C-2

ERGODIC THEORY IN STATISTICAL MECHANICS

by

Michael Aizenman

Submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy
in the Belfer Graduate School of Science
Yeshiva University
New York

January 1975

The committee for this doctoral dissertation consisted of:

Joel L. Lebowitz, Ph.D., Chairman

Oscar E. Lanford III, Ph.D.

Oliver Penrose, Ph.D.

ACKNOWLEDGEMENTS

I am mostly indebted to Joel Lebowitz for suggesting the problems and for his highly valuable guidance. I am also very grateful to Sheldon Goldstein who collaborated in parts of this study and to Oscar Lanford for criticism and useful discussions. It is my pleasure to thank Oliver Penrose for his criticism and to Giovani Gallavotti, Donald Orenstein, Jörgen Moser, Meir Smorodinski and Benjamin Weiss for valuable discussions. I wish to express my gratitude to Ann Cooper for devotedly typing the manuscript and to my wife Marta for her support and for enriching my world. Finally, I would like to acknowledge the support of the Air Force Office of Scientific Research.

This work is dedicated to the memory of my dearest friends and colleagues in study

Gideon Cain and Zvi Shindel

who fell in battle in Israel, October 1973.

TABLE OF CONTENTS

Page

# I. Introduction

## 1. Introduction

For macroscopic systems, the collection of those quantities which are measurable in practice is very much smaller then the set of microscopic variables. It is an experimental fact that, when in equilibrium, macroscopic systems admit a thermodynamical description. Their dynamical behavior not far from equilibrium is well approximated by the kinetic theory, hydrodynamics, etc. The objective of statistical mechanics is to explain these features of macroscopic systems on the basis of the underlying microscopic mechanics. In particular: to find the basis for the universality of the "simple" macroscopic description of systems which microscopically are described by widely varying interactions, to relate the few macroscopic variables and parameters (some of which are "universal") to microscopic quantities, and to explain how the approach to equilibrium (in the way described by the kinetic theory, for example) concurs with the microscopic (reversible) dynamics.

The success has been limited, so far, mainly to the treatment of equilibrium phenomena. In the realm of the dynamical theory one still looks for a general formalism which would give physically relevant information on the basis of the microscopic dynamics and relate the two levels of description.

In this chapter we will briefly lay down the framework in which some properties of (finite) mechanical systems are described by ergodic theory. Doing this we are merely touching the tip of an iceberg and the interested reader is referred to [34,14,26] (where additional references are given) for a discussion of the ergodic properties of physical systems and their physical relevance, and to [20,51] for a more detailed discussion of the ergodic theoretical concepts.

## 2. Ensembles

A useful framework for a microscopic description of a (finite) system is the Hamiltonian dynamics on the system's phase space. This is obtained by a transformation from the "Newtonian" variables, positions and velocities, to positions and momenta which are realized as functions on the systems "phase space", $\Gamma$, [13].

A point in $\Gamma$ is given by the values of $(q_1,\ldots,q_{dN}, p_1,\ldots,p_{dN})$ where d is the dimension of the space in which a single particle is located, N is the number of particles, $q_i$ is a (cartesian) configurational coordinate of a particle and $p_i$ its canonically conjugate momentum. Thus $\Gamma$ may be identified with a subset of $R^{2dN}$, inheriting a differential structure and a measure $dq_1\ldots dq_{dN}\, dp_1\ldots dp_{dN}$ which in this context is called the Liouville measure.

The evolution in time of a classical system is described by a "canonical" point mapping in $\Gamma$, i.e. one induced by equations of the form:

$$\frac{d}{dt} q_i = \frac{\partial H}{\partial p_i} \quad , \quad \frac{d}{dt} p_i = \frac{-\partial H}{\partial q_i}$$

where $H = H(\underline{p}, \underline{q})$ is the Hamiltonian function.

The fundamental nature, for statistical mechanics, of this description follows from the fact that the Liouville measure is invariant under the canonical mappings (Liouville's theorem), in particular under the time evolution induced by any interaction.

This fact is sometimes mentioned in a heuristic justification of the method of "ensembles" which was introduced by Gibbs in equilibrium statistical mechanics. An ensemble is the collection of independent similar systems which are subject to certain macroscopic constraints, equipped with a probability measure which gives the distribution of their microscopic quantities. It is sometimes argued (very heuristically) [42] that, due to the property expressed in Liouville's theorem, the size of macroscopic systems (which contain the order of $10^{23}$ particles) and the nature of interactions (which are given by piecewise smooth functions on the phase space) it is the Liouville measure on the system's phase space which, when properly normalized, gives the experimental probability distribution of an equilibrium ensemble* . Further, it is argued that, for ensembles which are defined by constraints which experimentally single out a "pure phase", the distribution of the intensive variables with respect to the Liouville measure is sharply picked around a single value [28]. Thus, in order to compute the values of macroscopic observables for a given system, one may use the ensemble average on a proper ensemble. Such an approach is natural in a theory which deals with statistical predictions; however the boldness of Gibbs was to apply it to the description of any given system.

The method of ensembles, together with the proper limit for large systems, has been successfully used in equilibrium statistical mechanics. The limit which is used there is the "thermodynamic limit" in which the "additive" quantities (volume, number of particles and energy) increase to infinity while their densities are kept (approximately) constant. Sharp results are, of course, obtained only in the limit, for which formal techniques are now being developed.

Since the problems of equilibrium and nonequilibrium statistical

---

*The "microcanonical ensemble", which is represented by surface in $\Gamma$, of constant energy, should be thought of as a limit of ensembles for which the energy is constrained to a small interval. For such an ensemble one should take the surface measure which is induced, in the above limit, by the Liouville measure.

mechanics are not entirely independent, one may hope that a formal theory of infinite systems, which uses the ensemble method, may provide a useful tool for the study of nonequilibrium properties as well. Among the recent investigations along this line one finds the study of the $C^*$ algebras of observables [ 9] and the study of ergodic properties of infinite systems [17]. For certain results, other directions, which involve different macroscopic limits, were followed (see Lanford [29]).

3. Ergodicity

The description of a mechanical system in terms of a measure space with a measure preserving transformation, and in particular the attempt to obtain another, purely dynamical, explanation for the success of the method of ensembles, stimulated the development of the ergodic theory. This has benefitted both mathematics and physics. The concepts and results obtained are of great interest to the study of dynamical properties of physical systems.

In the context of abstract ergodic theory, a dynamical systems is a triple $(X, \mu, T)$ of a space $X$ (equipped with a $\sigma$-algebra which we will usually omit in our notation), a probability measure $\mu$ and a measure preserving point transformation $T$ (which in our applications is invertible) or a measure preserving flow, which is a measurable group of transformations $\{T_t\}_{t \in R}$.

A transformation $T$ defines also mappings of functions $(f)$ and measures $(\nu)$ on $X$ by:

$$(Tf)\ (x) = f(Tx)$$

$$(T\nu)\ (dx) = \nu(T^{-1}(dx))$$

These satisfy:

$$\int (Tf)\ (x)\nu(dx) = \int f(x)\ (T\nu)\ (dx)\ .$$

Many of the properties of a flow may be obtained by studying the discrete transformation $T = T_1$.

Probably the first result within the realm of the ergodic theory is Poincaré's recurrence theorem:

In a dynamical system $(X, \mu, T)$ almost any point (meaning that the set of the exceptional points has zero measure) of any measurable set $A$, $\mu(A) > 0$, will, under the action of the iterates of $T$, return to $A$ infinitely often.

This, initially surprising, result seems to indicate that it is hopeless to explain the approach to equilibrium within the framework of the Hamiltonian dynamics of finite closed systems. Of course, the flaws in this argument are that the time of such a recurrence is extremely large and that its dependence on the precise microscopic description is very unstable. Thus, on this time scale the accumulated perturbations from the environment make the microscopic description, in terms of a Hamiltonian flow, unrealistic.

Another, more practical, way out of this paradox is to assume that macroscopic states are described by probability measures ("dynamical ensembles") and to study their time evolution. We will say no more about the justification of this assumption and take it as the formal definition of a state.

Another significant result is <u>von Neumann's ergodic theorem:</u>

If $(X, \mu, T)$ is a dynamical system and $f, g \in L^2(\mu)$ then the following limit exists:

$$\lim_{t \to \infty} 1/t \sum_{i=0}^{t-1} \int f \cdot T^i g \ d\mu = \int f \cdot Pg \ d\mu$$

(for a flow:

$$\lim_{t \to \infty} 1/t \int_0^t du \int f \cdot T_u g \ d\mu = \int f \cdot Pg \ d\mu)$$

where $P$ is the orthogonal projection (in $L^2(\mu)$) on the subspace of functions which are invariant under $T$.

It is not difficult to see that the subspace of invariant functions includes only the constant (a.e.) functions if and only if $X$ has no nontrivial (i.e. not of $\mu$ measure 0 or 1) measurable invariant subsets. Such systems are called <u>ergodic</u> and for them the above limit takes the form

$$\lim_{t \to \infty} 1/t \sum_{i=0}^{t-1} \int f \cdot T^i g \; d\mu = \int f \, d\mu \cdot \int g \, d\mu \; .^{*}$$

There is a correspondence between such properties of a stationary state and the dynamical properties of a non equilibrium state which is not singular with respect to it. In particular, if the initial state is given by an absolutely continuous measure, $\mu'$, with respect to an ergodic state $\mu$; i.e. one of the form

$$\mu'(d\gamma) = f(\gamma)\, \mu(d\gamma), \; f \in L^1(\mu) \quad \int f d\mu = 1,$$

---

* A stronger ergodic theorem, due to Birkhoff [6] states that in a dynamical system, $(X, \mu, T_t)$, for every $f \in L^1(\mu)$ the time average exists for almost every point (a.e.) $x \in X$:

$$\lim_{t \to \infty} 1/t \int_0^t f(T_t x)\, dt = E(f(x) \,|\, \mathcal{J}) \qquad \text{(a.e)}$$

Here $E(f|\mathcal{J})$ is the conditional expectation (function) on the $\sigma$-algebra of invariant sets.

This result is often mentioned as another, dynamical, justification of the use of equilibrium ensembles, since if one accepts the idea that physical measurements have long duration with respect to the "microscopic time scale" and in effect are results of averaging in time then, for ergodic systems, the result of an observation taken "at" almost any point of the ensemble is the ensemble average. However, while for some physical systems the ergodicity (of the microcanonical ensemble) may be shown or reasonably postulated, the other assumption is too strong at its face value. Worse: the acceptance of such an assumption on the nature of physical measurement would (for finite systems) rule out the possibility of observing any dynamics, since it implies that all the quantities which are measurable in practice are given by invariant functions on the phase space.
Nevertheless, ergodicity does imply, even without further dynamical justification, the uniqueness of an equilibrium ensemble (assuming that those given by singular measures, with respect to the microcanonical ensemble, are unrealistic).

then, the time average of any measurement will exist and be equal to the value which corresponds to $\mu$.

For Hamiltonian systems the energy (i.e. the Hamiltonian function) is always a constant of the motion. If there are no other measurable invariants, except for functions of the energy, then almost all the microcanonical ensembles are ergodic with respect to the time evolution. For such systems the equilibrium measures on the microcanonical ensembles are unique (assuming that those given by measures which are singular, with respect to the above limit of the Liouville measure, are unrealistic). This follows from the fact that an ergodic system does not admit another invariant probability measure which is absolutely continuous with respect to the one given.

4. Mixing

An ergodic system may have a stronger dynamical property: mixing. This is defined by the existence of the limit:

$$\lim_{t \to \infty} \int f \cdot T_t g \, d\mu = \int f d\mu \cdot \int g d\mu \quad *$$

for any $f, g \in L^2(\mu)$.

* In general, if such a limit exists then, by the ergodic theorem, it is given by $\int f(Pg)d\mu$. The existence of such a limit, $\forall f,g \in L^2(\mu)$, is, by itself, an interesting property (Prosser mixing).

Mixing implies ergodicity. Its meaning is that any two given measurements, if performed successivly with a large time delay, become independent.

If the equilibrium state ($\mu$) of a system is mixing, then if one"prepares" the system in a non stationary state ($\mu'$), which is not singular with respect to $\mu$ , the dynamical system will approach equilibrium with respect to any given finite set of measurements (i.e. "weakly"):

$$\lim_{t \to \infty} \int f \, d(T_{-t}\mu') = \int f \, d\mu$$

(the state of the system after the time t is given by $T_{-t}\mu'$). This, macroscopic, approach to equilibrium is consistent with reversible dynamics and formally occurs backward in time as well (although the corresponding physical experiment is almost unfeasible).

This coexistence of different features of the macroscopic and microscopic behavior is related to the instability of the microscopic dynamics. Indeed, if a system is mixing then so is the "product" of two replicas of it, which implies that a pair of points chosen at random from any small set (of positive measure) will, after sufficiently large time, be independently distributed over the phase space.

Ergodicity and mixing may be formulated as spectral properties of the unitary transformation in $L^2(\mu)$ which is induced by T. Further properties of a dynamical system are obtained by studying its dynamics through "coarse grained" measurements.

5. Process representations of dynamics

A physical measurement is always performed with an apparatus of some finite resolution. An idealized description, which maintains this basic feature, is a discrete valued measurement to which there corresponds an essentially finite partition of the phase space. Such an element of structure was introduced by Kolmogorov [23] to the study of ergodic properties of abstract dynamical systems.

A partition Q of X is a collection $\{Q_i\}$ of disjoint sets which covers X. The partitions of a given set are partially ordered by the relation $Q < P$; P "finer" than Q or, equivalently, Q coarser than P. For a collection of partitions $\{Q_{(n)}\}$ one defines the lattice operations:

$\bigvee_n Q_{(n)}$ - the coarsest partition which is finer than $Q_{(n)}$, $\forall n$, (its elements are intersections of sets in $Q_{(n)}$).

$\bigwedge_n Q_{(n)}$ - the finest partition which is coarser than all the $Q_{(n)}$.

The measure spaces which we will consider will be Lebesgue spaces [19]. For these, the correspondence between sub-$\sigma$-algebras and partitions extends to include even those whose typical elements (fibers) have zero measure. The corresponding measurable partitions are discussed in [40], let us only remark here that in such systems the sub-$\sigma$-algebra which corresponds to $\bigvee_{n=-\infty}^{\infty} Q_{(n)}$ is the full $\sigma$-algebra if the partitions $\{Q_{(n)}\}$ separate the points of a full measure subset of X.

The <u>entropy</u> of a countable partition (see [4] and [40]) is defined as

$$H(Q) = -\sum_i \mu(Q_i) \ln \mu(Q_i)$$

and its <u>conditional entropy</u>, given the partition P, is defined by

$$H(Q\|P) = H(Q \vee P) - H(P).$$

This turns out to be equal to the average entropy of the partitions induced by Q on the elements of P ( when properly normalized):

$$H(Q\|P) = \sum_j \mu(P_j)\, H(Q|P_j) = -\sum_j \mu(P_j) \sum_i \mu(Q_i|P_j) \ln \mu(Q_i|P_j) \quad (\geq 0)$$

with $\mu(Q_i|P_j) = \mu(P_i \wedge Q_j)/\mu(Q_j)$.

The following properties of the entropy justify its interpretation as the measure of "information" contained in a partition (or of the average amount of "uncertainty" removable by the corresponding measurement):

1) $H(Q \vee P) \leq H(P) + H(Q)$ (which implies $H(Q\|P) \leq H(Q)$) with equality holding if and only if P and Q are (pairwise) independent.

2) $H(Q\|P) \leq H(R\|P)$ if $Q \leq R$

3) $H(Q\|P) \geq H(Q\|R)$ if $P \leq R$

4) $H(P \vee Q\|R) = H(P\|R) + H(Q\|P \vee R)$

One denotes (unfortunately) by $T_t Q$ the partition whose elements are $\{T_t Q_i\}$; it corresponds to the "measurement of Q" performed at the time -t.

The sequence of results of a measurement repeated at integral times defines a <u>process</u>, (Q,T), conveniently described by means of the partition which corresponds to the measurement. The process distribution is given by a time invariant measure, determined by the dynamical system. In particular, such a process is itself a dynamical system, where the time evolution is given by a shift of the sequence.

We say that Q is a <u>generating partition</u> if to each sequence there corresponds at most a single point in X (possibly, after excluding a set of zero measure), equivalently: if $\bigvee_{n=-\infty}^{\infty} T^n Q$ corresponds to the full $\sigma$-algebra. If Q is a generating partition for $(X, \mu, T)$ then this is isormorphic to the process (Q,T) when it is viewed as a dynamical system.

The <u>entropy of the process (Q,T)</u> is defined as

$$H(Q,T) = \lim_{n \to \infty} 1/n \; H(Q \vee T^{-1} Q \vee \ldots \vee T^{-(n-1)} Q)$$

(the existence of the limit follows from the general properties of entropy), and is equal to:

$$= H(Q \| \bigvee_{n=-1}^{\infty} T^n Q)$$

(for invertible transformations the same results are obtained after replacing T by $T^{-1}$).

Thus $H(Q,T)$ measures both the rate in which information is generated by the process and the process instability. In particular, $H(Q,T) = 0$ iff $Q < \bigwedge_n (\bigvee_{K=-\infty}^{n} T^K Q)$, in which case the process is deterministic, i.e. the knowledge of all the, arbitrarily remote, past results of the particular measurement is sufficient to determine all the future outcomes.

The entropy of T is defined as

$$H(T) = \sup_Q H(Q,T).$$

An interesting result is the Kolmogorov-Sinai theorem:

If Q is a generating partition for $(X, \mu, T)$ then

$$H(Q,T) = H(T).$$

## 6. K-systems

A particularly interesting property is that of a K-system (after Kolmogorov). Its usefulness stems from the fact that it may be defined by different conditions which, as it turns out [51], are equivalent. Among these:

1) A K-system has a completely positive entropy, i.e. $H(Q,T) > 0$ for any, nontrivial, partition of X whose entropy is finite.

2) the tail of any partition of finite entropy, i.e. $\bigwedge_{n=-\infty}^{\infty} (\bigvee_{K=-\infty}^{n} T^K Q)$, is the trivial partition (mod 0).

3) There exists a generating partition, for $(X, \mu, T)$, whose tail is trivial.

In fact, one may show [51], that any dynamical system possesses a T-invariant sub-$\sigma$-algebra (i.e. a "factor") which is the tail $\sigma$-algebra for each generating partition of finite entropy (f.e.), and which includes the tail $\sigma$-algebra of any other (f.e.) partition. For K systems this factor (which may be studied by means of a single generating partition) is trivial.

K systems are also highly mixing:

$$\mu(A_1 \wedge \alpha_{t_1} A_1 \wedge \ldots \wedge \alpha_{t_n} A_n) \xrightarrow[\text{as } \min |t_i - t_j| \to \infty]{} \prod_{i=1}^{n} \mu(A_i).$$

One may observe that certain process properties, like the process entropy and the tail $\sigma$-algebra, are shared by all the generating partitions, and are therefore of great interest as properties of the dynamical system.

## 7. Bernoulli systems

The "strongest" property of a dynamical system is the existence of a generating partition whose iterates under T are jointly independent (a Bernoulli partition). Such a system is called a Bernoulli-systems and is isomorphic to the dynamical system obtained by the shift on a process of independent random variables (as the one which describes the sequence of outcomes of a roulette wheel).

A weaker property of a partition is the "weak-Bernoulli" property whose general meaning is that the full future process becomes, after certain time delay, independent of the full past (see sec.(III. 5)). The existence of a generating partition which is weakly Bernoulli implies that the system is a Bernoulli system.

Although only exceptional partitions of a Bernoulli system are Bernoulli, there are other properties which are shared by all the processes obtained from such a system. One such property is that the process may be approximated by a "finite coding" of a Bernoulli process, in the sense that the two differ only very infrequently (another, is the very-weak Bernoulli property which will not be discussed here, [46,53]). $(X,\mu,T_t)$ is a Bernoulli flow if $(X,\mu,T_1)$ is a Bernoulli system; which, as it turns out, implies that $\forall \tau \in \mathbb{R}$ $(X,\mu,T_\tau)$ is a Bernoulli system.

The study of Bernoulli systems has advanced only recently and owes many of its concepts and results, including those mentioned above, to D. Ornstein. In particular, it was shown by him that a Bernoulli system (discrete or a flow) is completely characterized (i.e. up to an isomorphism) by its entropy.

These results and a wealth of other information about Bernoulli systems may be found in [39], [48] and [51].

II. Ideal Gas in the Thermodynamic Limit

1. Introduction

In this work we study ergodic properties of certain infinite systems of interacting particles. As an illustrative example we discuss first the ideal gas in the thermodynamic limit whose ergodic properties are well known*. This will serve both as an example of a framework in which infinite systems are studied and as a demonstration of the difference in origin and in significance of ergodic properties between finite and infinite systems. Some of the tools used here may be applied to other systems of non interacting particles as well [16].

A more general discussion of equilibrium states of infinite systems and the existence of a time evolution which satisfies requirements which correspond to the system being a limit of finite classical systems may be found in [29] (see also [7,8,43,24,25 and 27]). This general formalism is much simplified when one deals with systems whose particles interact with simpler potentials like those which will be considered in the coming chapters.

The system which we consider here is an idealized model of the dilute gas. Its elementary constituents are assumed to be non interacting identical point particles which move freely on a line. The dimensionality of the space will not play any role in this discussion. The formalism which will be described is suitable for a probabilistic description of local observables for an infinite system in which the density of particles is finite.

*Ergodic properties of the infinite ideal gas were first studied by Sinai (who proved it to be a K-system). Here we follow a simpler representation introduced by O.E.Lanford III.

## 2. Phase space and time evolution

A point in the phase space of a classical system corresponds to an assignment of values to all its canonical coordinates. We will not exploit fully the canonical structure of the phase space and it will be convenient to use positions and velocity (rather than momentum) as the degrees of freedom of a single particle.

Let $\Gamma = \mathbb{R} \otimes \mathbb{R}$ denote the phase space of a particle on the line. The phase space of the infinite ideal gas is $X' = \Gamma^{Z}_{(sym.)}$ A point $x \in X'$ may be thought of as an equivalence class, under permutations, of sequences

$$\{x_i\}_i \quad , \quad x_i \in \Gamma \ \forall i \in Z$$

or as an unordered countable collection of points in the one particle phase space.

Let us denote by

$$\alpha_t \ \Gamma \to \Gamma \ , \ \alpha_t((x,v)) = (x + tv, \ v)$$

the time evolution of a free particle on the line. Since the ideal gas consists of non interacting particles, the time evolution, $S_t$, on $X'$ is induced by $\alpha_t$ in the following way

$$\forall \{x_i\}_i \in X' \quad , \quad S_t(\{x_i\}) = \{\alpha_t(x_i)\}_i$$

which defines a flow $S_t\colon X' \to X'$.

Since we are interested in the infinite ideal gas as an approximation to a large volume gas system it is natural to confine our discussion to the subset $X \subset X'$ of locally finite configurations (i.e.

$$X = \{x \in X' \mid \forall \text{ bounded } \Lambda \subset \Gamma : x \cap \Lambda \text{ is a finite set } \}).$$

Although $X$ is not invariant under $S_t$, it contains an invariant subset which does have the full measure with respect to the states which we will be considering.

Let $F_n$ be the set of Borel measurable symmetrical functions on $\Gamma^n$ with spacially bounded support, and let $F = \bigoplus_{n=0}^{\infty} F_n$. Denote by $\Sigma : F \to R^X$ the linear mapping defined by

$$(\Sigma f_n)(\{x_i\}) = \sum_{(i_1 < i_2 < \ldots < i_n)} f_n(x_{i_1}, \ldots, x_{i_n}) \;, \quad f_n \in F_n, \{x_i\} \in X$$

Let $A \subset \Gamma$ be a Borel set. We will denote by $N_A : X \to N$ the "occupational number" function

$$N_A(x) = \text{card.}\ (A \cap x)$$

(or, extending the definition of $\Sigma$, $N_A = \Sigma\, \chi_A$ where $\chi_A \in F_1$ is the characteristic function of $A$).

The functions $\Sigma f$, $f \in F$ may represent local observables whose expectation values will be given by a "state".

3. <u>States</u>

Equip $X$ with the weak topology generated by functions of the form $\Sigma g$ where $g$ is a continous function on $\Gamma$ of spacially bounded support. With respect to this topology:

1) $x_n \to x$ iff $x_n \to x$ on any bounded domain whose boundary does not contain a point in $x$.

2) The collection of subsets of $X$ of the form:

$\{x \in X \mid N_A(x) \geq k\}$, $A \subset \Gamma$ open set

and $\{x \in X \mid N_B(x) \leq i\}$, $B \subset \Gamma$ closed

forms a basis of open sets.

3) For any $g_n$, continuous function on $\Gamma^n$ of bounded support, $\Sigma\, g_n$ is continuous.

4) $N_A(x)$, for a bounded domain $A \subset \Gamma$, is continuous at $x \in X$ which have no point on the boundary of $A$.

Denote by $B(X)$ the corresponding, quasi local, Borel $\sigma$-algebra. Similarly, for a bounded domain $A \subset \Gamma$, let $B_A(X)$ be the local $\sigma$-algebra generated by functions in $F$ with support in $A$.

A state is a measure on $B(X)$. Notice that the functions $\Sigma f$, $f \in F$, are $B(X)$ measurable, their integral being the expectation value of the corresponding observables in the "ensemble" described by the state.

In particular we will be interested in states which are invariant under the time evolution $S_t$. Due to the lack of interactions, the ideal gas admits invariant states which have the disjoint independence property, meaning that for disjoint $A, B \subset \Gamma$ $B_A(X)$ and $B_B(X)$ are independent.

Examples of such states are:

1) Let $\gamma$ be a locally finite Borel measure on $\Gamma$ invariant under $\alpha_t$. We will denote by $\mu_\gamma$ the state under which:

a) $B_A(X)$ and $B_B(X)$ are independent for any disjoint Borel sets $A, B \subset \Gamma$,

and b) $\mu_\gamma(\{x \in X \mid N_A(x) = k\}) = \frac{\gamma(A)^k}{k!} e^{-\gamma(A)}$

These two conditions are known to be consistent ( b) is the Poisson distribution) and they clearly define an $S_t$ invariant state. We will refer to the state $\mu_\gamma$ as the <u>Poisson construction on $(\Gamma, \gamma)$</u>.

For one dimensional systems $\gamma$ has to be of the form

$$\gamma(dqdv) = \rho \, dq \, \nu(dv) + \rho_o(dq) \, \delta(dv)$$

where $\nu(\cdot)$ is a probability distribution, $\delta(\cdot)$ assigns probability 1 to $\{v=0\}$ and $\rho_o$ is a locally finite measure on $R$. $\mu_\gamma$ will be invariant under space translations only if $\rho_o(dq) = C \, dq$ for some $C \geq 0$, and then the second term in the last equation can be absorbed in the first one. Such states correspond to particles being "independently distributed" on the line, with uniform (expected) density $\rho$ and identical velocity distribution $\nu(\cdot)$.

This category includes the Gibbs equilibrium states $\mu_{\rho,\beta}$ parametrized by the density $\rho$ and the inverse temperature $\beta$, for which

$$\nu(dv) = (\beta m/2\pi)^{\frac{1}{2}} e^{-\beta m v^2/2} dv.$$

2) Let $\{\gamma_1, \gamma_2, \ldots\}$ be a collection of $\alpha_t$ invariant measures on $\Gamma$ s.t.

$$\sum_{k=0}^{\infty} k\, \gamma_k\, ([0,1] \otimes R) < \infty.$$

One can construct an $S_t$ invariant state by letting clusters of k identically placed particles, $k = 1,2,\ldots$, have the Poisson distribution which corresponds to $\gamma_k$ . Such a state will have the disjoint independence property.

In fact, any invariant state on $B(X)$ which has the disjoint independence property may be obtained by the above construction.

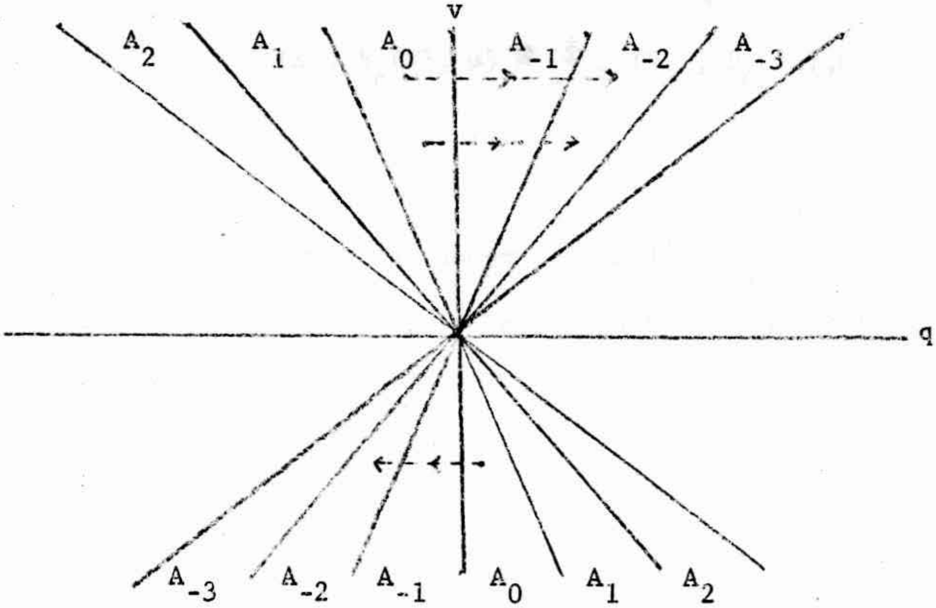
4. Ergodic Properties

Let us denote $C = \{(q,v) \in \Gamma \mid v = 0\}$ , and define the function $\hat{t}((q,v)) = -q/v$ on $\Gamma \setminus C$ . $\hat{t}$ is the time at which a particle at $(q,v)$ will reach the origin under the time evolution $\alpha_t$ .

$$\hat{t}(\alpha_1(q,v)) = \hat{t}((q,v)) - 1$$

Consider the partition $\{\ldots, A_{-1}, A_o, A_1, \ldots\}$ of $\Gamma \setminus C$ such that

$$(q,v) \in A_k \text{ iff } k \leq \hat{t}((q,v)) < k + 1$$




(fig.1 )

$\Gamma = C \cup (\bigcup_{k=-\infty}^{\infty} A_k)$ (a union of disjoint Borel sets), therefore $X$ is isomorphic to the product

$$X \approx X_c \otimes \bigotimes_{k=-\infty}^{\infty} X_{A_k}$$

where $X_A$ is the space of locally finite configurations in $A$.

If $\mu$ has the disjoint independence property $\{B_{A_k}(X)\}$ are jointly independent and $(X, B(X), \mu)$ is isomorphic, as a measure space, to the product

$$(X, B(X), \mu) \approx (X_c, B_c(X), \mu) \otimes \bigotimes_{k=-\infty}^{\infty} (X_{A_k}, B_{A_k}(X), \mu)$$

Further, C is invariant under $\alpha_t$ and $\alpha_1(A_k) = A_{k-1}$ (see fig. 1 ) therefore the mapping $S_t$ is the identity on the factor

$$(X_c, B_c(X), \mu)$$

and $S_1$ acts on

$$(X_{A_k}, B_{A_k}(X), \mu)$$

as an isomorphism onto

$$(X_{A_{k-1}}, B_{A_{k-1}}(X), \mu).$$

We have just shown that the dynamical system $(X, S_t, \mu)$ factorizes to a factor, on which $S_t$ is the identity, and a Bernoulli shift (of infinite entropy). The first factor is trivial whenever $\mu$ assigns zero measure to configurations which have a particle of zero velocity. Therefore:

<u>Theorem</u> The dynamical system $(X, \mu_\gamma, S_t)$ is Bernoulli iff

$$\gamma(\{v=0\}) = 0$$

If $\gamma(\{v=0\}) > 0$ the system is not ergodic, however $S_t$ is a Bernoulli flow with respect to almost every ergodic component of $\mu_\gamma$.

To summarize the above proof notice that $B_{A_o}(X)$, which is an independent generator for the Bernoulli factor, corresponds to a measurable partition of $X$ by the coordinates of the particles which cross the origin (q-0) during the time interval $[0,1)$ . The Bernoulli factor is isomorphic to the process obtained by observing the particles which cross the origin (a hyper-plane for a space of higher dimensionality) at each time. This process is Bernoulli since no particle visits the origin twice and, under $\mu_\gamma$, the coordinates of different particles are independent.

5. Relation to the ergodic properties of finite ideal gas systems.

In order to clarify the meaning of the ergodic properties of the infinite system it may be instructive to compare its ergodic structure with that of the finite volume systems.

The finite volume ideal gas consists of a finite number of particles in a box (interval) $\Lambda$ . Its phase space is

$$X_\Lambda = \bigcup_{n=0}^{\infty} \Gamma^n_{\Lambda(\text{sym.})}$$

where $\Gamma_\Lambda = \Lambda \otimes R$ is the phase space of a particle in $\Lambda$ . Under the time evolution $S_t^{(\Lambda)}$ each particle moves independently reflecting elastically from the boundaries.

Notice that the time evolution in the limiting, infinite volume , system will not be altered if one changes the nature of the collisions with the boundary (while keeping them local) of the finite system.

Denote by $B(X_\Lambda)=\bigcup_{n=0}^{\infty} B(\Gamma^n_{\Lambda(sym.)})$ the natural Borel $\sigma$-algebra on $X_\Lambda$. A given state, $\mu$, of the infinite system may be obtained by a weak limit of different sequences of states $\mu_{\Lambda_i}$ on $B(X_{\Lambda_i})$, $\Lambda_1 \subset \Lambda_2 \subset \ldots$. Notice, however, that under the mapping

$$\Pi_\Lambda : X \rightarrow X_\Lambda$$

$$\Pi_\Lambda(x) = x \cap \Lambda$$

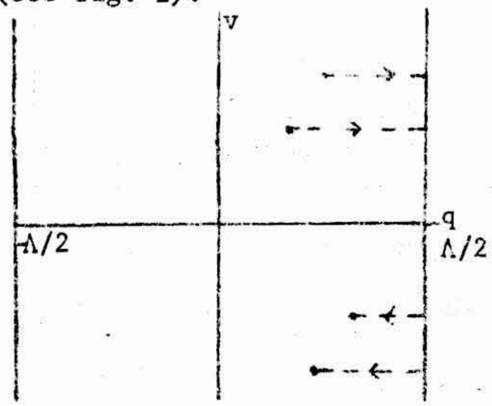
$B(X_\Lambda)$ is homomorphic to $B_\Lambda(x)$. Further, if the state $\mu$ has the disjoint independence property, the projected measure $\mu_\Lambda = \Pi_\Lambda^{-1}\mu$ is invariant under $S_t^{(\Lambda)}$ (*). It seems natural therefore to consider the systems $(X_\Lambda, \mu_\Lambda, S_t^{(\Lambda)})$ as the finite version of $(X,\mu,S_t)$.

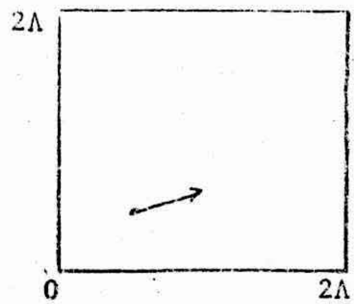
The above finite systems are not ergodic since the number of particles and their individual energies are constants of the motion. For a given number, N, of particles in the box one may pick N functionally independent <u>symmetrical</u> functions of $(v_1^2 \ldots, v_N^2)$ as measurable constants of the motion. The motion on the invariant surfaces is

---

(*) we are considering here only states invariant under reflection of velocities. In one dimension this restriction can be removed by considering systems with periodic boundaries. In higher dimension other conditions would appear.

isomorphic (with the projected measure) to a flow on $T^N$ (N dimensional torus) with velocity vector proportional to $(|v_1|, \ldots, |v_N|)$ (see fig. 2).



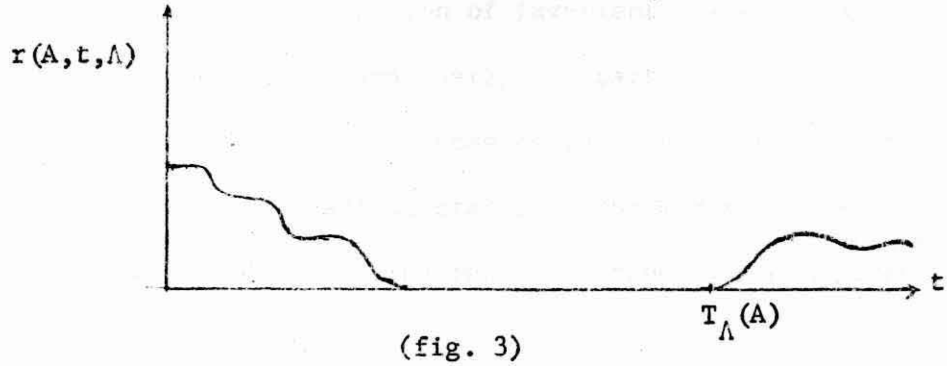

(fig. 2)

In order to understand how such a non ergodic system leads to a Bernoulli flow in the thermodynamic limit let us recall that we are dealing with a quasi-local state.

The strictly local observables of the form $N_A$, where $A \subset \Gamma$ is a compact region, generate a dense set in $L^2(\mu)$. Now, let $A \subset \Gamma$ be a compact set such that $A \cap C = \emptyset$. Consider the correlation:

$$r(A,t,\Lambda) = \mu_\Lambda(N_A \cdot S_t^{(\Lambda)}(N_A)) - [\mu_\Lambda(N_A)]^2 =$$
$$= \mu_\Lambda(N_A \cdot N_{\alpha_t^{(\Lambda)} A}) - [\mu_\Lambda(N_A)]^2 .$$

For a given $\Lambda$ (large enough) $r(A,t,\Lambda)$ will eventually decrease to zero and remain there until a time $T_\Lambda(A)$, which is the minimal time in which a particle "in" A returns to it after bouncing from a wall (see fig. 3). $T_\Lambda(A)$ increases with $\Lambda$ and is infinite in the thermodynamical limit.




(fig. 3)

Therefore, while dealing with a strictly local observable, in a large system the non ergodicity exhibits itself only after a large time. This becomes infinite in the thermodynamical limit in which the system is mixing (in fact Bernoulli). In this respect the thermodynamical limit can be taken to represent a large finite system with some random collisions on the boundary, due to interactions with the environment.

We have seen that strong ergodic properties of the infinite system do not correspond to similar properties of the finite system(*). There is a way, however, in which the non ergodicity

---

(*) we have considered here a particular "natural" sequence of states $\mu_{\Lambda_i}$ on $X_{\Lambda_i}$, which converge on local observables to $\mu$. Nevertheless, one may choose a sequence of <u>ergodic</u> states on $X_{\Lambda_i}$ with the same limit. For example, states characterized by $N_i \approx \rho|\Lambda_i|$ and velocities $(v_{i,1},..,v_{i,N_i})$ whose distribution converges to $\nu(v)$, converge to $\mu_{\rho,\nu}$.

of the finite systems is reflected in properties of the limit.

As a result of the "integrability" of the finite system it possesses a large collection of invariant states at a given (approximate) density and energy per particle. This, together with the "consistency" of those states, is closely related to the multitude of invariant states of the infinite ideal gas, which have the disjoint independence property and a given density and energy per particle.

III. Ergodic Properties of an Infinite One Dimensional Hard Rod System

1. Introduction

The explanation of the good ergodic properties of the infinite ideal gas is simple: local disturbances 'fly off' unhindered to infinity where they are not longer observable (with respect to measures natural from the point of view of statistical mechanics, i.e., measures concentrated on local observables). Formally the proof for the infinite ideal gas is obtained by showing that the flow is isomorphic to the process obtained by observing the particles which at any moment cross a given hyperplane (a point for one dimensional systems). The absence of interactions play there a double role:

1) The fact that the "information" "flows" unperturbed guarantees that all of it eventually gets recorded by the local observations (on the hyperplane), in a way which enables one to reconstruct the phase-space description at the time .

2) The observed "information" does not return to the hyperplane, making observations at different times independent.

It is plausible that some infinite systems of interacting particles will no longer admit a generating "local observation". This would not rule out strong ergodic properties; their proof however would require different methods which may, in fact, lead to stronger results [16]. In those systems which do admit representation by a process constructed on local observations, the interaction will induce a dependence among the observations at different times, possibly preventing strong ergodic properties (of the local observation).

It is with an eye to understanding the latter behavior that we consider the effect of a hard core interaction in an infinite one dimensional gas. The motion of a "velocity pulse" in this system is a combination of a steady flow with discrete independent jumps back and forth. Pulses with effective velocity 0 would reappear infinitely often at any place on the line. It is shown that an infinite system of hard rods for which the effective velocities are bounded away from some neighborhood of 0 is Bernoulli. This extends a result of Sinai [50] who showed (considering a Gibbs state) that a one dimensional system of hard rods is a K-system. We also clear up some points in Sinai's proof.

2. Description of the System and the Main Result

Let $X$ denote the phase space of an infinite system of hard rods of diameter $d>0$ . $x \in X$ is a countable collection of pairs $x = \{(x_\alpha, v_\alpha)\}_\alpha$ , where $x_\alpha$ is the position of the left corner (or any other fixed point on it) of a rod and $v_\alpha$ its velocity. Let $\mu$ denote the translationally invariant measure on $X$ under which:

1) The free distances between consecutive rods (given that the origin is covered) are jointly independent and identically distributed, with an exponential distribution of parameter $\rho > 0$. $\rho = n/(1 - nd)$ where $n$ is the average particle density.

2) The velocities are independent and identically distributed with probability measure $\nu$ , which has a finite first moment.

Let $S_t$ denote the flow on X under which each particle in $x \in X$ moves freely $\frac{dx_\alpha}{dt} = v_\alpha$, $\frac{dv_\alpha}{dt} = 0$ except for elastic collisions. By an argument similar to Sinai's [50] (used for systems with a Maxwellian velocity distribution) it may be shown that $\{S_t\}$ is well defined on a set of full measure.

For convenience, reference will be made to velocity pulses, whose positions are the positions of rods but which are understood to exchange rods in a collision. A pulse of velocity v moves at this velocity except for moments of collision, when it jumps the distance d in the direction of the other colliding particle. However, the "free distance" between two pulses (obtained by subtracting the total length of rods between them) behaves linearly in time.

Moreover, for a given position of a pulse, the "free distances" to other pulses are distributed along the line, with a Poisson distribution. Therefore, for given velocity and position of a pulse, the collisions it undergoes at different times are independent.

Lemma (2.1) There is a measurable set $\tilde{X} \subset X$, $\mu(\tilde{X}) = 1$, such that $\forall\ x \in \tilde{X}$ the following holds; let v be the velocity of a pulse in x, then:

1) During the motion induced on it by $\{S_t x\}$, the pulse crosses the origin.

2) The average velocity of the pulse for the time interval $[0,t]$, approaches

$$v_{eff}(v) = v + \rho d(v - E(v)) \quad , \quad \text{as } t \to \pm\infty$$

The lemma can be proven by showing that properties 1) and 2) hold, with probability 1, for each pulse separately (labeling each pulse by $\ell_\alpha = [x_\alpha/d]$) . This can be easily done with the help of the previous remarks.

Note that for pulses of velocity $v \neq 0$ 1) follows from 2), while for pulses of velocity $v = 0$ 1) holds as a result of the fluctuations in the number of collisions.

Let $v_o = \frac{\rho d}{1+\rho d} E(v)$ . By lemma (2.1), pulses of velocity $v_o$ propagate with the effective velocity (for long times) 0. Since the effective velocity contains a part (of positive variance) which is due to independent collisions, pulses of this velocity recur infinitely often at any place on the line.

With the help of lemma (2.1) one can generalize Sinai's result [50] to obtain

<u>Theorem (2.2):</u> $(X,\mu,S)^+$ $(d > 0)$ is a K-system for any velocity probability distribution $\nu$ .

The proof will not be given here. Let us remark however that Sinai's proof carries over to systems with $\nu(v = v_o) = 0$. Other systems are covered by a modified argument.

---

$^+$ We will write S in place of $S_1$

The main result which we prove is

Theorem (2.3): The dynamical system $(X,\mu,S_t)$ for which the velocity distribution $\nu$ satisfies

$$\nu(|v - v_o| < \delta) = 0,$$

for some $\delta > 0$ and $v_o = \frac{\rho d}{1+\rho d} E(v)$, is a Bernoulli flow.

3. Reduced Description

As already indicated, we are interested in a representation of the system by a process generated by local observations. However, since collisions occur at any interval on the line, the reappearance of a pulse at the origin depends on the distribution of particles elsewhere, inducing a complicated dependence among local observations taken at different times. In the following "reduced description" (which is limited to one dimensional systems) the discontinuities in the trajectory of a pulse result from effects which take place at the origin.

For any $x \in X$, let us label the particles so that their positions at time $t = 0^-$ satisfy

$$\ldots x_{-2} < x_{-1} < 0 \leq x_o < x_1 \ldots$$

taking the limit $t \to 0^-$ for each inequality separately.

Definition (3.1): Let $(x_1, v_1)$ be the position and velocity of the i-th particle in $x \in X$. Its reduced position is given by

$q_i = x_i - id$ .

Clearly, an equivalent description of $x \in X$ is given by $y = \{(q_i, v_i)\}$ , the enumerated collection of the reduced positions and velocities of the induced particles. Note that the unenumerated collection $\{(q_\alpha, v_\alpha)\}$ may not determine $x \in X$ uniquely.

Definition (3.2): The reduced phase space, Y, is the class of enumerated collections $y = \{(q_i, v_i)\}_{i \in Z}$ for which

1) $q_i \leq q_{i+1}$ $\forall i \in Z$
2) $0 \leq q_o$
3) $q_{-1} \leq d$
4) the replacement $q_i \to q_i + tv_i$, $t = 0^-$ , makes each of the above inequalities strict (except if $v_i = v_{i+1}$ or $v_o = 0$ ).

The mapping $\varphi: X \to Y$ which carries $x = \{(x_\alpha, v_\alpha)\}$ to the enumerated collection of the reduced positions and velocities of its particles, ordered as above, is 1 - 1 and onto. By a harmless abuse of notation, let $\mu$ and $S_t$ denote the measure and the flow induced on Y by the correspondence $\varphi$ .

Lemma (3.3): With respect to the measure $\mu$ , the distribution of the reduced positions and velocities (ignoring the labeling) of those pulses of $y \in Y$ which lie in $R \setminus [0,d]$ is isomorphic to the Poisson distribution of points, with density $\rho$ , over $(R \setminus [0,d], \ell) \otimes (R, \nu)$ ($\ell$ being the Lebesgue measure) and is independent of the distribution and labeling of pulses in $[0,d]$.

**Proof**: Let $q_{\overset{+}{(-)}} = \underset{(\max)}{\min} \{q_i \mid q_i \underset{(<0)}{>} d\}$

The lemma follows from the observation that the distributions of both $q_+ - d$ and $|q_-|$ are exponential with parameter $\rho$, independent of the configuration in $[0,d]$ and of each other.

We remark that $\mu$ has the following realization: Let $\bar{Y} = Y_+ \otimes Y_-$ be the product of two independent Poisson distributions of points with density $\rho$, $Y_+$ over $([0,\infty), \ell) \otimes (R,\nu)$, whose left particle is labeled "0"; and $Y_-$ over $((-\infty, d], \ell) \otimes (R,\nu)$, whose right particle is labeled "-1". $(Y,\mu)$ is isomorphic to $(\bar{Y}',\mu')$, where $\bar{Y}' = \{y \in \bar{Y} | q_o \geq q_{-1}\}$ and $\mu'$ is the induced probability measure on $\bar{Y}'$.

Let $y \in Y$. We denote by $N(t,y)$ the directed number of crossings of the origin by particles in $x = \varphi^{-1}(y)$, during the time interval $[0,t)$, counting crossings from left (right) as positive (negative). $K(y)$ will denote the index of the first particle whose reduced position (in $S_{-0}y$) is non negative.

Consider now the motion of the pulses in $y \in Y$ induced by the flow $S_t$. The pulses move at their characteristic velocity, exchanging indices at collisions, except for moments at which a particle crosses the origin (in $x$), when the reduced positions of all the pulses are shifted by $\mp d$ and their index values change by $\pm 1$, depending on the direction of the crossing. The reduced distance traveled by a pulse $(q,v) \in y \in Y$ during the time interval $[0,1)$ is thus equal to $tv-dN(t,y)$.

<u>Lemma (3.4):</u> For almost any $y \in Y$

$$\lim_{t \to \infty} \frac{N(t,y)}{t} = \frac{\rho}{1+\rho d} E(v), \quad \text{for } t \in Z$$

<u>Proof</u>: Since our system is ergodic, Theorem (2.2), it follows that,

$$N(t,y)/t = \frac{1}{t} \sum_{i=0}^{t-1} N(1,S_i y) \xrightarrow[(t \to \infty)]{} E(N(1,y)) \quad \text{almost surely (a.s)}$$

To compute $E(N(1,y))$ we observe that the average velocity, in the reduced description, of a pulse of velocity $v$ is

$$[tv - dN(t,y)]/t \xrightarrow[(t \to \infty)]{} v - dE(N(1,y)),$$

which corresponds to velocity $[v-dE(N(1,\cdot))]\cdot(1+\rho d)$ in terms of "real" distance. Comparing the above with Lemma (2.1) we obtain

$$E(N(1,\cdot)) = \frac{\rho}{1+\rho d} E(v)$$

4. <u>Process Description</u>

<u>Definition (4.1):</u> We will call a pulse $(q,v) \in y \in Y$ <u>marked at time t</u> if one of the following holds

a) $0 \leq q + tv - dN(t,y) \leq d$

b) $q + tv - dN(t,y) < 0$ and $q + (t+1)v - dN(t+1,y) \geq 0$

c) $q + tv - dN(t,y) > d$ and $q + (t+1)v-dN(t + 1, y) \leq d$

We now define a partition of the phase space which will be used to represent our system by a process. In order to apply approximation techniques coarser partitions will be defined as well.

Definition (4.2):

1) Denote by $\eta$ the partition of $Y$ generated by:

a) $N(1,y)$

b) $K(y)$ and

c) the reduced positions and velocities of those pulses of y which are marked at the time $t = 0$.

2) Denote by $\eta^{(k)}$ ( $k \in Z$) the partition whose typical element $C^{(k)}(\bar{y}) \in \eta^{(k)}$ is the collection of the phase space points $y \in Y$ for which

a) $N(1,y) = N(1,\bar{y})$

b) $K(y) = K(\bar{y})$, and

c) there is a 1-1 correspondence between the pulses marked at the time $t = 0$ in y and $\bar{y}$, such that at the times $t = 0,1$ the corresponding pulses are at the same distances from the origin, measured in intervals of the size $d/2^k$ .

In essence, $\eta$ partitions Y (and therefore X) by the characteristics of the pulses which in the reduced description appear in the interval $[0,d]$ during the time $[0,1]$, disregarding those which do appear but eventually cross back. The pulses have to be observed in an interval since their trajectories have discontinuities of the size d.

It is an advantage of the reduced description that $\eta(x)$ contains the full information about those pulses of $x$ which are in a certain region (which depends on $\eta(x)$) of the one particle phase space, and is independent of the characteristics of the pulses elsewhere. To show this we need the following lemma.

Denote by $\alpha^{\pm}(n,t)$ the regions in the one particle phase space, defined by

$$\alpha^{\binom{+}{-}}(n,t) = \{(q,v) \mid q \underset{(\geq)}{\leq} 0,\ q + vt - dn \underset{(<)}{>} 0\}$$

Lemma 4.3: Let $y \in Y$ and let $m = N(t,y)$. If $y$ and $\bar{y} \in Y$ possess the same occupation numbers for pulses in the regions $\alpha^{\pm}(m,t)$ and $\alpha^{\pm}(m+1,t)$, and $K(y) = K(\bar{y})$ then $N(t,\bar{y}) = m$.

**Proof:** Denote by $\hat{N}_y(\beta)$ the number of pulses in $y \in Y$ which occupy a given region $\beta$ in the one particle phase space.

Notice that for $m = N(t,y)$, $\alpha^{\binom{+}{-}}(m,t)$ is the region of those pulses whose reduced positions change from non-positive (non-negative) in $y$ to positive (negative) in $S_t y$.

Keeping in mind the fact that the index values get readjusted each time a particle (in the unreduced description) crosses the origin one obtains

$$K(S_t y) = K(y) - \hat{N}_y(\alpha^+(m,t)) + \hat{N}_y(\alpha^-(m,t)) + m,$$

therefore

$$K(y) - \hat{N}_y(\alpha^+(m,t)) + \hat{N}_y(\alpha^-(m,t)) + m \leq 0.$$

Similarly, $K(y)-\hat{N}_y(\alpha^+(m+1,t))+\hat{N}_y(\alpha^-(m+1,t)) + m$ is the index of the first pulse in $S_{t-0}\,y$ whose reduced position is not smaller than d, therefore

$$K(y) - \hat{N}_y(\alpha^+(m+1,t)) + \hat{N}_y(\alpha^-(m+1,t)) + (m+1) > 0.$$

Consider now the function (defined on $Z$ )

$$f_y(n) = K(y) - \hat{N}_y(\alpha^+(n,t)) + \hat{N}_y(\alpha^-(n,t)) + n$$

Since $\alpha^+(n,t) \supset \alpha^+(n+1,t)$ and $\alpha^-(n,t) \subset \alpha^-(n+1,t)$, $f(n)$ is strictly increasing. Moreover, by the above inequalities, $N(t,y)$ is the unique solution of

$$f_y(n) \leq 0, \quad f_y(n+1) > 0.$$

Now, by the conditions of the lemma,

$$f_{\bar{y}}(m) = f_y(m) \leq 0 \quad \text{and} \quad f_{\bar{y}}(m+1) = f_y(m+1) > 0,$$

implying $N(t,y) = N(t,\bar{y})$.

Corollary (4.4): Let the region $\alpha(n)$ of the one particle phase space be given by

$$\alpha(n) = \alpha^{+}(n,1) \cup \alpha^{-}(n+1,1) \cup ([0,d] \otimes R).$$

If $N(1,y) = n$, then $\eta^{k}(y)$ is independent of the characteristics of those pulses which occupy $\alpha(n)^{c}$ (the complement of $\alpha(n)$). Notice that $\eta(y)$ contains the full information regarding the pulses in $\alpha(n)$ .

The above independence will be used to establish strong ergodic properties for the process defined by $\eta^{(k)}$ .

5. Tools Used

In proving the Bernoulli property we will make use of the method developed by Ornstein which utilizes the following results [39,48]:

Lemma (5.1): The dynamical system $(X,\mu,\{T_t\})$ is Bernoulli if $(X,\mu,T_1)$ is a Bernoulli shift.

Lemma (5.2): If $A_1 < A_2 < A_3 < \ldots$ form an increasing sequence of T-invariant $\sigma$-algebras, if $\overset{\infty}{\underset{0}{\vee}} A_k = B$ , and if for each n, $(X, A_n, \mu, T)$ is a Bernoulli shift then $(X, B, \mu, T)$ is a Bernoulli shift.

This lemma enables one to use results obtained for processes defined by countable partitions. Given a generating partition P, the dynamical system $(X, \mu, T)$ is isomorphic to the process $(T,P)$

with the induced measure.

Definition (5.3): The partition $P = \{P_i\}$ is $\varepsilon$-independent of $Q = \{Q_i\}$ if there is a class C of sets in Q such that

a) $\mu(\cup C) \geq 1 - \varepsilon$

b) $\sum_i |\mu(P_i|Q_j) - \mu(P_i)| \leq \varepsilon \qquad Q_j \in C$

Definition (5.4): A partition P is called weakly-Bernoulli for an automorphism T if given $\varepsilon > 0$ there exists an N such that for all $m \geq 1$ :

$$\bigvee_{-(N+m)}^{-N} T^i P \text{ is } \varepsilon\text{-independent of } \bigvee_{o}^{m} T^i P$$

Lemma (5.5): If the partition P is weakly-Bernoulli for the automorphism T then (T,P) is a Bernoulli process.

## 6. Proof of the Main Result

Let us restate the main theorem (2.3), using the notation of the previous sections.

Theorem (1.6): The dynamical system $(Y, \mu, S_t)$ for which

$$\nu(|v-v_o| < \delta) = 0 \text{ for some } \delta > 0 \text{ and } v_o = \frac{\rho d}{1+\rho d} E(v)$$

is a Bernoulli system.

The proof will consist of several steps. The "times" t mentioned throughout this section are to be understood as integral.

Lemma (6.2): For a given $\delta > 0$, let the "good" sets be given by

$$G_t^{(\stackrel{+}{-})} = \{y \in Y \mid \sup_{\substack{t'>t \\ (t'<-t)}} \{(|dN(t',y) - t'v_o| + d)/|t'|\} < \delta\}$$

Then $\forall \epsilon > 0, \exists\, t(\epsilon,\delta) > 0$ for which $\mu(G_t^+), \mu(G_t^-) > 1 - \frac{(\epsilon)^2}{10}$

Proof: By lemma (3.4) $\lim_{|t| \to \infty} dN(t,y)/t = v_o$ for almost every $y \in Y$, therefore $\sup_{|t'|>t} \{(|dN(t',y) - t'v_o| + d)/t'\} \xrightarrow[(t \to \infty)]{} 0$ a.s. implying convergence in probability, which is lemma (6.2).

For the system under consideration $\nu(|v-v_o| < \delta) = 0$ ; let us assume therefore that there are no pulses of velocity $v$, $|v-v_o| < \delta$ (in fact, we are confining the discussion to a subset of $Y$ of full measure).

Let $G_t = G_t^+ \cap G_t^-$

Remark (6.3): $\forall y \in G_t$, no pulse of $y$ whose reduced position was in $[0,d]$ at a time $t' \in (-\infty,-t]$ will reappear in $[0,d]$ at a time $t'' \in [t,\infty)$ or at $t'' = 0$ *. This can be easily shown, remembering that the reduced distance traveled by a pulse of velocity $v$ between the times $t'$ to $t''$ is

$$v(t''-t') - d(N(t'',y) - N(t',y))$$

Definition (6.4): Denote by $\zeta$ the partition generated by

1) $K(y)$

2) the number of velocity pulses in $[0,d]$ for which

---

* Here the times are not to be understood as necessarily integral.

$v > v_o + \delta$ and the number of those for which $v < v_o - \delta$.

We now come to the key step in the argument:

Lemma (6.5): Let $a$ be a set measurable with respect to $\overset{\infty}{\underset{t}{\vee}} S^i\eta (t > 0)$, $b$ be measurable with respect to $\overset{-(t+1)}{\underset{-\infty}{\vee}} S^i\eta$, and $\zeta_k \in \zeta$. Then

$$\mu(a|b,\zeta_k,G_t) = \mu(a|\zeta_k,G_t)$$

Proof: Note that (by virtue of corollary (4.4) and lemma (4.3)) $a \cap G_t^+$ and $b \cap G_t^-$ depend on the distribution of pulses in two corresponding regions in the one particle phase space, whose intersection (after taking out the slice $R \otimes (v_o - \delta, v_o + \delta)$) is $[0,d] \otimes R$. Further, the dependence of $a \cap G_t^+$ and of $b \cap G_t^-$ on the distribution of pulses in $[0,d] \otimes R$ is only through the variables used to define the partition $\zeta$ (def.(6.4)). Therefore, by lemma (3.3), on a given element $\zeta_k \in \zeta$, $a \cap G_t^+$ and $b \cap G_t^-$ are independent:

a) $\mu(a \cap G_t^+ \mid \zeta_k, b \cap G_t^-) = \mu(a \cap G_t^+|\zeta_k)$.

In particular, by choosing $a = Y$ and then $b = Y$, we obtain

b) $\mu(G^+|\zeta_k \cap b \cap G_t^-) = \mu(G_t^+|\zeta_k)$

and

c) $\mu(G_t^+|\zeta_k) = \mu(G_t^+|\zeta_k, G_t^-)$.

The lemma follows now from a), b), and c).

We observe therefore that our system exhibits an approximate Markov property. The proof that it is Bernoulli will follow in a way similar to a proof that a K Markov system is Bernoulli (actually, for this end, mixing could replace the K property). We thus first consider the space Y equipped with the $\sigma$-algebras $\bigvee_{i=-\infty}^{\infty} S^i\eta^{(k)}$.

Theorem (6.6):

For any integer k, the dynamical system $(Y, \bigvee_{-\infty}^{\infty} S^i\eta^{(k)}, \mu, S)$ for which the velocity distribution $\nu$ satisfies

$$\nu(|v-v_o| < \delta) = 0 \text{ , for some } \delta > 0 \text{ and } v_o = \frac{\rho d}{1+\rho d} E(v),$$

is a Bernoulli-shift.

Proof: By lemma (5.5) it is enough to show that the partition $\eta^{(k)}$ is weakly-Bernoulli under S.

Let $\epsilon > 0$ be given. Take $t_1 = t_1(\epsilon,\delta)$ as defined in lemma (6.2) and let $\tilde{\zeta} = \zeta \vee \{G_{t_1}, G_{t_1}^c\}$.

Because of the K-property of our system $t_2(\epsilon) > 0$ such that $\tilde{\zeta}$ is $\frac{\epsilon}{10}$ -independent of

$$\bigvee_{i=-(t_1+t_2+m)}^{-(t_1+t_2+1)} S^i\eta^{(k)} \quad , \qquad \forall m \geq 1$$ +

We now claim that $\forall\ m \geq 1$ , the partition

$$\bigvee_{t_1}^{t_1+m} S^i\eta^{(k)} = \{a_\ell^{(m)}\}$$

is $\epsilon$-independent of

$$\bigvee_{i=-(t_1+t_2+m)}^{-(t_1+t_2+1)} S^i\eta^{(k)} = \{b_j^{(m)}\}.$$

To see this, note that by virtue of lemma (6.5) (omitting the superscript m)

$$\mu(a_\ell | b_j, \zeta_k) = \mu(a_\ell | \bar{\zeta}_k)$$

$\forall j, \ell$ and $\bar{\zeta}_k \subset G_{t_1}$ an element of $\bar{\zeta}$.

---

+This is so (see [6]) because the partitions $\eta^{(k)}$ all have finite entropy (since the velocity distribution has a finite first moment). We wish to point out, however, that the argument which we give does not depend in an essential way upon the finiteness of the entropy of $\eta^{(k)}$. We could easily find an increasing sequence $\eta_\ell^{(k)}$ $\eta^{(k)}$ of finite partitions whose supremum is $\eta^{(k)}$; our argument could be applied exactly as it stands to the $\eta_\ell^{(k)}$, so that Theorem (6.6) would be valid with $\eta_\ell^{(k)}$ in place of $\eta^{(k)}$ . Theorem (6.6) would then, itself, follow from an application of Lemma (5.2).

This implies that

$$\mu(a_\ell | b_j) = \sum_{\bar{\zeta}_k \subset G_{t_1}} \mu(a_\ell | \bar{\zeta}_k) \cdot \mu(\bar{\zeta}_k | b_j) +$$

$$+ \mu(a_\ell | b_j, G^c_{t_1}) \cdot \mu(G^c_{t_1} | b_j).$$

Therefore

$$\sum_i |\mu(a_\ell | b_j) - \mu(a_\ell)| \leq \sum_{\bar{\zeta}_k \subset G_{t_1}} | \mu(\bar{\zeta}_k | b_j) - \mu(\bar{\zeta}_k)| +$$

$$+ \mu(G^c_{t_1} | b_j) + \mu(G^c_{t_1})$$

Now, 1) the above sum is smaller than $\frac{\epsilon}{10}$ except possibly for sets $b_j$ whose total measure is smaller than $\frac{\epsilon}{10}$ .

2) $\mu(G^c_{t_1}) < \frac{\epsilon}{10}^2$

3) 2) implies that $\mu(G^c_{t_1} | b_j) < \frac{\epsilon}{10}$ , except for a collection of sets whose total measure is less than $\frac{\epsilon}{10}$ . Summing the above estimates we obtain

$$\sum_\ell | \mu(a^{(m)}_\ell | b^{(m)}_j) - \mu(a^m_\ell)| < \epsilon$$

for a collection of elements $b^{(m)}_j$ whose total measure exceeds $1 - \epsilon$ , proving the claim.

Since $t_1(\epsilon)$ , $t_2(\epsilon)$ are independent of m, $\eta^{(k)}$ is weakly-Bernoulli.

In order to apply theorem (6.6) to the proof of (6.1) we need the following lemma

Lemma (6.7): $\eta$ is a generating partition.

Proof: Observe that knowing $N(t,y)$ and the characteristics of a pulse in $S^t y$ enables one to find the (reduced) position of the pulse in $y \in Y$. Since, by lemma (2.1), each pulse is marked at some time (with probability 1) $\bigvee_{i=-\infty}^{\infty} S^i \eta$ separates points in $Y$(mod 0).

Proof of theorem (6.1):

By the previous lemma, the processes considered in theorem (6.6) have the property required for application of lemma (5.2), by which (6.1) follows.

Q.E.D.

Conclusion

In summary, the essential ideas of the proof are:

1) The infinite hard rod system admits a representation by generating K-process obtained by observing the pulses close to the origin (in the reduced description, for convenience).

2) Rods which cross the origin tend to draw back pulses which crossed before them. $dN(t,\cdot)/t$ is the random velocity with which pulses are "pursued" by the origin (this has a clearer meaning in the reduced description).

3) Due to the random character of the above velocity (which tends to $v_o$ in the limit $t \to \pm\infty$) $\forall\, \delta, \epsilon > 0, \exists\, t > 0$ for which,

with probability $> 1-\varepsilon$ , no pulse which appeared at the origin before $-t$ could reappear after the time $t$ with the exception of those whose velocity is $\delta$ close to $v_o$.

4) When viewed as a process, systems from which pulses slow relative to $v_o$ are excluded exhibit an approximate Markov property.

5) A generating family of processes, which are constructed on countable partitions and have an approximate Markov property, can be found. The Bernoulli property for these is proved in a way similar to the proof for a Markov K-process.

7. A Clarification of the Proof of Sinai's Result

Sinai [1] constructed a K-partition ($\zeta_o$ in his notation) for a one dimensional hard rod gas with an infinite number of degrees of freedom. However, Remark 4 in his article is incorrect as stated, leaving the proof of the generating property of the partition incomplete.

$t$ is called there a moment of intersection of zero of the trajectory of point $x \in X$ , if either

1) $x_i(t) = 0$ for some $i$ (there is a pulse in $S_t x$ which is crossing the origin), or

2) $t$ is a moment of collision of two rods which at this time are on opposite sides of the origin (and therefore two pulses in $S_t x$ are jumping across the origin).

The intersection time, velocities and positions of the pulses involved are called the characteristics of the intersection.

One still has to show that the generated partition $(\bigvee_t S_t \zeta_o)$ , which corresponds to partitioning by the characteristics of all the intersections, completely separates the set of phase space points for which any pulse has a moment of intersection of the origin (which was shown to be a subset of full measure). This can be done using the reduced description.

The reduced position of a pulse in x can be obtained from its reduced position in $S_t x$ by

$$q(o) = q(t) - tv + d\,N(t,x).$$

Further, by looking for the first, in terms of lowest $|t|$, intersection of zero of a pulse in x one can determine whether its index is positive or negative. In this method, provided each pulse eventually crosses the origin, one can reconstruct the reduced description of x from the characteristics of all the intersections, which therefore determine it uniquely (mod. 0).

## IV. Mixture of several components of equal masses

### 1. Introduction

As a first step towards understanding the ergodic properties of systems which contain a mixture of interacting particles of several types, we will discuss now the consequences of a particular type of "constants" of the motion which are present in such one dimensional systems of non penetrating particles.

The simplest system with the above properties is a random mixture of several components of equal mass particles, which are marked by different colors. The point particles move freely except for elastic collisions by which, upon impact, they exchange velocities.

One immediately notices that under this dynamics the order of the colors of particles, to the right and left of a given particle, is invariant. Such "invariants" appear in any one dimensional random mixture of non penetrating particles, for example in a mixture of two types of hard core particles of different mass.

While for finite systems the order of colors is a valid constant of the motion, it is not clear that in an infinite system it gives rise to measurable invariants which are non trivial with respect to a state which has good clustering (mixing under space translations) properties. A special feature of infinite systems is that there is no time-invariant measurable way of picking a reference particle (except if there are particles which are effectively confined to a certain region). Any function on the phase space may be approximated by one which depends on the colors of only those particles which are in some bounded volume. Since the local combination of colors keeps changing (while globally the sequence of colors is just shifting randomly), one is led to expect good mixing properties. In fact, it will be shown that the time evolution of such infinite systems is a K-flow.

The recurrence of particles whose colors were observed in the past leads to a kind of long memory which poses difficulties in deciding if the system has the Bernoulli property. We will mention a "simpler" dynamical K-system (of finite entropy), with some similar features, for which the Bernoulli property is still an open question.

2. The phase space

The degrees of freedom of a single particle consists now of a triple

$$(q,v,i) \in \Gamma \otimes \mathcal{K}$$

where $\mathcal{K} = \{1,2,\ldots,h\}$ is a finite collection of "colors". The "natural" phase space of the infinite system, $Y$, consists of locally finite collections of points in $\Gamma \otimes \mathcal{K}$. It will be convenient, however, to separate the color degrees of freedom from the positions and velocities, whose time evolution is independent of colors. Further, the one dimensional system of non penetrating <u>identical</u> particles of equal mass is indistinguishable from the ideal gas (Ch.II).

We will consider a point in the phase space $Y$ to be given by a point in $X$, the phase space of the ideal gas, and a double infinite sequence

$$d = (\ldots, d_{-1}, d_o, d_1, \ldots) \in D = \aleph^{\mathbb{Z}}$$

which describes the colors of particles. These are ordered by their positions with respect to the origin ($q=0$) so that $d_o$ is the color of the first particle to the right of it. In case of ambiguity in the order it is defined to be "continuous from below" under the time evolution (see sec. (III.3)).

Therefore $Y = X \otimes D$. It can be easily seen that the quasi local topology on Y, when it is defined as at the beginning of this section (with the discrete topology on $\aleph$), is the same as the direct product of the quasi-local topology on $X$ with the product topology on $D$.

<u>Time evolution</u>

We will denote by $U_t$ the flow on Y which corresponds to the evolution in time of the system, as described in the proceeding sections.

Let $(x,d) \in Y = X \otimes D$ and let $U_t(x,d) = (x',d')$. As already mentioned, $x' = S_t x$ where $S_t$ is the time evolution of the ideal gas defined on $X$.

Collisions of particles do not alter the order of colors on the line. This is changed only when a particle crosses the origin $(q=0)$ increasing, or decreasing, the "index" of all the particles by 1.

Let us denote by $m(t,x)$ the "net" number of crossings of the origin during the time interval $[0,t)$, counting those where a particle crosses from right (left) as positive (negative). Then $d' = T^{m(t,x)} d$, with the shift $T$ being defined on $D$ by

$$(T(\ldots,d_o,d_1,\ldots))_i = d_{i+1} \quad .$$

The flow $U_t : Y \to Y$ has therefore the skew product structure [51]:

$$U_t(x,d) = (S_t x,\ T^{m(t,x)} d) \quad .$$

The group property of $\{U_t\}_{t \in R}$ is guaranteed by:

$$m(t,x) = m(t_1,x) + m(t-t_1, S_{t_1} x) \quad , \quad 0 \leq t_1 \leq t.$$

### 3. Invariant states

Definition: Let $\nu(\cdot)$ be a probability measure on R and $\tilde{\rho}$ a measure on $\mathcal{K} = \{1,..,h\}$ given by the weights $(\rho_1,\ldots,\rho_h)$ . We will denote by $\mu_{\tilde{\rho},\nu}$ the measure on $\mathcal{B}(Y)$ which corresponds to the Poisson construction on $(\Gamma \otimes \mathcal{K}, \gamma \otimes \tilde{\rho})$ , where $\gamma$ is the measure on $\Gamma$ given by

$$\gamma(dq\ dv) = dq\ \nu(dv).$$

The Poisson construction was discussed in sec. (II.3) . In the state given by $\mu_{\tilde{\rho},\nu}$ the particles are "independently" distributed on the line, with the uniform (average) density $\rho_i$ for the i-th color, and have independent velocities of identical distribution .

Claim: The measure $\mu_{\tilde{\rho},\nu_h}$ (as above) factorizes to a product of $\mu_{\rho,\nu}$ on X , $\rho = \sum_{i=1} \rho_i$ being the total density, and the product measure, $\rho^*$ , on $D = \mathcal{K}^Z$ induced by the probability vector $\left(\frac{\rho_1}{\rho},\ldots,\frac{\rho_1}{\rho}\right)$ on $\mathcal{K}$ .

We will omit the proof of the claim which is quite straightforward. An important consequence is

Corrollary: The state (i.e. measure) $\mu_{\tilde{\rho},\nu}$ is invariant under the time evolution $S_t$ .

Proof: $U_t$ is a skew product of $S_t$ on X , which preserves the measure $\mu_{\rho,\nu}$ , with the shift T on $D = \mathcal{K}^Z$ under which any product measure is preserved. Therefore $U_t$ preserves the product of the above measures on $X \otimes D = Y$ .

4. Ergodic Properties

Using the notation introduced in the proceeding sections, our main result is:

Theorem: For any $\tilde{\rho}$ and $\nu(\cdot)$ which is continuous at zero ( $\nu(\{0\}) = 0$), the dynamical system

$$(Y, \mu_{\tilde{\rho},\nu}, U_t)$$

is a K-flow.

Proof: To prove the theorem it is enough to show the existence of a measurable partition $\eta_o$ such that

i) $\eta_t \overset{\text{def}}{=} U_t \eta_o \supset \eta_o$ for $t > 0$,

ii) $\eta_\infty = \bigvee_{t \in R} \eta_t$ partitions (mod. 0) to single points.

and iii) $\eta_{-\infty} = \bigwedge_{t \in R} \eta_t$ is a trivial partition.

The following properties of the above system will be used in the proof:

a) The system is a skew product

$$(Y, \mu_{\tilde{\rho},\nu}) = (X, \mu_{\rho,\nu}) \otimes (D, \rho^*)$$

and

$$(U_t(x,d)) = (S_t x, T^{m(t,x)} d)$$

b) The "base" system $(X, \mu_{\rho,\nu}, S_t)$ is a Bernoulli flow (see sec.( II. 4)).

c) $\forall t > 0$ , the function $m(t,\cdot)$ is measurable with respect to an independent generator for the discrete transformation $S_t$ and

$$E\{[m(t,\cdot) - E(m(t,\cdot))]^2\} > 0 .$$

We will denote by $\xi_{t_1}^{t_2}$ the partition of $X$ according to the velocity pulses in $x \in X$ which cross the origin in the time interval $[t_1,t_2)$. It follows from our discussion in sec.(II.4) that $\xi_o^t$ is an independent generator for $S_t$. $m(t,\cdot)$, which is the "net" number of crossings of the origin, is clearly measurable with respect to $\xi_o^t$, hence the above statement holds true.

d) $(D,\rho^*,T)$ is a Bernoulli shift, as follows from the product structure of $\rho^*$.

Now, let $\alpha_o$ be the partition of $D = \{(\ldots,d_o,d_1,\ldots) | d_i \in \aleph)\}$ generated by $d_o$. $\alpha_n \overset{\text{def}}{=} T^n \alpha_o$ is then generated by $d_{-n}$ and $\alpha \overset{\text{def}}{=} \bigvee_{v \in Z} \alpha_n$ is the fully separating partition of D.

$\xi_{t_1}^{t_2}$ was defined above. Notice that

$$U_t \, \xi_{t_1}^{t_2} = \xi_{t_1 - t}^{t_2 - t} .$$

We will denote by $\zeta_o = \bigvee_{t > o} \xi_o^t$ the partition of $X$ generated the future crossings of the origin. Clearly, $\zeta_o$ is a generating K-partition (i.e. it satisfies the above i) - iii)) with respect to $S_t$.

We claim now that $\eta_o \overset{\text{def.}}{=} \zeta_o \otimes \alpha$ is a generating K-partition of Y with respect to $U_t$.

i) In a harmless abuse of notation we will define $S_t : Y \to Y$ and $T: Y \to Y$ by

$$S_t(x,d) = (S_t x,d) \quad \text{and} \quad T(x,d) = (x,Td).$$

Notice that $\forall\, t > 0$ :

$$U_{-t} = [U_t]^{-1} = [S_t\, T^{m(t,x)}]^{-1} =$$
$$= [T^{m(t,S_{-t}x)} S_t]^{-1} = S_{-t}\, T^{-m(t,S_{-t}x)} \quad .$$

Let us consider an atom of the partition $\eta_t$ , for $t > 0$.

$$\eta_t(x,d) = U_t \eta_o(U_{-t}(x,d)) =$$
$$= U_t \eta_o(S_{-t}x, T^{-m(t,S_{-t}x)} d) =$$
$$= S_t T^{m(t,\cdot)} [\zeta_o(S_{-t}x) \otimes \alpha(T^{-m(t,S_{-t}x)} d)]$$

$m(t,\cdot)$ , for $t > 0$, is measurable with respect to $\zeta_o$ therefore on all the points of the above set it assumes the value $m(t,S_{-t}x)$. Hence

$$\eta_t(x,d) = [S_t\, \zeta_o(S_{-t}x)] \otimes [T^{m(t,S_{-t}x)}\, \alpha(T^{-m(t,S_t x)} d)]$$

and, since the partition $\alpha$ is invariant under $T_t$ :

$$\eta_t(x,d) = \zeta_t(x) \otimes \alpha(d)$$

It follows that

$$t > 0: \eta_t = \zeta_t \otimes \alpha \supset \zeta_o \otimes \alpha = \eta_o$$

which proves i).

ii) By i):

$$\eta_\infty = \bigvee_{t \in R} \eta_t = \bigvee_{t \in R} (\zeta_t \otimes \alpha)$$

hence

$$\eta_\infty = \zeta_\infty \otimes \alpha .$$

$\zeta_\infty$ separates points in X (mod.0) as does $\alpha$ in D, therefore $\eta_\infty$ is the full partition of Y.

iii) In order to prove that $\eta_{-\infty}$ is a trivial partition it is enough to show that

(*) $\mu(C|\eta_{-\infty}(x)) = \mu(C)$, at almost every (a.e) $(x,d) \in Y$, for any measurable set $C \subset X$ (for convenience we will drop the subscripts of $\mu_{\rho,\nu}$).

In fact it is enough to show (*) for any $C$ which is measurable with respect to $\xi_{-m}^{m} \otimes \alpha$ for some $m > 0$. Since the partition $\eta_{-\infty}$ is invariant under $U_t$

$$\mu(C|\eta_{-\infty}(x,d)) = \mu(U_m C| \eta_{-\infty}(U_m(x,d)))$$

It is sufficient, therefore to show (*) for any $C$ measurable with respect to $U_m(\xi_{-m}^{m} \otimes \alpha)$ for some $m > 0$. However, by an argument similar to that used in i)

$$\bigcup_m (\xi^m_{-m} \otimes \alpha) = \xi^o_{-2m} \otimes \alpha$$

In conclusion, it suffices to prove (*) for any $C = B \otimes A$ where $A \in D$ is measurable with respect to $\bigcup_{|k|<n} \alpha_k$ and $B \in X$is measurable with respect to $\xi^o_{-m}$, for some $n, m > 0$.

For this end one may use Doob's theorem [ 6] from which follows that (since $\eta_{-t} \underset{(t \to -\infty)}{\downarrow} \eta_{-\infty}$).

$$\mu(C|\eta_{-\infty}(x)) \overset{a.e.}{=} \lim_{t \to \infty} \mu(C|\eta_{-t}(x,d)).$$

Let $B \otimes A \subset Y$ be as above and denote $\bar{A} = X \otimes A$ and $\bar{B} = B \otimes D$. For $t > 0$:

$$\mu(A \otimes B|\eta_{-t}(x,d)) = \mu(\bar{A}|\eta_{-t}(x,d)) \cdot \mu(\bar{B}|\bar{A},\eta_{-t}(x,d)) =$$

$$= \mu(\bar{B}) \cdot \mu(\bar{A}|\eta_{-t}(x,d)) ,$$

since $\bar{A}$ and $\eta_{-t}(x,d)$ are measurable with respect to $\eta_o$ which is independent of $\xi^o_{-n}$. It remains to show that

$$\lim_{t \to \infty} \mu(\bar{A}|\eta_{-t}(x,d)) \overset{a.e.}{=} \mu(\bar{A}).$$

Now, for $t > 0$,

$$\mu(\bar{A}|\eta_{-t}(x,d)) \overset{(a.e.)}{=\!=} \mu(U_t\bar{A}|\eta_o(U_t(x,d)))$$

$$= \mu(U_t\bar{A}|\zeta_o(S_t x) \otimes \alpha(T^{m(t,x)}d))$$

This is to be understood as a relation between two different functions on Y. It follows from the relation

$$\eta_{-t}(x,d) = U_{-t}\,\eta_o(U_t(x,d)).$$

Let us define

$$\mathcal{a}_d = \{i \in Z|T^{-i}\, d \in A\}$$

Since $U_t A = T^{m(t,\cdot)}A$, a valid version of the above conditional expectation is given by

$$\sum_{j \in \mathcal{a}_d + m(t,x)} \mu_{\rho,\nu}(\{y \in Y|m(t,y) = j\}|\zeta_{-t}(S_t x))$$

which may formally be written as

$$\mu_{\rho,\nu}(\{m(t,\cdot) - m(t,x) \in \mathcal{a}_d\}|\zeta_{-t}(S_t x)).$$

Therefore, using the above formal notation, (the existence of the limit follows from Doob's theorem)

$$\lim_{t\to\infty} \mu(\bar{A}|\eta_{-t}(x,d)) =$$
$$= \lim_{t\to\infty} \mu_{\rho,\nu}(\{m(t,\cdot)-m(t,x) \in \mathcal{a}_d \setminus [-\ell,\ell]\}|\zeta_{-t}(S_t x))$$
$$+ \lim_{t\to\infty} \mu_{\rho,\nu}(\{m(t,\cdot)-m(t,x) \in \mathcal{a}_d \cap [-\ell,\ell]\}|\zeta_{-t}(S_t x)) \quad ,$$

for any $\ell \in Z_+$. The second summand is easily seen to converge to zero, for almost any $(x,d) \in Y$. The first, when viewed as a function on D, is measurable with respect to $\bigvee_{|k|>\ell} \alpha_k$, for any $x \in X$. Since the "bilateral tail" $\bigwedge_{\ell>0} \bigvee_{|k|>\ell} \alpha_k$ is trivial, it follows that $\mu(\bar{A}|\eta_{-\infty}(x,d)$ does not depend on d. Thus for almost every $(x,d) \in Y$:

$$\mu(\bar{A}|\eta_{-\infty}(x,d)) = \int \mu(\bar{A}|\eta_{-\infty}(x,d))\, d\rho^*(d) =$$

$$= \lim_{t\to\infty} \int \mu(\bar{A}|\eta_{-t}(x,d))\, d\rho^*(d) = \lim_{t\to\infty} \int \mu(U_t\bar{A}|\zeta_o(S_t x)\otimes \alpha(T^{m(t,x)}d))d\rho^*(d)$$

$$= \lim_{t \to \infty} \mu(U_t \bar{A} | \zeta_o(S_t x)) = \lim_{t \to \infty} \mu(\bar{A} | \zeta_{-t}(x)) = \mu(\bar{A}) \quad ,$$

which concludes the proof of iii).

Q.E.D.

<u>An extension of the result:</u>

The proof of the K-property was based on the properties a)-d) of our system. This, however, was not a very economical way and it may be observed that the same result would hold under more relaxed conditions than b)-d).

In particular the same proof may be applied to systems with the properties a) and d) for which

b') the base is a K-flow

c') $m(t,\cdot)$ is measurable with respect to $\xi_o^t$ (which transform under $U_t$ as above) such that

$\zeta_o = \bigcup_{t>0} \xi_o^t$ is a generating K-partition and

$$P(\{|m(t,\cdot) - m(t,x)| < \ell\} \,|\xi_{-t}(x)) \xrightarrow[(t \to \infty)]{} 0, \forall \ell > 0 \text{ and a.e. } x \in X.$$

This set of conditions seems to have wider applications to infinite systems. For example, b') and c') are satisfied by the hard rod systems discussed in Ch.III (even without the restriction on $\nu(\cdot)$ used for the Bernoulli property, see Th. (2.2)).

We observe again that the existence of constants of the motion in the finite systems (here the sequence of colors) does not exclude, for reasons discussed in the introduction, strong ergodic properties of the thermodynamic limit. However, those invariants lead to the non uniqueness of the macroscopic state. In fact, every $T$ invariant measure on $D$ may be used in the construction of a $U_t$ invariant measure on $Y$ .

5. Bernoulli property, an open question

Whether the system $(Y, \mu_{\tilde{\rho},\nu}, U_t)$ is always a Bernoulli flow is still an open question. In the less interesting case one may prove:

Proposition: The dynamical system $(Y, \mu_{\tilde{\rho},\nu}, U_t)$ (as above) for which

$$\int v \; \nu(dv) \neq 0$$

is a Bernoulli flow.

The detailed proof will not be given here, let us remark however that

$$\int v \; \nu(dv) = \int m(1,\cdot) \; d\mu_{\tilde{\rho},\nu}$$

and in the above system the "random walk" described by $m(t,x)$ is transient. It is not difficult to show that the partition

$$\xi_o^1 \otimes \left( \bigvee_{i=1}^{m(1,x)} \alpha_i \right)$$

generates and is weakly Bernoulli under $U_1$ .

In the symmetrical case,

$$\int m(1,\cdot) \; d\mu_{\tilde{\rho},\nu} = 0 \; ,$$

the partition (of finite entropy) generated by $m(1,\cdot)$ and $\alpha_1$ is easily seen not to be weakly Bernoulli. While this does not exclude the possibility of its being very weakly Bernoulli, it points to a longer memory of the system with respect to "natural" local partitions, which results from the recurring property of the symmetrical random shift.

The question of the Bernoulli property in such situations (which occur in other systems as well) may be clarified if one would understand the properties of the skew product of a Bernoulli shift $(\bar{X},T)$ with itself

$$\bar{Y} = \bar{X} \otimes \bar{X} \qquad \text{(with the product measure)}$$

$$U(x_1,x_2) = (Tx_1, T^{n(x_1)} x_2)$$

where $E(n(\cdot)) = 0$ and $n(x) = \pm 1$ defines a generating Bernoulli partition of $(\bar{X},T)$.

Such a system is K, by a proof similar to that given in sec. 3, and differs from $(Y, \mu_{\underset{\sim}{\rho},\nu}, U_t)$ (perhaps not significantly) in that $n(\cdot)$ does not have an infinitely divisible distribution as $m(1,\cdot)$ does. The question of the Bernoulli property for $(\bar{Y},T)$ is still open.

## 6. Some Concluding Remarks on Ergodic Properties of Infinite Systems of Interacting Particles

Within the framework in which we have been working, probabilistic description of quasi local observables of infinite systems, the ergodic properties depend on the mechanism of dissipation of local information. In general, one may expect two types of such mechanism to be present: one by which the local information "wanders" off to infinity and that of local dissipation.

In the two examples which were studied here the local mechanism was essentially absent. This vague statement may be supported by the fact that the corresponding finite systems have poor ergodic properties, in particular their Kolmogorov-entropy is zero. Yet, these examples may be helpful for the understanding of the role played by the first mechanism. While in the ideal gas local information is steadily flowing to infinity, the interactions which we considered cause some of it to "wander" off to infinity in a "random walk" fashion. In this sense there is a similarity between the system of hard rods and the mixture which was considered in this chapter. In both the time evolution may formally be described as carried in two steps: first a simple steady flow, like the time evolution of the ideal gas, and then a shift in a random direction. In the mixture of several components the shift acted on a separate factor (i.e. the space of colors) while in the system of hard rods it was represented by jumps of the "origin" in the reduced description. In a more general interaction (like the one obtained in a mixture of particles of different masses) the information will also get locally dissipated or "spread" over neighboring particles.

In both systems which we have considered this mechanism leads to the K-property but posses somewhat similar difficulties in deciding if the systems are Bernoulli. The conditions which we imposed on the volocity distribution in the h.r.s. in order to obtain the Bernoulli property amount to requiring that the above "random walk" of local information be transient. In this respect it is worth mentioning that the "random walk systems" considered in [16] which also have a similar mechanism, and may be constructed in different dimensions, are Bernoulli in dimensions $\geq 3$ in which the random walk becomes transient.

For systems in which the space translations have good ergodic properties with respect to local partitions (which might be the general case for systems with interactions of finite range, [11]) the existence of a generating partition constructed on local observations may be an indicator of the strong presence of the first mechanism of dissipation of the local information. Although the examples which have been studied are not sufficiently general, one is tempted to speculate that such systems will have the K-property (or even the Bernoulli property in three dimensional systems).

With respect to the abstract ergodic properties of the time evolution, the escape to infinity of local information may screen the existence of a local mechanism of dissipation which might be even more interesting for the explanation of "good thermodynamical behavior" of the system. In this respect (and in face of the isomorphism theorem of Bernoulli systems) it may be more rewarding, from the physical point of view, to study the ergodic properties of the time evolution with respect to strictly local partitions or, as was suggested by S. Goldstein, to study the ergodic properties of the combined group of space and time translations, see [15,17].

## V . On Stability of Equilibrium States

### 1. Introduction

Physical systems may have many stationary states. These are described by measures on the phase space which are invariant under the time evolution. However, it has been widely accepted, following the founders of statistical mechanics, that equilibrium phenomena of a large system can be described by assuming that it is in one of its thermodynamical equilibrium (Gibbs) states (these are parametrised by only few macroscopic quantities such as energy per particle, density, etc.).

For large systems the equilibrium states were shown [22,43] to have maximal entropy under the proper conditions and it was heuristically argued that small perturbations, due to interactions with the environment, will bring a dynamical system close to equilibrium. This suggests another, dynamical, characterization of the equilibrium states as those which are stable under small local perturbations of the dynamics.

In the case of infinite quantum systems, Haag, Kastler and Trych-Pohlmeyer (HKP) showed [18] that the equilibrium (K.M.S.) states may indeed be characterized by a condition of "stability" under arbitrary local perturbation. Motivated by this result, we discuss the applicability of stability to the characterization of equilibrium states of finite and infinite classical systems. The argument of HKP may be adapted to prove a similar result for infinite classical systems [1]. For the ideal gas however we obtain a positive result under weaker assumptions than those used in the general case.

## 2. Equilibrium States as Special Stationary States

Macroscopic states of physical systems are assumed, in statistical mechanics, to be given by probability measures ($\omega$) on the phase space ($\Gamma$ ) appropriate to the system. To describe a stationary state the measure must be invariant under the time evolution. Since the energy (Hamiltionian) H of a finite system of particles is always a constant of the motion, a measure whose density with respect to the Liouville measure (which is invariant under any Hamiltonian dynamics) is given by a function of energy will always be stationary. A state of this form is completely characterized by the distribution of energy. In the heuristic justification of the validity of thermodynamical description for large systems it is usually assumed that equilibrium states are of the above type.

If the time evolution is ergodic* on almost all the energy surfaces, equipped with their natural (microcanonical) measures, then, indeed, any non singular (with respect to the Liouville surface measure) stationary state is an equilibrium state in the above sense. However, if the system possesses additional 'smooth' constants of the

* This application of ergodicity should not be confused with the different question of justification of the use of ensembles.

motion then there will also be stationary states whose densities depend on those invariants. This is the case with integrable systems; for example, the ideal gas is a box (sec.(II.5)) where the individual energies are invariant, or the one particle system of an elastic ball constrained to move in a round disc, where both the energy and the angular momentum are constants of the motion.

It will be shown here that, in the generic case, among the stationary states of a finite system only the equilibrium states (in the above sense) are stable under small perturbations of the dynamics.

As we have seen in the infinite ideal gas (sec.II ), the non uniqueness of a stationary state with given density and energy per particle may persist in infinite systems, even if one requires the state to be translationally invariant and have good clustering (i.e., mixing under space translations) properties. The quasi local states of infinite systems are suitable for description of local phenomena for which the surrounding gas acts as heat and particle reservoir. Thus one would expect, and indeed it was shown (Lanford [28]), that different (pure*) equilibrium ensembles (in the above sense) of a large system would produce, in the thermodynamical limit, the Gibbs grand canonical ensemble for strictly local observables (with proper boundary dependence). Accordingly, stability under local perturbations may be expected, in the case of infinite systems, to single out the Gibbs states. This will be demonstrated

---

* i.e. for which the distribution of energy is concentrated around a single value.

here for the ideal gas.

3. Stability Conditions

The notion of stability which we wish to use is similar to that used by HKP and may be described roughly as follows: Let $\omega^{o}$ be the stationary state given by the function $f = f(H)$. If we perturb $H$ slightly to obtain a new Hamiltonian $H_\lambda = H + \lambda h$, we obtain a new time evolution $T_t^{\lambda h}$ ($\equiv T_t^{\lambda}$) for which there exists a measure $\omega^{\lambda h}$ ($\equiv \omega^{\lambda}$) (given by the function $f(H + \lambda h)$) which is stationary under $T_t^{\lambda}$ and "close" to $\omega^{o}$. We will say that any state $\omega$ stationary under $T_t$ is stable if there exists such a family $\omega^{\lambda h}$ which is close to $\omega$ for all (sufficiently nice) perturbations h. A state $\omega$ which fails to be stable in this sense should not be regarded as "physical" because an arbitrarily small error in our knowledge of H could imply that $\omega$ does not even approximate a state stationary under the actual Hamiltonian time evolution.

To obtain a precise formulation of stability we must decide exactly how $\omega^{\lambda}$ is to be close to $\omega$. Since the only use of the measure (or ensemble) is to obtain expectation values of physical observables, i.e. of functions $A(x)$, which (by the very nature of physical observations) may be assumed to be smooth functions of $x, x \in \Gamma$, closeness should refer to such expectation values. We shall write $\omega(A)$ and $\omega^{\lambda}(A)$ for the expectation value of A, with respect to the measures $\omega$ and $\omega^{\lambda}$, and will assume throughout that H and all perturbations are $\in C^{2}(\Gamma)$ and that h is bounded. Some possibilities are:

i) $\omega^\lambda \to \omega$ in norm, i.e.

$$|\omega^\lambda (A) - \omega (A)| \leq \epsilon(\lambda) \|A\|,$$

where $\lim_{\lambda \to 0} \epsilon(\lambda) = 0$, $A \in C(\Gamma)$, the bounded continuous functions on the phase space $\Gamma$ of the finite system, and $\|A\| = \sup_{x \in \Gamma} |A(x)|$,

ii) $\omega^\lambda \xrightarrow[\lambda \to 0]{} \omega$ weakly, i.e. $\omega^\lambda (A) \xrightarrow[\lambda \to 0]{} \omega (A)$ for all $A \in C(\Gamma)$.

Clearly i) implies ii). It is also worth noting that there is a natural dynamical formulation of stability which is equivalent to i)

i') $T_t^{\lambda h} \omega$ remains close (in norm) to $\omega$ uniformly in t, for any perturbation h, when $\lambda$, is sufficiently small, i.e.,

$$|\omega(T_t^\lambda A) - \omega (A)| < \epsilon(\lambda) \|A\|$$

for all $A \in C(\Gamma)$ and all t.

To prove equivalence we note that i') follows from i) because

$$|\omega(T_t^\lambda A) - \omega(A)| \leq |\omega(T_t^\lambda A) - \omega^\lambda(T_t^\lambda A)| + |\omega^\lambda(A) - \omega(A)| \leq 2\,\epsilon(\lambda) \|A\|,$$

since $\omega^\lambda(T_t^\lambda A) = \omega^\lambda(A)$ by the stationarity of $\omega^\lambda$ under the perturbed evolution and $\|T_t^\lambda A\| = \|A\|$. Conversely, if i') holds we may construct $\omega^\lambda$ norm close to $\omega$ as a weak limit point of the time averages $\vec{\omega}_T^\lambda$ of the measures $T_t^\lambda\, \omega$ ($\vec{\omega}_T^\lambda = 1/T \int_0^T dt\, T_t^\lambda \omega$).

Condition i') may be called dynamical stability: Suppose a perturbation $\lambda h$ is added to H at some time, say $t = 0$, then $\omega$ will change with time for $t > 0$. If however $\omega$ satisfies i') and $\lambda$ is small then the expectation values of physical observables will also be changed only slightly even after very long times. (This remains true also if the initial state is not exactly $\omega$ but some state $\omega'$ which is close to $\omega$ in norm.)

These conditions have quantum counterparts: one replaces $C(\Gamma)$ in the above by the C*-algebra $B(\mathcal{H})$ of bounded operators on the Hilbert space $\mathcal{H}$ corresponding to the finite quantum system - of a finite number of particles in a finite volume. $\omega$ and $\omega^\lambda$ correspond to normal states on $B(\mathcal{H})$ (i.e., positive linear functionals $\tilde{\omega}$ of the form $A \to \mathrm{tr}\,(A\rho)$, $A \in B(\mathcal{H})$, where $\rho \in B(\mathcal{H})$ is positive and $\mathrm{tr}(\rho) = 1$) which are invariant under the one-parameter groups $T_t$ and $T_t^{\lambda h}$ generated by the Hamiltonians $H$ and $H + \lambda h$, $h \in B(\mathcal{H})$, respectively. For finite systems $H$ has discrete spectrum and corresponding to states of the form $f(H)$ for classical systems one has the invariant states given by $\rho = f(H)$ (e.g., $\rho = e^{-\beta H} / \mathrm{Tr}\, e^{-\beta H}$) for quantum systems.

For both the classical and quantum finite systems, a state given by a (reasonable) function $f(H)$ will satisfy i) and ii) and thus, also i'). In the quantum case a state is stationary if and only if $[\rho, H]$ $(\equiv \rho H - H\rho) = 0$, so that if $H$ has nondegenerate spectrum $\rho$ must clearly be of the desired form. Even if $H$ is degenerate the restriction of $\rho$ to each energy level must still be the identity if ii) is to be satisfied, since any splitting of an energy level may be achieved by the appropriate choice of perturbation [18]. In the classical situation we need stronger conditions than i) and ii) to obtain a general result. Before introducing such a condition, in sec. 5, we shall, in the next section, investigate some consequences which follow solely from the "weak stability" condition ii).

## 4. Some Consequences of the Weak Stability (Finite Systems)

Stability ii), which is the weakest condition mentioned, has already strong implications. We study them first on the finite systems, which are the subject of this and the following section.

Proposition 1: Let $\omega$ be weakly stable under the perturbation h as in ii), i.e. there exists a collection $\omega^{\lambda h}$ of states invariant under the dynamics generated by $H + \lambda h$ which converge weakly to $\omega$. Then $\omega^{\lambda h}(Q)$ is differentiable at $\lambda = 0$ on observables of the form $Q = \{H,B\}$ (the Poisson Bracket (P.B.) of H with B) for some $B \in C_o^1(\Gamma)$ ($C^1$ functions of compact support)[9] and

$$\frac{d}{d\lambda}\,\omega^{\lambda h}\,(\{H,B\})\Big|_{\lambda=0} = -\omega(\{h,B\}). \tag{3.1}$$

In particular if B is a constant of the motion, $\{H,B\} = 0$, then

$$\omega\,(\{h,B\}) = 0. \tag{3.2}$$

Proof: For any $B \in C_o^1(\Gamma)$ the perturbed states satisfy

$$0 = \frac{d}{dt}\,\omega^{\lambda h}(T_t^{\lambda}B)\Big|_{t=0} = \omega^{\lambda h}(\{H + \lambda h, B\}),$$

or

---

* This collection includes observables of the form $\alpha_t A - A$, for $t \in \mathbb{R}$ and $A \in C_o^1(\Gamma)$ (since $T_t A - A = \{H, \int_o^t T_u(A)\,du\}$) and is, therefore, dense on the orthogonal complement (in $L^2(\mu_\omega)$) of the measurable constants of the motion.

$$\frac{1}{\lambda}\,\omega^{\lambda h}(\{H,B\}) = -\omega^{\lambda h}(\{h,B\}).$$

The weak continuity of $\omega^{\lambda h}$ at $\lambda = 0$ implies therefore the existence of the limit

$$\lim_{\lambda \to 0} \frac{1}{\lambda}\,\omega^{\lambda h}(\{H,B\}) = -\lim_{\lambda \to 0} \omega^{\lambda h}(\{h,B\})$$

$$= -\omega(\{h,B\}).$$

Since, by stationarity,

$$\omega(\{H,B\}) = 0,.$$

the above limit is the weak derivative of $\omega^{\lambda h}$ on $Q = \{H,B\}$.

<u>Proposition 2:</u> If $\omega$ satisfies stability ii) and is given by a $C^1(\Gamma)$ density $\rho$, then

$$\{\rho,B\} = 0 \tag{4.3}$$

for any $B \in C_o^1(\Gamma)$ such that $\{H,B\} = 0$.

<u>Proof</u>: By Proposition 1 $\{H,B\} = 0$ implies $\omega(\{h,B\}) = 0$ for any $h \in C_o^2$. In terms of $\rho$ we thus have, using well known properties of the P.B.,

$$0 = \int dx\, \rho\{h,B\} =$$
$$= \int dx\, \{\rho h,B\} - \int dx\, h\{\rho,B\} =$$
$$= -\int dx\, h\{\rho,B\}.$$

Since h is arbitrary this implies (4.3).

We have thus obtained a simple condition on $\omega$, (4.2) and (4.3), necessary for stability ii).

The above arguments can be reproduced for quantum systems, with the understanding that $\{\ ,\ \}$ stands for the commutator. According to (4.3) a state of a quantum system, given by a density operator $\rho$, is stable ii) only if $\rho$ commutes with all operators which commute with the Hamiltonian H. Since H has discrete spectrum it follows simply that $\rho$ is a function of H.

No such general result can be expected for classical systems as may be seen be considering integrable systems for which the Kolmogorov-Arnold-Moser (KAM) theorem [35,4] is applicable. It can be shown, see remark at end of sec. 4, that for such systems even the stronger stability condition i) is not sufficient to insure the desired result $\rho = f(H)$.

The difference between classical and quantum systems appears to be due to the lack of a sufficient number of global constants of the motion in the classical case. This prevents fuller exploitation of Proposition 2 whose usefulness depends on the existence of an abundance of invariants. Even integrable systems, if they satisfy the conditions of the KAM theorem, have only a "limited" number of such constants (i.e. n constants when $\Gamma$ is a 2n dimensional space). This shows up in the requirement for the KAM theorem that the frequencies be incommensurable[4] which reduces the number of smooth invariants, e.g., for two uncoupled oscillators there exists a function of the two phases which is a (smooth) invariant iff the frequencies are commensurable. Indeed, we shall now prove that in the extreme case of a periodic system weak stability alone implies that $\rho = f(H)$. We shall consider this case explicitly, despite its limited applicability, to illustrate the method used in the next section for more "typical" systems.

Proposition 3: Let $\omega$ be a state of a periodic system, given by a $C^1(\Gamma)$ density $\rho$. If $\omega$ is weakly stable (i.e. satisfies stability ii)) then locally (away from fixed points) $\rho$ is a function of H, i.e.

$$\text{grad } \rho \text{ is parallel to grad } H. \tag{4.4}$$

Proof: Denote by $\tau$ the period of the system. Then, for any $A \in C_0^1$,

$$\overline{A}(x) = \int_0^\tau dt\, A(T_t x)$$

is a constant of the motion. (Since $\{H,\overline{A}\} = \int_0^\tau dt\, \{H, T_t A\} = \int_0^\tau dt\, \frac{d}{dt}(T_t A)$ $T_\tau A - A = 0$.) Proposition 2 now implies that

$$0 = \{\rho, \overline{A}\} = \{\rho, \int_0^\tau dt\, T_t A\} = \int_0^\tau dt\, \{\rho, T_t A\} = \int_0^\tau dt\, T_t\{\rho, A\}, \tag{4.5}$$

where we have used the invariance of $\rho$ under $T_t$. Assume now that grad $\rho$

is not parallel to grad H at some point x. One could then find on observable A, with support in a neighborhood of x, in which $\{\rho, A\} > 0$ along the orbit of x. This would contradict (4.5).

The typical (generic) integrable system is not periodic. Nevertheless its periodic points are dense in the phase space [4]. In the next section we show how to obtain a positive result for such systems at the price of imposing a somewhat stronger, and not so physical, requirement of stability on the equilibrium states.

## 5. A Stronger Stability Condition

As we have seen in propositions 1 and 2, the weak stability of a state $\omega$ enables one to define, for each smooth perturbation h of compact support, a functional $L_h$, in whose domain are observables of the form $Q = \{H, B\}$, by

$$L_h (\{H,B\}) = - \omega (\{h,B\}).$$

$L_h$ was shown there to be the weak derivative of the perturbed states $\omega^{\lambda h}$.

<u>Definition</u>: A state $\omega$ satisfies stability iii) if it is weakly stable and if, for each $h \in C_0^2$, the functional $L_h$ is given by a $C^2(\Gamma)$ function $f_h$, i.e.

$$L_h (\{H,B\}) = \int dx\, f_h(x) \{H,B\}.$$

When $\omega$ has a density $\rho$ then

$$\int dx\, f_h \{H,B\} = - \int dx\, \rho \{h,B\}.$$

This gives after integration by parts, assuming $\rho \in C^1(\Gamma)$,

$$-\int dx\ B\ \{H,f_h\} = \int dx\ B\ \{h,\rho\}.$$

Since this holds for, essentially, any B it implies

$$-\{H,f_h\} = \{h,\rho\}. \tag{5.1}$$

Thus, for states given by a density, stability iii) implies that for each perturbation h, there exists a $C^1(\Gamma)$ function $f_h$ which satisfies (4.1). This condition is satisfied by $\rho$ of the desired form, i.e. $\rho = f(H)$, $f \in C^1$, since

$$\{h,\rho\} = \{h,f(H)\} = f'(H)\ \{h,H\} = \{f'(H)\ h,H\}$$

and one may choose $f_h = f'(H)\ h$.

We will now show that in the generic case, the converse of the above statement is also true.

<u>Proposition 4</u>: Let $\omega$ satisfy stability iii) and be given by a $C^1$ density $\rho$. If periodic orbits (under $T_t$) are dense in $\Gamma$ and if the energy surfaces $S_E$ are connected then $\rho$ is a function of H.

<u>Proof</u>: Let $y \in \Gamma$ be a periodic point with period $\tau$. By stability iii), there corresponds to each $h \in C_0^2$ a $C^1(\Gamma)$ function $f_h$ such that

$$\{\rho,h\} = \{H,f_h\}.$$

Therefore, using the periodicity of the orbit through y, we obtain

$$\int_0^\tau du\ \{\rho,h\}(T_u y) = \int_0^\tau du\ \{H,f_h\}(T_u y) =$$

$$\int_0^\tau du \left(\frac{d}{du} f_h(T_u y)\right) = f_h(T_u y) - f_h(y) = 0$$

for any $h \in C_0^2$. By the same argument as in the proof of Proposition 3, we conclude that grad $\rho$ is parallel to grad H at y.

Since the periodic points are dense, the gradients of $\rho$ and of H are parallel everywhere. The connectedness of energy surfaces now implies that $\rho$ is a function of H.

Remark: The assumptions made in Proposition 4 cannot easily be weakened as may be seen by considering stability in integrable systems to which the KAM theorem is applicable [35,4]. (The ideal gas in a torus is such a system.) In these systems the phase space is decomposable into invariant (under $T_t$) tori "most" of which are stable under small (sufficiently smooth) perturbations h: i.e., except for a family of tori of total measure $\epsilon(\lambda)$, there corresponds to each $T_t$ - invariant torus M a uniformly close $T_t^{\lambda h}$ - invariant torus $M^\lambda$ (on which the $T_t^{\lambda h}$ time evolution uniformly approximates the $T_t$ evolution on M). Here $\epsilon(\lambda) \to 0$ as $\lambda \to 0$ and $M^\lambda$ is "differentiably close" to M. Hence for any $T_t$-stationary measure which is given by a smooth "function of the invariant tori" (i.e., a function of the "action variables" parameterizing the tori) one may use the correspondence $M \leftrightarrow M^\lambda$ to construct a $T_t^\lambda$-stationary measure $\omega^\lambda$ which is norm close to $\omega$ and even differentiably close. Thus, unless the use of perturbations to which KAM does not apply is allowed - in our argument h could be arbitrarily smooth - the Proposition will not hold if we replace in it stability iii) by stability ii) or even stability i). Stability iii), on the other hand, will rule out these cases because the derivative of $\omega^\lambda$ at $\lambda = 0$ may fail to be even a function and will certainly not be $C^1$. A positive result may however be possible if the $\omega^\lambda$ are required to be given by smooth functions, since this is almost certainly not the case for the $\omega^\lambda$ which can be constructed by the use of the KAM theorem.

## 6. Stable States of the Infinite Ideal Gas

We turn now to the application of stability conditions to states of a simple infinite system: the ideal gas. As explained in section 2, one may expect a stability condition to single out, among the "pure" stationary states, the Gibbs equilibrium states. This, indeed, will be shown.

The content of this section is independent of the previous two: the setup of an infinite system differs from the finite systems and the stability condition will be modified, essentially by restricting the perturbations to be local. However, there will be some similarities in the approach and in the results.

Notation: We will denote by $\Gamma = R^d \otimes R^d$ the one particle phase space and by $\alpha_t$ the free time evolution on it (generated by $H_1 = \frac{1}{2}m\, |p_1|^2$). Correspondingly:

$$\Gamma^n = \underbrace{\Gamma \otimes \Gamma \otimes \ldots \otimes \Gamma}_{n}$$

$$\alpha_t^n = \alpha_t \otimes \ldots \otimes \alpha_t$$

and $H_n((x)_n) = \sum_{i=1}^{n} H_1(x_i)$.

$K_n^i$ is the space of $C^i$ symmetric functions on $\Gamma^n$ with compact support and $K^i = \overset{\infty}{\underset{i=1}{\oplus}} K_n^i$ .

The phase space of the infinite system, X, is the collection of locally finite configurations in $\Gamma$. For $f_n \in K_n^o, \Sigma f_n$ will denote the function

$$(\Sigma f_n): X \to R$$

$$(\Sigma f_n)(\{x_i\}) = \sum_{(i_1 < i_2 \ldots < i_n)} f_n(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) .$$

For $v \in K^2$, $H + \Sigma v$ (with $H = \Sigma H_1$) is the generator of the time evolution $S_t^{H+\Sigma v}$ on X, with respect to which

$$\frac{d}{dt} S_t^{H+\Sigma v} \Sigma f \Big|_{t=0} = \{H + \Sigma v, \Sigma f\} .$$

It is understood that the P.B. is to be taken with respect to only those particles which lie in the common support.

X is equipped with a $\sigma$-algebra $\mathcal{B}(X)$ described in sec. (II. 2) with respect to which functions in $\Sigma K^o$ are measurable. A state $\omega$ is a linear functional on $\Sigma K^o$ which corresponds to a measure $\mu_\omega$ on $\mathcal{B}(X)$:

$$\omega(\Sigma f) = \int_{\xi \in X} (\Sigma f)(\xi)\, \mu_\omega(d\xi)$$

If, for a state $\omega$, $\Sigma f$ is integrable for all $f \in K$, then

$$\hat{\rho}_\omega : K \to R,$$

$$\hat{\rho}_\omega(f) = \omega(\Sigma f)$$

defines a positive linear functional on K and hence a measure $\hat{\rho}_n$ on $\Gamma^n$, $n = 1,2,\ldots$ . If all the $\hat{\rho}_n$ are absolutely continuous with respect to the Lebesgue measure, we have, for $n \in \mathbb{Z}_+$,

$$d\hat{\rho}_n = (1/n!)\rho_n\,(x_1,\ldots x_n)\;dx_1\ldots dx_n$$

for some "correlation functions" $\rho_n((x)_n)$ with

$$dx_i = dx_{i,1}\cdot\ldots\cdot dx_{i,d}\;d\rho_{i,1}\cdot\ldots\cdot d\rho_{i,d}\;.$$

Now, any state given by correlation functions of the form

$$\rho_n((x)_n) = \prod_{i=1}^{n} f(\underline{p}_i) \qquad f \in L'\,(R^d)$$

is invariant under $S_t (= S_t^H)$ and possesses the space clustering property:

$$\rho_n((q_1,p_1),\ldots,(q_n,p_n)) \to \rho_1((q_1,p_1)) \ldots \rho_1((q_n,p_n)) \text{ (weakly)}$$

$$\text{as } \min |q_i - q_j| \to \infty$$

One would like, however, to single out the Gibbs states $\omega_{\rho,\beta}$, characterized by correlation functions of the form $\rho_n((x)_n) = \prod_{i=1}^{n} \rho\, e^{-\beta H_1(x_i)}$, as those proper for the description of a system which is in equilibrium while subjected to weak local perturbations by the environment. The following is an adaptation of the weak stability condition to infinite systems, where it is reasonable to require it with respect to local perturbations only.

Definition: We call a state $\omega$, of an infinite system, weakly stable (under local perturbations) if for each (repulsive) local smooth perturbation $\Sigma v \geq 0$, $v \in K^2$ *, there exists a collection of states $\omega_\lambda$ (for $0 \leq \lambda < \Delta$ with some $\Delta > 0$) which:

1) are invariant under the perturbed dynamics "generated" by $H + \lambda\, \Sigma v$

2) converge weakly, on $\Sigma K$, to $\omega$

3) "relax" under the free time evolution, i.e. the limits

$$\lim_{t \to \pm\infty} \omega_\lambda(S_t\, (\Sigma f)) \quad , \quad f \in K$$

exist (in which case they are equal).

---

* We restrict the allowed perturbations in order to avoid a null definition: too attractive potentials will result in a collapse of the infinite system under which no state is stable. A more physical restriction would be to allow only the so called "stable potentials" [43]. Technically those restrictions would have no influence on the proof (in one direction) and would be ignored.

Notice that 3) is satisfied independently of 1) and 2) if the dynamical system $(X, \omega, S_t)$ is mixing (or even "Prosser-mixing", see sec.(I.4)) and $\omega_\lambda$ are absolutely continuous with respect to $\omega$.

We will show now that, due to the transient character of the one particle time evolution, the ideal gas possesses a large class of "canonical" transformations which commute with the free time evolution. As we have seen in the previous sections, for such systems the weak stability is already a strong condition. Although the "generators" of these transformations are no longer functions on the system's phase space, they still satisfy similar equations as far as their P.B. with strictly local observables are concerned.

Let's denote by $K_n^S$ the subspace of functions in $K_n^2$ whose support is bounded away from the fixed points of $\alpha_t^n$ (i.e. $\underline{p} = 0$). For $g_n \in K_n^S$ denote

$$\hat{g}_n((x)_n) = \int_{-\infty}^{\infty} dt\, g_n(\alpha_t^n (x)_n).$$

It is easy to see that $\hat{g}_n((x)_n)$ is a finite $C^2$ function of $\Gamma^n_{\text{sym.}}$, although not of compact support. Further, it has the following properties:

1) $\hat{g}((x)_n)$ is invariant under $\alpha_t^n$ .

2) The functions $\hat{g}((x)_n)$ obtained in the above way may locally (away from fixed points of $\alpha_t^n$) be any constants of the motion.

To see this, notice that (for the I.G.) if $g_n \in K_n^S$ and if $f_n \in C^2(\Gamma^n_{sym.})$ satisfies $\{f_n, H_n\} = 0$ along the orbit of the support of $g_n$ then

$$f_n \cdot g_n \in K_n^S \text{ and } \widehat{f_n \cdot g_n} = f_n \, \hat{g}_n \; .$$

3) For any $v_n \in K_n^1$ and $g_m \in K_m^S$, the formal P.B.

$$\{\Sigma v, \Sigma \hat{g}\} = \int_{-\infty}^{\infty} dt \, \{\Sigma v, S_t \, \Sigma g\}$$

is given by a finite function on X. Further, it satisfies the usual rule if one applies the same convention as in $\{\Sigma v, H\}$:

$$\{\Sigma v_m, \Sigma \hat{g}_n\} = \sum_{K=1}^{m \wedge n} \frac{(m+n-K)!}{K!(m-K)!(n-K)!} \; \Sigma P \, \{v_m((x)_K, (y)_{m-K}), \hat{g}_n((x)_K (Z)_{n-K})\}$$

(P is the normalized symmetrization operator). Notice that the summands are functions of compact support.

Using techniques reminiscent of propositions 1 and 2, we shall obtain now useful conditions necessary for the weak stability of a state $\varpi$.

Proposition 5: Let $\varpi$ be a weakly stable state (of the I.G.). Then, for each $v \in K^2$ and $g_n \in K_n^S$,

$$\omega(\{\Sigma v_m, \Sigma\hat{g}\}) = 0$$

Proof: By stability with respect to $\Sigma v$:

$$0 = \omega_{\lambda v}(\{H + \lambda\Sigma v,\ S_t\ \Sigma g\}).$$

Thus

$$-\frac{1}{\lambda}\int_{-T}^{T} \omega_{\lambda v}\ (\{H, S_t\ \Sigma g\})\ dt = \int_{-T}^{T} \omega_{\lambda v}\ (\Sigma v, \Sigma\alpha_t^{(n)} g)\ dt$$

or

$$-\frac{1}{\lambda}\left[\omega_{\lambda v}\ (S_T g) - \omega_{\lambda v}\ (S_{-T} g)\right] = \omega_{\lambda v}\ (\Sigma v, \Sigma\int_{-T}^{T}\alpha_t^{(n)} g\ dt\})$$

By the "relaxation" property, this converges to

$$0 = \omega_{\lambda v}\ (\{\Sigma v, \Sigma\hat{g}\}).$$

By a previous remark we may apply now the weak convergence of $\omega_\lambda$ to obtain:

$$\omega(\{\Sigma v, \Sigma\hat{g}\}) = 0.$$

Lemma 6: Let $\omega$ be a weakly stable state of the I.G.. If $\omega$ possesses $C^1$ correlation functions then, locally, these are functions of the energy.

Proof: It is sufficient to show that, for each $n \in \mathbb{Z}_+$, grad $\rho_n$ is parallel to grad $H_n$. To show this, it is enough to prove that whenever $\{f_n, H_n\}((\bar{x})_n) = 0$ at some $(\bar{x})_n \in \Gamma^n$, then $\{f_n, \rho_n\}((\bar{x})_n) = 0$. By a previous remark, this certainly follows from the following claim:

If $K \leq m$ and $g_K \in K_n^S$ then

$$(*) \qquad \{\rho_n(x_1,\ldots,x_n),\ \hat{g}(x_1,\ldots,x_K)\} = 0\ .$$

We shall now prove $(*)$, under the assumptions of the Lemma, by induction on K.

1) Let $n \geq 1$. Since $\omega$ is a weakly stable state, proposition 5 implies that, for each $v_n \in K_n^2$ and $g_1 \in K_1^S$ :

$$\omega(\{\Sigma v_n, \Sigma \hat{g}_1\}) = 0.$$

Expressing it in terms of the n-th correlation function we obtain:

$$0 = \int d(x)_n\ \rho_n((x)_n)\ \{v_n((x)_n),\ \hat{g}(x_1)\} =$$

$$= -\int d(x)_n\ v_n \{\rho_n((x)_n),\ \hat{g}(x_1)\}.$$

Since the only essential restriction on $v_n$ is its symmetry, we obtain

$$0 = P\{\rho_n((x)_n),\ \hat{g}(x_i)\} = \frac{1}{n}\sum_{i=1}^{n} \{\rho_n((x)_n),\ \hat{g}(x_i)\}$$

If all the momenta of $(\bar{x})_n$ are different (which implies $\alpha_t^1 x_i \neq x_j\ \forall t,\ i \neq j$) then $\hat{g}(\bar{x}_i)$ is independent from $\hat{g}(\bar{x}_j)$; therefore

$$\{\rho_n((x)_n)\ ,\ \hat{g}(x_i)\} = 0 \qquad 1 \leq i \leq n.$$

By continuity, this holds for any $x \in \Gamma^n$.

2) For a given $m > 1$ assume (*) to hold for each n and $K < m$ and let $g_m \in K_m^S$. For any $v_n \in K_n^2$

$$0 = \omega(\Sigma v_n, \Sigma \hat{g}_m) =$$

$$= \sum_{k=1}^{m} \frac{1}{k!(m-k)!(n-k)!} \int d(x)_k\, d(y)_{n-k}\, d(Z)_{m-k}\, \rho_{m+n-k}((x)_k,(y)_{n-k},(Z)_{m-k})$$

$$\{v_n((x)_k,\ (y)_{n-k}),\ \hat{g}_m((x)_k,\ (Z)_{m-k})\}$$

$$= \sum_{k=1}^{m} \frac{1}{k!(m-k)!(n-k)!} \int d(x)_k\, d(y)_{n-k}\, d(Z)_{m-k}\, v_n\, ((x)_k,\ (y)_{n-k})$$

$$\{\rho_{m+n-k}\ ((x)_k,\ (y)_{n-k},\ (Z)_{m-k}),\ \hat{g}_m\ ((x)_k,\ (Z)_{m-k})\}_{(x)}$$

The subscript on $\{,\}_{(x)}$ indicates that the P.B. is to be taken with respect to $(x)_k$ only. By the induction assumption the P.B. vanishes for $k < m$, leaving us with

$$0 = \int d(x)_m \, d(y)_{n-m} \, v_n((x)_m, (y)_{n-m}) \, \{\rho_n((x)_m, (y)_{n-m}), \hat{g}_m((x)_m)\}$$

The arbitrariness of $v_n$ and the continuity of $\{\rho,\hat{g}\}$ imply (as in 1))

$$\{\rho_n((x)_m, (y)_{n-m}), \hat{g}_m((x)_m)\} = 0.$$

The above claim (*) follows now by induction.

Since the energy surfaces in $\Gamma^n$ are connected, with the exception of $\Gamma'$ in one dimensional systems, it follows that the correlation functions are globally functions of the energy only, with the possible exception of $\rho_1$, which in one dimension may depend on $\underline{P}$ (rather than on $|P|$).

The stability condition which we are considering is linear in $\omega$, therefore it is clear that it does not single out Gibbs equilibrium states. However, we will see now that among the "pure" states they are, indeed, characterized by stability.

<u>Proposition 7:</u> A state, $\omega$, of the ideal gas which:

1) possesses $C^1$ correlation functions

2) is weakly stable under $V = \Sigma(K_1^2 \cup K_2^2)$ (smooth local external perturbations and (repulsive) local pair interactions)

and:3) is ergodic with respect to the time evolution (equivalently, space translations)

is a Gibbs state.

<u>Proof:</u> Let us use the "purity" of the state $\omega$ combined with a result of the previous lemma, to obtain a relation between correlation functions of different orders.

Ergodicity with respect to the time evolution implies,

$\forall\ v_n \in K_n^0\ f_1 \in K_1^0$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dt\ \omega(\Sigma v_n\ \alpha_t\ \Sigma f_1) = \omega(\Sigma v_n)\ \omega(\Sigma f_1)$$

Or, in terms of correlation functions

$$\lim_{T\to\infty} 1/T \int_0^T dt \Big\{ 1/n! \int d\ (x)_n\ n v_n((x)_n)\ \alpha_t^1\ f_1(x_n)$$

$$+ 1/(n+1)! \int d(x)_{n+1}\ \rho_{n+1}((x)_{n+1})\ (n+1)\ v_n((x)_n)\ \alpha_t^1\ f_1(x_{n+1})\Big\} =$$

$$= 1/n! \int d(x)_n\ \rho_n((x)_n)\ v_n((x)_n) \cdot \int dy\ \rho_1(y)\ f_1(y)$$

Notice that, since the supports of $v_n$ and $\alpha_t^1\ f_1$ separate, the first integral vanishes as $t \to \infty$ (bounded convergence th.). Moreover, by a result of lemma 6

$$\rho_m((x)_m) = \rho_m(x_1,\ldots,\alpha_t x_m) \qquad m \in Z^+ , \ t \in \mathbb{R}$$

implying that the second integral is independent of t. Therefore, for any $n \in Z_+$,

$$\rho_{n+1}((x)_{n+1}) = \rho_n((x)_n)\,\rho_1(x_{n+1}) \quad ,$$

which implies

$$\rho_n((x)_n) = \prod_i \rho_i(x_i) \ .$$

(The same result would follow from ergodicity of $\omega$ with respect to space translations.)

Now, by the previous lemma

$$\{\rho_1(x), P_\alpha\} = 0 \qquad \text{for } 1 \leq \alpha \leq d \quad \text{(d-the dimensionality of the system)}$$

from which it follows that $\rho_1(x) = \varphi_1(\underline{p})$.

The energy surfaces on $\Gamma^2$ are connected (which is not the case with $\Gamma^1$ for $d = 1$), so Lemma 6 implies that

$$\rho_2(x_1,x_2) = \varphi_2(|p_1|^2 + |p_2|^2) \quad *$$

Combining these results:

$$\varphi_1(\underline{p}_1)\,\varphi_1(\underline{p}_2) = \varphi_2(|p_1|^2 + |p_2|^2)$$

which can be satisfied only if

$$\rho_1(x) = \rho\, e^{-\beta|p|^2} \quad \text{for some } \rho \geq 0 \text{ and } \beta > 0 \;;$$

$\beta > 0$ since the corresponding measure on X is concentrated on locally finite configurations. Therefore

---

* Alternatively, one could argue that (by lemma 6)

$$\{\ell,\ \rho_2(x_1,x_2)\} = 0$$

for any component of the "mixed angular momentum" in $\Gamma^n$, i.e.

$$\ell = p_{i_1,\alpha_1}\, q_{i_2,\alpha_2} - p_{i_2,\alpha_2}\, q_{i_1,\alpha_1}$$

with some $1 \leq i_1, i_2 \leq 2$ and $1 \leq \alpha_1, \alpha_2 \leq d$.

However, $\rho_2 = \rho_2(p_1,p_2)$ (as above) and we have just concluded that $\rho_2$ is invariant under the canonical transformation generated by $\ell$. In the product momentum space this is a rotation which for $i_1 = i_2$ corresponds to a change in the direction of motion of one particle and for $i_1 \neq i_2$ corresponds to a coupled change of the momenta of the two particles, which preserves the total energy. Since $\rho_2(p_1,p_2)$ is invariant under a general rotation in the product momentum space, it is of the form

$$\rho_2(x_1,x_2) = \varphi(p_1^2 + p_2^2).$$

$$\rho_n((x)_n) = \prod_{i=1}^{n} \rho\, e^{-\beta |p_1|^2}$$

and $\omega$ is a Gibbs state.

Q.E.D.

With respect to the converse of this proposition, it is easy to see that the Gibbs states are stable under local perturbations given by "stable" potentials [43]; i.e. those for which

$$(\Sigma v)\ (x_1, \ldots, x_n) \geq -nB \text{ , with some } B > 0.$$

For these $e^{-\beta\lambda\Sigma_v} \in L^1\ (\omega)$ and the states $\omega_\lambda(\cdot) = \omega(e^{-\beta\lambda\Sigma_v}\ \cdot\ )$ satisfy the requirements.

References


[1] M. Aizenman, G. Gallavotti, S. Goldstein and J.L. Lebowitz (in preparation).

[2] M. Aizenman, S. Goldstein and J.L. Lebowitz, "Ergodic Properties of a One-Dimensional System of Hard Rods with An Infinite Number of Degrees of Freedom", Commun. Math. Phys., (to appear).

[3] R.L. Adler and P.C. Shields, "Skew products of Bernoulli shifts with rotations", Israel J. Math. (to appear).

[4] V.I. Arnold and A. Avez, Ergodic Problems of Classical Mechanics, Benjamin (1968).

[5] P. Billingsley, Ergodic Theory and Information, Wiley, New York (1965).

[6] L. Breiman, Probability, Addison-Wesley, Reading, Massachusetts (1968).

[7] R.L. Dobrushin, "Gibbsian probability field for lattice systems with pair interactions", Funct. Anal. Appl. 2, 292 (1968).

[8] R.L. Dobrushin, "The question of uniqueness of a Gibbsian probability field and problems of phase transition", Funct. Anal. Appl. 2, 302 (1968).

[9] G.G. Emch, Algebraic Methods in Statistical Mechanics and Quantum Field Theory, Wiley, New York (1972).

[10] G. Gallavotti, "Modern theory of the Billiard, An Introduction" (to appear).

[11] G. Gallavotti, "Ising model and Bernoulli schemes in one dimension", Commun. Math. Phys. 32, 183 (1973).

[12] G. Gallavotti, contribution in Dynamical Systems, Proceedings of the Fifth Battelle Rencontres 1974, Lecture notes, Springer-Verlag (to appear).

[13] H. Goldstein, Classical Mechanics, Addison-Wesley, Reading, Massachusetts (1950).

[14] S. Goldstein, Doctoral dissertation, Yeshiva University New York (1974).

[15] S. Goldstein, "Space-time properties of systems of infinitly many particles", Commun. Math. Phys., (to appear).

[16] S. Goldstein and J.L. Lebowitz, "Ergodic Properties of an Infinite System of Particles Moving Independently in a Periodic Field", Commun. Math. Phys. 37, 1 (1974).

[17] S. Goldstein, J.L. Lebowitz and M. Aizenman, "Ergodic Properties of Infinite Systems", contribution in Dynamical Systems, Proceedings of the Fifth Battelle Rencontres 1974, Lecture notes, Springer-Verlag (to appear).

[18] R. Haag, D. Kastler and E. Trych-Pohlmeyer, Comm. Math. Phys. (to appear).

[19] P.R. Halmos, Measure Theory, Van Nostrand Reinhold, New York (1950).

[20] P.R. Halmos, Lectures on Ergodic Theory, Chelsea, New York (1956).

[21] Y. Katznelson and B. Weiss, "Commuting measure preserving transformations", Israel J. Math. 12, 161 (1972).

[22] A.I. Khinchin, Mathematical Foundations of Statistical Mechanics, Dover, New York (1949).

[23] A.N. Kolmogorov, "A new metric invariant of transitive automorphisms and flows of Lebesgue spaces", Dokl. Akad. Nauk. SSSR 19, No. 5, 861-864 (1958).

[24] O.E.Lanford III, "The classical mechanics of one-dimensional systems of infinitely many particles; I. An existence theorem", Commun. Math. Phys. 9, 176 (1968).

[25] O.E. Lanford III, "The classical mechanics of one-dimensional systems of infinitely many particles; II. Kinetic theory", Commun. Math. Phys. 11, 257 (1969).

[26] O.E. Lanford III, "Ergodic theory and approach to equilibrium for finite and infinite systems", contribution in Boltzmann Equations (Theory & Applications) Proceeding. Symposium, Vienna, Sept. 1972, Springer-Verlag (1973).

[27] O.E. Lanford III, "Classical mechanics of systems of infinitely many particles", (to appear).

[28] O.E. Lanford III, Statistical Mechanics and Mathematical Problems, Proc. Battelle Rencontres 1971, Lecture Notes in Physics 20, Springer-Verlag (1973).

[29] O.E. Lanford III, contribution in Dynamical Systems, Proceedings of the Fifth Battelle Rencontres 1974, Lecture notes, Springer-Verlag (to appear).

[30] O.E. Lanford III and J.L. Lebowitz, "Time Evolution and Ergodic Properties of Harmonic Systems", contribution to Dynamical Systems, Proceedings of the Fifth Battelle Rencontres 1974, Lecture notes, Springer-Verlag (to appear)

[31] O.E. Lanford III, and D. Ruelle, "Observables at Infinitely and States with Short Range Correlations in Statistical Mechanics", Commun. Math. Phys. 13, 194 (1969).

[32] J.L. Lebowitz, "Hamiltonian Flows and Rigorous Results in Non-equilibrium Statistical Mechanics", contribution in Statistical Mechanics, New Concepts, New Problems, New Applications, Univ. of Chicago Pr. (1972).

[33] J.L. Lebowitz, M. Aizenman and S. Goldstein, "On The Stability of Equilibrium States of Finite Classical Systems", (to appear).

[34] J.L. Lebowitz and O. Penrose, "Modern Ergodic Theory", Physics Today, V.26, N.2 (1973).

[35] J. Moser, contribution in Dynamical Systems, Proceedings of the Fifth Battelle Rencontres 1974, Lecture Notes, Springer-Verlag (to appear).

[36] J. Moser, "Three Integrable Hamiltonian Systems Connected With Iso-spectral Deformations" (to appear).

[37] D.S. Ornstein, "The Isomorphism Theorem for Bernoulli Flows", Advances in Math. 10, 124 (1973).

[38] D.S. Ornstein, "Imbedding Bernoulli Shifts in Flows", Contributions to Ergodic Theory and Probability, Lecture Notes in Math., Springer-Verlag, Berlin, 178 (1970).

[39] D.S. Ornstein, Ergodic Theory, Randomness, and Dynamical Systems, Yale University Press, New Haven (1974).

[40] W. Parry, Entropy and Generators in Ergodic Theory, Benjamin, New York (1969).

[41] O. de Pazzis, "Ergodic properties of a semi-infinite hard rods system", Commun. Math. Phys. 22, 121 (1971).

[42] O. Penrose, Foundations of statistical mechanics, Pergamon, Oxford, (1970).

[43] D. Ruelle, Statistical Mechanics. Rigorous Results, Benjamin, New York (1967).

[44] D. Ruelle, contribution in Statistical Mechanics and Quantum Field Theory: Summer School of Theoretical Physics, Les Houches, 1970, Gordon and Breach, New York (1971).

[45] D. Ruelle "Superstable interactions in classical statistical mechanics", Commun. Math. Phys. 18, 127 (1970).

[46] V.A. Rohlin, "On the fundamental ideas of measure theory", Amer. Math. Soc. Transl. (1) 10, 1-54 (1962).

[47] V.A. Rohlin, "Selected topics in the metric theory of dynamical systems" Amer. Math. Soc. Transl. (2) 49, 171 (1966).

[48] P. Shields, The Theory of Bernoulli Shifts, University of Chicago Press (1973).

[49] Ya. G. Sinai, "Dynamical systems with elastic reflections", Russ. Math. Surveys 25, 137 (1970).

[50] Ya. G. Sinai, "Ergodic properties of a gas of one-dimensional hard rods with an infinite number of degrees of freedom", Funct. Anal. Appl. 6, 35 (1972).

[51] M. Smorodinsky, Ergodic Theory, Entropy, Springer Lecture Notes 214 (1970).

[52] K.L. Volkovysskii and Ya. G. Sinai, "Ergodic properties of an ideal gas with an infinite number of degrees of freedom", Funct. Anal. Appl. 5, 185 (1971).