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On stability of fixed points and chaos in fractional systems

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In this paper, we propose a method to calculate asymptotically period two sinks and define the range of stability of fixed points for a variety of discrete fractional systems of the order $0 < \alpha < 2$. The method is tested on various forms of fractional generalizations of the standard and logistic maps. Based on our analysis, we make a conjecture that chaos is impossible in the corresponding continuous fractional systems. *Published by AIP Publishing.* https://doi.org/10.1063/1.5016437

Many natural (biological, physical, etc.) and social systems possess power-law memory and can be described by the fractional differential/difference equations. Nonlinearity is an important property of these systems. Behavior of such systems can be very different from the behavior of the corresponding systems with no memory. Previous research on the issues of the first bifurcations and the stability of fractional systems mostly addressed the question of sufficient conditions. In this paper, we propose the equations that allow calculations of the coordinates of the asymptotically stable period two sinks and the values of nonlinearity and memory parameters defining the first bifurcation form the stable fixed points to the $T = 2$ sinks.

I. INTRODUCTION

It is generally understood that socioeconomic and biological systems are systems with memory. Specific analysis showing that the memory in financial and socioeconomic systems obeys the power law can be found in papers[1–3] and sources cited in these papers. Power-law in human memory was investigated in Refs. 4–9: the accuracy on memory tasks decays as a power law $\sim t^{-\beta}$, with $0 < \beta < 1$ and, with respect to human learning, it is shown in Ref. 10 that the reduction in reaction times that comes with practice is a power function of the number of training trials. Power-law adaptation has been used to describe the dynamics of biological systems in papers.[8,11–15]

The importance and origin of the memory in biological systems can be related to the presence of memory at the level of individual cells: it has been shown recently that processing of external stimuli by individual neurons can be described by fractional differentiation.[16–18] The orders of fractional derivatives α derived for different types of neurons fall within the interval [0,1], which implies power-law memory $\sim t^{\beta}$ with power $\beta = 1 - \alpha$, $\beta \in [-1, 0]$. For neocortical pyramidal neurons, the order of the fractional derivative is quite small: $\alpha \approx 0.15$.

Viscoelastic properties of the human organ tissues are best described by fractional differential equations with time fractional derivatives, which imply the power-law memory (see, e.g., references in Ref. 19). In most of the biological systems with the power-law behavior, the power β is between -1 and 1 ($0 < \alpha < 2$).

Among the fundamental scientific problems driving interest and research in fractional dynamics are the origin of memory and a possibility of memory being present in the very basic equations of Physics. Could it be that the fundamental laws describing fields and particles are not memory less and are governed by fractional differential/difference equations?

Because most of the social, biological, and physical systems are nonlinear, it is important to look for the fundamental differences in the behavior of nonlinear systems with and without memory. Let us list some of the differences.

- Trajectories in continuous fractional systems of orders less than two may intersect (see, e.g., Fig. 2 form Ref. 19) and chaotic attractors may overlap [see e.g., Fig. 4(f) from Ref. 20].
- As a result, the Poincaré-Bendixson Theorem does not apply to fractional systems and even in continuous systems of the order $\alpha < 2$ non-existence of chaos is only a conjecture (see Refs. 19 and 21).
- Periodic sinks may exist only in asymptotic sense and asymptotically attracting points may not belong to their own basins of attraction (see Refs. 20, 22, and 24). A trajectory starting from an asymptotically attracting point jumps out of this point and may end up in a different asymptotically attracting point.
- The way in which a trajectory approaches an attracting point depends on its origin. Trajectories originating from the basin of attraction may converge faster (as $x_n \sim n^{-1-\alpha}$ for the fractional Riemann-Liouville standard map, see Fig. 1 from Ref. 20) than trajectories originating from the chaotic sea (as $x_n \sim n^{-\alpha}$).
- Cascade of bifurcations type trajectories (CBTT) exists only in fractional systems. The periodicity of such trajectories changes with time: they may start converging to the period 2^n sink, but then bifurcate and start converging to the period 2^{n+1} sink and so on. CBTT may end its evolutions converging to the period 2^{n+m} sink [Fig. 1(a)] or in chaos [Fig. 1(b)].[22,23]

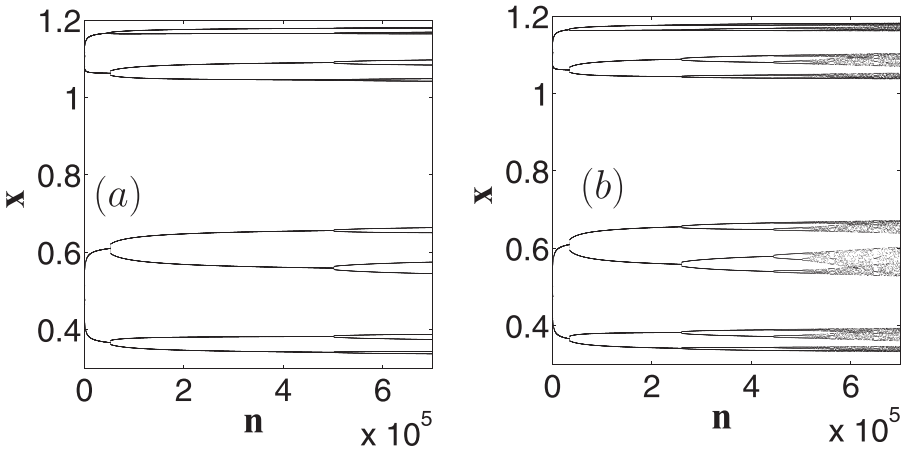

FIG. 1. Two examples of cascade of bifurcations type trajectories in the Caputo logistic α-family of maps [Eq.(22) with $h=1$ and $G_K(x)=x-Kx(1-x)$] with $\alpha=0.1$ and $x_0=0.001$: (a) for the nonlinearity parameter $K=22.37$ the last bifurcation from the period $T=8$ to the period $T=16$ occurs after approximately 5×10^5 iterations; (b) when $K=22.423$ the trajectory becomes chaotic after approximately 5×10^5 iterations.

- Continuous and discrete fractional systems may not have periodic solutions except fixed points (see e.g., Refs. 25–31). Instead they may have asymptotically periodic solutions.
- Fractional extensions of the volume preserving systems are not volume preserving. If the order of a fractional system is less than the order of the corresponding integer system, then behavior of the system is similar to the behavior of the corresponding integer system with dissipation.[32] Correspondingly, the types of attractors which may exist in fractional systems include sinks, limiting cycles, and chaotic attractors.[24,33–36]

A particular problem related to the differentiation between fractional systems and integer ones, the first bifurcation on CBTT, and related problems of stability of fixed points in discrete fractional systems and transition to chaos in continuous fractional systems are considered in this paper.

The stability of fractional systems was investigated in numerous papers based on various methods (Lyapunov's direct and indirect methods, Lyapunov function, Routh-Hurwitz criterion,…). Here, we'll list only some of the research papers, reviews, and books on the topic. The paper Ref. 37 is the most cited article on the stability of linear fractional differential equations. In application to the stability of nonlinear fractional differential equations, we'll mention papers.[38–44] Some of the results on the stability of discrete fractional systems can be found in papers.[45–50] The reviews on the topic include papers[51–53] and books.[54,55] Almost all results obtained in the cited papers define sufficient conditions of stability and do not allow calculation of the ranges of nonlinearity parameters and orders of derivatives for which fixed points are stable.

In this paper, we derive the algebraic equations to calculate asymptotically period two sinks of discrete fractional systems, which define the conditions of their appearance, and conjecture that these equations define the values of nonlinearity parameters and orders of derivatives for which fixed points become unstable. This conjecture is numerically verified for the fractional standard and logistic maps. This paper is a continuation of the research on general properties of fractional systems based on the properties of fractional maps.[19,20,22–24,33,34,45,56–64] In Sec. II, we review the most common forms of fractional maps. In Sec. III, we derive the equations defining the ranges of nonlinearity parameters and orders of derivatives for which fixed points are stable. Section IV presents the summary of our results.

II. FRACTIONAL/FRACTIONAL DIFFERENCE MAPS

In this section, some essential definitions and theorems are presented.

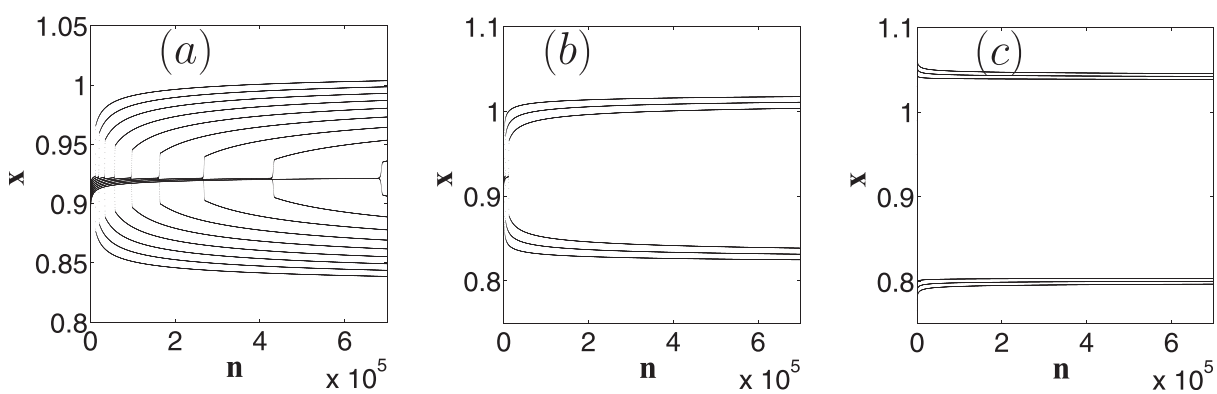

FIG. 2. Asymptotically period two trajectories for the Caputo logistic α-family of maps with $\alpha=0.1$ and $K=15.5$: (a) nine trajectories with the initial conditions $x_0=0.29+0.04i$, $i=0,1,\ldots,8$ ($i=0$ corresponds to the rightmost bifurcation); (b) $x_0=0.61+0.06i$, $i=1,2,3$; (c) $x_0=0.95+0.04i$, $i=1,2,3$. As $n\to\infty$ all trajectories converge to the limiting values $x_{lim1}=0.80629$ and $x_{lim2}=1.036030$ [see Eq. (61)]. The unstable fixed point is $x_{lim0}=(K-1)/K=0.93548$.

A. Fractional integrals and derivatives

In this paper, we will use the definition of fractional integral introduced by Liouville, which is a generalization of the Cauchy formula for the n-fold integral

$$ {}_aI_t^p x(t) = \frac{1}{\Gamma(p)} \int_a^t \frac{x(\tau)d\tau}{(t-\tau)^{1-p}}, \tag{1} $$

where p is a real number, $\Gamma()$ is the gamma function, and we'll assume $a = 0$.

The left-sided Riemann-Liouville fractional derivative ${}_0D_t^\alpha x(t)$ is defined for $t > 0$ (Refs. 65–67) as

$$ \begin{aligned} {}_0D_t^\alpha x(t) &= D_t^n \, {}_0I_t^{n-\alpha} x(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{x(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}, \end{aligned} \tag{2} $$

where $n-1 \le \alpha < n$, $n \in \mathbb{Z}$, $D_t^n = d^n/dt^n$.

In the definition of the left-sided Caputo derivative, the order of integration and differentiation in Eq. (2) is switched[66]

$$ \begin{aligned} {}_0^CD_t^\alpha x(t) &= {}_0I_t^{n-\alpha} D_t^n x(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{D_\tau^n x(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}, \quad (n-1 < \alpha \le n). \end{aligned} \tag{3} $$

B. Fractional sums and differences

In this paper, we will use the proposed by Miller and Ross sum/difference operator,[68] which is a generalization of the forward difference operator

$$ \Delta x(t) = x(t+1) - x(t), \tag{4} $$

(see below) and call it simply the fractional sum/difference operator. Nabla fractional sum/difference operator, which is the generalization of the backward difference $\nabla x(t) = x(t) - x(t-1)$[69] is not considered in this paper.

The fractional sum ($\alpha > 0$)/difference ($\alpha < 0$) operator defined in Ref. 68

$$ {}_a\Delta_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s), \tag{5} $$

is a fractional generalization of the n-fold summation formula[58,69]

$$ \begin{aligned} {}_a\Delta_t^{-n} f(t) &= \frac{1}{(n-1)!} \sum_{s=a}^{t-n} (t-s-1)^{(n-1)} f(s) \\ &= \sum_{s^0=a}^{t-n} \sum_{s^1=a}^{s^0} \cdots \sum_{s^{n-1}=a}^{s^{n-2}} f(s^{n-1}), \end{aligned} \tag{6} $$

where $n \in \mathbb{N}$. In Eq. (5), f is defined on $\mathbb{N}_a$ and ${}_a\Delta_t^{-\alpha}$ on $\mathbb{N}_{a+\alpha}$, where $\mathbb{N}_t = \{t, t+1, t+2, \ldots\}$. The falling factorial $t^{(\alpha)}$ is defined as

$$ t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}, \quad t \ne -1, -2, -3, \ldots \tag{7} $$

and is asymptotically a power function

$$ \lim_{t\to\infty} \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)t^\alpha} = 1, \quad \alpha \in \mathbb{R}. \tag{8} $$

For $\alpha > 0$ and $m-1 < \alpha \le m$, the fractional (left) Riemann-Liouville difference operator is defined (see Refs. 70 and 71) as

$$ \begin{aligned} {}_a\Delta_t^\alpha x(t) &= \Delta_a^m \Delta_t^{-(m-\alpha)} x(t) \\ &= \frac{1}{\Gamma(m-\alpha)} \Delta^m \sum_{s=a}^{t-(m-\alpha)} (t-s-1)^{(m-\alpha-1)} x(s), \end{aligned} \tag{9} $$

and the fractional (left) Caputo-like difference operator (see Ref. 72) as

$$ \begin{aligned} {}_a^C\Delta_t^\alpha x(t) &= {}_a\Delta_t^{-(m-\alpha)} \Delta^m x(t) \\ &= \frac{1}{\Gamma(m-\alpha)} \sum_{s=a}^{t-(m-\alpha)} (t-s-1)^{(m-\alpha-1)} \Delta^m x(s). \end{aligned} \tag{10} $$

Due to the fact that ${}_a\Delta_t^\lambda$ in the limit $\lambda \to 0$ approaches the identity operator (see Refs. 58 and 68) the definition Eq. (10) can be extended to all real $\alpha \ge 0$ with ${}_a^C\Delta_t^m x(t) = \Delta^m x(t)$ for $m \in \mathbb{N}_0$.

Fractional h-difference operators, which are generalizations of the fractional difference operators, were introduced in Refs. 73 and 74. The h-sum operator is defined as

$$ ({}_a\Delta_h^{-\alpha} f)(t) = \frac{h}{\Gamma(\alpha)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}-\alpha} (t-(s+1)h)_h^{(\alpha-1)} f(sh), \tag{11} $$

where $\alpha \ge 0$, $({}_a\Delta_h^0 f)(t) = f(t)$, f is defined on $(h\mathbb{N})_a$, and ${}_a\Delta_h^{-\alpha}$ on $(h\mathbb{N})_{a+\alpha h}$. $(h\mathbb{N})_t = \{t, t+h, t+2h, \ldots\}$. The h-factorial $t_h^{(\alpha)}$ is defined as

$$ t_h^{(\alpha)} = h^\alpha \frac{\Gamma\left(\frac{t}{h}+1\right)}{\Gamma\left(\frac{t}{h}+1-\alpha\right)} = h^\alpha \left(\frac{t}{h}\right)^{(\alpha)}, \tag{12} $$

where $t/h \ne -1, -2, -3, \ldots$. With $m = \lceil \alpha \rceil$, the Riemann-Liouville (left) h-difference is defined as

$$ \begin{aligned} \left({}_a\Delta_h^\alpha x\right)(t) &= \left(\Delta_h^m \left({}_a\Delta_h^{-(m-\alpha)} x\right)\right)(t) = \frac{h}{\Gamma(m-\alpha)} \\ &\quad \times \Delta_h^m \sum_{s=\frac{a}{h}}^{\frac{t}{h}-(m-\alpha)} (t-(s+1)h)_h^{(m-\alpha-1)} x(sh), \end{aligned} \tag{13} $$

and the Caputo (left) h-difference is defined as

$$ \begin{aligned} \left({}_a\Delta_{h,*}^\alpha x\right)(t) &= \left({}_a\Delta_h^{-(m-\alpha)} \left(\Delta_h^m x\right)\right)(t) = \frac{h}{\Gamma(m-\alpha)} \\ &\quad \times \sum_{s=\frac{a}{h}}^{\frac{t}{h}-(m-\alpha)} (t-(s+1)h)_h^{(m-\alpha-1)} (\Delta_h^m x)(sh), \end{aligned} \tag{14} $$

where $(\Delta_h^m x))(t)$ is the mth power of the forward h-difference operator

$$(\Delta_h x)(t) = \frac{x(t+h) - x(t)}{h}. \tag{15}$$

As it has been noted in Refs. 73 and 74, due to the convergence of solutions of fractional Riemann-Liouville h-difference equations when $h \to 0$ to solutions of the corresponding differential equations, they can be used to solve fractional Riemann-Liouville differential equations numerically.

C. Fractional maps

Maps with power-law memory can be introduced directly as a particular form of maps with memory (see papers[19,45] which contain references and discussions on the topic). The most general form of the convolution-type map with power-law memory introduced in Ref. 19 can be written as

$$x_n = \sum_{k=1}^{\lceil\alpha\rceil-1} \frac{c_k}{\Gamma(\alpha-k+1)}(nh)^{\alpha-k} + \frac{h^\alpha}{\Gamma(\alpha)}\sum_{k=0}^{n-1}(n-k)^{\alpha-1}G_K(x_k), \tag{16}$$

where $\alpha \geq 0$, K is a parameter, and h is a constant time step between the time instants $t_n = a + nh$ and t_{n+1}. We assume that $G_K(x)$ is a nonlinear function and K is the nonlinearity parameter characterizing nonlinearity of the function and nonlinear properties of the corresponding system. For a physical interpretation of this formula, we consider a system whose state is defined by the variable $x(t)$ and evolution by the continuous function $G_K(x)$. The value of the state variable at the time t_n, $x_n = x(t_n)$, is a weighted total of the functions $G_K(x_k)$ from the values of this variable at the past time instants $t_k = a + kh$, $0 \leq k < n$, $t_k = kh$. The weights are the times between the time instants t_n and t_k to the fractional power $\alpha - 1$. Equation (16) in the limit $h \to 0+$ yields the Volterra integral equation of the second kind

$$x(t) = \sum_{k=1}^{\lceil\alpha\rceil-1} \frac{c_k}{\Gamma(\alpha-k+1)}(t-a)^{\alpha-k} + \frac{1}{\Gamma(\alpha)}\int_a^t \frac{G_K(\tau, x(\tau))d\tau}{(t-\tau)^{1-\alpha}}. \ (t > a). \tag{17}$$

This equation is equivalent to the fractional differential equation with the Riemann-Liouville or Grünvald-Letnikov fractional derivative[19,75,76]

$${}_a^{RL/GL}D_t^\alpha x(t) = G_k(t, x(t)), \quad 0 < \alpha, \tag{18}$$

with the initial conditions

$$\left({}_a^{RL/GL}D_t^{\alpha-k}x\right)(a+) = c_k, \ \ k = 1, 2, \ldots, \lceil\alpha\rceil. \tag{19}$$

For $\alpha \notin \mathbb{N}$, we assume $c_{\lceil\alpha\rceil} = 0$, which corresponds to a finite value of $x(a)$.

The same map, Eq. (16), called the universal map, represents the solution of the fractional generalization of the differential equation of a periodically (with the period h) kicked system (see Refs. 23, 33, 34, and 60–63 for the fractional universal maps and Ref. 77 in regular dynamics).

To derive the equations of the fractional universal map, which we will call the universal α-family of maps (α-FM) for $\alpha \geq 0$, we start with the differential equation

$$\frac{d^\alpha x}{dt^\alpha} + G_K(x(t-\Delta h))\sum_{k=-\infty}^{\infty}\delta\left(\frac{t}{h} - (k+\varepsilon)\right) = 0, \tag{20}$$

where $\varepsilon > \Delta > 0$, $\alpha \in \mathbb{R}$, $\alpha > 0$, and consider it as $\varepsilon \to 0$. The initial conditions should correspond to the type of the fractional derivative used in Eq. (20). The case $\alpha = 2$, $\Delta = 0$, and $G_K(x) = KG(x)$ corresponds to the equation whose integration yields the regular universal map.

Integration of Eq. (20) with the Riemann-Liouville fractional derivative ${}_0D_t^\alpha x(t)$ and the initial conditions

$$({}_0D_t^{\alpha-k}x)(0+) = c_k, \tag{21}$$

where $k = 1, \ldots, N$ and $N = \lceil\alpha\rceil$ yields the Riemann-Liouville universal α-FM Eq. (16).

Integration of Eq. (20) with the Caputo fractional derivative ${}_0^CD_t^\alpha x(t)$ and the initial conditions $(D_t^k x)(0+) = b_k$, $k = 0, \ldots, N-1$ yields the Caputo universal α-FM

$$x_{n+1} = \sum_{k=0}^{N-1}\frac{b_k}{k!}h^k(n+1)^k - \frac{h^\alpha}{\Gamma(\alpha)}\sum_{k=0}^{n}G_K(x_k)(n-k+1)^{\alpha-1}. \tag{22}$$

In this paper, we'll refer to the map Eq. (16), the RL universal α-FM, as the Riemann-Liouville universal map with power-law memory or the Riemann-Liouville universal fractional map; we'll call the Caputo universal α-FM, Eq. (22), the Caputo universal map with power-law memory or the Caputo universal fractional map.

In the case of integer α, the universal map converges to $x_n = 0$ for $\alpha = 0$, $x_{n+1} = x_n - hG_K(x_n)$ for $\alpha = 1$, and for $\alpha = N = 2$ with $p_{n+1} = (x_{n+1} - x_n)/h$

$$\begin{cases} p_{n+1} = p_n - hG_K(x_n), & n \geq 0, \\ x_{n+1} = x_n + hp_{n+1}, & n \geq 0. \end{cases} \tag{23}$$

N-dimensional, with $N \geq 2$, universal maps are investigated in Ref. 23, where it is shown that they are volume preserving.

D. Universal fractional difference map

In what follows, we will consider fractional Caputo difference maps—the only fractional difference maps whose behavior has been investigated. The following theorem[56,58,59,64,78] is essential to derive the universal fractional difference map

Theorem 1. *For $\alpha \in \mathbb{R}$, $\alpha \geq 0$, the Caputo-like h-difference equation*

$$({}_0\Delta_{h,*}^\alpha x)(t) = -G_K(x(t + (\alpha-1)h)), \tag{24}$$

where $t \in (h\mathbb{N})_m$, with the initial conditions

$$({}_0\Delta_h^k x)(0) = c_k, \quad k = 0, 1, \ldots, m-1, \quad m = \lceil \alpha \rceil, \tag{25}$$

is equivalent to the map with h-factorial-law memory

$$x_{n+1} = \sum_{k=0}^{m-1} \frac{c_k}{k!}((n+1)h)_h^{(k)} - \frac{h^\alpha}{\Gamma(\alpha)} \sum_{s=0}^{n+1-m} (n-s-m+\alpha)^{(\alpha-1)} G_K(x_{s+m-1}), \tag{26}$$

where $x_k = x(kh)$, which is called the h-difference Caputo universal α-family of maps.

In the case of integer α, the fractional difference universal map converges to $x_{n+1} = -G_K(x_n)$ for $\alpha = 0$, $x_{n+1} = x_n - hG_K(x_n)$ for $\alpha = 1$, and for $\alpha = N = 2$ with $p_{n+1} = (x_{n+1} - x_n)/h$

$$\begin{cases} p_{n+1} = p_n - hG_K(x_n), \ n \geq 1, \quad p_1 = p_0, \\ x_{n+1} = x_n + hp_{n+1}, \quad n \geq 0. \end{cases} \tag{27}$$

N-dimensional, with $N \geq 2$, difference universal maps are volume preserving.[56]

All the above considered universal maps in the case $\alpha = 2$ yield the standard map if $G_K(x) = K \sin(x)$ (harmonic nonlinearity) and we will call them the standard α-families of maps. When $G_K(x) = x - Kx(1-x)$ (quadratic nonlinearity) in the one-dimensional case, all maps yield the regular logistic map and we will call them the logistic α-families of maps.

III. PERIOD TWO SINKS AND STABILITY OF FIXED POINTS

In fractional systems not only the speed of convergence of trajectories to the periodic sinks but also the way in which convergence occurs depends on the initial conditions. As $n \to \infty$, all trajectories in Fig. 2 converge to the same period two ($T = 2$) sink [as in Fig. 2(c)], but for small values of the initial conditions x_0 all trajectories first converge to the $T = 1$ trajectory which then bifurcates and turns into the $T = 2$ sink converging to its limiting value. As x_0 increases, the bifurcation point n_{bif} gradually evolves from the right to the left [Fig. 2(a)]. Ignoring this feature may result (as in Ref. 64 and some other papers) in very messy bifurcation diagrams.

In this paper, we consider the asymptotic stability of periodic points. A periodic point is asymptotically stable if there exists an open set such that all trajectories with initial conditions from this set converge to this point as $t \to \infty$. It is known from the study of the ordinary nonlinear dynamical systems that as a nonlinearity parameter increases the system bifurcates. This means that at the point (value of the parameter) of birth of the $T = 2^{n+1}$ sink, the $T = 2^n$ sink becomes unstable. In this section, we will investigate the $T = 2$ sinks of discrete fractional systems and apply our results to analyze the stability of the systems' fixed points.

A. 0 < α < 1

When $0 < \alpha < 1$, all forms of the universal α-family of maps introduced in this paper, Eqs. (16), (22), and (26), can be written in the form

$$x_{n+1} = x_0 - \sum_{k=0}^{n} \tilde{G}(x_k) U_\alpha(n-k+1). \tag{28}$$

In this formula, $\tilde{G}(x) = h^\alpha G_K(x)/\Gamma(\alpha)$ and x_0 is the initial condition [$x_0 = 0$ in Eq. (16)]. In fractional maps, Eqs. (16) and (22)

$$U_\alpha(n) = n^{\alpha-1}, \quad U_\alpha(1) = 1, \tag{29}$$

and in fractional difference maps, Eq. (26)

$$U_\alpha(n) = (n+\alpha-2)^{(\alpha-1)}, \quad U_\alpha(1) = (\alpha-1)^{(\alpha-1)} = \Gamma(\alpha). \tag{30}$$

For $n = 2N$, Eq. (28) can be written (after subtracting x_{2N}) as

$$x_{2N+1} = x_{2N} - \tilde{G}(x_{2N}) U_\alpha(1) + \sum_{n=1}^{N} \tilde{G}(x_{2N-2n+1})(U_\alpha(2n-1) - U_\alpha(2n)) + \sum_{n=1}^{N} \tilde{G}(x_{2N-2n})(U_\alpha(2n) - U_\alpha(2n+1)). \tag{31}$$

The terms $U_\alpha(2n-1) - U_\alpha(2n)$ are of the order $n^{\alpha-2}$. If we assume that in the limit $n \to \infty$ the period, $T = 2$ sink exists

$$x_o = \lim_{n\to\infty} x_{2n+1}, \quad x_e = \lim_{n\to\infty} x_{2n}, \tag{32}$$

then the series in Eq. (31) converge absolutely. In the limit $n \to \infty$, Eq. (31) converges to

$$x_o - x_e = [\tilde{G}(x_o) - \tilde{G}(x_e)] W_\alpha, \tag{33}$$

where W_α is a converging series

$$W_\alpha = \sum_{n=1}^{\infty} [U_\alpha(2n-1) - U_\alpha(2n)], \tag{34}$$

which can be computed numerically with $U_\alpha(n)$ defined either by Eq. (29) or by Eq. (30).

Now, instead of subtracting, let us add x_{2N} to x_{2N+1}

$$x_{2N+1} + x_{2N} = 2x_0 - \sum_{n=1}^{2N} [\tilde{G}(x_{2N-n+1}) + \tilde{G}(x_{2N-n})] \times U_\alpha(n) - \tilde{G}(x_0) U_\alpha(2N+1). \tag{35}$$

If the $T = 2$ sink exists, then, in the limit $n \to \infty$, the left-hand side (LHS) of Eq. (35), as well as the first term on the right-hand side (RHS) and the last term of this equation, is finite. Expressions in the brackets in Eq. (35) tend to the limit $\tilde{G}(x_o) + \tilde{G}(x_e)$. Because the series $\sum_{n=1}^{\infty} U_\alpha(n)$ is diverging, the only case in which Eq. (35) can be true is when

$$\tilde{G}(x_o) + \tilde{G}(x_e) = 0. \tag{36}$$

Equations which define the existence and value of the asymptotic $T = 2$ sink can be written as

$$\begin{cases} G_K(x_o) + G_K(x_e) = 0, \\ x_o - x_e = \dfrac{W_\alpha}{\Gamma(\alpha)} h^\alpha [G_K(x_o) - G_K(x_e)]. \end{cases} \tag{37}$$

- It is easy to see that the fixed point defined by the equation $G_K(x_o) = 0$ is a solution of the system Eq. (37).
- As it was mentioned above, when $h \to 0$, fractional difference equations converge to the corresponding fractional differential equations. As $h \to 0$, the second equation from the system Eq. (37) leads to $x_o - x_e \to 0$. This implies that in fractional differential equations of the order $0 < \alpha < 1$ transition from a fixed point to periodic trajectories will never happen. A strict proof of the impossibility of periodic trajectories (except fixed points) in autonomous fractional systems described by the fractional differential equation

$$\frac{d^\alpha x}{dt^\alpha} = G_K(x(t)), \quad 0 < \alpha < 1, \tag{38}$$

with the Caputo or Riemann-Liouville fractional derivative was given in Ref. 27 (Theorem 9 there). Impossibility of the fixed-point bifurcations and the fact that in regular dynamics transition to chaos occurs through cascades of the period doubling bifurcations are additional arguments supporting mentioned in Sec. I conjecture

Conjecture 2. *Chaos does not exist in continuous fractional systems of the orders* $0 < \alpha < 1$.

B. $1 < \alpha < 2$

For $1 < \alpha < 2$ map equations Eqs. (16), (22), and (26) can be written in the form

$$x_{n+1} = x_0 + f(\alpha)[h(n+1)]^\beta p_0 - h\sum_{k=0}^{n} \tilde{G}(x_k) U_\alpha(n-k+1) + h f_1(n). \tag{39}$$

Here, $\tilde{G}(x) = h^{\alpha-1} G_K(x)/\Gamma(\alpha)$, x_0, and $U(n)$ are defined the same way as in Eqs. (28), (29), and (30), p_0 is the initial momentum (b_k or c_k in corresponding formulae), β is equal to 1 in Eqs. (22) and (26) and $\alpha - 1$ in Eq. (16) $f(\alpha)$ is 1 in Eqs. (22) and (26) and $1/\Gamma(\alpha)$ in Eq. (16), and $f_1(n) = 0$ in Eqs. (16) and (22) and $f_1(n) = h^{\alpha-1} G(x_0)(n-1+\alpha)^{(\alpha-1)}/\Gamma(\alpha) \sim n^{\alpha-1}$ in Eq. (26).

If we define

$$p_{n+1} = \frac{x_{n+1} - x_n}{h}, \tag{40}$$

then, taking into account that $U_\alpha(0) = 0$, from Eq. (39) follows:

$$p_{n+1} = \tilde{f}(n) p_0 - \sum_{k=0}^{n} \tilde{G}(x_k) \tilde{U}_\alpha(n-k+1) + f_1(n) - f_1(n-1), \tag{41}$$

where

$$\tilde{U}_\alpha(n) = U_\alpha(n) - U_\alpha(n-1) = \begin{cases} n^{\alpha-1} - (n-1)^{\alpha-1} \sim n^{\alpha-2} \\ \text{and } \tilde{U}_\alpha(1) = 1 \text{ in Eqs. (16), (22);} \\ (n+\alpha-2)^{(\alpha-1)} - (n+\alpha-3)^{(\alpha-1)} \\ = (\alpha-1)(n+\alpha-3)^{(\alpha-2)} \\ = (\alpha-1) U_{\alpha-1}(n) \sim n^{\alpha-2} \\ \text{and } \tilde{U}_\alpha(1) = \Gamma(\alpha) \text{ in Eq. (26).} \end{cases} \tag{42}$$

$f_1(n) - f_1(n-1) = 0$ in Eqs. (16) and (22) and $f_1(n) - f_1(n-1) \sim n^{\alpha-1}$ in Eq. (26), $\tilde{f}(n) = 1$ in Eqs. (22) and (26) and $\tilde{f}(n) \sim n^{\alpha-2}$ in Eq. (16). Note that the definitions of $\tilde{U}_\alpha(1)$ in Eq. (42) and $U_\alpha(1)$ in Eqs. (29) and (30) are identical.

Assuming existence of the $T = 2$ sink and limits x_o and x_e are defined by Eq. (32), the limiting values for p are defined by

$$p_o = \lim_{n\to\infty} p_{2n+1} = \lim_{n\to\infty} \frac{x_{2n+1} - x_{2n}}{h} = \frac{x_o - x_e}{h} \quad \text{and} \quad p_e = \lim_{n\to\infty} p_{2n} = -p_o. \tag{43}$$

As in the derivation of Eqs. (33) and (36), if we add and subtract expressions for p_{2N+1} and p_{2N}, we'll arrive at relations

$$p_o - p_e = [\tilde{G}(x_o) - \tilde{G}(x_e)] \tilde{W}_\alpha \tag{44}$$

and

$$\tilde{G}(x_o) + \tilde{G}(x_e) = 0, \tag{45}$$

where $\tilde{W}_\alpha$ is a converging series

$$\tilde{W}_\alpha = \sum_{n=1}^{\infty} [\tilde{U}_\alpha(2n-1) - \tilde{U}_\alpha(2n)]. \tag{46}$$

Let us note that with $U_\alpha(n) = n^{\alpha-1}$, as defined in Eq. (29), $\tilde{W}$ is identical to introduced in Ref. 22 $V_{\alpha l}$ defined as

$$\tilde{W}_\alpha = V_{\alpha l} = \sum_{n=1}^{\infty} (-1)^{n+1} \left[n^{\alpha-1} - (n-1)^{\alpha-1} \right]. \tag{47}$$

A high accuracy algorithm for calculating $V_{\alpha l}$ is presented in Appendix to Ref. 24. For $U(n)$ defined by Eq. (30), $\tilde{W}$ was calculated in Ref. 56. Taking into account that converging series Eq. (46) can be written as

$$\tilde{W}_\alpha = \tilde{U}_{\alpha 1} - \sum_{n=1}^{\infty} [\tilde{U}_\alpha(2n) - \tilde{U}_\alpha(2n+1)], \tag{48}$$

where

$$\tilde{U}_{\alpha 1} = \begin{cases} 1 \text{ in Eqs. (16), (22),} \\ \Gamma(\alpha) \text{ in Eq. (26),} \end{cases} \tag{49}$$

and using the absolute convergence of series Eq. (34) [and, correspondingly, the series on the first line of Eq. (51) below], for $0 < \alpha < 1$ we can write

$$\begin{aligned}\tilde{W}_\alpha &= \tilde{U}_{\alpha 1} - \sum_{n=1}^{\infty}\{[U_\alpha(2n) - U_\alpha(2n-1)] \\ &\quad -[U_\alpha(2n+1) - U_\alpha(2n)]\} \\ &= \tilde{U}_{\alpha 1} + \sum_{n=1}^{\infty}[U_\alpha(2n-1) - U_\alpha(2n)] \\ &\quad -\sum_{n=1}^{\infty}[U_\alpha(2n) - U_\alpha(2n+1)] = W_\alpha + \tilde{U}_{\alpha 1} \\ &\quad -U_\alpha(2) + U_\alpha(3) - U_\alpha(4) + U_\alpha(5) - \ldots = 2W_\alpha.\end{aligned} \tag{50}$$

Let us notice that in fractional difference maps Eq. (48) can be written as

$$\begin{aligned}\tilde{W}_\alpha &= (\alpha-1)\Gamma(\alpha-1) - (\alpha-1)\sum_{n=1}^{\infty}[U_{\alpha-1}(2n) \\ &\quad -U_{\alpha-1}(2n+1)] = (\alpha-1)W_{\alpha-1} = \frac{\alpha-1}{2}\tilde{W}_{\alpha-1}.\end{aligned} \tag{51}$$

Finally, the equations which define the existence and value of the asymptotic $T=2$ sink for $0<\alpha<2$ can be written as

$$\begin{cases} G_K(x_o) + G_K(x_e) = 0, \\ x_o - x_e = \dfrac{\tilde{W}_\alpha}{2\Gamma(\alpha)} h^\alpha [G_K(x_o) - G_K(x_e)], \end{cases} \tag{52}$$

where $\tilde{W}_\alpha$ is defined by Eqs. (48) and (49). Notice that according to Eq. (48) $\tilde{W}_1 = 1$.

- As in the case $0<\alpha<1$, for $1<\alpha<2$ the fixed point, defined by the equation $G_K(x_o) = 0$ is a solution of the system Eq. (52).
- As $h \to 0$, fractional difference equations converge to the corresponding fractional differential equations and $x_o - x_e \to 0$, which implies that in fractional differential equations of the order $1<\alpha<2$ transition from a fixed point to periodic trajectories will never happen. All arguments supporting Conjecture 2 can be repeated to support the stronger (mentioned in Sec. I) conjecture:

Conjecture 3. *Chaos does not exist in continuous fractional systems of the orders* $0<\alpha<2$.

C. Examples

Now we will consider application of the results from this section to the fractional and fractional difference standard $[G_K(x) = K\sin(x)]$ and logistic $[G_K(x) = x - Kx(1-x)]$ α-families of maps introduced at the end of Sec. II.

1. Standard α-families of maps

With $G_K(x) = K\sin(x)$, all the above-considered forms of the universal map for $\alpha=2$ converge to the regular standard map and they are called the standard α-families of maps. These families of maps are usually considered on a torus (mod 2π). The first equation of the system Eq. (52) yields

$$\sin\frac{x_o + x_e}{2}\cos\frac{x_o - x_e}{2} = 0, \tag{53}$$

which on $x \in [-\pi, \pi]$ yields two solutions

$$\begin{aligned}&\text{symmetric point } x_{osy} = -x_{esy} \quad \text{and} \\ &\text{shift} - \pi \text{ point } x_{osh} = x_{esh} - \pi.\end{aligned} \tag{54}$$

Then, the second equation of Eq. (52) yields the equation which together with Eq. (54) defines two $T=2$ sinks for $0<\alpha<2$

$$\sin x_{osy} = \frac{2\Gamma(\alpha)}{\tilde{W}_\alpha h^\alpha K} x_{osy} \tag{55}$$

and

$$\sin x_{osh} = \frac{\pi\Gamma(\alpha)}{\tilde{W}_\alpha h^\alpha K}. \tag{56}$$

The symmetric $T=2$ sink appears when

$$h^\alpha|K| > h^\alpha|K_{C1s}| = \frac{2\Gamma(\alpha)}{\tilde{W}_\alpha}, \tag{57}$$

and the shift-π $T=2$ sink appears when

$$h^\alpha|K| > \frac{\pi}{2}h^\alpha|K_{C1s}|. \tag{58}$$

2. Logistic α-families of maps

With $G_K(x) = x - Kx(1-x)$, all the above-considered forms of the universal map for $\alpha=1$ converge to the regular logistic map and they are called the logistic α-families of maps. The system Eq. (52) becomes

$$\begin{cases} (1-K)(x_o + x_e) + K(x_o^2 + x_e^2) = 0, \\ x_o - x_e = \dfrac{\tilde{W}_\alpha}{2\Gamma(\alpha)} h^\alpha (x_o - x_e)[1 - K + (x_o + x_e)]. \end{cases} \tag{59}$$

Two fixed point solutions with $x_o = x_e$ are $x_o = 0$, stable for $K<1$, and $x_o = (K-1)/K$.

The $T=2$ sink is defined by the equation

$$x_o^2 - \left(\frac{2\Gamma(\alpha)}{\tilde{W}Kh^\alpha} + \frac{K-1}{K}\right)x_o + \frac{2\Gamma^2(\alpha)}{(\tilde{W}Kh^\alpha)^2} + \frac{(K-1)\Gamma(\alpha)}{\tilde{W}K^2h^\alpha} = 0, \tag{60}$$

which has solutions

$$x_o = \frac{K_{C1s} + K - 1 \pm \sqrt{(K-1)^2 - K_{C1s}^2}}{2K}, \tag{61}$$

defined when

$$K \geq 1 + \frac{2\Gamma(\alpha)}{\tilde{W}h^\alpha} = 1 + K_{C1s} \quad \text{or} \quad K \leq 1 - \frac{2\Gamma(\alpha)}{\tilde{W}h^\alpha} = 1 - K_{C1s}. \tag{62}$$

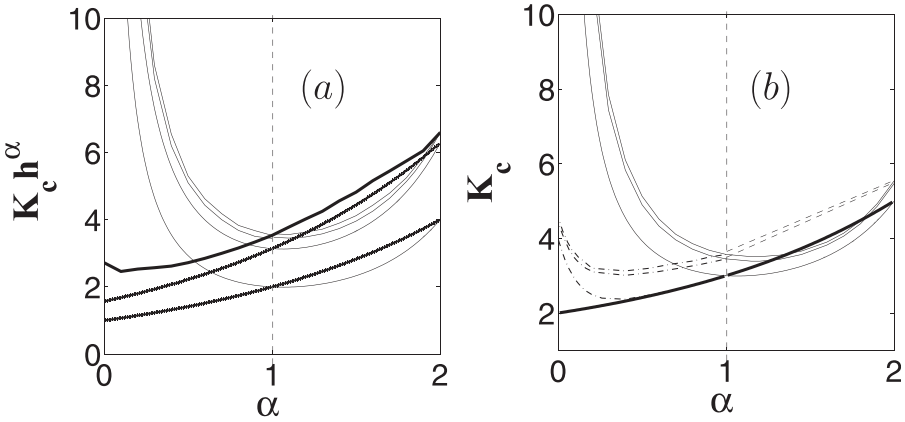

FIG. 3. 2D bifurcation diagrams for fractional (solid thing lines) and fractional difference (bold and dashed lines) Caputo standard (a) and $h=1$ logistic (b) maps. The first bifurcation, transition from the stable fixed point to the stable period two ($T=2$) sink, occurs on the bottom curves. $T=2$ sink (in the case of standard α-families of maps antisymmetric T = 2 sink with $x_{n+1} = -x_n$) is stable between the bottom and the middle curves. Transition to chaos occurs on the top curves. For the standard fractional map, transition from $T=2$ to $T=4$ sink occurs on the line below the top line (the third from the bottom line). Period doubling bifurcations leading to chaos occur in the narrow band between the two top curves. All bottom curves, as well as the next to the bottom in (a), are obtained using formulae Eqs. (57), (58), and (62). Two dashed lines for $1 < \alpha < 2$ in (b) are obtained by interpolation. The remaining lines are results of the direct numerical simulations. Stability of the fixed point for the fractional difference logistic α-family of maps is calculated using both Eq. (62) (bold solid line) and the direct numerical simulations (a dashed line branching from the solid line). The difference is due to the slow, as $n^{-\alpha}$ (see Ref. 56), convergence of trajectories to the $T=2$ sink for small α (x vs. K, fixed α, bifurcation diagrams used to find the first bifurcation were calculated on trajectories after 5000 iterations).

The first inequality of Eq. (62) was derived in Ref. 24 for $h=1$ and $1 < \alpha < 2$. In this paper, we consider $K > 0$ and $h \leq 1$. It follows from the definition, Eq. (48), that $\tilde{W} < \tilde{U}_1$, which is either 1 or $\Gamma(\alpha)$, and it is known that $\Gamma(\alpha) > 0.885$ for $\alpha > 0$. Then, $2\Gamma(\alpha)/(\tilde{W}h^\alpha) > 1$ and we may ignore the second of the inequalities in Eq. (62). We may also note that the fixed point $x = (K-1)/K$ is stable when

$$1 \leq K < K_{C1l} = 1 + \frac{2\Gamma(\alpha)}{\tilde{W}h^\alpha} = 1 + K_{C1s}. \tag{63}$$

IV. CONCLUSION

The main result of this paper is Eq. (52) which defines coordinates of the asymptotic period two sinks for the fractional and fractional difference universal maps of the orders $0 < \alpha < 2$. The conditions of the existence of a solution for this equation define the conditions of the stability (instability) of the maps' fixed points.

Figures 3(a) and 3(b), the two-dimensional bifurcation diagrams, present results of the computer simulations of the fractional and fractional difference standard and logistic maps. Low curves on these diagrams are obtained using Eqs. (57), (58), and (62). They are in good agreement with the results (also used to calculate all other curves) obtained by the direct numerical simulations to calculate x vs. K bifurcation diagrams for various $\alpha \in (0,2)$ after 5000 iterations. The slight difference in Fig. 3(b) for the fractional difference logistic map for $\alpha < 0.2$ is probably due to the slow, as $\sim n^{-\alpha}$, convergence of trajectories to the fixed points. This confirms the validity of Eq. (52) to calculate the coordinates of the asymptotic $T=2$ sinks and the points of the first bifurcations for the discrete fractional/fractional difference maps. The continuous limits of the discrete maps considered in this paper are fractional differential equations and from the consideration presented in this paper we may conclude that chaos is impossible in systems described by equations

$$\frac{d^\alpha x}{dt^\alpha} = f(x), \tag{64}$$

with $0 < \alpha < 2$.

There are still many unanswered questions related to the behavior of fractional systems. They include:

- What is the nature and the corresponding analytical description of the bifurcations on a single trajectory of a fractional system?
- What kind of self-similarity can be found in CBTT?
- How to describe a self-similar behavior corresponding to the bifurcation diagrams of fractional systems? Can constants similar to the Feigenbaum constants be found?
- Can cascade of bifurcations type trajectories be found in continuous systems?

This paper is a small step in the investigation of the fractional dynamical systems and we hope that subsequent works will lead to the more complete description of fractional (with power-law memory) systems which have many applications in biological, social, and physical systems.

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