</u> if there exists an increasing sequence $\{A_n\}$ of

G-invariant Σ-subsets of X such that $T(A_n \cap R_o)$ is bounded for any n and $\bigcup_n A_n = X$. It is clear that $\tau_{\nu,\rho}$ and the periodic Lorentz gas are built over systems of periodic-σ-bounded-type. We will prove

Theorem 5.8: If τ is of periodic-σ-bounded-type, then we have for the (S_∞, T_∞) entropy of τ_∞

$$h(G_\infty) = h_\infty(T_\infty) = (1/\|R\|)\ h((\tau_R)_\infty) = \mu(R_o)\ h(\tau_R).$$

Proof: Let Σ_{A_n} be the (invariant) sub-σ-algebra of Σ associated with the measurable partition of X into the set X-A_n and the points of A_n. Let τ^{A_n} be the factor of τ with respect to Σ_{A_n}. Then τ^{A_n} is "essentially" of periodic-bounded-type, since it can be expressed as the direct sum $\tau^{A_n} = \bar{\tau}^{A_n} \oplus 1^{A_n}$, where $\bar{\tau}^{A_n}$ is the restriction of τ to A_n and 1^{A_n} is the system consisting of a single (invariant) "point" (of infinite measure)-the set X-A_n. Furthermore $\tau_\infty^{A_n} = \bar{\tau}_\infty^{A_n} \otimes 1_\infty^{A_n}$. Now, since $\bar{\tau}^{A_n}$ is of periodic-bounded-type we can apply to it Theorem 5.7 to obtain

$$h(\bar{G}_\infty^{A_n}) = h_\infty(\bar{T}_\infty^{A_n}) = (1/\|R\|)\ h((\bar{\tau}_R^{A_n})_\infty) =$$

$$= \mu(A_n \cap R_o)\ h(\bar{\tau}_R^{A_n}).$$

We further have

$$h(G_\infty^{A_n}) = h(\bar{G}_\infty^{A_n});\ h((\tau_R^{A_n})_\infty) = h((\bar{\tau}_R^{A_n})_\infty);$$

and

$$h(\tau_R^{A_n}) = (\mu(A_n \cap R_o)/\mu(R_o))\, h(\bar{\tau}_R^{A_n}).$$

Recalling the remarks immediately prior to Lemma 5.5, the desired result follows upon taking the limit $n \to \infty$ (using, for example, the definition of a measurable partition and a diagonalization argument).

We now calculate the space-time entropy of the $\tau_{\nu,\rho}$, the periodic Lorentz gas, and systems of a similar nature[25]. Let τ be the infinite volume one particle system of $\tau_{\nu,\rho}$ (so that $\tau_\infty = \tau_{\nu,\rho}$) or of the Lorentz gas. Let τ_v be the component of τ at speed v. τ is the direct integral of its components τ_v. We have shown that the space-time entropy of τ_∞ is $\rho\, h(\tau_{R_o})$. Using the representation of the entropy of a direct integral as the integral of the entropies of the components, we have

$$h(\tau_{R_o}) = \int_o^\infty \tilde{\nu}(dv)\, h(\tau_{v,R_o}) \quad ,$$

where $\tilde{\nu}$ is the distribution of the speed of a particle induced by μ. (For $\tau_{\nu,\rho}$, $\tilde{\nu}$ is twice the restriction of ν to $\mathbb{R}^+$.) We prove this representation for the systems under consideration as follows:

We will use the notation of the proof of Theorem 5.8. Let

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25. The Lorentz gas, of course, has two dimensional translational symmetry; the generalization of our method to a larger number of dimensions is straightforward; we mention only that one must extend the lexicographical ordering, used in several places, to a lattice of a larger number of dimensions.

$A_v = \{x \in X \mid v(x) \leq v\}$, where $v(x)$ is the speed characteristic of the point x. We have

$$h(\tau_{R_o}) = \lim_{v\to\infty} h(\tau_{R_o}^{A_v}) .$$

Now $h(\tau_{R_o}^{A_v}) = \sup_{\text{finite } P\in\Sigma_{A_v} \cap R_o} h(P, T_{R_o}^{A_v}) = \sup_{\text{finite } P\in\Sigma_{A_v} \cap R_o} h(P\vee\gamma_v, T_{R_o}^{A_v})$,

where γ_v is the partition of A_v according to the number of collisions with the obstacles between t = 0 and t = 1. γ_v is clearly a finite partition. We claim that v(x) is measurable (mod 0) with respect to $\gamma^-_{T_{R_o}^{A_v}}$. This is obvious for $\tau_{\nu,\rho}$, and follows from the ergodicity of τ_{v,R_o} for the Lorentz gas: Ergodicity implies that on each surface X_v of constant speed v, $v \in \mathbb{R}^+$, the asymptotic number of collisions per unit time is constant, a.e., It is clear that this time average is proportional to the speed characteristic of the surface. The claim is established with the observation that this constant cannot vanish, since, again by ergodicity, it equals the expected value of the number of collisions between t = 0 and t = 1, which does not vanish. Thus ,

$$h(P \vee \gamma_v, T_{R_o}^{A_v}) = H(P \vee \gamma_v \| \bigvee_{j=1}^{\infty} (T_{R_o}^{A_v})^{-j} (P \vee \gamma_v))$$
$$= \int_o^v \tilde{\nu}(dv')\, h((P \vee \gamma_v)| X_{v'}, T_{v',R_o})^{26} .$$

26. For conditional entropy with respect to an arbitrary measurable partition,see reference [33].

Taking the supremum over $P \in \Sigma_{A_v} \cap R_o$ we obtain

$$h(\tau_{R_o}^{A_v}) = \int_o^v \tilde{\nu}(dv')\, h(\tau_{v',R_o})$$

by Lebesgue's bounded convergence theorem. Now letting $v \to \infty$ we obtain the desired result.

Finally we use the formula of Abramov to obtain a simple expression for the space-time entropy of τ_∞. If we denote by $(X_{R_o}, \mu_{R_o}, \tilde{T}_{t,R_o})$ the flow on the surface of unit speed, then the flow on the surface of speed v is isomorphic to $(X_{R_o}, \mu_{R_o}, \tilde{T}_{vt,R_o})$, so that

$$h(\tau_{v,R_o}) = h(\tilde{T}_{v,R_o}) = v\, h(\tilde{T}_{1,R_o}) = v\, h(\tau_{1,R_o}).$$

Thus the space-time entropy of τ_∞ equals $\rho\, h(\tau_{1,R_o}) \int_o^\infty \tilde{\nu}(dv)v$ $= \rho \langle v \rangle_\nu\, h(\tau_{1,R_o})$, consistent with our interpretation of it as representing the loss of information (due to "collisions") per unit volume per unit time.

We note in particular that, since one easily verifies that for $\tau_{\nu,\rho}$ we have $h(\tau_{1,R_o}) = \log 2$, the space-time entropy of $\tau_{\nu,\rho}$ is $\rho \langle v \rangle_\nu \log 2$.

6. Concluding remarks

The results of the previous section, in addition to indicating that the space-time entropy of the systems we have considered has a natural interpretation, establish that the time entropy per unit volume is an invariant of our expanded framework, at least for the class of systems of the kind considered. We also have the desirable result that two such systems in which "dissipation" per unit volume occurs at different rates cannot be isomorphic.

We would like all these results to extend to general translation invariant equilibrium states of Hamiltonian systems. The notion of time entropy per unit volume could be defined by using, say, sequences of cubes with either reflecting or periodic boundary conditions [21]. We might then expect the space-time entropy of the equilibrium states of these systems to equal the time entropy per unit volume, so that the local rate of dissipation would be invariant in a larger class of systems, including all systems of physical significance.

We have found that a system may be Bernoulli under both space translations and time evolution separately without being a space-time K-system, much less space-time Bernoulli (e.g., the ideal gas). We have not found any models of realistic systems which are space-time Bernoulli, though we can give a characterization of such systems which makes clearer what is involved. It is clear that if a system is Bernoulli under the space-time group, it is Bernoulli under space translations and possesses an S-invariant independent generator for T. The converse is also true: Indeed, since factors of Bernoulli shifts are Bernoulli [28], any S-invariant independent generator for T can be expressed in the form ξ_S, with ξ, $S\xi$, $S^2\xi$,...forming an independent sequence of partitions, so that ξ is, in fact, an independent generator for (S,T).

We give a simple example of a class of (S,T)-Bernoulli systems: Let $(B,\widetilde{T})$ be a Bernoulli scheme. Let $X = B^{\mathbb{Z}}$,[27] $T = \widetilde{T}^{\mathbb{Z}}$ and let S act in the obvious way as a translation on $B^{\mathbb{Z}}$. It is clear that $\underset{i}{\otimes} Q_i$ ($Q_i = B$ for $i \neq 0$, and $Q_o = P$, an independent generator for $\widetilde{T}$) is an independent generator for (S,T).

Finally, we observe that though, e.g., $\tau_{\hat{\beta},\rho}$ clearly exhibits better thermodynamic behavior than, say, the ideal gas, a non-equilibrium velocity distribution for $\tau_{\hat{\beta},\rho}$ does not approach, as $t \to \infty$, the appropriate Maxwellian distribution. This is not at all to be unexpected because velocities are, perhaps, not very "natural" within the framework of discrete symmetry (spatial for $\tau_{\hat{\beta},\rho}$). The question of interest would be the behavior of the velocity distribution in systems with continuous symmetry (G = full space-time group) and strong (say, K or Bernoulli) G-ergodic properties. Systems with continuous symmetry can be obtained in a natural way from systems of interacting particles,

27. We are, of course, here forming a measure theoretic product.

which we would hope to exhibit the appropriate behavior of velocity distribution functions.

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