EXISTENCE AND COMPLETENESS OF WAVE OPERATORS IN TWO HILBERT SPACES

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TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

1.1 Quantum mechanics developed in the first quarter of the twentieth century as scientists studied the motions of electrons and other constituents of atoms. At first they assumed that the laws and concepts of classical mechanics and electromagnetism controlled the motion of atoms also. According to classical ideas an electron in accelerated motion around the nucleus of an atom would radiate energy. As the electron lost energy it would decrease its action. The system might also absorb radiation in which case the energy and action would increase. However, the classical treatment could not account for the precise radiations which observation detected in the hydrogen atom.

The inadequacies of classical mechanics led Planck and Bohr to modify the theory. In trying to account for the theory of incandescence Planck introduced the idea of discrete quantum states into physics. Whereas it was previously assumed that particles moved along a continuous path and could end up in any position, this theory related the position of the electron to changes in quantum states. Planck assumed that when radiant heat or energy is emitted it takes the form of sudden bursts, each representing a quantity of energy equal to the product of a universal constant and the frequency of vibration in the emitted radiation. He represented the universal constant by the letter h and called it a quantum of action. Today the constant is referred to as Planck's constant. Similarly Planck assumed that energy is absorbed in discrete amounts.

In 1913 Bohr published his theory on the spectra of the hydrogen atom. In this paper he applied Planck's ideas to the nuclear atom model put forth by Rutherford. Bohr assumed that atoms exist only in sharply defined states or levels whose energy differs by fixed amounts. There are no intermediate states. He also assumed that radiation occurs only through transition between two stationary states.

This original formulation of quantum mechanics by Planck and Bohr was soon found to be inadequate. It did not accurately describe the precise energy levels of the quantum states, or why certain drops in quantum states were more numerous than others. No means were given to calculate the probability of a particle being in a particular state. The theory could also not account for differences between experimental observation of the states, and the anticipated results of the theory.

Some of the attempts made to refine the original quantum theory include the matrix method of Heisenberg, and the wave and quantum mechanics of DeBroglie and Schrodinger. Although these theories approach quantum mechanics in different ways, they have all been shown to be mathematically equivalent.

In 1924 DeBroglie postulated that the duality between the wave and particle nature of light is also applicable to the behavior of particles such as electrons and protons. Later in 1926, Schrodinger elaborated the theory into the formal mathematical structure of wave mechanics. Schrodinger showed that matter waves must satisfy a partial differential equation subject to certain boundary conditions, and acceptable solutions may be obtained only for certain values of the energy. Concurrently Heisenberg was giving a different formulation of the laws of mechanics. Heisenberg felt that the theory should deal only with observable quantities. Because an atom could only be observed by making it change from one stationary state to another, observables with their initial and final states were represented by a matrix. Schrodinger showed that Heisenberg's matrix method was equivalent to wave mechanics.

There remained some defects of principle in Schrodinger's theory. For one, the spin of an electron had not been explained. For another the theory was not in harmony with the theory of relativity. Dirac and Neumann simultaneously gave more general formulations of quantum mechanics that resolved these difficulties. Dirac's theory implied the existence of particles called positrons, identical to the electron except for the sign of the electric charge.

In 1926 Max Born suggested that the intensity of the DeBroglie wave associated with an electron, or the square of the state function amplitude, is proportional to the probability of finding the electrons in that place in space for which the intensity is calculated. The electron is most likely to be found where the square of state function has most amplitude. This probability model was a new concept in physics. From the time of Newton classical physics had assumed that there was a course for all motions in nature. Born now said that the most we can hope to know about atomic processes is a set of possible outcomes of various experiments, and the probability of their occurences in repeated experiments.

Heisenberg developed this idea in his uncertainty principle. This principle states that the act of observing position and momentum of particles interferes with the motion of the particle. It is impossible to determine both accurately at the same time. This statistical theory has some problems in that it endows particles with infinite energy, but much of it is still used today.

In the years since 1930 many more microscopic particles have been discovered. Thus quantum theory has become more complicated and much additional work is still being done.

1.2 In this section we shall examine some of the mathematical concepts used in quantum theory. The probability model of the atom led to the definition of a function ψ called the state function, such that the probability that a particle is in an interval I is given by

$$\int_I |\psi(x,t)|^2 dx$$

where x represents the position of the particle and t the time.

The position and other quantities that may be measured for a particle are called observables. Associated with any observable quantity a there is an operator A such that the expected or average value $\overline{a}$ of a is given by $(A\psi,\psi)$. In the applications of the theory that follow we shall examine two types of observables. One of them is momentum. The momentum of a particle is defined to be mass times velocity. We may represent this as $p = m\frac{dx}{dt}$ where p is the momentum and m the mass. It can be shown that the expected value of p is given by $\overline{p} = \int_{-\infty}^{\infty} (L\psi)\,\overline{\psi(x,t)}\,dx = (L\psi,\psi)$ where $L\psi = -ih\psi'(x)$ and h is Planck's constant. One may choose units so that the constant h is 1.

In the other applications we shall examine the kinetic energy of a particle. The kinetic energy T of a particle is given by $T = \frac{p^2}{2m}$. The expected value of T is then $\frac{1}{2m}(L^2\psi,\psi)$. Choosing units such that m = 1 we define the kinetic energy operator $H_0\psi = L^2\psi = -\psi''(x)$. Total energy is the sum of the kinetic energy and the potential energy. Letting v(x) represent potential energy, we let $H = H_0+v$ be the total energy operator. H is called the Hamiltonian. With a suitable choice of units the state function satisfies Schrodinger's equation $i\psi'(t) = H\psi$.

In scattering experiments we examine the behavior of a particle both before and after being shot into a region where it is subject to potential forces. The potential forces are assumed to be negligible both before and after the particle enters the region. The potential is often a target that the particle interacts with for only a short duration of the experiment. Most of the time the particle is too far from the target for it to have any effect. Thus at a sufficient interval of time before and after hitting a target the particle is governed by H_0. When the particle gets close to the target its movement is governed by H. If there is a state function $\psi-(t)$ to represent the motion of the particle long before entering the region, we call it an incoming asymptotic state. Then $\psi-$ satisfies $i\psi'- = H_0\psi-$. Similarly we let $\psi+(t)$ be the state function representing the motion of the particle long after entering a region. The function $\psi+$ is called the outgoing asymptotic state and satisfies $i\psi+' = H_0\psi+$. The function ψ should satisfy $i\psi' = H\psi$ where $\|\psi(t) - \psi\pm(t)\| \to 0$ as $t \to \pm\infty$. We call a function ψ possessing both incoming and outgoing asymptotic states, a scattering state. In experimental physics it is important that there be as many scattering states as possible. Defining $e^{-itH}\psi = \int_{-\infty}^{\infty} e^{-it\lambda} dE(\lambda)\psi$ where $E(\lambda)$ is the spectral family it can be shown that $\psi(0) = \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}\psi\pm(0)$.

The above discussion motivates the definition of the wave operators $W\pm\psi = \lim_{t\to\pm\infty} e^{itH}\, e^{-itH_0}\psi$ where H and H_0 are self adjoint operators on Hilbert spaces and ψ the state function. In physics the wave operator exists if a particle does not get trapped within a target. Besides its application with short range potentials such as the target discussed above, the wave operators play a large role in theoretical mathematics. One important question of mathematics is when the wave operator exists. In general one must impose rather strong restrictions in order that the wave operator exist. The Hilbert spaces must be infinite dimensional and in any significant application H must have a pure continuous spectrum as well. It is unrealistic to expect scattering for an eigenelement.

Another question asked about the wave operator is whether it is complete. The completeness of the wave operator tells us whether the motion of a particle is essentially the same long before and long after interacting with a target. We say that a wave operator is complete if every $\psi \epsilon H_c(H_0)$ is the value at t=0 of incoming and outgoing asymptotic states for scattering states where $H_c(H_0) = \{f/E(u)f \rightarrow E(\lambda)f$ when $u \rightarrow \lambda\}$ and $E(\lambda)$ is the spectral family.

In many scattering problems it is necessary to consider self adjoint operators H_0,H operating in different Hilbert spaces $\mathcal{H}_0,\mathcal{H}$ respectively. Such problems arise, for example with wave equations and Maxwell's equation. We then need to define an operator J mapping $\mathcal{H}_0$ to $\mathcal{H}$. The wave operator is defined as $W\pm\psi = \lim_{t\rightarrow\pm\infty} e^{itH}J\ e^{-itH_0}\psi$, and the existence and completeness of the wave operator have the same physical significance as the wave operator discussed above. The two Hilbert spaces enable us to study the motion of a particle when it goes from a region of unrestricted motion to a region of restricted motion. For example a particle may be shot into a region with a wall. In some of the applications that follow we shall consider this situation in one dimension.

CHAPTER II

PRINCIPAL RESULTS

The main theorem of this chapter states conditions for the existence of a self-adjoint operator H, given H_0,A,B,J as defined below. It also notes some of the properties of this operator. In the next chapter we shall give examples of operators that satisfy this theorem and for which the wave operator exists and is complete.

If I_1 and I_2 are open subsets of the real line we shall write $I_1 \subset\subset I_2$ if I_1 is bounded and $\overline{I}_1 \subset I_2$.

We let $\mathcal{H}_0$,$\mathcal{H}$ and K be Hilbert spaces, H_0 a self-adjoint operator on $\mathcal{H}_0$ and J a bounded linear operator from $\mathcal{H}_0$ to $\mathcal{H}$ such that R(J) is dense in $\mathcal{H}$. Let A and B be linear operators from $\mathcal{H}_0$ to K such that the domain $D(H_0) \subset D(A) \cap D(B)$ and J^*J maps $D(H_0)$ to D(B). We shall assume the following

(2.1) I. $\mathrm{Im}[(JH_0u,Ju) + (Au,BJ^*Ju)] = 0,\ u \in D(H_0)$

II. $B(AR_0(\overline{z}))^*$ is closable. Letting $Q_0(z)$ be the closure of $B(AR_0(\overline{z}))^*$ and $G_0(z) = I + Q_0(z)$ where $R_0(z) = (H_0-z)^{-1}$, we assume that $G_0(z)$ is bounded and has a bounded inverse on K for z in an open set and its conjugate image.

III. There exists an open subset Λ of $\mathbb{R}$ such that $C\Lambda$ has measure 0 and for each $I \subset\subset \Lambda$ there is a constant C_I such that

$$a(\|AR_0(s+ia)\|^2 + \|BR_0(s+ia)\|^2) \leq C_I,\ a>0,\ s \in I$$

IV. $(B(BJ_0R_0(z))^*)^* = BJ_0(BR_0(\overline{z}))^*$ where

$J_0 = J^*J$.

Theorem 2.1: There exists a self-adjoint operator H on $\mathcal{H}$ with the following properties. H is uniquely determined by (a) on R(J).

(a) The resolvent $R(z) = (H-z)^{-1}$ exists for z in an open set and its conjugate image and satisfies the second resolvent equation.

$$R(z)J - JR_0(z) = -(BJ^*R(\overline{z}))^*AR_0(z) \tag{2.2}$$

(2.3) (b) $HJ \supset J(H_0+B^*A)$

We now proceed with the proof of Theorem 2.1. We define

$$T(z) = R_0(z) - (AR_0(\overline{z}))^*G_0(z)^{-1}BR_0(z). \tag{2.4}$$

Note that

$$T(z)^* = R_0(\overline{z}) - (BR_0(z))^*G_0^*(\overline{z})^{-1}AR_0(\overline{z}) \tag{2.5}$$

In proving Theorem 2.1 we shall make use of the following lemmas.

Lemma 2.1: T(z) is injective and satisfies the first resolvent equation

$$T(z) - T(\zeta) = (z-\zeta)\, T(z)\, T(\zeta)$$

Proof: Multiplying (2.4) on the left by B and taking the closure we have

$$BT(z) = BR_0(z) - Q_0(z)G_0(z)^{-1}BR_0(z)$$

$$= (I-Q_0(z)G_0(z)^{-1})\ BR_0(z)$$

$$= (I-(I+Q_0(z)-I)\ G_0(z)^{-1})\ BR_0(z)$$

$$= (I-I+G_0(z)^{-1})BR_0(z)$$

$$= G_0(z)^{-1}BR_0(z)$$

Thus we have

$$BT(z) = G_0(z)^{-1}BR_0(z) \tag{2.6}$$

We can now show that $T(z)$ is one-one. We let $T(z)u = 0$. Then $u \in D(BR_0(z))$ and by (2.6) $BR_0(z)u = 0$. By (2.4) $R_0(z)u = 0$. Hence $u=0$ and $T(z)$ is one-one.

We now verify that $T(z)$ satisfies the first resolvent equation. We have

$$(z-\zeta)T(z)T(\zeta)$$

$$= (z-\zeta)R_0(z)R_0(\zeta)$$

$$- (z-\zeta)R_0(z)(AR_0(\overline{\zeta}))^*G_0(\zeta)^{-1}BR_0(\zeta)$$

$$- (z-\zeta)(AR_0(\overline{z}))^*G_0(z)^{-1}BR_0(z)R_0(\zeta)$$

$$+ (z-\zeta)(AR_0(\overline{z}))^*G_0(z)^{-1}BR_0(z)(AR_0(\overline{\zeta}))^*G_0(\zeta)^{-1}BR_0(\zeta).$$

Since $R_0(z)$ satisfies the first resolvent equation $R_0(z) - R_0(\zeta) = (z-\zeta)R_0(z)R_0(\zeta)$ we have

$$
\begin{aligned}
&(z-\zeta)\ T(z)\ T(\zeta) \\
&= R_0(z) - R_0(\zeta) \\
&\quad - (AR_0(\overline{z}))^*G_0(\zeta)^{-1}BR_0(\zeta) + (AR_0(\overline{\zeta}))^*G_0(\zeta)^{-1}BR_0(\zeta) \\
&\quad - (AR_0(\overline{z}))^*G_0(z)^{-1}BR_0(z) + (AR_0(\overline{z}))^*G_0(z)^{-1}BR_0(\zeta) \\
&\quad + (AR_0(\overline{z})^*G_0(z)^{-1}B(AR_0(\overline{z}))^*G_0(\zeta)^{-1}BR_0(\zeta) \\
&\quad - (AR_0(\overline{z}))^*G_0(z)^{-1}B(AR_0(\overline{\zeta}))^*G_0(\zeta)^{-1}BR_0(\zeta).
\end{aligned}
$$

Note that

$$G_0(z)^{-1}Q_0(z)G_0(\zeta)^{-1} = G_0(\zeta)^{-1} - G_0(z)^{-1}G_0(\zeta)^{-1}$$

and that

$$G_0(z)^{-1}Q_0(\zeta)G_0(\zeta)^{-1} = G_0(z)^{-1} - G_0(z)^{-1}G_0(\zeta)^{-1}.$$

Hence the last two terms are equivalent to

$$
\begin{aligned}
&(AR_0(\overline{z}))^*G_0(\zeta)^{-1}BR_0(\zeta) - (AR_0(\overline{z}))^*G_0(z)^{-1}G_0(\zeta)^{-1}BR_0(\zeta) \\
&- (AR_0(\overline{z}))^*G_0(z)^{-1}BR_0(\zeta) + (AR_0(\overline{z}))^*G_0(z)^{-1}G_0(\zeta)^{-1}BR_0(\zeta).
\end{aligned}
$$

After cancelling out appropriate terms we have

$$
\begin{aligned}
(z-\zeta)(T(z)-T(\zeta)) &= R_0(z) - R_0(\zeta) - (AR_0(\overline{z}))^*G_0(z)^{-1}BR_0(z) \\
&\quad + (AR_0(\overline{\zeta}))^*G_0(\zeta)^{-1}BR_0(\zeta) \\
&= T(z) - T(\zeta).
\end{aligned}
$$

□

<u>Lemma 2.2</u>: $T(z)^*$ is injective.

Proof: Multiplying (2.5) on the left by A and taking the closure we have

$$AT(z)^* = AR_0(\bar{z}) - Q_0^*(\bar{z})G_0^*(\bar{z})^{-1}AR_0(\bar{z})$$

$$= (I-Q_0^*(\bar{z})G_0^*(\bar{z})^{-1})AR_0(\bar{z})$$

$$= (I-(Q_0^*(\bar{z})+I-I)G_0^*(\bar{z})^{-1})AR_0(\bar{z})$$

$$= (I-I+G_0^*(\bar{z})^{-1})AR_0(\bar{z})$$

$$= G_0^*(\bar{z})^{-1}AR_0(\bar{z}).$$

Thus we have

(2.7) $$AT(z)^* = G_0^*(\bar{z})^{-1}AR_0(\bar{z}).$$

Now let $T(z)^*u = 0$. Then by (2.7) $AR_0(\bar{z})u = 0$ and by (2.5) $R_0(\bar{z})u = 0$. Hence $u=0$. □

Lemma 2.3: $J^*JT(z)^* = T(\bar{z})J^*J$

Proof: Recall that by hypothesis $(B(BJ_0R_0(z))^*)^* = BJ_0(BR_0(\bar{z}))^*$. Thus $BJ_0(BR_0(\bar{z}))^*$ is closed. Since J_0 maps $D(H_0)$ to $D(B)$ these operators are densely defined, and hence $B(BJ_0R_0(z))^*$ and $BJ_0R_0(z)$ are closable. We shall denote their closures by $[B(BJ_0R_0(z))^*]$ and $[BJ_0R_0(z)]$.

Now define $S(z) = G_0(z)BJ_0(BR_0(\bar{z}))^*$. We shall prove that

(2.8) $S(\bar{z})^* = S(z)$. First note that (2.1) implies that

(2.9) $$J_0R_0(z) - R_0(z)J_0 = (BJ_0R_0(\bar{z}))^*AR_0(z) - (AR_0(\bar{z}))^*[BJ_0R_0(z)].$$

Now

$$S(\bar{z})^* = [B(BJ_0R_0(\bar{z}))^*]G_0(\bar{z})^*$$

$$= [B(BJ_0R_0(\bar{z}))^*](1+Q_0(\bar{z})^*)$$

$$= (B(B(R_0(z)J_0+(BJ_0R_0(\bar{z}))^*AR_0(z)))^*)^*$$

$$= (B(B(J_0R_0(z)+(AR_0(\bar{z}))^*\ [BJ_0R_0(z)]))^*)^*$$

$$= S(z) \text{ by } (2.9).$$

Next note that

$$G_0(z)[BJ_0R_0(z)]$$

$$= (1+Q_0(z))[BJ_0R_0(z)]$$

$$= [BJ_0R_0(z)] + [B(AR_0(\bar{z}))^*][BJ_0R_0(z)]$$

$$= BR_0(z)J_0 + [B(BJ_0R_0(\bar{z}))^*]AR_0(z)$$

Thus we have

$$G_0(z)[BJ_0R_0(z)] = BR_0(z)J_0 + [B(BJ_0R_0(\bar{z}))^*]AR_0(z). \tag{2.10}$$

Now let

$$F(z) = (AR_0(\bar{z}))^*([BJ_0R_0(z)] - G_0(z)^{-1}BR_0(z)J_0). \tag{2.11}$$

We shall prove that $F(z)^* = F(\bar{z})$. By (2.10) we have

$$F(z) = (AR_0(\bar{z}))^*([BJ_0R_0(z)] - G_0(z)^{-1}(G_0(z) \times [BJ_0R_0(z)] - [B(BJ_0R_0(\bar{z}))^*]AR_0(z))) \tag{2.12}$$

$$= (AR_0(\overline{z}))^*([BJ_0R_0(z)] - [BJ_0R_0(z)] +$$

$$G_0(z)^{-1}[B(BJ_0R_0(\overline{z}))^*]AR_0(z))$$

$$= (AR_0(\overline{z}))^*G_0(z)^{-1}[B(BJ_0R_0(\overline{z}))^*]AR_0(z)$$

$$= F(\overline{z})^* \text{ by } (2.8).$$

Next note that

$$T(z)J_0 = R_0(z)J_0 - (AR_0(\overline{z}))^*G_0(z)^{-1}BR_0(z)J_0$$

$$= R_0(z)J_0 + F(z) - (AR_0(\overline{z}))^*[BJ_0R_0(z)] \text{ by } (2.11).$$

Thus $T(\overline{z})J_0 = R_0(\overline{z})J_0 + F(\overline{z}) - (AR_0(z))^*[BJ_0R_0(\overline{z})]$.

Similarly

$$J_0T(\overline{z})^* = J_0R_0(z) - J_0(BR_0(\overline{z}))^*G_0^*(z)^{-1}AR_0(z).$$

By (2.11)

$$F(\overline{z})^* = (BJ_0R_0(\overline{z}))^*AR_0(z) - J_0(BR_0(\overline{z}))^*G_0^*(z)^{-1}AR_0(z).$$

Thus $J_0T(\overline{z})^* = J_0R_0(z) + F(\overline{z})^* - (BJ_0R_0(\overline{z}))^*AR_0(z)$

$$= J_0R_0(z) + F(z) - (BJ_0R_0(\overline{z}))^*AR_0(z)$$

$$= R_0(z)J_0 + F(z) - (AR_0(\overline{z}))^*[BJ_0R_0(z)].$$

By (2.9) we have $J_0T(\overline{z})^* = T(z)J_0$. This proves Lemma 2.3. □

Lemma 2.4: If $v \in D(T^*)$ then $\mathrm{Im}(Jv, JT^*v) = 0$.

Proof: Lemma 2.4 is equivalent to

(2.13) $\quad (J(T^*-z)v,Jv) = (Jv,J(T^*-\bar{z})v)$,

Let $u_1 = (T^*-z)v$ and $u_2 = (T^*-\bar{z})v$. Then $v = T^*(z)u_1 = T^*(\bar{z})u_2$.

The expression in (2.13) is equivalent to $(Ju_1, J\,T^*(\bar{z})u_2) =$

$(JT^*(\bar{z})u_1,Ju_2)$ or $(u_1,J^*JT^*(\bar{z})u_2) = (u_1,T(\bar{z})J^*Ju_2)$. This

last equality is satisfied by Lemma 2.3. □

<u>Lemma 2.5</u>: $T^*(z)$ satisfies

(2.14) $\quad \|JT^*(z)u\| \le C\,\|Ju\|$, z non real

<u>Proof</u>: Equivalently we shall show that

$\|Jv\|^2 \le C\,\|J(T^*-z)v\|^2$. Letting $z = x + iy$ we have

$$\|J(T^*-z)v\|^2 = \|JT^*v\|^2 - 2x\mathrm{Re}(Jv,JT^*v) + 2y\mathrm{Im}(Jv,JT^*v) + (x^2+y^2)\,\|Jv\|^2$$

$$= \|JT^*v\|^2 - 2x\mathrm{Re}(Jv,JT^*v) + x^2\|Jv\|^2 + y^2\|Jv\|^2$$

$$\ge (\|JT^*v\|^2 - 2|x|\,\|Jv\|\;\|JT^*v\| + x^2\|Jv\|^2) + y^2\|Jv\|^2$$

$$= (\|JT^*v\| - |x|\,\|Jv\|)^2 + y^2\,\|Jv\|^2 \ge y^2\|Jv\|^2$$

This proves the lemma. □

Now put

(2.15) $\quad R(z)Ju = JT^*(z)u$. By Lemma 2.5 $Ju = 0$ implies that $JT^*(z)u = 0$. Therefore $R(z)Ju = 0$ by (2.15). This implies that $R(z)$ is well defined on $R(J)$. We can now define $R(z)$ on $\overline{R(J)}$. Since J has dense range, for any $u \in \mathcal{H}$ we may set

(2.16) $u = \lim_{n\to\infty} Ju_n$ and define

(2.17) $R(z)u = \lim_{n\to\infty} R(z)Ju_n$. Since

$\|R(z)Ju_n - R(z)Ju_m\| = \|JT^*(z)u_n - JT^*(z)u_m\| = \|JT^*(z)(u_n-u_m)\|$

$\leq C\|J(u_n-u_m)\|$ by Lemma 2.5 and this last expression approaches zero by (2.16), we see that the limit in (2.17) exists. To show that it is unique let

(2.18) $u = \lim_{n\to\infty} Ju_n = \lim_{n\to\infty} Jv_n$ and

(2.19) $w_1 = \lim_{n\to\infty} JT^*(z)u_n$, $w_2 = \lim_{n\to\infty} JT^*(z)v_n$.

Then $\|w_1-w_2\| = \|w_1-JT^*(z)u_n + JT^*(z)u_n - JT^*(z)v_n + JT^*(z)v_n - w_2\|$

$\leq \|w_1 - JT^*(z)u_n\| + \|JT^*(z)u_n - JT^*(z)v_n\| + \|JT^*(z)v_n - w_2\|$.

The first and third terms approach zero by (2.19). Moreover,

$\|JT^*(z)u_n - JT^*(z)v_n\| \leq C\|Ju_n - Jv_n\|$ by Lemma 2.5, and

$\|Ju_n - Jv_n\| \to 0$ by (2.18). Hence $w_1 = w_2$ and the limit is unique. In addition we show that $u = 0$ implies that

$R(z)u = 0$. Let $\lim_{n\to\infty} Ju_n = 0$. Then $R(z)u = \lim_{n\to\infty} JT^*(z)u_k$

but $\|JT^*(z)u_k\| \leq C\|Ju_k\| \to 0$. So $R(z)$ is well defined

on $\overline{R(J)}$. By (2.15) $\|R(z)Ju\| = \|JT^*(z)u\| \leq C\|Ju\|$ by Lemma 2.5 so we see that $R(z)$ is a bounded operator on $R(J)$. Clearly it is also bounded on $\overline{R(J)} = \mathcal{H}$.

Lemma 2.6: R(z) has nullity zero.

Proof: Let $R(z)Ju = 0$. Then $JT^*(z)u = 0$ by (2.15) and $T(z)J^*Ju = 0$ by Lemma 2.3. Since $T(z)$ and J^* are injective we have $Ju = 0$. To show that $R(z)$ is one-one on $\overline{R(J)}$ we let $w = \lim_{k\to\infty} Ju_k$ and $R(z)w = 0$. Then $\lim_{k\to\infty} R(z)Ju_k = 0$. This implies by (2.16) that $\lim_{k\to\infty} JT^*(z)u_k = 0$. Hence by Lemma 2.3 $\lim_{k\to\infty} T(z)J^*Ju_k = 0$. Thus we have $T(z)J^*w = 0$. But both $T(z)$ and J^* are one-one and hence $w = 0$. □

Lemma 2.7: R(z) satisfies the first resolvent equation.

Proof: On $R(J)$ we have

$$\begin{aligned} R(z)Ju - R(\zeta)Ju &= JT^*(z)u - JT^*(\zeta)u \\ &= J(T^*(z)u - T^*(\zeta)u) = (\zeta-z)JT^*(z)T^*(\zeta)u \\ &= (\zeta-z)R(z)JT^*(\zeta)u = (\zeta-z)R(z)R(\zeta)Ju. \end{aligned}$$

Similarly on $\overline{R(J)}$ let $w = \lim_{k\to\infty} Ju_k$.

Then $R(z)w - R(\zeta)w = \lim_{k\to\infty} R(z)Ju_k - R(\zeta)Ju_k$.

$$\begin{aligned} &= \lim_{k\to\infty} (\zeta-z)R(z)R(\zeta)Ju_k \\ &= (\zeta-z)R(z)R(\zeta)w \end{aligned}$$ □

Lemma 2.8: $R(z)^* = R(\overline{z})$

Proof: We shall first show that Lemma 2.8 holds on $R(J)$.

We have

$$R(z)^*Ju = J^{*-1}J^*R(z)^*Ju$$

$$= J^{*-1}T(\bar{z})J^*Ju$$

$$= J^{*-1}J^*JT(z)^*u$$

$$= JT(z)^*u$$

$$= R(\bar{z})Ju \text{ by (2.15) and Lemma 2.3.}$$

On $R(J)$ let $w_1 = \lim_{k\to\infty} Ju_k$. Then $R(z)^*w = \lim_{k\to\infty} R(z)^*Ju_k$

$$= \lim_{k\to\infty} R(\bar{z})Ju_k = R(\bar{z})w. \qquad \square$$

<u>Lemma 2.9</u>: $R(z)$ has dense range.

<u>Proof</u>: We shall first prove that it suffices to show that $R(z)^*$ is one-one. Let $R(z)^*$ be one-one and $(R(z)u,v) = 0$ for all $u \in D(R(z))$. By the definition of the adjoint $v \in D(R(z))^*$ and $R(z)^*v = 0$. This implies that $v = 0$. Thus $R(z)$ has dense range.

We shall now show that $R(z)^*$ is one-one. Since $R(z)^* = R(\bar{z})$ by Lemma 2.8, and $R(z)$ is one-one by Lemma 2.6, we have that $R(z)^*$ is also one-one. This proves the lemma. $\square$

By Lemmas 2.5 through 2.9 we may define $H - z = R(z)^{-1}$ where H has dense domain. Since $R(z)$ satisfies the first resolvent equation this determines H uniquely. Since $R(z)$ is closed, H is closed. In addition $(H-z)^* = (R(z)^{-1})^* = (R(z)^*)^{-1} = R(\bar{z})^{-1} = H - \bar{z}$. Hence $H^* = H$ and H is self adjoint.

Before proving that $R(z)$ satisfies the second resolvent equation, we shall prove that $T^*(z)$ satisfies the second resolvent equations

$$(2.20) \qquad T(z)^* - R_0(\bar{z}) = -(BR_0(z))^*AT(z)^*$$

and

$$(2.21) \qquad T(z)^* - R_0(\bar{z}) = -(BT(z))^*AR_0(\bar{z})$$

We have by 2.5

$$(2.22) \qquad T^*(\bar{z}) - R_0(\bar{z}) = -(BR_0(z))^*G_0{}^*(\bar{z})^{-1}AR_0(\bar{z})$$

$$= -(BR_0(z))^*AT(z)^* \text{ by } (2.7).$$

By the definition of $T(z)$ we also have

$$T(z) - R_0(z) = -(AR_0(\bar{z}))^*G_0(z)^{-1}BR_0(z)$$

$$= -(AR_0(\bar{z}))^* BT(z) \text{ by } (2.6).$$

Taking the adjoint

$$T(z)^* - R_0(\bar{z}) = -(BT(z))^* AR_0(\bar{z}) ,$$

This proves (2.21).

We can now prove that $R(z)$ satisfies (2.2). We have

$$R(z)Ju - JR_0(z)u$$

$$= JT^*(z)u - JR_0(z)u$$

$$= J(T^*(z)u - R_0(z)u)$$

$$= -J(BT(\overline{z}))^*AR_0(z)$$

$$= -(BT(\overline{z})J^*)^*\ AR_0(z)$$

$= -(BJ^*R(\overline{z}))^*AR_0(z)$ and this proves the second resolvent equation.

We are now ready to prove that $HJ \supset J(H_0+B^*A)$. Let $u \in D(J(H_0+B^*A))$. Since $u \in D(H_0)$ we may let $v = (H_0-z)u$. Applying (2.2)

$$R(z)Jv - JR_0(z)v = -(BJ^*R(\overline{z}))^*\ AR_0(z)v.$$

Thus

$$R(z)Jv - Ju = -(BJ^*R(\overline{z}))^*\ A\,R_0\,(z)v.$$

Hence $\qquad Ju = R(z)Jv + (BJ^*R(\overline{z}))^*Au$

$\qquad\qquad = R(z)Jv + R(z)JB^*Au$ for $u \in D(J(H_0+B^*A))$.

So $Ju \in D(H)$ and $(H-z)Ju = Jv + JB^*Au$.

$\qquad\qquad = J(H_0-z)u + JB^*Au.$

This gives $HJ \supset J(H_0+B^*A)$.

We can now show that (2.2) determines H uniquely on R(J). We have by (2.2) and (2.5)

$$R(z)Ju = JR_0(z)u - (BJ^*R(\overline{z}))^*AR_0(z)u$$

$$= (J - (BJ^*\ R(\overline{z}))^*A)R_0(z)u$$

$$= (J - (BJ^*R(\overline{z}))^*A)(T(\overline{z})^* + (BR_0(\overline{z}))^*\ AT(\overline{z})^*) \text{ by (2.20).}$$

Thus $R(z)Ju = JT^*(z)u + J(BR_0(\overline{z}))^* AT(\overline{z})^*$

$$- (BJ^*R(\overline{z}))^*AT(\overline{z})^*$$

$$- (BJ^*R(\overline{z}))^* A(BR_0(\overline{z}))^* AT(\overline{z})^* .$$

The last three terms are equivalent to

$$J(BR_0(\overline{z}))^* AT(\overline{z})^* - (BJ^*R(\overline{z}))^* G_0^*(z)AT(\overline{z})^*$$

$$= J(BR_0(\overline{z}))^* AT(\overline{z})^* - J(BT(\overline{z}))^* AR_0(z)$$

and this is zero by (2.20) and (2.21). Thus $R(z)Ju = JT^*(z)u$.

CHAPTER III

APPLICATIONS

In this chapter we give a few applications of Theorem 2.1 and show that the wave operator $Wf = \lim_{t\to\infty} e^{itH} J e^{-itH_0} f$ exists and is complete.

3.1 We shall prove the following theorem.

Theorem 3.1: Let H_0 be the self-adjoint operator associated with D^2 in $\mathcal{H}_0 = L^2(-\infty,\infty)$ where $D = \frac{1}{i}\frac{d}{dx}$. We let Ju be the restriction of u to $(0,\infty)$. Thus J maps $\mathcal{H}_0$ to $\mathcal{H} = L^2(0,\infty)$. Take $Au = \{-u'(0+), q_1 u\}$ and $Bv = \{v(0+), q_2 v\}$ where q_1 and q_2 are real valued functions in L^2 and $A,B: \mathcal{H}_0 \to \mathbb{C} \oplus L^2(-\infty,\infty)$ with

$$D(A) = \{u\in L^2/q_1 u, u'\in L^2 \text{ and } \lim_{x\to 0+} u'(x) \text{ exists}\},$$

$$D(B) = \{u\in L^2/q_2 u, u'\in L^2\}.$$

Then there exists an operator H with the properties of Theorem 2.1.

Proof: To prove the existence of H we first note that since $q_1, q_2 \in L^2$

$$\sup_x \int_x^{x+1} |q_i(y)|^2 dy < \infty \qquad \text{for } i = 1,2$$

and thus $D(H_0) \subset D(q_1) \cap D(q_2)$ [5, p. 35]. If $u\in D(H_0)$ then $u, u'' \in L^2$ and thus $u'\in L^2$. Hence $\lim_{x\to 0+} u'(x)$ and $\lim_{x\to 0+} u(x)$ exists [5, p. 38].

We have that $D(H_0) \subset D(A) \cap D(B)$. Next note that

$$(Ju,v) = \int_0^\infty u(x)\overline{v(x)}dx = (u,J^*v)$$

Thus
$$J^*V = \begin{cases} v \text{ if } x > 0 \\ 0 \text{ if } x < 0 \end{cases}$$

and
$$J^*J = \chi_{(0,\infty)}$$

We now verify (2.1)

$$\text{Im}[JH_0u,Ju) + (Au,BJ^*Ju)]$$

$$(3.1) \qquad = \text{Im}\left[\int_0^\infty (D^2u(x))\overline{u(x)}dx - u'(0+)u(0+) + \int_0^\infty q_1(x)q_2(x)|u(x)|^2dx\right].$$

The first term in (3.1) is $-\text{Im}\int_0^\infty u''(x)\overline{u(x)}dx$,

Integrating by parts this is

$$(3.2) \qquad -\text{Im}(u'(x)\overline{u(x)}\Big|_0^\infty - \int_0^\infty |u'(x)|^2dx).$$ Since $u, u' \in L^2$ the first term in (3.2) is $\text{Im } u'(0+)\overline{u(0+)}$. Since $\int_0^\infty |u'(x)|^2dx$ is real the second term in (3.2) is zero. Thus the sum of the first two terms in (3.1) is zero. The third term in (3.1) is zero because q_1 and q_2 are real valued functions. Thus we have that (3.1) equals zero and this proves (2.1).

In verifying some of the remaining conditions of Theorem 2.1 we shall use the formula [5, p. 160].

$$(3.3) \qquad R_0(z)u(x) = \frac{1}{2ki}\int_{-\infty}^\infty e^{ik|x-y|}u(y)dy$$ where $k^2 = z$ and $\text{Im } k > 0$.

Note that $(-\bar{k})^2 = \bar{z}$ and $\mathrm{Im}(-\bar{k}) > 0$ so

$$R_0(\bar{z})u(x) = -\frac{1}{2\bar{k}i}\int_{-\infty}^{\infty} e^{-i\bar{k}|x-y|}u(y)dy .$$

Also note that

$$\frac{d}{dx}R_0(z)u = \frac{d}{dx}\left(\frac{1}{2ki}\int_{-\infty}^{\infty} e^{ik|x-y|}u(y)dy\right)$$

$$= \frac{1}{2ki}\frac{d}{dx}\left[\int_{-\infty}^{x} e^{ik(x-y)}u(y)dy + \int_{x}^{\infty} e^{ik(y-x)}u(y)dy\right]$$

$$= \frac{1}{2ki}\left[e^{ikx}e^{-ikx}u(x) + ik\int_{-\infty}^{x} e^{ik(x-y)}u(y)dy\right.$$

$$\left. - e^{-ikx}e^{ikx}u(x) - ik\int_{x}^{\infty} e^{ik(y-x)}u(y)dy\right] .$$

Hence

(3.4) $$\frac{d}{dx}R_0(z)u = \frac{1}{2}\left[\int_{-\infty}^{x} e^{ik(x-y)}u(y)dy - \int_{x}^{\infty} e^{ik(y-x)}u(y)dy\right]$$

and

(3.5) $$\frac{d}{dx}R_0(\bar{z})u = \frac{1}{2}\left[\int_{-\infty}^{x} e^{-i\bar{k}(x-y)}u(y)dy - \int_{x}^{\infty} e^{-i\bar{k}(y-x)}u(y)dy\right].$$

We have $$AR_0(\bar{z})u = \{(-R_0(\bar{z})u)'(0+),\ q_1R_0(\bar{z})u\}$$

$$= \{-\frac{1}{2}\left[\int_{-\infty}^{0} e^{i\bar{k}y}u(y)dy - \int_{0}^{\infty} e^{-i\bar{k}y}\right.$$

$$\left. u(y)dy,\ q_1R_0(\bar{z})u\right\} .$$

Thus $$(A\,R_0(\bar{z})u, \{v_1,v_2\}) = \frac{-\bar{v}_1}{2}\int_{-\infty}^{0} e^{i\bar{k}y}u(y)dy + \frac{\bar{v}_1}{2}\int_{0}^{\infty} e^{-i\bar{k}y}u(y)dy$$

$$+ (q_1R_0(\bar{z})u, v_2)$$

We can now show that $a \| AR_0(s+ia)\|^2 \le C_I$.

We have $AR_0(z)u$

$$= \left\{-\frac{1}{2}\left[\int_{-\infty}^{0} e^{-iky}u(y)dy - \int_0^\infty e^{iky}u(y)dy\right],\ q_1R_0(z)u\right\}$$

Thus

$$\|AR_0(z)u\|^2 = \frac{1}{4}\left|\int_{-\infty}^{0} e^{-iky}u(y)dy - \int_0^\infty e^{iky}u(y)dy\right|^2$$

$$+ \|q_1R_0(z)u\|^2$$

(3.6)
$$\le \frac{1}{2}\left|\int_{-\infty}^{0} e^{-iky}u(y)dy\right|^2 + \frac{1}{2}\left|\int_0^\infty e^{iky}u(y)dy\right|^2$$

$$+ \|q_1R_0(z)u\|^2$$

The first term in (3.6) is bounded by

$$\frac{1}{2}\left(\int_{-\infty}^{0} e^{\eta y}|u(y)|dy\right)^2 \le \frac{1}{2}\int_{-\infty}^{0} e^{\eta y}dy \int_{-\infty}^{0} e^{\eta y}|u(y)|^2dy$$

$\le \frac{1}{2\eta}\int_{-\infty}^{0} e^{\eta y}|u(y)|^2dy$ where $k = \sigma + i\eta$ and $z = s + ia$

$= \sigma^2 - \eta^2 + i(2\sigma\eta)$. Since $a = 2\sigma\eta$ this is

$$\frac{\sigma}{a}\int_{-\infty}^{0} e^{\eta y}|u(y)|^2\, dy \le \frac{\sigma}{a}\|u\|^2.$$

The second term in (3.6) is bounded by

$$\frac{1}{2}\left(\int_0^\infty e^{-\eta y}|u(y)|dy\right)^2 \le \frac{1}{2}\left(\int_0^\infty e^{-\eta y}dy\right)\int_0^\infty e^{-\eta y}|u(y)|^2dy$$

$$\leq \frac{1}{2\eta} \|u\|^2 = \frac{\sigma}{a} \|u\|^2.$$

The last term in (3.6) is

$$\frac{1}{4|k|^2} \int |q_1(x)|^2 \left|\int_{-\infty}^{\infty} e^{ik|x-y|} u(y)dy\right|^2 dx$$

$$\leq \frac{1}{4|k|^2} \int |q_1(x)|^2 \left(\int e^{-2\eta|x-y|} dy\right)\left(\int |u(y)|^2 dy\right) dx$$

$$\leq \frac{1}{4|k|^2\eta} \|q_1\|^2 \|u\|.$$

Since $|\sigma| \leq |k|$ this is bounded by $\frac{1}{4|k|\sigma\eta} \|q_1\|^2 \|u\|^2$.

But $a = 2\sigma\eta$ so this is $\frac{1}{2a|k|}\|q_1\|^2\|u\|^2$. If we let $\Lambda = R - \{0\}$

then $I \subset\subset \Lambda$ must be bounded away from zero. Letting $C > 0$

be a lower bound for $|s|$ in I we have $|z|^2 = s^2 + a^2 \geq s^2 \geq C^2$.

Thus $|z| \geq |C|$ and $|k| \geq |C|^{\frac{1}{2}}$. Hence $\frac{1}{|k|} \leq |C|^{-\frac{1}{2}}$. We have

that (3.6) is bounded by

$$\frac{\sigma}{a}\|u\|^2 + \frac{\sigma}{a}\|u\|^2 + \frac{1}{2a|C|^{\frac{1}{2}}} \|q_1\|^2\|u\|^2 = \frac{C_I}{a} \|u\|^2$$

and therefore

$$a\|AR_0(s+ia)\|^2 \leq C_I .$$

We now show that $a\|BR_0(s+ia)\|^2 \leq C_I$.

We have

$$(3.7) \qquad \|BR_0(s+ia)u\|^2 = \left|\frac{1}{2ki}\int_{-\infty}^{\infty} e^{ik|y|}u(y)dy\right|^2$$

$$+ \|q_2R_0(z)u\|^2$$

The first term on the right hand side of (3.7) is bounded by

$$\frac{1}{4|k|^2}\left(\int_{-\infty}^{\infty} e^{-\eta y}|u(y)|dy\right)^2$$

$$\leq \frac{1}{4|k|^2}\int_{-\infty}^{\infty} e^{-2\eta y}dy \int_{-\infty}^{\infty}|u(y)|^2 dy$$

$$= \frac{1}{4|k|^2\eta}\|u\|^2$$

$$\leq \frac{1}{4|k|\sigma\eta}\|u\|^2$$

$$\leq \frac{1}{2a|k|}\|u\|^2 \quad .$$

The last term of (3.7) is bounded by

$$\frac{1}{4|k|^2}\left(\int_{-\infty}^{\infty} e^{-\eta y}|u(y)|dy\right)^2$$

This proves the third condition of Theorem 2.1.

Next we show that $B(AR_0(\bar{z}))^*$ is closable. We have already shown that

$$(AR_0(\bar{z})u, \{v_1,v_2\}) = \frac{-\bar{v}_1}{2}\int_{-\infty}^{0} e^{i\bar{k}y}u(y)dy + \frac{\bar{v}_1}{2}\int_{0}^{\infty} e^{-i\bar{k}y}u(y)dy$$

$$+ (q_1R_0(\overline{z})u,\ v_2)$$

$$= (u, \frac{-v_1}{2}\chi_{(-\infty,0)}e^{-iky} + \frac{v_1}{2}\chi_{(0,\infty)}e^{iky} + R_0(z)q_1v_2)\ .$$

So $(AR_0(\overline{z}))^*\{v_1,v_2\} = \dfrac{-v_1}{2}\,e^{-iky}\,\chi_{(-\infty,0)} + \dfrac{v_1}{2}\chi_{(0,\infty)}e^{iky} + R_0(z)q_1v_2.$

Applying B we have

$$(3.8)\qquad B(AR_0(\overline{z}))^*\{v_1,v_2\} = \Big\{\frac{v_1}{2} + \frac{1}{2ki}\int_{-\infty}^{\infty} e^{ik|y|}q_1(y)v_2(y)dy,$$

$$\frac{-q_2}{2}\,v_1e^{-iky}\,\chi_{(-\infty,0)} + \frac{q_2}{2}\,v_1\chi_{(0,\infty)}e^{iky}$$

$$+ q_2R_0(z)q_1v_2\Big\}.$$

Hence

$$\|B(AR_0(\overline{z}))^*\{v_1,v_2\}\| = (|\frac{v_1}{2} + \frac{1}{2ki}\int_{-\infty}^{\infty} e^{ik|y|}q_1(y)v_2(y)dy|^2$$

$$+ \left\|\frac{-q_2}{2}\,v_1e^{-iky}\,\chi_{(-\infty,0)} + \frac{q_2}{2}\,v_1\,\chi_{(0,\infty)}e^{iky} + q_2R_0(z)q_1v_2\right\|^2)^{\frac{1}{2}}$$

$$\leq |\frac{v_1}{2} + \frac{1}{2ki}\int_{-\infty}^{\infty} e^{ik|y|}q_1(y)v_2(y)dy|$$

$$+ \left\|\frac{-q_2}{2}\,v_1e^{-iky}\,\chi_{(-\infty,0)} + \frac{q_2}{2}\,v_1\,\chi_{(0,\infty)}e^{iky} + q_2R_0(z)q_1v_2\right\|$$

$$(3.9)\qquad \leq \frac{1}{2}|v_1| + \frac{1}{2|k|}\int_{-\infty}^{\infty} e^{-t|y|}|q_1(y)v_2(y)|dy$$

$$+ \left\|\frac{q_2}{2} v_1 e^{-iky} \chi_{(-\infty,0)}\right\| + \left\|\frac{q_2}{2} v_1 \chi_{(0,\infty)} e^{iky}\right\| + \left\|q_2 R_0(z) q_1 v_2\right\|$$

where $k = s + it$. The second term in (3.9) is

$$\le \frac{1}{2|k|} \int_{-\infty}^{\infty} |q_1(y)|\,|v_2(y)|\,dy \le \frac{1}{2|k|} \|q_1\|\,\|v_2\| = C_1(z)\|v_2\|$$

where $C_1(z) \to 0$ as $\operatorname{Im} z \to \infty$. The square of the third term in (3.9) is

$$\frac{v_1^2}{4} \int_{-\infty}^{0} |q_2(y)|^2 e^{2ty} dy$$

$$= \frac{v_1^2}{4} \left[\int_{-\infty}^{-\varepsilon} |q_2(y)|^2 e^{2ty} dy + \int_{-\varepsilon}^{0} |q_2(y)|^2 e^{2ty} dy\right]$$

$$\le \frac{v_1^2}{4} \left[e^{-2t\varepsilon}\|q_2\|^2 + \int_{-\varepsilon}^{0} |q_2(y)|^2 dy\right].$$

Letting $\varepsilon = \frac{1}{\sqrt{t}}$ and $C_\varepsilon = \int_{-\varepsilon}^{0} |q_2(y)|^2 dy$ we note that

$C_\varepsilon \to 0$ as $\varepsilon \to 0$. We have

$$\left\|\frac{q_2}{2} v_1 e^{-iky} \chi_{(-\infty,0)}\right\| \le \frac{|v_1|}{2} \sqrt{e^{-2\sqrt{t}}\|q^2\|^2 + C_\varepsilon}$$

$$= C_2(z)|v_1| \text{ where } C_2(z) \to 0 \text{ as } t \to \infty.$$

The square of the fourth term in (3.9) is bounded by

$$\frac{v_1^2}{4} \int_{0}^{\infty} |q_2(y)|^2 e^{-2ty} dy$$

$$= \frac{v_1^2}{4} \left[\int_{0}^{\varepsilon} |q_2(y)|^2 e^{-2ty} dy + \int_{\varepsilon}^{\infty} |q_2(y)|^2 e^{-2ty} dy\right]$$

$$\leq \frac{v_1{}^2}{4}\left[\int_0^{\varepsilon}|q_2(y)|^2 dy + e^{-2t\varepsilon}\int_{\varepsilon}^{\infty}|q_2(y)|^2 dy\right.$$

$$\leq \frac{v_1{}^2}{4}\left[\int_0^{\varepsilon}|q_2(y)|^2 dy + e^{-2t\varepsilon}\|q_2\|^2\right].$$

Now letting $C_\varepsilon = \int_0^{\varepsilon}|q_2(y)|^2 dy$ we note that $C_\varepsilon \to 0$ as $\varepsilon \to 0$.

We have

$$\left\|\frac{q_2}{2} v_1 \chi_{(0,\infty)} e^{iky}\right\| \leq C_3(z)|v_1|$$

where $C_3(z) \to 0$ as $t \to \infty$. We shall show that the last norm in (3.9) is less than or equal to $C_4(z)\|v_2\|$ where $C_4(z) \to 0$ as $\operatorname{Im} z \to \infty$. By (3.3)

$$|(R_o(z)q_1u, q_2v)|$$

$$= \left|\frac{1}{2k}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik|x-y|} q_1(y)u(y)dy\ \overline{q_2(x)v(x)}\ dx\right.$$

$$\leq \frac{1}{2|k|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-t|x-y|}|q_1(y)v(x)||q_2(x)u(y)|dxdy$$

$$\leq \frac{1}{2|k|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|q_1(y)||v(x)||q_2(x)u(y)|dxdy$$

$$\leq \frac{1}{2|k|}\left(\int\int|q_1(y)v(x)|^2 dxdy\right)^{\frac{1}{2}}\left(\int\int|q_2(x)u(y)|^2 dxdy\right)^{\frac{1}{2}}$$

$$= \frac{1}{2|k|}\ \|q_1\|\ \|v\|\ \|q_2\|\ \|u\| \quad .$$

Hence $\|q_2R_0(z)q_1v_2\| \leq \dfrac{1}{2|z|^{1/2}}\|q_1\|\ \|q_2\|\ \|v_2\| = C_4(z)\|v_2\|$.

Thus $\| B(AR_0(z))^*\{v_1,v_2\}\| \le \frac{1}{2}|v_1| + C_1(z)\|v_2\|$

$$+ C_2(z)|v_1| + C_3(z)|v_1| + C_4(z)\|v_2\|$$

$$= |v_1|(\tfrac{1}{2}+C_5(z)) + \|v_2\| C_6(z)$$

where $C_i(z) \to 0$ as $t \to \infty$. We have

$$\| B(AR_0(\bar{z}))^*\{v_1,v_2\}\|$$

$$\le \sqrt{|v_1|^2+\|v_2\|^2}\ \sqrt{(\tfrac{1}{2}+C_5(z))^2+C_6(z)^2}$$

$$= \|\{v_1,v_2\}\|\ \sqrt{(\tfrac{1}{2}+C_5(z))^2+C_6(z)^2}$$

Note that the constant on the right is less than one for t sufficiently large. This shows that $B(AR_0(\bar{z}))^*$ is bounded and therefore closable, that $G_0(z)$ is bounded and that $G_0(z)^{-1}$ is bounded for $\operatorname{Im} z$ sufficiently large.

In addition we can show that $G_0(z)$ is the sum of an invertible operator and a compact operator. We have

$$(I+B(AR_0(z))^*)\{v_1,v_2\}$$

$$= \left\{\frac{3v_1}{2} + \frac{1}{2ki}\int_{-\infty}^{\infty} e^{ik|y|} q_1(y)v_2(y)\,dy,\right.$$

$$\left. - \frac{q_2}{2} v_1 e^{-iky}\chi_{(-\infty,0)} + \frac{q_2}{2} v_1\chi_{(0,\infty)}e^{iky} + (I+q_2R_0(z)q_1v_2\right\}$$

$$(3.10)\qquad \left\{\frac{3v_1}{2},\ v_2\right\} + \left\{\frac{1}{2ki}\int_{-\infty}^{\infty} e^{ik|y|} q_1(y)v_2(y)dy,\right.$$

$$\left. \frac{-q_2}{2} v_1 e^{-iky}\chi_{(-\infty,0)} + \frac{q_2}{2} v_1 \chi_{(0,\infty)} e^{iky} + q_2 R_0(z) q_1 v_2 \right\} .$$

The operator $\left\{\frac{3v_1}{2},\ v_2\right\}$ is invertible. We shall show that the second operator is compact. Let $\|v_{2,n}\| \le C$. Then

$$\left| \frac{1}{2ki}\int_{-\infty}^{\infty} e^{ik|y|} q_1(y)v_{2,n}(y)dy \right| \le \frac{1}{2|k|}\int_{-\infty}^{\infty} e^{-t|y|} |q_1(y)v_{2,n}(y)|dy$$

$$\le \frac{1}{2|k|} \|q_1\|\|v_{2,n}\| \le C_z \|v_{2,n}\| \text{ where}$$

C_z is a constant depending on z. Hence

$$\frac{1}{2ki} \int_{-\infty}^{\infty} e^{iky} q_1(y)v_{2,n}(y)dy$$

has a convergent subsequence.

Now let $|v_{1,n}| \le C$. Then a subsequence of $v_{1,n}$ converges to v_1 and

$$\left\| \frac{q_2}{2} e^{-iky}\chi_{(-\infty,0)}(v_{1,n}-v_1) \right\| \le |v_{1,n}-v_1| \left\| \frac{q_2}{2} e^{-iky}\chi_{(-\infty,0)} \right\|$$

and this approaches zero as $n \to \infty$. Similarly if $|v_{1,n}| \le C$ then $\frac{q_2}{2} v_{1,n}\chi_{(0,\infty)}e^{iky}$ has a convergent subsequence.

Since $q_2R_0(z)$ is compact [5, p. 185] $q_2R_0(z)q_1 =$

$q_2|R_0(i)|^{\frac{1}{2}}\ (q_1|H_0-i|R_0(z))^*$ is the product of a compact operator and a bounded operator and hence it is compact. Thus (3.10) is the sum of an invertible operator and a compact operator. By the first resolvent equation and the fact that $G_0(z)$ is continuous, as shall be shown later on, we have that $G_0(z)$ is analytic. Thus $G_0(z)^{-1}$ is bounded and defined everywhere outside of some finite discrete set.

We now verify that $B(BJ_0R_0(z))^* = BJ_0(BR_0(\bar{z}))^*$. We shall compute $B(BJ_0R_0(z))^*$. We have

$$J_0R_0(z) = \frac{\chi_{(0,\infty)}}{2ki} \int_{-\infty}^{\infty} e^{ik|x-y|}u(y)dy.$$

Thus

$$BJ_0R_0(z) = \left\{\frac{1}{2ki} \int_{-\infty}^{\infty} e^{ik|y|}u(y)dy, q_2\chi_{(0,\infty)}R_0(z)u\right\},$$

Hence

$$(BJ_0R_0(z)u,\ \{v_1,v_2\}) = \frac{\bar{v}_1}{2ki} \int_{-\infty}^{\infty} e^{ik|y|}u(y)dy + (q_2\chi_{(0,\infty)}R_0(z)u,V_2)$$

$$= (u,\ \frac{-v_1}{2\bar{k}i}\ e^{-i\bar{k}|y|} + R_0(\bar{z})\chi_{(0,\infty)}q_2v_2).$$

Consequently

$$(BJ_0R_0(z))^*\{v_1,v_2\} = \frac{-v_1}{2\bar{k}i}\ e^{-i\bar{k}|y|} + R_0(\bar{z})\chi_{(0,\infty)}q_2v_2.$$

Since

$$R_0(\bar{z})u = \frac{-1}{2\bar{k}i} \int_{-\infty}^{\infty} e^{-i\bar{k}|x-y|}u(y)dy$$

we have

$$B[BJ_0R_0(z)]^*\{v_1,v_2\} = \left\{\frac{-v_1}{2\bar{k}i} - \frac{1}{2\bar{k}i}\int_{-\infty}^{\infty} e^{-i\bar{k}|y|}\chi_{(0,\infty)}(y)q_2(y)v_2(y)dy,\right.$$

$$\left.\frac{-q_2}{2\bar{k}i}v_1e^{-i\bar{k}|y|} + q_2R_0(\bar{z})\chi_{(0,\infty)}q_2v_2\right\}$$

Let $B[BJ_0R_0(z)]^* = \{F_{11}v_1 + F_{12}v_2,\ F_{21}v_1 + F_{22}v_2\}$ where

$$F_{11}v_1 = \frac{-v_1}{2\bar{k}i},\ F_{12}v_2 = -\frac{1}{2\bar{k}i}\int_{-\infty}^{\infty} e^{-i\bar{k}|y|}\chi_{(0,\infty)}(y)q_2(y)v_2(y)dy,$$

$$F_{21}v_1 = \frac{-q_2v_1}{2\bar{k}i}e^{-i\bar{k}|y|} \text{ and } F_{22}v_2 = q_2R_0(\bar{z})\chi_{(0,\infty)}q_2v_2.$$

We now compute $BJ_0(BR_0(\bar{z}))^*$. We have

$$BR_0(\bar{z})u = \left\{-\frac{1}{2\bar{k}i}\int_{-\infty}^{\infty} e^{-i\bar{k}|y|}u(y)dy,\ q_2R_0(\bar{z})u\right\}$$

and

$$(BR_0(\bar{z})u,\ \{v_1,v_2\}) = \frac{-\bar{v}_1}{2\bar{k}i}\int_{-\infty}^{\infty} e^{-i\bar{k}|y|}u(y)dy + (q_2R_0(\bar{z})u,\ v_2)$$

$$= (u,\ \frac{v_1}{2ki}e^{ik|y|} + R_0(z)q_2v_2).$$

Thus $(BR_0(\bar{z}))^* = \frac{v_1}{2ki}e^{ik|y|} + R_0(z)q_2v_2$ and

$$J_0(BR_0(\bar{z}))^* = \chi_{(0,\infty)}\frac{v_1}{2ki}e^{ik|y|} + \chi_{(0,\infty)}R_0(z)q_2v_2.$$

Consequently

$$BJ_0(BR_0(\bar{z}))^* = \left\{\frac{v_1}{2ki} + \frac{1}{2ki}\int_{-\infty}^{\infty} e^{ik|y|}q_2(y)v_2(y)dy,\right.$$

$$\left. q_2\chi_{(0,\infty)}\frac{v_1}{2ki}e^{ik|y|} + q_2\chi_{(0,\infty)}R_0q_2v_2\right\}.$$

$$= \{G_{11}v_1 + G_{12}v_2,\ G_{21}v_1 + G_{22}v_2\}$$

where

$$G_{11}v_1 = \frac{v_1}{2ki},\quad G_{12}v_2 = \frac{1}{2ki}\int_{-\infty}^{\infty} e^{ik|y|}q_2(y)v_2(y)dy,$$

$$G_{21}v_1 = q_2\chi_{(0,\infty)}\frac{v_1}{2ki}e^{ik|y|} \quad\text{and}\quad G_{22}v_2 = q_2\chi_{(0,\infty)}R_0(z)q_2v_2\,.$$

To show that $(B(BJ_0R_0(z))^*)^* = BJ_0(BR_0(\bar{z}))^*$ it suffices to show that $F_{ij}^* = G_{ji}$. It is clear that $F_{11}^* = G_{11}$ and that $F_{22}^* = G_{22}$. We have

$$(F_{12}v_2,v_1)_{\mathbb{C}} = \frac{-\bar{v}_1}{2\bar{k}i}\int_{-\infty}^{\infty} e^{-i\bar{k}|y|}\chi_{(0,\infty)}(y)q_2(y)v_2(y)dy$$

$$= (v_2,\ \frac{v_1}{2ki}e^{ik|y|}\chi_{(0,\infty)}q_2)_{L^2} = (v_2,G_{21}v_1).$$

Hence $F_{12}^* = G_{21}$. We also have

$$(F_{21}v_1,v_2)_{L^2} = -\int_{-\infty}^{\infty}\frac{q_2v_1}{2\bar{k}i}e^{-i\bar{k}|y|}\overline{v_2(y)dy}$$

$$= v_1 \overline{\int_{-\infty}^{\infty} \frac{q_2}{2ki} e^{ik|y|} v_2(y)dy}$$

$$= (v_1, \int_{-\infty}^{\infty} \frac{q_2}{2ki} e^{ik|y|} v_2(y)dy)_{\mathbb{C}} = (v_1, G_{12}v_2).$$

Hence $F_{21}{}^* = G_{21}$ and the last condition is satisfied. □

To show that the wave operator $W_+ \; u = \lim_{t\to\infty} e^{itH}e^{-itH_0}u$ exists it suffices to show [6, p. 176] that there is a number t_0 such that

(3.11) $\int_{t_0}^{\infty} \|Ae^{-itH_0}u\|dt < \infty$ and that

(3.12) $\lim_{t\to\infty}\|Je^{-itH_0}u\|$ converges to a limit as $t \to \infty$,

We have

$$\|Ae^{-itH_0}u\| \le |(e^{-itH_0} u)'(0+)| + \|q_1 e^{-itH_0} u\| .$$

Hence (3.11) is equivalent to showing that

(3.13) $\int |(e^{-itH_0} u)'(0+)|dt < \infty$

and that

(3.14) $\int \|q_1 e^{-itH_0} u\|dt < \infty$

We shall first show these integrals to be finite for the set of functions whose fourier transforms are of the form $\psi_s(k) = ke^{-k^2-iks}$, s real. Linear combinations of such functions are dense in L^2 [5, p. 114]. Hence the wave operator will exist on a dense set. Since the domain of the wave operator is a closed subspace, the wave operator exists everywhere. Setting $u = \psi_s$ we have

$$(3.15) \qquad e^{-itH_0} u = \frac{i(x-s)}{(2+2it)^{3/2}} \exp\left\{\frac{-(x-s)^2}{4(1+it)}\right\}. \qquad [5, p. 115]$$

Differentiating with respect to χ we get

$$(3.16) \qquad \frac{d}{dx} e^{-itH_0} u = \left(\frac{i}{(2+2it)^{3/2}} - \frac{i(x-s)^2}{(2+2it)^{5/2}}\right) \exp\left\{\frac{-(x-s)^2}{4(1+it)}\right\}$$

and

$$\left|\frac{d}{dx} e^{-itH_0} u\right| \le \left[\frac{1}{(4+4t^2)^{3/4}} + \frac{(x-s)^2}{(4+4t^2)^{5/4}}\right] \exp\left\{\frac{-(x-s)^2}{4(1+t^2)}\right\} .$$

Hence

$$\left|\frac{d}{dx} e^{-itH_0} u(0+)\right| \le \left[\frac{1}{(4+4t^2)^{3/4}} + \frac{s^2}{(4+4t^2)^{5/4}}\right] \exp\left\{\frac{-s^2}{4(1+t^2)}\right\}$$

$$= \frac{1}{(4+4t^2)^{3/4}} \exp\left\{\frac{-s^2}{4(1+t^2}\right\} + \frac{s^2}{(4+4t^2)^{5/4}} \exp\left\{\frac{-s^2}{4(1+t^2)}\right\} .$$

Note that

$$\lim_{t\to\infty} \frac{t^{3/2}}{(4+4t^2)^{3/4}} \exp\left\{\frac{-s^2}{4(1+t^2)}\right\} = \frac{1}{4^{3/4}}$$

and that

$$\lim_{t\to\infty} \frac{t^{5/2}s^2}{(4+4t^2)^{3/4}} \exp\left\{\frac{-s^2}{r(1+t^2)}\right\} = \frac{s^2}{4^{5/4}} ,$$

Hence $\int \left|\frac{d}{dx} e^{-itH_0} u(0+)\right| dt$ exists.

For $u = \psi_s$ we also have that $\lim_{t\to\infty} \| Je^{-itH_0} u \|^2 = \| Ju \|^2$.

We have the following theorem on the existence of the wave operator.

<u>Theorem 3.2</u>: If $(1+|x|)^{\alpha}\, q_1(x)$ is in L^2 for some $\alpha > \frac{1}{2}$ then the wave operator exists. Before proving Theorem 3.2 we shall prove the following Lemmas.

<u>Lemma 3.1</u>: If for each real s there is an a(s) such that

$$(3.17) \qquad \int_{a(s)}^{\infty} t^{-3/2} \left(\int_{-\infty}^{\infty} |q_1(x)(x-s)|^2 \exp\left\{ \frac{-(x-s)^2}{2(1+t^2)} \right\} dx \right)^{\frac{1}{2}} dt < \infty$$

with the inner integral finite for each $t > a(s)$.
then the wave operator exists for all $u \in L^2$.

<u>Proof</u>: By (3.15) we have for $u = \psi_s$

$$\| q_1 e^{-itH_0} u \|^2 = (4+4t^2)^{-3/2} \int_{\infty}^{\infty} |q_1(x)(x-s)|^2 \exp \left\{ \frac{-(x-s)^2}{2(1+t^2)} \right\} dt$$

By hypothesis this is finite for all real s and $t > a$.

Hence $e^{-itH_0} u \in D(A)$ for all such s, t. Moreover $e^{-itH_0} u \in D(\mathcal{H}_0)$.

We have

$$\int \| q_1 e^{-itH_0} u \| dt$$

$$= \int (4+4t^2)^{-3/4} \left[\int_{-\infty}^{\infty} |q_1(x)(x-s)|^2 \exp \left\{ \frac{-(x-s)^2}{2(1+t^2)} \right\} dx \right]^{\frac{1}{2}} dt \quad .$$

By (3.17) this integral exists. Hence the wave operator exists for all $u \in L^2$.

Lemma 3.2: If $x > 0$ and $0 > c > 1$ then $e^{-x} \le \frac{1}{x^c}$

Proof: The Taylor expansion for e^x is $1 + x + \frac{x^2}{2} + \ldots$

Hence $e^x \ge x$. Thus for $x > 0$ $e^{-x} \le \frac{1}{x}$. Now take $x \ge 1$.

Then for $c < 1$ $x \ge x^c$ and $\frac{1}{x} \le \frac{1}{x^c}$. Next take $0 < x < 1$.

Then $e^{-x} < 1 < \frac{1}{x^c}$ if $c > 0$. □

We now proceed with the proof of Theorem 3.2. Note that $\frac{1}{2} < \alpha < 1$ implies that $0 < 2(1-\alpha) < 1$. Hence by Lemma 3.2

$$e^{\frac{-(x-s)^2}{1+t^2}} \le \left[\frac{1+t^2}{(x-s)^2}\right]^{2(1-\alpha)} .$$

Thus

$$e^{\frac{-(x-s)^2}{2(1+t^2)}} \le \left[\frac{1+t^2}{(x-s)^2}\right]^{1-\alpha} = (1+t^2)^{1-\alpha}|x-s|^{2\alpha-2} .$$

Also note that $|x-s| \le (1+|x|)(1+|s|)$. We have

$$\int_1^\infty t^{-3/2} \left(\int_{-\infty}^\infty |q_1(x)(x-s)|^2 \exp\left\{\frac{-(x-s)^2}{2(1+t^2)}\right\} dx\right)^{\frac{1}{2}} dt$$

$$\le \int_1^\infty t^{-3/2} \left(\int_{-\infty}^\infty |q_1(x)|^2(1+|x|)^2(1+|s|)^2(1+t^2)^{1-\alpha}|x-s|^{2\alpha-2}dx\right)^{\frac{1}{2}} dt$$

$$\le \int_1^\infty t^{-3/2+1-\alpha}dt \left(\int_{-\infty}^\infty |q_1(x)|^2(1+|x|)^{2\alpha}(1+|s|)^{2\alpha}dx\right)^{\frac{1}{2}} .$$

By hypothesis the integral in x is finite. The integral in t is finite for $\alpha > \frac{1}{2}$. □

We have the following corollary to Theorem 3.2.

Corollary 3.1: If there is a $\beta > 1$ such that $q_1(x) \leq \frac{M}{|x|^\beta}$ for $|x|$ large then the wave operator exists.

Proof: It suffices to show that the conditions of Theorem 3.2 are satisfied for x large. Let $\alpha = \frac{\beta}{2}$. Then $\alpha > \frac{1}{2}$ and

$$\int q_1(x)^2(1+|x|)^{2\alpha}dx \leq M^2 \int \frac{(1+|x|)^{2\alpha}}{|x|^{2\beta}}\, dx$$

$$\leq M^2 2^{2\alpha} \int |x|^{-2\alpha}dx$$

and this is finite for x large if $2\alpha < -1$ or $\alpha > \frac{1}{2}$. □

To show the completeness of the wave operator we use a theorem by Schechter [7] which states that if the following six conditions hold then the wave operator is complete.

1. There exists a Banach space K and linear operators A from $\mathcal{H}_0$ to K and B from $\mathcal{H}_0$ to K' such that $D(H_0) \subset D(A) \cap D(B)$, $D(H) \subset D(BJ^*)$ and $(Ju,Hv)-(JH_0u,v) = (Au,BJ^*v)$ $u \in D(H_0)$, $v \in D(H)$.

2. There exists an open set Λ of R such that $C\Lambda$ has measure zero and for each $I \subset\subset \Lambda$ there is a constant C_I such that $a\|AR_0(s+ia)\|^2 +$

$$a\| BR_0(s+ia)\|^2 \le C_I \quad 0 < a < 1,\ s\in I$$

3. The operators $Q_0(z) = [B(AR_0(\bar{z}))^*]$, $G_0(z) = 1-Q_0(z)$ and $G_0(z)^{-1}$ are bounded and everywhere defined on K' for Im $z \neq 0$.

4. For each $I \subset\subset \Lambda$, $Q_0(z)$ is uniformly continuous in the region $W_I = \{z = s+ia/0 < a < 1,\ s\in I\}$.

5. There is a z_1 such that $BR_0(z)(AR_0(z_1))^*$ is a compact operator on K' when Im $z \neq 0$.

6. $(B(BJ_0R_0(z))^*)^* = BJ_0(BR_0(\bar{z}))^*$

We now apply this theorem to our example with $H_0, H, A, B, G_0(z)$ defined as above. Condition 1 is a simple consequence of Theorem 2.1. We have already verified conditions 2, 3 and 6.

Since

$$(3.18) \qquad B(AR_0(z))^*\{v_1,v_2\} = \left\{\frac{v_1}{2} + \frac{1}{2ki}\int_{-\infty}^{\infty} e^{ik|y|}q_1(y)v_2(y)dy, \right.$$

$$\left. -\frac{q_2}{2}v_1e^{-iky}\chi_{(-\infty,0)} + \frac{q_2}{2}v_1\chi_{(0,\infty)}e^{iky} + q_2R_0(z)q_1v_2\right\}$$

by (3.8), for the fourth condition it suffices to show that each of these operators is uniformly continuous in W_I. The first of these operators, $\frac{v_1}{2}$, is clearly uniformly continuous. Since q_1 and q_2 are in L^2, the last operator, $q_2R_0(z)q_1$ is uniformly continuous in any W_I for I bounded and away from the origin [5, p. 184]. To show that the second operator in (3.18) is uniformly continuous in W_I, we first show that it is uniformly continuous if

$$(3.19) \qquad \int_{-\infty}^{\infty} (1+|x|)^2 q_1(x)^2 dx < \infty.$$

Let z,z' be points in W_I and let $k^2 = z, (k')^2 = z'$ and Im k,k' > 0. Set $f(k) = \frac{e^{ik|y|}}{k}$. Then $f'(k) = \frac{(ik|y|-1)}{k^2} e^{ik|y|}$. Suppose I = (c,d) with c > 0. If $k = \alpha + i\eta$, $z = s + ia$ and $0 < s = \alpha^2 - \eta^2$ then $\alpha > c^{1/2}$ and consequently

$$|f'(k)| \leq \frac{(1+|ky|)}{C}$$

If $k(\theta) = (1-\theta)k' + \theta k$, $0 \leq \theta \leq 1$ then Re $k(\theta) \geq C^{1/2}$ in W_I and $f(k) - f(k') = f(k(1)) - f(k(0))$

$$= \int_0^1 f'(k(\theta))k'(\theta)d\theta = (k-k') \int_0^1 f'(k(\theta))d\theta.$$

Hence

$$|f(k)-f(k')| \leq |k-k'| \frac{(1+M|y|)}{C}$$

where M is an upper bound for $|z|^{1/2}$ in W_I. Thus

$$\left|\frac{1}{2ki} \int_{-\infty}^{\infty} e^{ik|y|} q_1(y)v_2(y)dy - \frac{1}{2k'i} \int_{-\infty}^{\infty} e^{ik'|y|} q_1(y)v_2 dy\right|$$

$$\leq \frac{1}{2c}|k-k'| \int_{-\infty}^{\infty} (1+M|y|)|q_1(y)v_2(y)|dy$$

$$\leq \frac{1}{2c}|k-k'| \int_{-\infty}^{\infty} (1+M|y|)^2 (q_1(y))^2 dy \|v_2\| .$$

This proves that the second operator in (3.18) is uniformly continuous if (3.19) holds. Now for each n set

$$q_{1,n}(x) = \begin{cases} q_1(x) & \text{if } |x| \leq n \\ 0 & \text{if } |x| > n \end{cases}$$

and

$$F_n v_2 = \frac{1}{2ki} \int_{-\infty}^{\infty} e^{ik|y|} q_{1,n}(y) v_2(y) dy$$

$$F_0(z) v_2 = \frac{1}{2ki} \int_{-\infty}^{\infty} e^{ik|y|} q_1(y) v_2(y) dy$$

Then

$$|F_n(z)v_2 - F_0(z)v_2| = \left|\frac{1}{2ki} \int_{-\infty}^{\infty} e^{ik|y|} (q_{1,n}(y) - q_1(y)) v_2(y) dy\right|$$

By Holder's inequality this is bounded by

$$\frac{1}{2|k|} \sqrt{\int_{-\infty}^{\infty} |q_{1,n}(y) - q_1(y)|^2 dy \int_{-\infty}^{\infty} |v_2(y)|^2 dy}$$

$$\leq \frac{1}{2|k|} \|q_{1,n} - q_1\| \|v_2\| \leq \frac{1}{2C^{1/2}} \|q_{1,n} - q_1\| \|v\| .$$

Since $\|q_{1,n}-q_1\| \to 0$, $F_n(z)$ converges to $F_0(z)$ uniformly in

W_I. On the other hand $q_{1,n}$ satisfies (3.19) for each n. Hence $F_0(z)$ is uniformly continuous in W_I for each n. This gives the desired result.

Now let $F(z)v_1 : \mathbb{C} \to L^2$ be the third operator in (3.18). Then

$$\|F(z)v_1 - F(z')v_1\| = \left\|\frac{q_2}{2} v_1 \chi_{(-\infty,0)} (e^{-iky} - e^{-ik'y})\right\|.$$

If we set $f(k) = e^{-iky}$ then $f'(k) = -iye^{-iky}$. Then $|f'(k)| \le |y|e^{\eta y} \le |y|$ for $y \le 0$. If $k(\theta) = (1-\theta)k' + \theta k$ then $f(k) - f(k') = (k-k') \int_0^1 f'(k(\theta))d\theta$. Hence $|f(k) - f(k')| \le |k-k'||y|$.

Consequently $\|F(z)v_1 - F(z')v_1\|$

$$\le \left\|\frac{q_2}{2} v_1 \chi_{(-\infty,0)} |k-k'||y|\right\|$$

$$= \frac{1}{2}|v_1||k-k'| \left(\int_{-\infty}^{0} |y|^2 q_2(y)^2 dy\right)^{\frac{1}{2}}.$$

This proves that $F(z)$ is uniformly continuous if

$$\int_{-\infty}^{0} (1+|y|)^2 q_2(y)^2 dy < \infty \tag{3.20}$$

Now set $q_{2,n}(x) = \begin{cases} q_2(x) & \text{if } |x| \le n \\ 0 & \text{if } |x| > n \end{cases}$

and $F_n(v_1) = \frac{q_{2,n}}{2} v_1 e^{-iky} \chi_{(-\infty,0)}$. Then

$$\|F_n(z)v_1 - F(z)v_1\| \leq \left\|\frac{q_{2,n}}{2} v_1 e^{-iky}\chi_{(-\infty,0)} - \frac{q_2}{2} v_1 e^{-iky}\chi_{(-\infty,0)}\right\|$$

$$\leq \frac{1}{2}|v_1| \sqrt{\int_{-\infty}^{0} (q_{2,n}-q_2)^2 e^{-iky} dy}$$

$$\leq \frac{1}{2}|v_1| \|q_{2,n}-q_2\| .$$

Thus $F_n(z)$ converges to $F(z)$ uniformly in W_I. Each $q_{2,n}$ satisfies (3.20). Hence $F_n(z)$ is uniformily continuous in W_I for each I. This gives the desired result. The fourth operator in (3.18) may similarly be shown to be uniformly continuous. This gives the fourth completeness condition.

To verify the fifth condition we show that $BR_0(z)$ is a compact operator and recall that we have already shown that $AR_0(z)$ is bounded. To show the compactness of $BR_0(z)$ we have

$$BR_0(z) = \left\{\frac{1}{2ki} \int_{-\infty}^{\infty} e^{ik|y|} u(y)dy,\ q_2 R_0(z)u\right\}$$

Since $q_2 \in L^2$, $q_2 R_0(z)$ is a compact [5, p. 184] operator on L^2. To show that the first operator is compact let $u_n \in L^2$ be a sequence such that $\|u_n\| \leq C$. Then

$$\left|\frac{1}{2ki}\right| \int_{-\infty}^{\infty} \left|e^{ik|y|}\right| |u_n(y)| dy \leq \frac{1}{2|k|} \int_{-\infty}^{\infty} e^{-\eta y} |u_n(y)| dy$$

$$\leq \frac{1}{2|k|} \left(\int_{-\infty}^{\infty} e^{-2\eta|y|} dy\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |u_n(y)|^2 dy\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{2|k|\eta} \|u_n\| \leq \frac{C}{2|k|\eta}.$$

Since $\frac{1}{2ki} \int_{-\infty}^{\infty} e^{ik|y|} u_n(y)dy$ is a bounded sequence in $\mathbb{C}$ it has a convergent subsequence and hence the operator $BR_0(z)$ is compact. We have proved the following Theorem.

Theorem 3.3: Let the operators H_0, H, A, B, J be as in Theorem 3.1. If $(1+|x|)^{\alpha} q_1(x)$ is in L^2 for some $\alpha > \frac{1}{2}$ then the wave operator exists and is complete.

3.2 In our next application we shall prove

Theorem 3.4: Let H_0 be the self adjoint operator associated with D^2 in $\mathcal{H}_0 = L^2(-\infty,\infty)$ where $D = \frac{1}{i}\frac{d}{dx}$. We let Ju be the restriction of u to $(0,\infty)$. Thus J maps $\mathcal{H}_0$ to $\mathcal{H} = L^2(0,\infty)$. Then take

$$Au = \{-u(0+),\ q_1 u\} \quad \text{and}$$

$$Bv = \{v'(0+),\ q_2 v\}$$

where q_1 and q_2 are real valued functions in L^2 and $A, B : \mathcal{H}_0 \to \mathbb{C} \oplus L^2(-\infty,\infty)$ and

$$D(A) = \{u \in L^2/u',\ q_1 u \in L^2\} \quad \text{and}$$

$$D(B) = \{u \in L^2/u',\ q_2 u \in L^2 \quad \text{and} \quad \lim_{x \to 0+} u'(x) \text{ exists}\}.$$

Then there exists an operator H with the properties of Theorem 2.1.

Proof: We first note that $D(H_0) \subset D(A) \cap D(B)$ as in the first application. We next verify (2.1). We have

$$\operatorname{Im}[JH_0u, Ju) + (Au, BJ^*Ju)]$$

$$= \operatorname{Im}\left[\int_0^\infty (D^2u(x))\overline{u(x)}\,dx - u(0+)\overline{u'(0+)} + \int_0^\infty q_1(x)q_2(x)|u(x)|^2 dx.\right.$$

As in the previous application the last term is zero and the first term is $-\operatorname{Im} u'(0+)\overline{u(0+)} = \operatorname{Im} \overline{u'(0+)}u(0+)$. This is the negative of the second term above so $\operatorname{Im}[JH_0u, Ju) + (Au, BJ^*Ju)] = 0$.

Since A of this application is the negative of B in our previous application, and our present B is the negative of A of the previous application, we have

$$a(\|AR_0(s+ia)\|^2 + \|BR_0(s+ia)\|^2) \leq C_I, \quad a > 0, \quad s \in I.$$

We shall now prove that $B(AR_o(\overline{z}))^*$ is closable and that $G_0(z)$ is bounded. We have

$$AR_0(\overline{z})u = \left\{\frac{1}{2\overline{k}i} \int_{-\infty}^{\infty} e^{-i\overline{k}|y|}u(y)\,dy,\ q_1R_0(\overline{z})u\right\} \in \mathbb{C} \oplus L^2$$

and

$$(AR_0(\overline{z})u, \{v_1, v_2\}) = \frac{\overline{v}_1}{2\overline{k}i} \int_{-\infty}^{\infty} e^{-i\overline{k}|y|}u(y)\,dy + (q_1R_0(\overline{z})u, v_2)$$

$$= (u, \frac{v_1}{-2ki} e^{ik|y|} + R_0(z)q_1v_2).$$

Hence $(AR_0(z))^* \{v_1,v_2\} = \frac{v_1}{-2ki} e^{ik|y|} + R_0(z)q_1v_2.$

To compute $B(AR_0(z))^*$ note that $\frac{d}{dy}\left[\frac{-v_1}{-2ki} e^{iky}\right] =$

$\frac{-v_1}{2}e^{iky}$. Evaluated at 0 this function is $\frac{-v_1}{2}$. In the

previous application we showed that

$$\frac{d}{dx} R_0(z)u(x) = \frac{1}{2}\left[\int_{-\infty}^{x} e^{ik(x-y)}u(y)dy - \int_{x}^{\infty} e^{ik(y-x)}u(y)dy\right]$$

Hence

$$\frac{d}{dx} R_0(z)q_1v_2(0) = \frac{1}{2}\left[\int_{-\infty}^{0} e^{-iky}q_1(y)v_2(y)dy - \int_{0}^{\infty} e^{iky}q_1(y)v_2(y)dy\right].$$

Consequently

$$B(AR_0(\overline{z}))^*\{v_1,v_2\}$$

$$= \left\{\frac{1}{2}\ [-v_1 + \int_{-\infty}^{0} e^{-iky}q_1(y)v_2(y)dy - \int_{0}^{\infty} e^{iky}q_1(y)v_2(y)dy],\right.$$

$$\left.\frac{-q_2}{2ki} v_1e^{ik|y|} + q_2R_0(z)q_1v_2\right\}.$$

We have

$$\|B(AR_0(\overline{z}))^*\{v_1,v_2\}\|$$

$$(3.21) \qquad \leq \tfrac{1}{2}|v_1| + \tfrac{1}{2}\left|\int_{-\infty}^{0} e^{-iky} q_1(y) v_2(y)\,dy\right|$$

$$+ \tfrac{1}{2}\left|\int_{0}^{\infty} e^{iky} q_1(y) v_2(y)\,dy\right| + \tfrac{1}{2|k|}\|q_2 v_1 e^{ik|y|}\| + \|q_2 R_0(z) q_1 v_2\|.$$

The second term in (3.21) is bounded by

$$\tfrac{1}{2}\int_{-\infty}^{0} e^{ty}|q_1(y)||v_2(y)|\,dy$$

where $k = s + it$

$$\leq \tfrac{1}{2}\left[\int_{-\infty}^{0} e^{ty}|q_1(y)|^2 dy\right]^{\frac{1}{2}} \left[\int_{-\infty}^{0} e^{ty}|v_2(y)|^2 dy\right]^{\frac{1}{2}}$$

$$\leq \tfrac{1}{2}\|v_2\| \left[\int_{-\infty}^{0} e^{ty}|q_1(y)|^2 dy\right]^{\frac{1}{2}}.$$

We have

$$\int_{-\infty}^{0} e^{ty}|q_1(y)|^2 dy = \int_{-\infty}^{-\varepsilon} e^{ty}|q_1(y)|^2 dy + \int_{-\varepsilon}^{0} e^{ty}|q_1(y)|^2 dy$$

$$\leq e^{-t\varepsilon}\int_{-\infty}^{-\varepsilon}|q_1(y)|^2 dy + \int_{-\varepsilon}^{0}|q_1(y)|^2 dy.$$

Letting $\varepsilon = \frac{1}{\sqrt{t}}$ this is bounded by $e^{-\sqrt{t}}\|q_1\|^2 + \int_{-\varepsilon}^{0}|q_1(y)|^2 dy$.

The second term approaches zero as $\varepsilon \to 0$. Hence

$\frac{1}{2}\left|\int_{-\infty}^{0} e^{-iky} q_1(y) v_2(y)\,dy\right| \leq C_1(z)\|v_2\|$ where $C_1(z) \to 0$ as

$\text{Im } z \to \infty$.

The third term in (3.21) is

$$\frac{1}{2}\left|\int_0^\infty e^{-ty} q_1(y) v_2(y)\,dy\right| \leq C_2(z)\|v_2\|$$

as in the previous calculation.

The square of the fourth term in (3.21) is bounded by

$$\frac{|v_1|^2}{4|k|^2}\int_{-\infty}^{\infty} q_2(y)^2 e^{-2ty}\,dy \leq \frac{|v_1|^2}{4|k|^2}\|q_2\|^2 = C_3{}^2(z)|v_1|^2$$

where $C_3(z) \to 0$ as $\operatorname{Im} z \to \infty$.

The last term in (3.21) is bounded by $C_4(z)\|v_2\|$ as in the previous application where $C_4(z) \to 0$ as $\operatorname{Im} z \to \infty$. Thus

$$\|B(AR_0(\overline{z}))^*\{v_1,v_2\}\| \leq \tfrac{1}{2}|v_1| + C_1(z)\|v_2\| + C_2(z)\|v_2\|$$

$$+ C_3(z)|v_1| + C_4(z)\|v_2\|$$

$$\leq (\tfrac{1}{2}+C_5(z))|v_1| + C_6(z)\|v_2\|$$

where $C_i(z) \to 0$ as $\operatorname{Im} z \to \infty$. This is bounded by

$$\sqrt{(\tfrac{1}{2}+C_5(z))^2 + C_6(z)^2}\;\|\{v_1,v_2\}\| .$$

Note that

$$\sqrt{(\tfrac{1}{2}+C_5(z))^2 + C_6(z)^2} < 1$$

for $\operatorname{Im} z$ sufficiently large. Hence $B(AR_0(\overline{z}))^*$ is closable,

$G_0(z)$ is bounded, and $G_0(z)$ has a bounded inverse for Im z sufficiently large.

We have $(I+B(AR_0(\bar{z}))^*)\{v_1,v_2\}$

$$= \left\{\frac{1}{2} v_1 + \frac{1}{2}\left[\int_{-\infty}^{0} e^{-iky} q_1(y)v_2(y)dy - \int_0^{\infty} e^{iky} q_1(y)v_2(y)dy\right],\right.$$

$$\left.\frac{-q_2}{2ki} v_1 e^{ik|y|} + (I+q_2R_0(z)q_1)v_2\right\}$$

$$= \left\{\frac{1}{2} v_1, v_2\right\} + \left\{\frac{1}{2}\left[\int_{-\infty}^{0} e^{-iky} q_1(y)v_2(y)dy - \int_0^{\infty} e^{iky} q_1(y)v_2(y)dy\right],\right.$$

$$\left.\frac{-q_2}{2ki} v_1 e^{ik|y|} + q_2R_0(z)q_1v_2\right\}.$$

The first operator is invertible and the second may be shown to be compact by the same methods used in the first application. Later on we shall show that $G_0(z)$ is also continuous. Using the first resolvent equation, it then follows that $G_0(z)$ is analytic outside of a set of measure zero. Thus $G_0(z)$ actually has a bounded inverse for all z outside of some finite discrete set.

We now verify that $(B(BJ_0R_0(z))^*)^* = BJ_0(BR_0(\bar{z}))^*$.

We first compute $B(BJ_0R_0(z))^*$. We have

$$BJ_0R_0(z) = \left\{\frac{1}{2}\left[\int_{-\infty}^{0} e^{-iky} u(y)dy - \int_0^{\infty} e^{iky} u(y)dy\right], q_2\chi_{(0,\infty)}R_0(z)u\right\}$$

and

$$(BJ_0R_0(z)u,\ \{v_1,v_2\}) = \frac{\overline{v}_1}{2}\left[\int_{-\infty}^{0} e^{-iky}u(y)dy - \int_0^{\infty} e^{iky}u(y)dy\right]$$

$$+ (q_2\chi_{(0,\infty)}R_0(z)u,\ v_2)$$

$$= (u,\ \frac{v_1}{2}\, e^{i\overline{k}y}\chi_{(-\infty,0)} - \frac{v_1}{2}\, e^{-i\overline{k}y}\chi_{(0,\infty)} + R_0(\overline{z})\chi_{(0,\infty)}q_2v_2)\ .$$

Consequently $(BJ_0R(z))^*$

$$= \frac{v_1}{2}\, e^{i\overline{k}y}\chi_{(-\infty,0)} - \frac{v_1}{2}\, e^{-i\overline{k}y}\chi_{(0,\infty)} + R_0(\overline{z})\chi_{(0,\infty)}q_2v_2.$$

We have $\frac{d}{dy}\left(\frac{v_1}{2}\, e^{i\overline{k}y}\chi_{(-\infty,0)}\right)(0+) = 0$ and

$$\frac{d}{dy}\left(\frac{-v_1}{2}\, e^{-i\overline{k}y}\right) = \frac{iv_1}{2}\,\overline{k}\,e^{-i\overline{k}y}\ .$$

Evaluated at 0 this is $\frac{i}{2}v_1\overline{k}$. We also have

$$\frac{d}{dx} R_0(\overline{z})\chi_{(0,\infty)}q_2v_2(0) = \frac{1}{2}\left[\int_{-\infty}^{0} e^{i\overline{k}y}\chi_{(0,\infty)}q_2(y)v_2(y)dy\right.$$

$$\left. - \int_0^{\infty} e^{-i\overline{k}y}\chi_{(0,\infty)}q_2(y)v_2(y)dy\right]$$

$$= -\frac{1}{2}\int_0^{\infty} e^{-i\overline{k}y}q_1(y)v_2(y)dy.$$

Hence

$$B(BJ_0R_0(z))^*\ \{v_1,v_2\}$$

$$= \left\{\frac{iv_1}{2}\,\overline{k} - \frac{1}{2}\int_0^{\infty} e^{-i\overline{k}y}q_2(y)v_2(y)dy,\right.$$

$$\frac{q_2}{2} v_1 e^{i\bar{k}y} \chi_{(-\infty,0)} - \frac{v_1}{2} q_2 e^{-i\bar{k}y} \chi_{(0,\infty)} + q_2 R_0(\bar{z}) \chi_{(0,\infty)} q_2 v_2 \Big\}$$

$$= \Big\{ F_{11} v_1 + F_{12} v_2,\ F_{21} v_1 + F_{22} v_2 \Big\}$$

where

$$F_{11} v_1 = \frac{i\, v_1 \bar{k}}{2},\quad F_{12} v_2 = -\frac{1}{2} \int_0^\infty e^{-i\bar{k}y} q_2(y) v_2(y)\, dy,$$

$$F_{21} v_i = \frac{q_2}{2} v_1 e^{i\bar{k}y} \chi_{(-\infty,0)} - \frac{v_1}{2} q_2 e^{-i\bar{k}y} \chi_{(0,\infty)} \quad \text{and}$$

$$F_{22} v_2 = q_2 R_0(\bar{z}) \chi_{(0,\infty)} q_2 v_2 \,.$$

We now compute $BJ_0(BR_0(\bar{z}))^*$. We have

$$BR_0(\bar{z}) u = \left\{ \frac{1}{2} \left[\int_{-\infty}^0 e^{i\bar{k}y} u(y)\, dy - \int_0^\infty e^{-i\bar{k}y} u(y)\, dy \right],\ q_2 R_0(\bar{z}) u \right\}.$$

Thus

$$(BR_0(\bar{z})u,\ \{v_1 v_2\}) = \frac{\bar{v}_1}{2} \left[\int_{-\infty}^0 e^{i\bar{k}y} u(y)\, dy - \int_0^\infty e^{-i\bar{k}y} u(y)\, dy \right]$$

$$+ (q_2 R_0(\bar{z}) u,\ v_2)$$

$$= (u,\ \frac{1}{2} v_1 \chi_{(-\infty,0)} e^{-iky} - \frac{1}{2} v_1 \chi_{(0,\infty)} e^{iky} + R(z) q_2 v_2).$$

Hence

$$(BR_0(\bar{z}))^* \{v_1, v_2\} = \frac{1}{2} v_1 \chi_{(-\infty,0)} e^{-iky} - \frac{1}{2} v_1 \chi_{(0,\infty)} e^{iky} + R(z) q_2 v_2.$$

and

$$J_0(BR_0(\overline{z}))^* = -\frac{1}{2}v_1\chi_{(0,\infty)}e^{iky} + \chi_{(0,\infty)}R_0(z)q_2v_2$$

Note that $\frac{d}{dy}(-\frac{1}{2}v_1e^{iky}) = -\frac{1}{2}kv_1e^{iky}$. Evaluated at 0+ this is $\frac{-ikv_1}{2}$. We have

$$BJ_0(BR_0(\overline{z}))^*\{v_1,v_2\}$$

$$\left\{\frac{-ikv_1}{2} + \frac{1}{2}\left[\int_{-\infty}^{0} e^{-iky}q_2(y)v_2(y)dy - \int_0^{\infty} e^{iky}q_2(y)v_2(y)dy\right],\right.$$

$$\left.\frac{-q_2}{2}v_1\chi_{(0,\infty)}e^{iky} + q_2\chi_{(0,\infty)}R_0(z)q_2v_2\right\}$$

$$= \{G_{11}v_1 + G_{12}v_2,\ G_{21}v_1 + G_{22}v_2\}$$

where

$$G_{11}v_1 = \frac{-ikv_1}{2},\ G_{12}v_2$$

$$= \frac{1}{2}\left[\int_{-\infty}^{0} e^{-iky}q_2(y)v_2(y)dy - \int_0^{\infty} e^{iky}q_2(y)v_2(y)dy\right],$$

$$G_{21}v_1 = \frac{-q_2}{2}v_1\chi_{(0,\infty)}e^{iky} \quad \text{and} \quad G_{22}v_2 = q_2\chi_{(0,\infty)}R_0(z)q_2v_2 .$$

To show that $(B(BJ_0R_0(z))^*)^* = BJ_0(BR_0(\overline{z}))^*$ it suffices to show that $F_{ij}^* = G_{ji}$. It is clear that $F_{11}^* = G_{11}$ and that $F_{22}^* = G_{22}$. We have $(F_{12}v_2,v_1)_{\mathbb{C}} = -\frac{1}{2}\overline{v}_1\int_0^{\infty} e^{-i\overline{k}y}q_2(y)v_2(y)dy$

$$= (v_2, -\frac{1}{2} v_1 e^{iky} q_2 \chi_{(0,\infty)})_{L^2}$$

$$= (v_2, G_{21} v_1).$$

Hence $F_{12}^* = G_{21}$. We have

$$(F_{21} v_1, v_2)_{L^2} = \int_{-\infty}^{\infty} \left[\frac{q_2}{2} v_1 e^{i\bar{k}y} \chi_{(-\infty,0)} - \frac{v_1}{2} q_2 e^{-i\bar{k}y} \chi_{(0,\infty)} \right] v_2(y)\,dy$$

$$= v_1 \overline{\int_{-\infty}^{\infty} (\frac{q_2}{2} e^{-iky} \chi_{(-\infty,0)} - \frac{q_2}{2} e^{iky} \chi_{(0,\infty)} v_2(y)\,dy}$$

$$= v_1 \frac{1}{2} \overline{\left[\int_{-\infty}^{0} e^{-iky} q_2(y) v_2(y)\,dy - \int_0^{\infty} q_2(y) e^{iky} v_2(y)\,dy \right]}$$

$$= (v_1, G_{12} v_2)_{\mathbb{C}} .$$

Hence $F_{21}^* = G_{12}$. □

To show that the wave operator exists we again give conditions under which $\int_{t_0}^{\infty} \| A e^{-itH_0} u \|\,dt \leq \infty$. We have

$$\int_{t_0}^{\infty} \| A e^{-itH_0} u \|\,dt \leq \int_{t_0}^{\infty} | e^{-itH_0} u(o+) |\,dt + \int_{t_0}^{\infty} q_1 e^{-itH_0} u \|\,dt .$$

The second integral is finite under the same conditions as in the previous application. We shall show that the first integral is finite for the set of functions whose fourier transforms are of the form $\psi_s(k) = k e^{-k^2 - iks}$ s real. Since linear combinations of these functions are dense in

L^2, the wave operator exists on a dense set. Since the domain of the wave operator is a closed subspace, the wave operator exists everywhere. Setting $u = \psi_s$ we have

$$|e^{-itH_0} u| = \frac{|x-s|}{(4+4t^2)^{3/4}} \exp\left\{\frac{-(x-s)^2}{4(1+t^2)}\right\} .$$

Hence

$$|e^{-itH_0} u(0+)| = \frac{|s|}{(4+4t^2)^{3/4}} \exp\left\{\frac{-s^2}{4(1+t^2)}\right\} .$$

Since

$$\lim_{t\to\infty} \frac{t^{3/2}\ |s|}{(4+4t^2)^{3/2}} \ \exp\left\{\frac{-s^2}{4(1+t^2)}\right\} = \frac{|s|}{4^{3/4}}$$

we have that $\int |e^{-itH_0} u(0+)|dt$ is finite. Since we also have that $\lim_{t\to\infty} \|Je^{-itH_0} u\|^2$ exists as in the first application, Theorem 3.2 and Corollary 3.1 apply in this application also.

In proving the completeness of the wave operator the second, third and last condition have already been verified. To prove the fourth condition we have $Q_0(z) \{v_1, v_2\} =$

(3.22) $$\frac{1}{2}\left[-v_1 + \int_{-\infty}^{0} e^{-iky} q_1(y)v_2(y)dy - \int_0^\infty e^{iky} q_1(y)v_2(y)dy,\right.$$

$$\left. - \frac{q_2}{2ki} v_1 e^{ik|y|} + q_2 R_0(z) q_1 v_2\right\} .$$

The operator $-\dfrac{v_1}{2}$ is clearly uniformly continuous in W_I.

The last operator may be shown to be uniformly continuous as in the last application. We showed in the previous application that

$$|e^{-iky} - e^{-ik'y}| \leq |k - k'||y|.$$

Thus

$$\left|\int_{-\infty}^{0} (e^{-iky} - e^{-ik'y})q_1(y)v_2(y)dy\right|$$

$$\leq |k - k'| \int_{-\infty}^{0} |y| \; |q_1(y)v_2(y)|dy$$

$$\leq |k - k'| \left(\int_{-\infty}^{0} |y|^2 |q_1(y)|^2 dy\right)^{\frac{1}{2}} \|v_2\|.$$

Hence if

$$\int_{-\infty}^{\infty} (1+|y|)^2 q_1(y)^2 dy < \infty \tag{3.23}$$

then the second operator in (3.22) is uniformly continuous in W_I. Letting

$$q_{1,n}(y) = \begin{cases} q_1 & \text{if } |x| \leq n \\ 0 & \text{if } |x| > n \end{cases}$$

and

$$F_n(z)v_2 = \int_{-\infty}^{0} e^{-iky} q_{1,n}(y)v_2(y)dy$$

$$F_0(z)v_2 = \int_{-\infty}^{0} e^{-iky} q_1(y)v_2(y)dy$$

Then

$$|F_n(z)v_2 - F_0(z)v_2| = |\int_{-\infty}^{0} e^{-iky}(q_{1,n}(y) - q_1(y))v_2(y)dy|$$

$$\leq \|q_{1,n} - q_1\|\|v_2\|.$$

Hence $F_n(z)$ converges to $F_0(z)$ uniformly in W_I. Each $q_{1,n}$ satisfies (3.23). Hence $F_0(z)$ is uniformly continuous in W_I.

The operator $\int_0^{\infty} e^{iky} q_1(y)v_2(y)dy$ may be similarly shown to be uniformly continuous in W_I.

In the previous application we showed that

$$\left| \frac{e^{ik|y|}}{k} - \frac{e^{ik'|y|}}{k'} \right| \leq \frac{(1+M|y|)}{C} |k-k'|$$

where $I = (c,d)$ and M is an upper bound for $|z|^{\frac{1}{2}}$ in W_I. Hence

$$\left\| \frac{q_2}{2ki} e^{ik|y|} - \frac{q_2 v_1}{2k'i} e^{ik'|y|} \right\| \leq$$

$$\frac{|v_1|}{C} |k-k'| \|(1+M|y|)q_2\|.$$

So if

(3.24) $\int (1+|y|)^2 q_2(y)^2 dy < \infty$ then the first operator is uniformly continuous in W_I. Now set

$$q_{2,n}(x) = \begin{cases} q_2(x) & \text{if } |x| \le n \\ 0 & \text{if } |x| > n \end{cases}$$

$$F_n(z)v_1 = \frac{q_{2,n}v_1 e^{ik|y|}}{2ki}$$

$$F_0(z)v_1 = \frac{q_2 v_1 e^{ik|y|}}{2ki} ,$$

Then $\|F_n(z) - F_0(z)\| = \left\|v_1\left(\frac{q_{2,n}}{2ki} - q_2\right) e^{ik|y|}\right\| \le \frac{|v_1|}{2|k|}\|q_{2,n}-q_2\|$.

Taking M to be an upper bound for $|z|^{-1/2} = \frac{1}{|k|}$ in W_I this is $\frac{M}{2}\|q_{2,n} - q_2\|\,|v_1|$. Thus $F_n(z)$ converges to $F_0(z)$ uniformly in W_I. In addition $q_{2,n}$ satisfies (3.24) for each n. Hence $F_0(z)$ is uniformly continuous in W_I for each n.

The fifth condition is verified as in the previous application. We have the following theorem.

Theorem 3.5: Let H_0, H, A, B, J be as in Theorem 3.4. If $(1+|x|)^\alpha < q_1(x)$ is in L^2 for some $\alpha > \frac{1}{2}$ then the wave operator exists and is complete.

3.3 In our next application we prove

Theorem 3.6: Let H_0 be the self adjoint operator associated with D^2 in $\mathcal{H}_0 = L^2_{(-\infty,\infty)}$, where $D = \frac{1}{i}\frac{d}{dx}$. We let Ju be the restriction of u to $(0,\infty)$. Thus J maps $\mathcal{H}_0$ to $\mathcal{H} = L^2_{(0,\infty)}$. Take

$$Au = \{-u'(0+) + \alpha u(0+),\ q_1 u\} \quad \alpha \text{ real}$$

$$Bv = \{\ v(0+),\ q_2 v\} \text{ where } q_1 \text{ and } q_2 \text{ are}$$

real valued functions in L^2 and $A,B : \mathcal{H}_0 \to \mathbb{C}\oplus L^2(-\infty,\infty)$ and

$$D(A) = \{u\in L^2/u', q_1 u\in L^2 \text{ and } \lim_{x\to 0+} u'(x) \text{ exists}\},$$

$$D(B) = \{u\in L^2/u',\ q_2 u\in L^2\}.$$

Then there exists an operator H with the properties of Theorem 2.1.

Note that our first application was the special case of this application where $\alpha = 0$.

Proof of Theorem 3.6: As in the first application we have $D(H_0) \subset D(A) \cap D(B)$. To verify (2.1) we have $\mathrm{Im}[(JH_0u,Ju) + (Au,BJ^*Ju)]$

$$= \mathrm{Im}\left[\int_0^\infty (D^2u(x))\overline{u(x)}dx + (-u'(0+) + \alpha u(0+))\overline{u(0+)} + \int_0^\infty q_1(x)q_2(x)|u(x)|^2dx\right].$$

As in the first application the first term is $\mathrm{Im}\; u'(0+)\overline{u(0+)}$. The second term is

$$-\mathrm{Im}\; u'(0+)\overline{u(0+)} + \alpha\; \mathrm{Im}|u(0+)|^2 = -\mathrm{Im}\; u'(0+)\overline{u(0+)}.$$

This is the negative of the first term. The third term is zero.

We shall now show that $a(\|AR_0(s+ia)\|^2 + \|BR_0(s+ia)\|^2) \leq C_I$, $a > 0$, $s \in I$. In our previous applications we showed that $a\|BR_0(s+ia)\|^2 \leq C_I$. It remains to show that $a\|AR_0(s+ia)\|^2 \leq C_I$. We have $AR_0(z)u = \{-(R_0(z)u)'(0+) + \alpha R_0(z)u(0+), q_1 u\}$. In the first application we showed that $|(R_0(z)u)'(0+)| \leq \frac{2\sigma}{a}\|u\|^2$ and that $|(R_0(z)u)'(0+)| \leq \frac{1}{a|k|}\|u\|^2$. The third term we showed to be bounded by $\frac{1}{2a|k|}\|u\|^2\|q_2\|^2$ and this proves that $a\|AR_0(s+ia)\|^2 \leq C_I$.

We can now show that $B(AR_0(\overline{z}))^*$ is closable and that $G_0(z)$ is bounded. We have

$$\begin{aligned} AR_0(\overline{z})u = {} & \{-(R_0(\overline{z})u)'(0+) + \alpha R_0(\overline{z})u(0+), q_1 u\} \\ = {} & -\frac{1}{2}\left[\int_{-\infty}^{0} e^{i\overline{k}y}u(y)\,dy - \int_0^\infty e^{-i\overline{k}y}u(y)\,dy\right] \\ & - \frac{\alpha}{2ki}\int_{-\infty}^{\infty} e^{-i\overline{k}|y|}u(y)\,dy,\; q_1 R_0(\overline{z})u\} . \end{aligned}$$

Thus $(AR_0(\bar{z})u, \{v_1,v_2\}) = -\frac{\bar{v}_1}{2}\int_{-\infty}^{0} e^{i\bar{k}y}u(y)dy + \frac{\bar{v}_1}{2}\int_0^{\infty} e^{-i\bar{k}y}u(y)dy$

$$- \frac{\alpha\bar{v}_1}{2\bar{k}i} \int_{-\infty}^{\infty} e^{-i\bar{k}|y|}u(y)dy + (q_1R_0(\bar{z})u, v_2)$$

$$= (u, \frac{-v_1}{2}\chi_{(-\infty,0)}e^{-iky} + \frac{v_1}{2}\chi_{(0,\infty)}e^{iky} + \frac{\alpha v_1}{2ki} e^{ik|y|} + R_0(z)q_1v_2).$$

So

$$(AR_0(\bar{z}))^*\{v_1,v_2\} = \frac{-v_1}{2}\chi_{(-\infty,0)}e^{-iky} + \frac{v_1}{2}\chi_{(0,\infty)}e^{iky} + \frac{\alpha v_1}{2ki} e^{ik|y|}$$

$$+ R_0(z)q_1v_2.$$

We can now compute $B(AR_0(\bar{z}))^*\{v_1,v_2\}$

$$= \left\{\frac{v_1}{2} + \frac{\alpha v_1}{2ki} + \frac{1}{2ki} \int_{-\infty}^{\infty} e^{ik|y|}q_1(y)v_2(y)dy,\right.$$

$$\left. - \frac{q_2v_1}{2}\chi_{(-\infty,0)}e^{-iky} + \frac{q_2v_1}{2}\chi_{(0,\infty)}e^{iky} + \frac{\alpha q_2v_1}{2ki} e^{ik|y|} + q_2R_0(z)q_1v_2\right\}.$$

Hence

$$\| B(AR_0(\bar{z}))^* \{v_1,v_2\}\|$$

$$\leq \left|\frac{v_1}{2} + \frac{\alpha v_1}{2ki} + \frac{1}{2ki} \int_{-\infty}^{\infty} e^{iky}q_1(y)v_2(y)dy\right|$$

$$+ \left\|\frac{-q_2}{2} v_1\chi_{(-\infty,0)}e^{-iky} + \frac{q_2}{2}v_1\chi_{(0,\infty)}e^{iky} + \frac{\alpha q_2v_1}{2ki} e^{ik|y|}\right.$$

$$+ q_2 R_0(z) q_1 v_2\|$$

$$\text{(3.25)} \qquad \leq \frac{1}{2}\,|v_1| + \frac{\alpha|v_1|}{2|k|} + \frac{1}{2|k|}\int_{-\infty}^{\infty} e^{-ty}|q_1(y)|\,|v_2(y)|\,dy$$

$$+ \left\|\frac{q_2}{2}\, v_1 \chi_{(-\infty,0)} e^{-iky}\right\| + \left\|\frac{q_2}{2}\, v_1 \chi_{(0,\infty)} e^{iky}\right\|$$

$$+ \left\|\frac{\alpha q_2 v_1}{2ki}\, e^{ik|y|}\right\| + \|q_2 R_0(z) q_1 v_2\| .$$

The third term is bounded by $C(z)\|v_2\|$ as in the first application, where $C(z) \to 0$ as $|\text{Im } z| \to \infty$. The fourth and fifth terms are bounded by $C(z)|v_1|$ as in the first application. The sixth term is bounded by $C(z)|v_1|$ as in the second application. Similarly the last term is bounded by $C(z)\|v_2\|$ as in the first application. The first term in (3.25) is bounded by $C(z)|v_1|$ as shown in the second application. Hence $\|B(AR_0(\overline{z}))^*\| < 1$ for Im z sufficiently large. Thus $B(AR_0(\overline{z}))^*$ is closable and $G_0(z)$ is bounded. In addition $G_0(z)$ has a bounded inverse for Im z sufficiently large. We also have that $G_0(z)$ is the sum of an invertible and a compact operator. Since it is also continuous, as we shall show later on, it is analytic. Thus $G_0(z)$ has a bounded inverse for all z outside of some finite discrete set.

In the first application we verified that $(B(BJ_0R_0(z))^*)^* = BJ_0(BR_0(\overline{z}))^*$. Thus the operator H of Theorem 2.1 exists.

We can now show the existence of the wave operator under the conditions given in the first application. We have

$$\int \| A\, e^{-itH_0}\, u \| dt$$

$$\leq \int \left|\frac{d}{dx}\, e^{-itH_0}\, u(0+)\right| dt + |\alpha| \int \left| e^{-itH_0}\, u(0+)\right| dt + \int \| q_1 e^{-itH_0}\, u \| dt.$$

Since each of the first two integrals exists as in the previous applications and (3.12) holds, the wave operator exists under the same conditions on q_1 and q_2 as given previously.

We can now show the completeness of the wave operator. Condition 1 is a consequence of Theorem 2.1. We have already verified conditions 2, 3 and 6.

Since $Q_0(z)\ \{v_1, v_2\}$

$$(3.26) \qquad = \left\{ \frac{v_1}{2} + \frac{\alpha v_1}{2ki} + \frac{1}{2ki} \int_{-\infty}^{\infty} e^{ik|y|} q_1(y) v_2(y) dy, \right.$$

$$-\frac{q_2}{2}\, v_1 \chi_{(-\infty,0)} e^{-iky} + \frac{q_2}{2}\, v_1 \chi_{(0,\infty)} e^{iky}$$

$$\left. + \frac{\alpha q_2}{2ki}\, v_1 e^{ik|y|} + q_2 R_0(z) q_1 v_2 \right\},$$

for the fourth condition it suffices to prove that each of these operators is uniformly continuous in W_I. The first, third, fifth and seventh terms of (3.26) were shown to be uniformly continuous in the first application. The sixth was shown to be uniformly continuous in the second application. Clearly, the second term is also uniformly continuous in W_I for I bounded and away from the origin.

For the fifth condition we verified that $BR_0(z)$ is compact in the first application. We have already shown that $AR_0(z)$ is bounded. We have the following theorem.

<u>Theorem 3.7</u>: Let H_0, H, A, B, J be as in Theorem 3.6. If $(1+|x|)^\alpha q_1(x)$ is in L^2 for some $\alpha > \frac{1}{2}$ then the wave operator exists and is complete.

3.4 In our next application we shall prove

<u>Theorem 3.8</u>: Let H_0 be the self adjoint operator associated with D in $\mathcal{H}_0 = L^2{}_{(-\infty,\infty)}$. We let Ju be the restriction of bu to $(0,\infty)$. Thus J maps $\mathcal{H}_0$ to $\mathcal{H} = L^2{}_{(0,\infty)}$. Take

$$Au = -i\ b^\theta u$$

$$Bv = \frac{b'}{b^{1+\theta}}\ v$$

where b is a real valued bounded function, $b^\theta \in L^2$ and $\frac{b'}{b^{1+\theta}} \in L^2$ and $b(0) = 0$. We take

$$D(A) = \{u \in L^2 / b^\theta u \in L^2\}$$

and

$$D(B) = \{u \in L^2 / \frac{b'}{b^{1+\theta}} \in L^2\} .$$

Then there exists an operator H as in Theorem 2.1.

<u>Proof</u>: Note that $(Ju,v) = \int_0^\infty b(x)u(x)\overline{v(x)}dx = (u,J^*v)$,

Thus

$$J^*V = \begin{cases} bv & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and

$$J^*J = \chi_{(0,\infty)} b^2.$$

We have

$$\mathrm{Im}[(JH_0u, Ju) + (Au, BJ^*Ju)]$$

$$= \mathrm{Im} \int_0^\infty b(x)^2 (Du(x))\overline{u(x)}dx - \mathrm{Im}\, i \int_0^\infty \frac{b^\theta(x)b'(x)}{b^{1+\theta}(x)} b^2(x)|u(x)|^2 dx$$

(3.27) $$= \mathrm{Im}(-i \int_0^\infty b(x)^2 u'(x)\overline{u(x)}dx) - \mathrm{Im}(i \int_0^\infty b(x)^2 |u(x)|^2 \frac{b'(x)}{b(x)}dx).$$

The second term in (3.27) is $-\mathrm{Re} \int_0^\infty b(x)|u(x)|^2 b'(x)dx.$

The first term in (3.27) is $-\mathrm{Re} \int_0^\infty b(x)^2 u'(x)\overline{u(x)}dx$

$$= -\frac{1}{2} \int_0^\infty b(x)^2 (\frac{d}{dx}|u(x)|^2)dx .$$

Integrating by parts this is

$$-\frac{1}{2}\, b(x)^2 |u(x)|^2 \Big|_0^\infty + \frac{1}{2} \int_0^\infty (\frac{d}{dx}\, b(x)^2)|u(x)|^2 dx.$$

Since $b(0) = 0$ this is

$$\frac{1}{2} \int_0^\infty |u(x)|^2 (\frac{d}{dx}\, b(x)^2)dx = \mathrm{Re} \int_0^\infty |u(x)|^2 b(x)b'(x)dx.$$

This is the negative of the second term in (3.27). Hence the sum in (3.27) is zero.

In verifying the remaining conditions of Theorem 2.1 we shall use a few lemmas.

Lemma 3.3: If $A(x) \in L^2$ and $Au(x) = A(x)u(x)$ then $D(H_0) \subset D(A)$.

Proof: Note that $\|Au\|^2 \leq C(\|H_0 u\|^2 + \|u\|^2)$, $u \in D(H_0)$ implies that $D(H_0) \subset D(A)$ for this says that $Au \in L^2$ whenever $u \in D(H_0)$

Now let $u \in D(H_0)$. Since $u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}\, \hat{u}(k)dk$ where $\hat{u}(k)$ is the Fourier transform of u, we have

$$|u(x)|^2 \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (1+k^2)|\hat{u}(k)|^2 dk \int_{-\infty}^{\infty} (1+k^2)^{-1} dk$$

$$\leq \sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} (1+k^2)|\hat{u}(k)|^2 dk$$

$$= \sqrt{\frac{\pi}{2}} \left[\int_{-\infty}^{\infty} |\hat{u}(k)|^2 dk + \int_{-\infty}^{\infty} k^2 |\hat{u}(k)|^2 dk \right].$$

The first term in this expression is $\int_{-\infty}^{\infty} |u(x)|^2 dx = \|u\|^2$ by Parseval's identify. Since $k\hat{u}(k) = -i[u'(x)]^{v}$, the second term is bounded by $\int_{-\infty}^{\infty} |[u'(x)]^{v}|^2 dx = \int_{-\infty}^{\infty} |u'(x)|^2 dx = \|H_0 u\|^2$ by Parseval's identity. Thus $|u(x)|^2 \leq C(\|u\|^2 + \|H_0 u\|^2)$. Multiplying both sides by A(x) and integrating

with respect to x we get $\|Au\|^2 \leq C\|A\|(\|u\|^2+\|H_0u\|^2)$.
Hence $u\in D(A)$. □

<u>Lemma 3.4</u>: $R_0(z)u(x) = i\int_{-\infty}^{x} e^{iz(x-y)}u(y)dy$ if $\text{Im } z > 0$

and

$$R_0(z)u(x) = -i\int_{x}^{\infty} e^{iz(x-y)}u(y)dy \quad \text{if Im } z < 0.$$

<u>Proof</u>: To prove the first formula we let $(D-z)v = u$.
Now

$$D(e^{-izx}v) = e^{-izx}(D-z)v = e^{-izx}u.$$

Hence

$$e^{-izx}v(x) = v(0) + i\int_0^x e^{-izy}u(y)dy.$$

Taking absolute values we have

(3.28) $|v(x)| = e^{-\eta x}|v(0) + i\int_0^x e^{-izy}u(y)dy|$ where
$z = \sigma + i\eta$. Since $u\in L^2$ the integral $\int_{-\infty}^{0} e^{-izy}dy$ exists.
Thus the only way that the right hand side of (3.28) exists is if the expression in the absolute value sign tends to 0 as $x \to \infty$. Thus

$$v(0) = i\int_{-\infty}^{0} e^{-izy}u(y)dy.$$

This gives

$$v(x) = i\int_{-\infty}^{x} e^{iz(x-y)}u(y)dy.$$

This proves the first formula of Lemma 3.4. The second formula may be proved similarly. □

Lemma 3.5: If $f(x) \in L^2$ then $\| fR_0(z)u\|^2 \leq \frac{1}{t}\|f\|^2\|u\|^2$ where $z = s + it$ and $t > 0$.

Proof: We have

$$\| fR_0(z)u \|^2 = \int_{-\infty}^{\infty} |f(x)|^2 \left|\int_{-\infty}^{x} e^{iz(x-y)} u(y)dy\right|^2 dx$$

$$\leq \int_{-\infty}^{\infty} |f(x)|^2 \left(\int_{-\infty}^{x} e^{-t(x-y)} |u(y)|dy\right)^2 dx$$

$$\leq \int_{-\infty}^{\infty} |f(x)|^2 \left(\int_{-\infty}^{x} e^{-t(x-y)} dy\right) \left(\int_{-\infty}^{x} e^{-t(x-y)} |u(y)|^2 dy\right) dx$$

$$\leq \frac{1}{t} \int_{-\infty}^{\infty} |f(x)|^2 \int_{-\infty}^{x} e^{-t(x-y)} |u(y)|^2 dy\, dx$$

$$\leq \frac{1}{t}\|f\|^2\|u\|^2.$$

Hence $\| fR_0(z)u\|^2 \leq \frac{1}{t}\|u\|^2$. □

For 2 operators A, B we say that B is compact relative to A or B is A compact if $D(A) \subset D(B)$ and $\|u_n\| + \|Au_n\| \leq C$ implies that $\{Bu_n\}$ has a convergent subsequence.

Lemma 3.6: If $q \in L^2$ than q is H_0 compact.

Proof: Recall that if $\psi \in D(H_0)$ then $|\psi(x)|^2 \leq C(\|\psi\|^2 + \|H_0\psi\|^2)$. Since $\psi'(x) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \hat{\psi}(k)kdk$, we have

$$|\psi'(x)|^2 \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k^2 |\hat{\psi}(k)|^2 dk = \frac{1}{\sqrt{2\pi}} \|H_0\psi\|^2.$$

We have

$$(3.29) \qquad |\psi(x)|^2 + |\psi'(x)|^2 \leq C(\|H_0\psi\|^2 + \|\psi\|^2).$$

Now suppose $\{\psi_n\} \in D(H_0)$ is a sequence such that $\|H_0\psi_n\| + \|\psi_n\| \leq C$. Then (3.29) shows that the $\psi_n(x)$ and $\psi_n'(x)$ are uniformly bounded. By the Arzela-Ascoli Theorem there is a subsequence of the ψ_n which converges uniformly in any bounded interval. If ψ_n denotes this subsequence and

$$q_{1,b}(x) = \begin{cases} q_1(x) & \text{if } |x| \leq b \\ 0 & \text{if } |x| > b \end{cases}$$

we have

$$(3.30) \qquad \|q_b(\psi_n - \psi_m)\|^2 = \int_{-b}^{b} |q(x)|^2 |\psi_n(x) - \psi_m(x)|^2 dx$$

$$\leq \max_{|x| \leq b} |\psi_n(x) - \psi_m(x)| \int_{-b}^{b} |q(x)|^2 dx \to 0 \text{ as } n,m \to \infty.$$

Next recall that since $q_b - q \in L^2$,

$$\|(q_b - q)(\psi_n - \psi_m)\|^2 \leq C_0 \varepsilon \|H_0(\psi_n - \psi_m)\|^2 + (1 + \tfrac{1}{\varepsilon}) C_0 \|\psi_n - \psi_m\|^2$$

where $C_0 = \sup_x \int_x^{x+1} |q(y)|^2 dy$. If $C_0 < \varepsilon^2 < 1$ then

$$(3.31) \qquad \|(q-q_b)(\psi_n-\psi_m)\|^2 \le \varepsilon\|H_0(\psi_n-\psi_m)\|^2 + 2\varepsilon\|\psi_n-\psi_m\|^2 = 4\varepsilon C.$$

Hence

$$\|q(\psi_n-\psi_m)\| \le \|(q-q_b)(\psi_n-\psi_m)\| + \|q_b(\psi_n-\psi_m)\| \to 0$$

by (3.30) and (3.31) and $q\psi_n$ converges. □

<u>Lemma 3.7</u>: If A is H_0 compact then $AR_0(z)$ is compact.

<u>Proof</u>: Let f_n be a bounded sequence in L^2. Then $R_0(z)f_n$ satisfies $\|R_0(z)f_n\| + \|H_0R_0(z)f_n\| \le C$. Thus $AR_0(z)f_n$ has a convergent subsequence and $AR_0(z)$ is compact. □

<u>Lemma 3.8</u>: If $g(z) = e^{izx}$ and Im $z > 0$ then $|g(z) - g(z')| \le |x||z - z'|$.

<u>Proof</u>: We have $g'(z) = ixe^{izx}$. Let $z = s + it$. Then $|g'(z)| \le |x|e^{-tx} \le |x|$. Now let $z(\theta) = (1-\theta)z' + \theta z$. Then

$$g(z) - g(z') = \int_0^1 g'(z(\theta))z'(\theta)d\theta = (z-z') \int_0^1 g'(z(\theta))d\theta.$$

So

$$|g(z) - g(z')| \le |z - z'| \int_0^1 g'(z(\theta))d\theta$$

$$\le |z - z'| \int_0^1 |x|d\theta = |z - z'||x|. \qquad \square$$

Lemma 3.9: If

$$(3.32) \qquad \int_{-\infty}^{\infty} (1+|x|)^2 (q_1(x)^2 + q_2(x)^2)\,dx < \infty \text{ then for each}$$

$I \subset\subset \Lambda$, $q_1 R_0(z) q_2$ is uniformly continuous in the region W_I $= \{z = s + ia / 0 < a < 1,\ s \in I\}$.

Proof: $|(q_1 R_0(z) q_2 u, v) - (q_1 R_0(z') q_2, v)|$

$$\leq \int_{-\infty}^{\infty} \int_{-\infty}^{x} |e^{iz(x-y)} - e^{iz'(x-y)}| |q_1(x)| |u(y)| |q_2(y) v(x)|\,dy\,dx$$

$$\leq |z - z'| \int_{-\infty}^{\infty}\int_{-\infty}^{x} |x - y| |q_1(x) u(y)| |q_2(y) v(x)|\,dy\,dx$$

$$\leq |z - z'| \int_{-\infty}^{\infty}\int_{-\infty}^{x} (1+|x|)(1+|y|) |q_1(x) u(y)| |q_2(y) v(x)|\,dy\,dx$$

$$\leq |z - z'| \left[\int_{-\infty}^{\infty}\int_{-\infty}^{x} (1+|x|)^2 |q_1(x) u(y)|^2 dy\,dx\right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty}\int_{-\infty}^{x} (1+|y|)^2 |q_2(y) v(x)|^2 dx\,dy\right]^{\frac{1}{2}}$$

$$\leq |z - z'| \|u\| \|v\| \int_{-\infty}^{\infty} (1+|x|)^2 (q_1(x)^2 + q_2(x)^2)\,dx$$

This proves the lemma. □

Lemma 3.10: If $q_1, q_2 \in L^2$ then $\|q_1 R_0(z) q_2 u\| \leq \|q_1\| \|q_2\| \|u\|$.

Proof: We have

$$|(R_0(z) q_2 u, q_1 v)| = \left|\int_{-\infty}^{\infty}\int_{-\infty}^{x} e^{iz(x-y)} q_2(y) u(y) q_1(x) v(x)\,dx\,dy\right|$$

$$\leq \int_{-\infty}^{\infty}\int_{-\infty}^{x} e^{-t(x-y)} |q_2(y)| |u(y)| |q_1(x)| |v(x)|\,dx\,dy$$

$$\leq \int |q_1(x)||v(x)|dx \quad \int |q_2(y)||u(y)|dy$$

$$\leq \|q_1\|\|q_2\|\|u\|\|v\|. \qquad \square$$

Lemma 3.11: If q_1 and q_2 are in L^2 $q_1R_0(z)q_2$ is uniformly continuous in W_I for I bounded and away from the origin.

Proof: For each n put $q_{1,n}(x) = q_1(x)$, $q_{2,n}(x) = q_2(x)$ if $|x| \leq n$ and $q_{1,n} = q_{2,n} = 0$ if $|x| > n$. Then

$$\|q_{1,n}R_0(z)q_{2,n} - q_1R_0(z)q_2\| \leq \|q_{1,n}R_0(z)q_{2,n} - q_{1,n}R_0(z)q_2\|$$

$$+ \|q_{1,n}R_0(z)q_2 - q_1R_0(z)q_2\|$$

$$\leq \|q_{1,n}\|\|q_{2,n} - q_2\| + \|q_2\|\|q_{1,n} - q_1\|$$

by Lemma 3.10, and this is bounded by $\|q_1\|\|q_{2,n} - q_2\| + \|q_2\|\|q_{1,n} - q_1\|$. Thus $q_{1,n}R_0(z)q_{2,n} \to q_1R_0(z)q_2$ uniformly in W_I. The operators q_1 and q_2 satisfy (3.32). Hence $q_{1,n}R_0(z)q_{2,n}$ is uniformly continuous in W_I. $\square$

Lemma 3.12: If f, $g \in L^2$ then $fR_0(z)g$ is closable, $I + [fR_0(z)g]$

is bounded, and $(I+[fR_0(z)g])^{-1}$ is bounded for Im z sufficiently large. If g is bounded then $I+[fR_0(z)g]$ has a bounded inverse for all z outside of some finite discrete set.

<u>Proof</u>: By Lemma 3.4 we have

$$|(R_0(z)gu,fv)| = \left|\int_{-\infty}^{\infty}\int_{-\infty}^{x} e^{iz(x-y)}g(y)u(y)dy\ f(x)\overline{v(x)}\ dx\right|$$

$$\leq \int_{-\infty}^{\infty}\int_{-\infty}^{x} e^{-t(x-y)}|f(x)u(y)||g(y)\overline{v(x)}|\,dyd\,x$$

where z = s + it

$$\leq \left[\int_{-\infty}^{\infty}\int_{-\infty}^{x} e^{-t(x-y)}|f(x)u(y)|^2 dy\,dx\right]^{\frac{1}{2}}$$

$$\times \left[\int_{-\infty}^{\infty}\int_{-\infty}^{x} e^{-t(x-y)}|g(y)v(x)|^2 dy\,dx\right]^{\frac{1}{2}}.$$

The square of the second term is

$$\int_{-\infty}^{\infty}|v(x)|^2\int_{-\infty}^{x} e^{-t(x-y)}|g(y)|^2 dy\,dx$$

$$= \int_{-\infty}^{\infty}|v(x)|^2\int_{-\infty}^{x-\varepsilon} e^{-t(x-y)}|g(y)|^2 dy\,dx$$

$$+ \int_{-\infty}^{\infty}|v(x)|^2\int_{x-\varepsilon}^{x} e^{-t(x-y)}|g(y)|^2 dy\,dx$$

$$\leq e^{-t\varepsilon}\|v\|^2\|g\|^2 + \int_{-\infty}^{\infty}|v(x)|^2\int_{x-\varepsilon}^{x}|g(y)|^2 dy\,dx \quad \text{if } t > 0.$$

Letting $\varepsilon = \frac{1}{\sqrt{t}}$, $t \to \infty$ and $C_\varepsilon = \sup_x \int_{x-\varepsilon}^{x} |g(y)|^2 dy$

we note that $C_\varepsilon \to 0$ as $\varepsilon \to 0$. We have

$$\int_{-\infty}^{\infty} |v(x)|^2 \int_{x-\varepsilon}^{x} |g(y)|^2 dy\, dx \leq C_\varepsilon \|v\|^2.$$

Similarly the square of the first integral is bounded by

$e^{-t\varepsilon}\|u\|^2\|f\|^2 + K_\varepsilon\|u\|^2$ where $K_\varepsilon = \sup_x \int_{x-\varepsilon}^{x} f(x)^2 dx \to 0$

as $\varepsilon \to 0$. Hence

$$|(R_0(z)gu,fv)| \leq \sqrt{e^{-t\varepsilon}\|f\|^2+K_\varepsilon}\ \sqrt{e^{-t\varepsilon}\|g\|^2+C_\varepsilon}\ \|u\|\|v\|$$

and we have $\|fR_0(z)g\| \leq C(z)\|u\|$ where $C(z) \to 0$ as $\operatorname{Im} z \to \infty$.

This proves that $fR_0(z)g$ is closable, $I + [fR_0(z)g]$ is bounded, and $I + [fR_0(z)g]$ has a bounded inverse for Im z sufficiently large.

We can now show that $fR_0(z)g$ is compact if g is bounded. Let $\|u_n\| \leq C$. Then $\|gu_n\| \leq C'$. Hence by Lemmas 3.6 - 3.7 $fR_0(z)g\,u_n$ has a convergent subsequence. Thus $fR_0(z)g$ is compact. By Lemma 3.11 and the first resolvent equation $fR_0(z)g$ is analytic outside of a set of measure zero. Thus the set of all z for which $I + [fR_0(z)g]$ has no bounded inverse is a finite discrete set. □

Since b^θ, $\frac{b'}{b^{1+\theta}} \in L^2$ and b^θ is bounded we have by Lemmas 3.3 - 3.7 that $D(H_0) \subset D(A) \cap D(B)$, that $a(\|AR_0(s+ia)\|^2 + \|BR_0(s+ia)\|^2) \leq C_I$. We also have that $B(AR_0(\bar{z}))^*$ is closable, $G_0(z)$ is bounded and $G_0(z)^{-1}$ is bounded and defined for all z outside of some finite discrete set. Thus the operator H of Theorem 2.1 exists.

To prove the existence of the wave operator, we first prove its existence for the functions u whose fourier transforms are of the form $\psi_s = ke^{-k^2-iks}$. Note that

$$[e^{-itk}\psi_s]^v = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} ke^{-k^2-iks} e^{-itk} dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(x-s-t)-k^2} kdk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-[k^2-ik(x-s-t)]} kdk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-[k-\frac{i}{2}(x-s-t)]^2-\frac{1}{4}(x-s-t)^2} kdk$$

$$= \frac{e^{-\frac{1}{4}(x-s-t)^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-[k-\frac{i}{2}(x-s-t)]^2} kdk .$$

Letting $z = k - \frac{i}{2}(x-s-t)$ this is

$$\frac{e^{-\frac{1}{4}(x-s-t)^2}}{\sqrt{2\pi}} \int e^{-z^2}(z+\tfrac{i}{2}(x-s-t)dz.$$

Since $\int ze^{-z^2}dz = 0$ and $\int e^{-z^2}dz = \sqrt{\pi}$ this is

$$\frac{i}{2\sqrt{2}}\, e^{-\frac{1}{4}(x-s-t)^2}(x-s-t).$$

We have

$$[e^{-itk}\psi s]^{v} = \frac{i}{2\sqrt{2}}\, e^{-\frac{1}{4}(x-s-t)^2}(x-s-t).$$

We next show that

$$\psi(t) = \frac{i}{2\sqrt{2}}\, e^{-\frac{1}{4}(x-s-t)^2}(x-s-t)$$

is a solution of the differential equation $i\psi'(t) = H\psi(t)$, $-\infty < t < \infty$, $\psi(0) = \psi_0$. Since $H = D$ this equation is equivalent to

$$-\psi'(t) = \frac{d}{dx}\psi.$$

We have

$$\psi'(t) = \frac{-i}{2\sqrt{2}}\left[e^{-\frac{1}{4}(x-s-t)^2} + \frac{1}{2}(x-s-t)^2\, e^{-\frac{1}{4}(x-s-t)^2}\right] = -\frac{d}{dx}\,\psi.$$

Since $e^{-itH_0}u$ is the unique solution of $i\psi'(t) = H\psi$ it follows that [5, p. 105] $e^{-itH_0}u = [e^{-itk}\psi_s]^{v}$. We have

$$|e^{-itH_0} u| = \frac{1}{2\sqrt{2}} e^{-\frac{1}{4}(x-s-t)^2} |x-s-t|$$

and

$$\| Ae^{-itH_0} u\| = \left[\frac{1}{8} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-s-t)^2} (x-s-t)^2 b^{2\theta}(x)dx\right]^{\frac{1}{2}} dt.$$

Thus

$$\int_a^{\infty} \| Ae^{-itH_0} u\| dt = \int_a^{\infty} \left[\frac{1}{8} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-s-t)^2} (x-s-t)^2 b^{2\theta}(x)dx\right]^{\frac{1}{2}} dt .$$

Hence for the functions of the form $\psi_s = ke^{-k^2-iks}$ (3.11) holds if

$$\int_a^{\infty} \left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-s-t)^2} (x-s-t)^2 b^{2\theta}(x)dx\right]^{\frac{1}{2}} dt < \infty . \tag{3.33}$$

In addition since b is bounded (3.12) holds. Hence the wave operator exists for the functions ψ_s if (3.33) holds. Since the functions ψ_s are dense in L^2, and the domain of the wave operator is closed, the wave operator exists for all functions satisfying (3.33).

Since b^{θ}, $\frac{b'}{b^{1+\theta}} \in L^2$ and b is bounded all the completeness conditions are satisfied by the Lemmas. We have the following theorem.

Theorem 3.9: Let H_0, H, A, B, J be as in Theorem 3.8. If there is an a such that

$$\int_a^{\infty} \left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-s-t)^2} (x-s-t)^2 b^{2\theta}(x)dx\right]^{\frac{1}{2}} dt < \infty$$

then the wave operator exists and is complete.

3.5 We shall prove the following Theorem.

<u>Theorem 3.10</u>: Let H_0 be the self adjoint operator on $L^2(-\infty,\infty)$ associated with $D = \frac{1}{i}\frac{d}{dx}$. Let $J = P^{-1}$ map $\mathcal{H}_0$ to $\mathcal{H}$ where $\mathcal{H} = L^2(R,P)$, $L^2(R,P)$ is $L^2(R)$ with the norm $\|u\|_{\mathcal{H}} = \|Pu\|$. Take $Au = Bu = o$. Then there exists an operator H such that $HJ \supset JH_0$. In addition the wave operator exists and is complete.

<u>Proof</u>: First note that $\|Ju\|_{\mathcal{H}} = \|P^{-1}u\|_{\mathcal{H}} = \|u\|_{L^2}$. Hence J is a bounded operator. Next note that since $Au = Bu = 0$ they are defined everywhere and hence $D(H_0) \subset D(A) \cap D(B)$.

We now verify (2.1). We have

$$\mathrm{Im}[(JH_0u,Ju)_{\mathcal{H}} + (Au,BJ^*Ju)_{\mathcal{H}}]$$

$$= \mathrm{Im}(JH_0u,Ju)_{\mathcal{H}}$$

$$= \mathrm{Im}(P^{-1}Du,P^{-1}u)_{\mathcal{H}}$$

$$= \mathrm{Im}(Du,u)_{L^2}$$

$$= \mathrm{Im}(-i)\int_{-\infty}^{\infty}\overline{u(x)}u'(x)dx$$

$$= -\mathrm{Re}\int_{-\infty}^{\infty}\overline{u(x)}u'(x)dx$$

$$= -\frac{1}{2}\int_{-\infty}^{\infty}\frac{d}{dx}|u(x)|^2 dx = -\frac{1}{2}|u(x)|^2 \Big|_{-\infty}^{\infty} = 0 .$$

Since $A = B = 0$ we have $a(\|AR_0(s+ia)\|^2 + \|BR_0(s+ia)\|^2) \le C_I$.

Also $B(AR_0(\bar{z}))^* = 0$. Hence it is closable and $G_0(z) = I$ is bounded and has a bounded inverse. Also note that $(B(BJ_0R_0(\bar{z}))^* = BJ_0(BR_0(\bar{z}))^* = 0$. Thus the conditions of Theorem 2.1 are satisfied and there exists an operator H such that $HJ \supset J(H_0+B^*A)$. Since $A = B = 0$, we have $HJ \supset JH_0$.

To show the existence of the wave operator note that $\int_{t_0}^{\infty}\|Ae^{-itH_0}u\|dt = 0$. Also note that $\lim_{t\to\infty}\|Je^{-itH_0}u\| = \|u\|$. Hence the wave operator exists.

To prove the completeness of the wave operator, we have already noted that $a(\|AR_0(s+ia)\|^2 + \|BR_0(s+ia)\|^2) \le C_I$ and that $(B(BJ_0R_0(z))^*)^* = BJ_0(BR_0(\bar{z}))^*$. We also have $Q_0(z) = 0$ is uniformly continuous, and $BR_0(z)(AR_0(z_1))^*$ is compact. Hence the wave operator is complete.

Now note that if $HJ \supset JH_0$ or equivalently if $HJ \supset JD$, then $u \in D(H_0)$ implies that $Ju \in D(H)$ and $HP^{-1} = P^{-1}Du$. Letting $v = P^{-1}u$ we have $Hv = P^{-1}DPv$. Thus if $(Pv), (Pv)' \in L^2$ then $v \in D(H)$ and $Hv = P^{-1}DPv$.

3.6 We shall prove

<u>Theorem 3.11</u>: Let H_0 be the self adjoint operator on $L^2(-\infty,\infty)$ associated with $D = \frac{1}{i}\frac{d}{dx}$. Let $J = P^{-1}$ map $\mathcal{H}_0$ to $\mathcal{H}$ where $\mathcal{H} = L^2(R,P)$, $L^2(R,P)$ is $L^2(R)$ with the norm $\|u\|_{\mathcal{H}} = \|Pu\|$.

Assume that P is a positive function and take

$$Au = \{\tau u,\ P^{-1}u\}$$

$$Bv = \{\tilde{\sigma}\, Dv,\ -P^{-2}(DP)v \quad \text{where } A,B : \mathcal{H}_0 \to L^2(-\infty,\infty)$$

$\oplus\ L^2(-\infty,\infty)$, σ, τ, $\tilde{\sigma}$ are functions such that $(P^{-1}-1) = \sigma\tau$.

and $\tilde{\sigma} = (P^{-1}+1)\sigma$ and

$$D(A) = \{u\in L^2/\tau u,\ P^{-1}u\in L^2\}$$

$$D(B) = \{v\in L^2/\tilde{\sigma}Dv,\ -P^{-2}(DP)v\in L^2\}.$$

Then the operator H of Theorem 2.1 exists if the following conditions are satisfied.

(3.34) 1. P^{-1}, $\tilde{\sigma}$, P', τ are bounded and in L^2.

2. $\sup_x (\tilde{\sigma}(x)\tau(x))^2 < \frac{1}{2}$

$\sup_x (\tilde{\sigma}(x)P(x)^{-1})^2 < \frac{1}{2}$

3. $a\|\tilde{\sigma}u + z\tilde{\sigma}R_0(s+ia)u\|^2 \le C_I$ for $a > 1$, and $s\in I$.

<u>Proof</u>: Note that since $\tilde{\sigma}$ is bounded and P is positive.

$|\sigma(x)| \le \dfrac{|\tilde{\sigma}(x)|}{P^{-1}(x)+1} \le |\tilde{\sigma}(x)|$. Thus σ is also bounded. Since τ and P^{-1} are also bounded we have that if $u\in D(H_0)$ then τu and $P^{-1}u\in L^2$. Thus $D(H_0)\subset D(A)$. To show that $D(H_0)\subset D(B)$ let $u\in D(H_0)$. Since $-P^{-2}(DP)$ is bounded we have that $-P^{-2}(DP)u\in L^2$. Since $u'\in L^2$ and $\tilde{\sigma}$ is bounded we have that $\tilde{\sigma}Du\in L^2$. This shows that $D(H_0)\subset D(B)$. Thus $D(H_0)\subset D(A)\cap D(B)$.

In the last application we showed that $\operatorname{Im}(JH_0u,Ju) = 0$ so to verify (2.1) it suffices to show that $\operatorname{Im}(Au,BJ^*Ju)_{L^2\oplus L^2} = 0$. Note that $(Ju,v)_{\mathcal{H}} = (P^{-1}u,v)_{\mathcal{H}} = (u,Pv)_{L^2}$. Hence $J^* = P$. We have $(Au,BJ^*Ju) = (Au,Bu)$

$$= (\tau u,\tilde{\sigma}Du) - (P^{-1}u,P^{-2}(DP)u)$$

$$= (u,(P^{-1}+1)\sigma\tau Du) + (u,P^{-1}(DP^{-1})u)$$

$$= (u,(P^{-2}-1)Du) + (u,P^{-1}(DP^{-1})u)$$

$$= (u,P^{-1}\ DP^{-1}u) - (u,Du)$$

$$= (P^{-1}u,DP^{-1}u) - (u,Du)$$

$$= \int P^{-1}(x)u(x)\overline{(DP^{-1}(x)u(x))} - \quad u(x)\overline{Du(x)}dx.$$

So

$$\operatorname{Im}(Au,BJ^*Ju) = \operatorname{Re}\int P^{-1}(x)u(x)\frac{d}{dx}\overline{(P^{-1}(x)u(x))}dx - \operatorname{Re}\int u(x)\overline{u'(x)}dx$$

$$= \frac{1}{2}\int\frac{d}{dx}|P^{-1}(x)u(x)|^2dx - \frac{1}{2}\int\frac{d}{dx}|u(x)|^2dx$$

$$= \frac{1}{2}\left|\frac{u(x)}{P(x)}\right|^2\Big|_{-\infty}^{\infty} - \frac{1}{2}|u(x)|^2\Big|_{-\infty}^{\infty} = 0 \quad \text{since } P^{-1}$$

is bounded and $u\in L^2(-\infty,\infty)$.

We have

$$(Au,\{v_1,v_2\})_{L^2\oplus L^2} = (\tau u,v_1) + (P^{-1}u,v_2)$$

$$= (u,\tau v_1) + (u,P^{-1}v_2)$$

$$= (u, \tau v_1 + P^{-1} v_2) .$$

Hence $A^* \{v_1, v_2\} = \tau v_1 + P^{-1} v_2$.

Now let $z = s + ia$. We have

$$\| AR_0(z)u \|^2 \leq \| \tau R_0(z)u \|^2 + \| P^{-1} R_0(z)u \|^2$$

$$\leq \frac{1}{t}(\| \tau \|^2 + \| P^{-1} \|^2) \| u \|^2$$

by Lemma 3.5 since τ, $P^{-1} \in L^2$.

Now note that

$$BR_0(z)u = \{\tilde{\sigma} DR_0(z)u, \ - (P^{-2})(DP)R_0(z)u\}$$

$$= \{\tilde{\sigma} u + z\tilde{\sigma} R_0(z)u, \ -P^{-2}(DP)R_0(z)u\} .$$

Hence

$$\| BR_0(z)u \|^2 \leq \| \tilde{\sigma} u + z\tilde{\sigma} R_0(z)u \|^2 + \| P^{-2}(DP)R_0(z)u \|^2 \tag{3.35}$$

$$\leq 2\| \tilde{\sigma} u \|^2 + 2|z|^2 \| \tilde{\sigma} R_0(z)u \|^2 + \| P^{-2}(DP)R_0(z)u \|^2 .$$

We have

$$\| \tilde{\sigma} u \|^2 \leq C \| u \|^2 \leq \frac{C}{t} \| u \|^2 \quad \text{for } o < t < 1.$$

By Lemma 3.5

$$|z|^2 \| \tilde{\sigma} R_0(z)u \|^2 \leq \frac{|z|^2}{t} \| \tilde{\sigma} \|^2 \| u \|^2 \leq \frac{C}{t} \| \tilde{\sigma} \|^2 \| u \|^2 \quad \text{for } 0 < t < 1, \ s \in I.$$

The last term in (3.35) is

$$\|P^{-2}(DP)R_0(z)u\|^2 \leq C\|P'R_0(z)u\|^2 \leq \frac{C}{t}\|P'\|^2\|u\|^2.$$

By our third hypothesis we have that

$$a(\|AR_0(s+ia)\|^2 + \|BR_0(s+ia)\|^2) \leq C_I \quad \text{for } a > 0 \text{ and } s \in I.$$

We shall now show that $B(AR_0(\bar{z}))^*$ is closable and that $G_0(z)$ is bounded and $G_0(z)$ has a bounded inverse for all z outside of some finite discrete set. We have

$$B(AR_0(\bar{z}))^*\{u_1,u_2\} =$$

$$\{\tilde{\sigma}\tau u_1 + \tilde{\sigma}P^{-1}u_2 + z\tilde{\sigma}R_0(z)\tau u_1 + z\tilde{\sigma}R_0(z)P^{-1}u_2,$$

$$-P^{-2}(DP)R_0(z)\tau u_1 - P^{-2}(DP)R_0(z)P^{-1}u_2\}.$$

Hence

$$\|B(AR_0(\bar{z}))^*\{u_1,u_2\}\|^2$$

$$= \|\tilde{\sigma}\tau u_1 + \tilde{\sigma}P^{-1}u_2 + z\tilde{\sigma}R_0(z)\tau u_1 + z\tilde{\sigma}R_0(z)P^{-1}u_2\|^2$$

$$+ \|P^{-2}(DP)R_0(z)\tau u_1 - P^{-2}(DP)R_0(z)P^{-1}u_2\|^2$$

$$\leq \frac{C}{t}\|\tau u_1\|^2 + \frac{C}{t}\|P^{-1}u_2\|^2$$

$$\leq \frac{C}{t}\|u_1\|^2 + \frac{C}{t}\|u_2\|^2$$

$= \frac{C}{t}\|\{u_1,u_2\}\|^2$, $z = s + it$. Thus $\|B(AR_0(\bar{z}))^*\| < 1$

for t sufficiently large. Hence $B(AR_0(\overline{z}))^*$ is closable, $G_0(z)$ is bounded, and $G_0(z)$ has a bounded inverse for z sufficiently large.

By the above we have that

$$I + B(AR_0(\overline{z}))^* \{u_1, u_2\}$$

$$= \{u_1, u_2\} + \{\tilde{\sigma}\tau u_1 + \tilde{\sigma}P^{-1}u_2, 0\} \tag{3.36}$$

$$+ \{z\tilde{\sigma}R_0(z) + z\tilde{\sigma}R_0(z)P^{-1}u_2,$$

$$-P^{-2}(DP)R_0(z)\tau u_1 - P^{-2}(DP)R_0(z)P^{-1}u_2\} .$$

We can show that the sum of the first two operators is invertible and that the third operator is compact. By (3.34) $\|\tilde{\sigma}\tau u_1\|^2 < \frac{1}{2}\|u_1\|^2$ and $\|\tilde{\sigma}P^{-1}u_2\|^2 < \frac{1}{2}\|u_2\|^2$. We have

$$\|\tilde{\sigma}\tau u_1 + \tilde{\sigma}P^{-1}u_2\|^2 \leq 2\|\tilde{\sigma}\tau u_1\|^2 + 2\|\tilde{\sigma}P^{-1}u_2\|^2$$

$$< \|u_1\|^2 + \|u_2\|^2 = \|\{u_1, u_2\}\|^2 .$$

Hence the sum of the first two operators in (3.36) is invertible. By hypothesis it can easilv be shown that the third operator in (3.36) is compact. Note that it is also analytic. Thus $G_0(z)$ has a bounded inverse for all z outside of some finite discrete set.

We can now show that $(B[BJ_0R_0(z)]^*)^* = BJ_0(BR_0(\overline{z}))^*$. Since $BR_0(z)u = \{\tilde{\sigma}DR_0(z)u, -P^{-2}(DP)R_0(z)u\}$ we have

$$(BR_0(z)u,v) = (\tilde{\sigma}DR_0(z)u,v) + (-P^{-2}(DP)R_0(z)u,v)$$

$$= (u,R_0(\overline{z})D(\tilde{\sigma}v) - R_0(\overline{z})(P^{-2})(DP)v).$$

Thus

$$(BR_0(z))^*v = R_0(\overline{z})D(\tilde{\sigma}v) - R_0(\overline{z})(P^{-2})(DP)v. \tag{3.37}$$

Consequently

$$B(BJ_0R_0(z))^*u = \{\tilde{\sigma}DR_0(\overline{z})D(\tilde{\sigma}u) - \tilde{\sigma}DR_0(\overline{z})(DP)(P^{-2})u,$$

$$-P^{-2}(DP)R_0(\overline{z})D\tilde{\sigma}u + P^{-2}(DP)R_0(\overline{z})(P^{-2})(DP)u\}$$

and

$$(B(BJ_0R_0(z))^*u,v)$$

$$= (u,\tilde{\sigma}DR_0(z)D\tilde{\sigma}v - (P^{-2})(DP)R_0(z)D\tilde{\sigma}v$$

$$- \tilde{\sigma}DR_0(z)(P^{-2})(DP)v + (P^{-2})(DP)R_0(z)(P^{-2})(DP)v)$$

$$= (u,\tilde{\sigma}D(R_0(z)D\tilde{\sigma}v - R_0(z)(P^{-2})(DP)v))$$

$$+ (u,(-P^{-2})(DP)(R_0(z)D\tilde{\sigma}v - R_0(z)(P^{-2})(DP)v))$$

$$= (u,B[R_0(z)D\tilde{\sigma}v - R_0(z)(P^{-2})(DP)v]).$$

Noting that by (3.37)

$$(BR_0(\bar{z}))^*v = R_0(z)D\tilde{\sigma}v - R_0(z)(P^{-2})(DP)v$$

this is

$$(u, B(BR_0(\bar{z}))^*v) = (u, BJ_0(BR_0(\bar{z}))^*v).$$

Thus

$$(B(BJ_0R_0(z))^*)^* = BJ_0(BR_0(\bar{z}))^*.$$

We have proved Theorem 3.11.

We now examine the existence of the wave operator. We have

$$\int_{t_0}^{\infty} \|Ae^{-itH_0}u\|dt \le \int_{t_0}^{\infty}\|\tau e^{-itH_0}u\|dt + \int_{t_0}^{\infty}\|P^{-1}e^{-itH_0}u\|dt.$$

Note that this is finite if

$$(3.38) \qquad \int_{t_0}^{\infty}\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-s-t)^2}(x-s-t)^2(\tau^2)dx\right)^{\frac{1}{2}}dt < \infty$$

and

$$(3.39) \qquad \int_{t_0}^{\infty}\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-s-t)^2}(x-s-t)^2(P^{-2})dx\right)^{\frac{1}{2}}dt < \infty.$$

We also have $\lim_{t\to\infty}\|Je^{-itH_0}u\| = \|u\|$. In addition all the completeness conditions are satisfied. We have the following theorem.

Theorem 3.12: If H_0, H, A, B, J are as in Theorem 3.11 and (3.38) and (3.39) hold, then the wave operator exists and is complete.

BIBLIOGRAPHY

[1] Tasio Kato, Perturbation Theory for Linear Operators, Springer-Verlag, 1976.

[2] Tosio Kato, Wave Operators and Similarity for Some Non self-adjoint Operators, Math. Annalen, 1962 (1966), 258-279.

[3] Martin Schechter, Completeness of Wave Operators in Two Hilbert Spaces, Ann. Inst. Henri Poincare, 30 (1979), 109-127.

[4] Martin Schechter, Nonhomogenious Elliptic Systems and Scattering, Tôhoku Mathematical Journal, 27 (1975), 601-616.

[5] Martin Schechter, Operator Methods in Quantum Mechanics, Elsevier North Holland, 1981.

[6] Martin Schechter, Scattering in Two Hilbert Spaces, J. London Math. Soc.(2), 19 (1979), 175-186.

[7] Martin Schechter, Self-Adjoint Realizations in Another Hilbert Space, American Journal of Math. (to appear).