Abstract

The first part of the thesis is devoted to radial symmetry and monotonicity of solutions for fractional elliptic and parabolic equations, we consider problems involving the n-dimensional fractional Laplacians including elliptic equations and parabolic equations. We also consider the problems involving fractional Monge-Ampére operators. The thesis is mostly devoted to presenting our original work on the progress obtained in the development of direct methods that can effectively deal with the above problems.

Part 1: Method of Moving Planes and Its Applications: Radial symmetry and monotonicity of solutions for fractional elliptic and parabolic equations and systems

It mainly includes the direct method of moving planes. We illustrate how the direct method of moving planes work by applying them to elliptic problems and parabolic problems:

We study fractional elliptic equations

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(u(x)), & x \in B_1(0), \\ u(x) \geq 0, & x \in B_1(0), \\ u(x) \equiv 0, & x \notin B_1(0), \end{cases}$$

under some conditions on f, we show that solutions $u(x)$ are radially symmetric and monotone decreasing about the origin.

We study fractional elliptic system:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}}u(x) = f(v(x)), & x \in B_1(0), \\ (-\Delta)^{\frac{\beta}{2}}v(x) = g(u(x)), & x \in B_1(0), \\ u(x), v(x) \geq 0, & x \in B_1(0), \\ u(x), v(x) \equiv 0, & x \notin B_1(0), \end{cases}$$

under some conditions on f and g, we show that solutions $u(x)$ and $v(x)$ are radially symmetric and monotone decreasing about the origin.

And we study fractional parabolic equations

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = f(t,|x|,u), & (x,t) \in B_1(0) \times (-\infty,\infty), \\ u(x,t) > 0, & (x,t) \in B_1(0) \times (-\infty,\infty), \\ u(x,t) \equiv 0, & x \notin B_1(0), \end{cases}$$

under some conditions on f, we show that solutions $u(x,t)$ are radially symmetric and monotone decreasing about the origin, where we use direct method of moving plane to show above.

We study fractional parabolic system:

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta)^{\frac{\alpha}{2}}u(x,t) = f(v(x,t)), & (x,t) \in B_1(0) \times (-\infty,\infty), \\ \frac{\partial v}{\partial t} + (-\Delta)^{\frac{\beta}{2}}v(x,t) = g(u(x,t)), & (x,t) \in B_1(0) \times (-\infty,\infty), \\ u(x,t), v(x,t) \geq 0, & (x,t) \in B_1(0) \times (-\infty,\infty), \\ u(x,t), v(x,t) \equiv 0, & x \notin B_1(0), \end{cases}$$

under some conditions on f and g, we show that solutions $u(x,t)$ and $v(x,t)$ are radially symmetric and monotone decreasing about the origin. We use the method of moving plane to show radial symmetry and monotonicity of solutions of those systems.

Part 2: Method of Sliding and Its Applications: Monotonicity and one-dimensional symmetry of solution of fractional parabolic and Monge-Ampére equations

We also show how the sliding method work by applying them to fractional parabolic equations and problems involving fractional Monge-Ampére operators.

We also study fractional parabolic equations in a bounded domain:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = f(t,|x|,u), & (x,t) \in \Omega \times (-\infty,\infty), \\ u(x,t) = \varphi(x,t), & (x,t) \in \Omega^c \times (-\infty,\infty), \end{cases}$$

We assume that u is monotone increasing in Ω^c and Ω^c, we also assume that f is non-increasing about u and is uniformly Lipschitz continuous in u, then $u(x,t)$ is monotone increasing with respect to x_n in Ω. We use the sliding method to show monotonicity and one-dimensional symmetry of the solution of this fractional parabolic equation in the bounded domain.

We also study problems involving fractional Monge-Ampére operators in bounded domains:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - D_s^\theta u(x,t) = f(t,|x|,u), & (x,t) \in \Omega \times (-\infty,\infty), \\ u(x,t) = \varphi(x,t), & (x,t) \in \Omega^c \times (-\infty,\infty), \end{cases}$$

We assume that u is monotone increasing in Ω^c and Ω^c, we also assume that f is non-increasing about u and is uniformly Lipschitz continuous in u, then $u(x,t)$ is monotone increasing with respect to x_n in Ω. We use the sliding method to show monotonicity and one-dimensional symmetry of solutions of fractional Monge-Ampére equations in bounded domains.

We also study problems involving fractional Monge-Ampére operators in unbounded domains, here, we proved the Gibbon's conjecture, Let $u(x,t)$ be an entire solution of:

$$\frac{\partial u}{\partial t}(x,t) - D_s^{\theta} u(x,t) = f(t,|x|,u),\ (x,t) \in \mathbb{R}^n \times (-\infty,\infty)$$

with condition

$$|u(x,t)| \leq 1$$

and

$$\lim_{x_n \to \pm\infty} u((x',x_n),t) \to \pm 1$$

uniformly in $x' = (x_1, \cdots, x_{n-1})$. Also, $f(t,|x|,u)$ is non-increasing near $u(x,t) = \pm 1$. Then u must be strictly increasing with respect to x_n, and it depends on x_n only.

We use the sliding method to show the monotonicity of solutions of fractional Monge-Ampére equations in unbounded domains. Also, we would show that u must be strictly increasing with respect to x_n and it depends on x_n only.

To this end, we introduce several new ideas and developed a systematic approach which may also be applied to investigate qualitative properties of solutions for many other fractional parabolic problems.

The second part of the thesis is devoted to free long flight in infinite horizon Lorentz Gas. In this work, we are interested in the length of a few consecutive long free flights in infinite horizon Lorentz Gas. In dimension $D = 2$, it is well known that a flight of length $T \gg 1$ is typically followed by a flight of length $C\sqrt{T}$. Here, we extend this result to any dimension D.

The main theorem we want to prove is:

In $D \geq 2$ and under some conditions. There exists a stochastic process $\mathcal{X}_1, \mathcal{X}_2, \ldots$ so that for any finite n and for any sets $A_i \subset \mathbb{R}$ with $Leb(\partial A_i) = 0$,

$$\lim_{T\to\infty} \frac{\nu_0(\tau_1 > T, \tau_i \in A_i T^{1/D^{i-1}}, i = 1, ..., n)}{\nu_0(\tau_1 > T)} = \mathbb{P}(\mathcal{X}_i \in A_i, i = 1, ..., n).$$

We divide our proof into some parts. For some special cases such that $D \geq 2$ and $n = 1$, we generate the theorem for special case such that:

$$\lim_{T\to\infty} \frac{\nu_0(\tau_1 > T, \tau_1 \in A_1 T, i = 1)}{\nu_0(\tau_1 > T)} = \mathbb{P}(\mathcal{X}_1 \in A_1) \sim \frac{1}{A}.$$

Let $D = 2$, $\mathcal{D} = \mathbb{R}^2 \setminus \cup_{z\in\mathbb{Z}^2} B(z, r)$ with $\sqrt{2}/4 < r < 1/2$. This condition ensures that principal corridors exist and they are parallel to coordinate hyperplanes. There exists a stochastic process $\mathcal{X}_1, \mathcal{X}_2, ...$ so that for any finite n and for any sets $A_i \subset \mathbb{R}$ with $Leb(\partial A_i) = 0$,

$$\lim_{T\to\infty} \frac{\nu_0(\tau_1 > T, \tau_i \in A_i T^{1/2^{i-1}}, i = 1, ..., n)}{\nu_0(\tau_1 > T)} = \mathbb{P}(\mathcal{X}_i \in A_i, i = 1, ..., n).$$

In higher dimensions, we follow by Marklof-Strömbergsson'theory to prove the theorem: There exists a continuous function $\Psi : \mathbb{R}_+ \to \mathbb{R}$ so that for all ξ,

$$\lim_{T\to\infty} (\hat{\nu} \times \lambda_T)(\tau > \xi T^{\frac{D-2}{D}}) = \int_\xi^\infty \Psi(\xi')d\xi'.$$

Followed by Marklof-Strömbergsson'theory but with some adjustments, there exists a continuous density function Φ so that for all $\underline{\xi}, \bar{\xi}$, $[a, b] \subset [0, 1]$

$$\lim_{T\to\infty} (\nu \times \lambda_T)(\tau \in [\underline{\xi} T^{\frac{D-2}{D}}, \bar{\xi} T^{\frac{D-2}{D}}], w \in [a, b]) = \int_{\underline{\xi}}^{\bar{\xi}} \int_a^b \Phi(\xi, w)dwd\xi.$$

The function Φ is explicitly given by

$$\Phi(\xi, w) = \begin{cases} \nu_y(\{M \in X_q(y) : (\mathbb{Z}^d + \alpha)M \cap (\Upsilon(0, \xi, 1) + z)\}) & \text{if } \alpha \in q^{-1}\mathbb{Z}^d \\ \nu_y(\{g \in X(y) : \mathbb{Z}^d g \cap \Upsilon(0, \xi, 1) + z) = \emptyset\}) & \text{if } \alpha \notin \mathbb{Q}^d \end{cases}$$

Based on this theorem, if we have one single collision, we would have the following result:

There exists a continuous density function Φ, ψ so that for all ξ, ξ',

$$\begin{aligned} \lim_{T\to\infty} (\nu \times \lambda_T)(\tau_1 \in [\underline{\xi} T^{\frac{D-2}{D}}, \bar{\xi} T^{\frac{D-2}{D}}], w_1 \in [a, b] \\ \tau_2 \in [\underline{\xi}' T^{\frac{D-2}{2D}}, \bar{\xi}' T^{\frac{D-2}{2D}}], w_2 \in [a', b']) \\ = \int_{\underline{\xi}}^{\bar{\xi}} \int_a^b \Phi(\xi, w) \int_{\underline{\xi}'}^{\bar{\xi}'} \int_{a'}^{b'} \psi(\xi', w', \xi, w) dw d\xi. \end{aligned}$$

where

$$\psi(\xi', w', \xi, w) = \Phi(\sqrt{\frac{\kappa}{8}} \frac{1}{\sqrt{\xi(1-w)}} \xi', \sqrt{\frac{8}{\kappa}} \sqrt{\xi(1-w)} w')$$

For k collisions, we would have the following:

There exists continuous density functions $\Phi, \psi, \cdots$ so that for all $\xi, \xi_2, \cdots, \xi_k$,

$$\begin{aligned} \lim_{T\to\infty} (\nu \times \lambda_T)(\tau_1 \in [\underline{\xi} T^{\frac{D-2}{D}}, \bar{\xi} T^{\frac{D-2}{D}}], w_1 \in [a, b], \cdots, \\ \tau_k \in [\underline{\xi_k} T^{\frac{D-2}{2D}}, \bar{\xi_k} T^{\frac{D-2}{2D}}], w_k \in [a_k, b_k]) \\ = \int_{\underline{\xi_1}}^{\bar{\xi_1}} \int_{a_1}^{b_1} \Phi(\xi_1, w_1) \int_{\underline{\xi_2}}^{\bar{\xi_2}} \int_{a_2}^{b_2} \psi(\xi_1, w_1, \xi_2, w_2) \\ \int_{\underline{\xi_3}}^{\bar{\xi_3}} \int_{a_3}^{b_3} \psi(\xi_2, w_2, \xi_3, w_3) \cdots \int_{\underline{\xi_k}}^{\bar{\xi_k}} \int_{a_k}^{b_k} \psi(\xi_{k-1}, w_{k-1}, \xi_k, w_k) \\ dw_k d\xi_k dw_{k-1} d\xi_{k-1} \cdots dw_1 d\xi_1 \end{aligned}$$

To this end, we hope the ideas employed here would be helpful for research in Long flights in Lorentz Gas.

Topics in Fractional Laplacian and Dynamical Systems

by

Xingyu Liu

Submitted in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in the Department of Mathematical Sciences

Yeshiva University

June 2023

The committee for this doctoral dissertation consists of:

Marian Gidea, Ph.D., Director of the Graduate studies, Department of Mathematical Sciences, Yeshiva University

Florin Catrina, Ph.D., Professor, Mathematics and Computer Sciences, St. John's University

Wenxiong Chen, Ph.D., Professor, Department of Mathematical Sciences, Yeshiva University (Ph.D. Advisor)

Peter Nandori, Ph.D., Assistant Professor, Department of Mathematical Sciences, Yeshiva University (Ph.D. Advisor)

Acknowledgements

In March 2017, I had got Master Offer from Yeshiva University, Math department. However, in Fall 2016, I was always busy with preparing GRE test while completing my courseworks as an undergraduate student, and I missed the email reminding of me to refresh my I-20, which represents my legal student identity in United States.

From December 2016 to Fall 2017, I did student identity reinstatement with the aid of my undergraduate school, and I was always waiting for result, since if I could not be back in my student identity before September 2017, I could not enroll in Yeshiva University and the Offer was invalid anymore. However, I waited until August 2017 and went to other country having no choice.

In Fall 2019, I came to Yeshiva University which I should be enrolled in Fall 2017.

These two years, I was always thinking, should I come back again?

However, when I came back, the things are not as smooth as I thought. Except my personal issues, the Covid virus broke out in Spring 2020. My Master classmate, a girl, who came back to home-country and waited for her PhD Offer in December 2019, could not come back that year.

As a girl approaching 30-year-old, in China, there is not so much time for a girl to wait for a chance to be a doctorate student. My classmate then became a high-school mathematical teacher and married soon. Never came back again. If I came back to my home-country in the end of 2019, I may miss my opportunity to become a doctorate student again.

If there is a boy being a doctorate student, he may always remember how much endeavor he had used and how much perspiration he had spent. But I am —— like a girl purchasing her love, after misses, you are still the one —— for my doctorate study.

Finally, I would like to express sincere thanks to all professors and staffs in Yeshiva

University. In 2019, when I faced personal difficulties, Dr. Marian Gidea and Dr. Peter Nandori gave me warm suggestions and care. In these three more years, Dr. Marian Gidea, Dr.Marini Antonella and Dr.Wenxiong Chen gave me different jobs which exercising my working ability, they also provided me with different research positions, enriching my graduate life.

I want to especially thank my advisor Dr.Wenxiong Chen and Dr.Peter Nandori for their constant guidance and support throughout these years. When I first met Dr. Chen in Fall 2020, I was amazed by his teaching style and his study area. He gave me a lot of valuable guides in both study and thesis, which form the foundation of PDE part I have completed so far. I met Dr.Nandori in 2019, he was my very initial advisor in my graduate study life, he also helped me with Master's thesis. We studied together for the Dynamical system part for this thesis.

I hope I could continue studying the areas in this thesis and become their proudest student after graduation.

Deep gratitude goes to my mother. I will always miss the Guinea pig that accompanied me in my studying time.

Contents

1 Introduction

1.1 Backgrounds

This dissertation focuses on studying nonlinear elliptic and parabolic partial differential equations, especially on fractional elliptic equations, fractional parabolic equations, fractional elliptic systems, fractional parabolic systems, also on parabolic Monge-Ampére operator.

The fractional Laplacian is a non-local operator that has gained much attention in recent years due to its ability to model diverse physical phenomena, such as anomalous diffusion, turbulence, and water waves. It is also of great interest in finance and probability theory. Here are a few examples about applications of fractional Laplacian:

1. Anomalous diffusion: The fractional Laplacian can be used to model anomalous diffusion, where the diffusion process is slower than normal. This phenomenon is observed in many physical systems, such as plasma physics [17], porous media [20], and biological systems [18].

2. Image processing: The fractional Laplacian can be used in image processing to remove noise and enhance image edges [32].

3. Financial mathematics: The fractional Laplacian is used in financial mathematics to model the behavior of asset prices and to price American options [28].

4. Nonlinear partial differential equations: The fractional Laplacian appears in many nonlinear partial differential equations, such as the fractional Allen-Cahn equation:

$$\frac{\partial u}{\partial t} = \epsilon^2(-\triangle)^\alpha u - f(u)$$

the fractional Burgers equation:

$$\frac{\partial u}{\partial t} + u \cdot \bigtriangledown u = \nu(-\triangle)^{\alpha}$$

and the fractional Navier-Stokes equations:

$$\frac{\partial u}{\partial t} + u \cdot \bigtriangledown u = -\frac{\bigtriangledown P}{\rho} + \nu(-\triangle)^{\alpha} u$$

where ρ is the density, P is the pressure.

5. Quantum mechanics: The fractional Laplacian appears in the study of relativistic quantum mechanics [26].

6. Geophysical fluid dynamics: The fractional Laplacian appears in the study of quasi-geostrophic flows and geographical fluid dynamics [1].

7. Population dynamics: The fractional Laplacian appears in the study of population dynamics, where it can be used to model the spread of infectious diseases [16].

In contrast to the usual differential operators, such as the regular Laplacian, the fractional Laplacian is a non-local operator, meaning that its value at a point depends on the values of the function in the whole space, rather than near that point. This non-locality gives rise to a number of unique mathematical properties that make the fractional Laplacian an important tool for modeling non-local phenomena.

The study of the fractional Laplacian and its applications is an active area of research, with many open questions and challenges. Over the past two decades, researchers have developed several methods for studying the fractional Laplacian and its properties, here we use the method of moving Planes.

The method of moving Planes is a technique used to study and prove certain qualitative properties, like the symmetry, existence/nonexistence and regularity properties for the solutions of various kinds of problems. The basic idea of the method is to consider a family of parallel planes, and to study how the solution to the PDE changes as the plane moves along a certain direction. By carefully choosing the direction of the plane and the parameters of the family of planes, one can often prove that the solution to the PDE is either symmetric with respect to some plane, or monotone in some direction. The method was first introduced by A.D. Alexandrov in the 1950s, and has since been used in many different areas of mathematics.

In the second part of the thesis, we also use sliding method to deal with fractional Laplacian equation and Fractional Monge-Ampére operators. Berestycki and Nirenberg [19] originally introduced the renowned sliding method to establish qualitative properties of positive solutions to local elliptic equations. Later on, a direct sliding method was developed by Wu and Chen [22], which has proved to be valuable in many applications such as deriving monotonicity, one-dimensional symmetry, uniqueness, and nonexistence of solutions to elliptic equations and systems involving fractional Laplacians and p-Laplacians. Detailed information can be found in [[40], [48], [23], [24]], and an exhaustive survey in [41]. This direct method avoids the need for classical extension methods established in [25] and overcomes the difficulties caused by the non-locality of fractional operators. Moreover, this direct sliding method can be applied to extend and prove Gibbons' conjecture in the settings of other fractional elliptic equations involving various nonlocal operators (cf. [[44], [43], [31], [29], [30]]). In contrast, there have been few studies on Gibbons' conjecture for entire solutions of parabolic equations, except for a recent article by Chen and Wu [42], in which they developed an appropriate sliding method to prove the Gibbons' conjecture for entire solutions of the following fractional reaction-diffusion equation.

We hold a strong conviction that the ideas and methods presented here can be readily employed to investigate diverse nonlocal problems that involve more comprehensive operators and nonlinearities.

1.2 Previous Methods for the Fractional Equations

In this section, we will introduce the commonly used definition of the fractional Laplacian in this section, followed by a more comprehensive account of the advancements made in both indirect and direct methods.

This fractional Laplacian is a pseudo-differential operator defined by

$$\begin{aligned} (-\Delta)^s u(x) &\equiv C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy \\ &\equiv C_{n,s} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy, \end{aligned} \tag{1.1}$$

for any real number $0 < s < 1$, where P.V. stands for the Cauchy Principal value.

Let

$$\mathcal{L}_{2s} \equiv \left\{ u : \mathbb{R}^n \to \mathbb{R} \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}} dx < +\infty \right\}.$$

Then the operator $(-\triangle)^s$ is well defined on the functions u in $\mathcal{L}_{2s} \cap C^{1,1}_{loc}$. One can see from the definition (1.1) that it is nonlocal. For example, consider $u(x) > 0$ in $B_1(0)$ and $u(x) \equiv 0$ in $\mathbb{R}^n \setminus B_1(0)$. Given any $x^o \in \mathbb{R}^n \setminus B_1(0)$, it's easy to see that all derivatives of u vanish at x^o.

In comparison, we have

$$\begin{aligned} (-\Delta)^s u(x^o) &\equiv C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x^o)-u(y)}{|x-y|^{n+2s}} dy \\ &\equiv C_{n,s} P.V \int_{B_1(0)} \frac{-u(y)}{|x-y|^{n+2s}} dy < 0. \end{aligned}$$

In other words, even u is identically 0 in a neighborhood of a point, $(-\triangle)^s u(x)$ still may not vanish.

Therefore, traditional methods on local differential operators, such as on Laplacian $-\triangle$

may not work on this nonlocal operator. Caffarelli and Silvestre [4] addressed this challenge by introducing the extension method, which transformed the nonlocal problem into a local one in higher dimensions.

For a function $u : \mathbb{R}^n \to \mathbb{R}$, let $U : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ become its extension such that

$$\begin{cases} div(y^{1-2s} \bigtriangledown U) = 0, & (x, y) \in \mathbb{R}^n \times [0, \infty), \\ U(x, 0) = u(x), & x \in \mathbb{R}^n \end{cases} \tag{1.2}$$

Then

$$(-\triangle)^s u(x) = -C_{n,s} \lim_{y \to 0^+} y^{1-2s} \frac{\partial U}{\partial y}, x \in \mathbb{R}^n$$

This extension method is a potent tool that has sparked significant interest in the study of equations involving the fractional Laplacian, resulting in a series of fruitful outcomes. (see [2],[10])

In both reference [2] and [10], when authors proved the fractional Laplacian problem on $(-\triangle)^s u$, they had to restrict the condition that $2s \geq 1$.

The reason for the usual constraint $2s > 1$ is that the method of moving planes requires this condition to be met when applied to the solutions U of the extended problem:

$$zdiv(y^{1-2s} \bigtriangledown U) = 0, (x, y) \in \mathbb{R}^n \times [0, \infty) \tag{1.3}$$

It appears that relaxing the condition $2s \geq 1$ is not possible if one intends to apply the method of moving planes to the extended equation. Then how about when $0 < 2s < 1$?

By taking into account the corresponding integral equation, this case can be effectively addressed. In [9], [11], the authors showed that if $u \in H^s(\mathbb{R}^n)$ is a positive weak solution of

$$(-\triangle)^s u = u^p(x), x \in \mathbb{R}^n \tag{1.4}$$

then it also satisfies the integral equation:

$$u(x) = C \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2s}} u^p(y) dy \tag{1.5}$$

They utilized the method of moving planes in integral forms to establish the radial symmetry in the critical case and the non-existence of positive solutions in the subcritical case for (1.5)

The equivalence between pseudo differential equation (1.4) and integral equation was also established in [49] by employing a Liouville theorem for $2s$-harmonic functions.

The above methods only apply to fractional Laplacian equations, there are many other non-local operators. In addition, when dealing with equations that feature uniformly elliptic nonlocal operators,

$$C_{n,s} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{a(x-z)(u(x)-u(z))}{|x-z|^{n+2s}} dz = f(x,u) \tag{1.6}$$

where

$$0 < c_0 \leq a(y) \leq C_1, y \in \mathbb{R}^n$$

And for equations that involve fully nonlinear nonlocal operators, such as the fractional p-Laplacian:

$$(-\triangle)_p^s u(x) = C_{n,s} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{|u(x)-u(z)|^{p-2}(u(x)-u(z))}{|x-z|^{n+sp}} dz$$

and more generally,

$$F_s(u(x)) \equiv C_{n,s} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{G(u(x)-u(z))}{|x-z|^{n+2s}} dz = f(x,u) \tag{1.7}$$

([3] has the introductions of these operators) To the best of our knowledge, there are no extension methods or integral equation methods that are effective for these types of operators. This serves as a motivation for us to develop direct approaches for general nonlocal operators.

In [8], Chen, Li and Yan Li introduced a direct method of moving planes for the fractional Laplacian, and utilized it to obtain symmetry, monotonicity, and non-existence of solutions for various semi-linear equations involving the fractional Laplacian. Furthermore, the direct approach is applicable in studying the qualitative characteristics of solutions for uniformly elliptic problems 1.6 and fully nonlinear problem 1.7 (see [33] and [14])

Based on this direct method of moving plane, in this paper we first study the following parabolic equations involving the fractional Laplacian:

$$\frac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = f(t,|x|,u),\ \ (x,t) \in B_1(0) \times (-\infty,\infty), \tag{1.8}$$

where $0 < s < 1$

For each fixed $t \in \mathbb{R}$, the fractional Laplacian acting on x is defined as

$$\begin{aligned} (-\Delta)^s u(x,t) &= C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x,t)-u(y,t)}{|x-y|^{n+2s}} dy \\ &= C_{n,s} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x,t)-u(y,t)}{|x-y|^{n+2s}} dy, \end{aligned}$$

where $P.V.$ stands for the Cauchy principal value. It is easy to see that for $u \in C^{1,1}_{loc} \cap \mathcal{L}_{2s}$,

$(-\Delta)^s u$ is well defined, where

$$\mathcal{L}_{2s} = \left\{ u(\cdot, t) \in L^1_{loc}(\mathbb{R}^n) \;\Big|\; \int_{\mathbb{R}^n} \frac{|u(x,t)|}{1+|x|^{n+2s}} dx < +\infty \right\}.$$

It is known that as $s \to 1$, the fractional Laplacian $(-\Delta)^s$ goes to the regular Laplacian $-\Delta$, in the sense that for each fixed x, $(-\Delta)^s u(x) \to -\Delta u(x)$.

We start with an equation whose domain is unit ball centered at origin with t in $(-\infty, \infty)$, in which case $u(x)$ is bounded. Assume that $u \in C^{1,1}_{loc}(\Omega)$ and is continuous on $\bar{\Omega}$. Assume $f(t, |x|, u)$ satisfies the following assumptions:

(f1) $f(t, |x|, u)$ is decreasing in $|x|$.

(f2) Assume that f is uniformly Lipschitz continuous in u. i.e:

$$|f(t, |x|, u) - f(t, |x|, v)| \leq c|u - v|, \;\; \forall (x, t) \in B_1(0) \times (-\infty, \infty)$$

The application of direct method of moving plane to study the parabolic equation 1.8 involving the fractional Laplacian is detailedly shown in section 5, before that, we would introduce the direct methods for the fractional equations in section 3 and 4.

1.3 Our Direct Methods for the Fractional Single Equations and Systems

In this section, I will introduce direct method of the moving planes. Detailed proofs will be given in the next chapter.

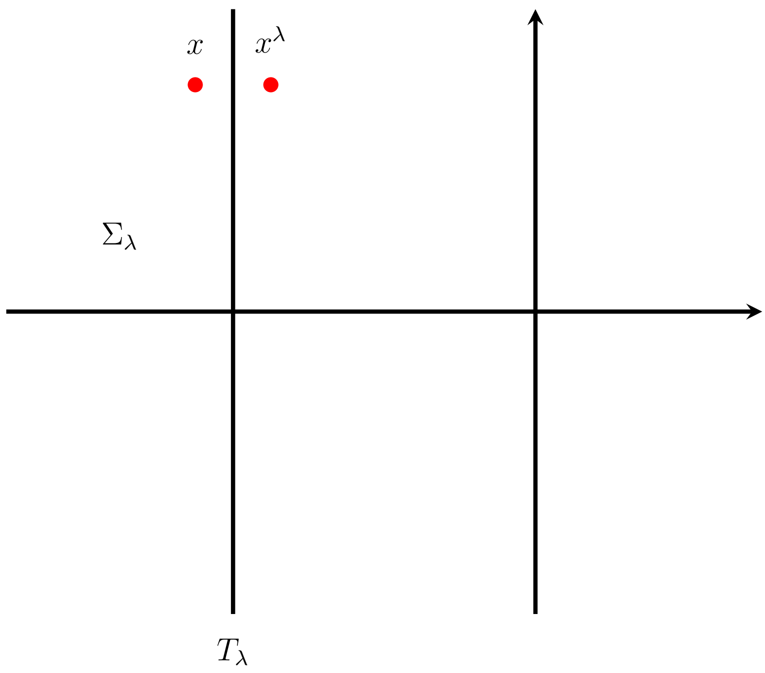


Figure 1: Moving Planes n-Dimensions.

1.3.1 Direct Method of the Moving Planes for single equations

In [5], Chen and Li have developed a systematic approach for applying the method of moving planes to nonlocal problems, whether on bounded or unbounded domains. Decades ago, Chen and Li introduced approaches for local elliptic operators in their publication [12] and summarized them in their book [7]. These approaches, including the narrow region principle and decay at infinity, have been widely used by researchers to solve various problems. In addition, they established a parallel system for the fractional Laplacian using elementary methods, making it easily applicable to a variety of nonlocal problems.

In our thesis, we apply and generalize this method from n-dimensional Euclidean space $\mathbb{R}^n$ to unit ball $B_1(0)$. To gain a general understanding of our method, let us consider a simple example in unit Ball $B_1(0)$.

Assume that $u(x)$ is a positive solution to some radially symmetric equation in the unit ball $B_1(0)$. In order to show that u is also radially symmetric, we first choose any direction to be the x_1 direction and let

$$T_\lambda = \{x \in \mathbb{R}^n \mid x_1 = \lambda \text{ for } \lambda \in \mathbb{R}\},$$

be the moving plane and

$$\Sigma_\lambda = \{x \in \mathbb{R}^n \mid x_1 < \lambda\}$$

be the region to the left of the plane, and

$$x^\lambda = \{(2\lambda - x_1, x') \mid x = (x_1, x') \in \mathbb{R}^n\}$$

and

$$\Omega_\lambda = \Sigma_\lambda \cap B_1(0)$$

be the intersection of $B_1(0)$ and Σ_λ.

Assume that u is a solution of pseudo differential equation $(-\triangle)^s u = f(x), x \in B_1(0)$. To compare the values of $u(x)$ with:

$$u_\lambda(x) = u(x^\lambda)$$

We denote

$$w_\lambda = u(x^\lambda) - u(x)$$

The first step is to show that for λ sufficiently negative, we have

$$w_\lambda(x) \geq 0, x \in \Omega_\lambda \tag{1.9}$$

This provides a starting point to move the plane. Then in the second step, we move the plane to the right as long as inequality (1.9) holds to its limiting position to show that u is symmetric about the limiting plane. A Narrow region principle is used to prove (1.9). Since w_λ is an anti-symmetric function:

$$w_\lambda(x) = -w_\lambda(x^\lambda)$$

we first prove a Narrow region principle for elliptic functions:

Narrow Region Principle for fractional elliptic equations

Theorem 1.1. *(Narrow region principle for an elliptic problem) Let $\Omega_\lambda = \Sigma_\lambda \cap B_1(0)$, Ω_λ is a bounded narrow region in Σ_λ, assume that $u(x) \in \left(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\right)$, if*

$$\begin{cases} (-\Delta)^s u(x) = f(u(x)), & u(x) > 0,\ x \in B_1(0), \\ u(x) \equiv 0, & x \notin B_1(0), \\ (-\Delta)^s w_\lambda(x) = c_\lambda w_\lambda(x), & x \in \Omega_\lambda \end{cases}$$

then for λ sufficiently close to -1, we have

$$w_\lambda(x) \geq 0,\ x \in \Omega_\lambda$$

The proof of this theorem involves using a contradiction argument at a negative minimum of w_λ, as one will observe.

Also, in the fractional parabolic equation, we use a Maximum principle as an ingredients.

Maximum Principle for fractional parabolic equations

Theorem 1.2. *(Maximum principle on a parabolic cylinder) Assume that*

$$\begin{cases} \frac{\partial w}{\partial t} + (-\Delta)^s w = c(x,t)w(x,t), & x \in \Omega_\lambda \times [\underline{t}, T], \\ w(x^\lambda, t) = -w(x,t), & x \in \Omega_\lambda \times [\underline{t}, T], \\ w(x,t) \geq 0, & x \in \Sigma_\lambda \backslash \Omega_\lambda \times [\underline{t}, T], \end{cases}$$

Then for λ sufficiently close to -1, we have

$$w(x,t) \geq \min\{0, \inf_{\Omega_\lambda \times [\underline{t},T]} w(x,\underline{t})\}, \ w \in [C^{1,1}_{loc}(\Omega_\lambda) \cap C(\bar{\Omega_\lambda}) \cap \mathcal{L}_{2s}] \times C^1([\underline{t}, T])$$

The proof of this theorem also involves using a contradiction argument at a negative minimum of w, as one will observe.

1.3.2 Direct Method of Moving Planes for the Systems

Considerable findings have been amassed for fractional systems that entail operators of the same order. For example, in [47], Systems characterized by comparable fractional orders and fairly comprehensive nonlinearities were analyzed by Yu. By utilizing the method of moving planes in integral forms, the author acquired symmetry for positive solutions. Such results on the system have also been proved in [27] and [15]. Nevertheless, there have been few presentations to date that address equations with varying orders. In [21], They introduced the iteration method as a novel approach to tackle such problems, which facilitates the establishment of various maximum principles - a crucial aspect for the method of moving planes - concerning fractional equations. The notation employed below is consistent with that introduced earlier.

Narrow Region Principle for fractional elliptic systems

Consider

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(v(x)), & x \in B_1(0), \\ (-\Delta)^{\frac{\beta}{2}} v(x) = g(u(x)), & x \in B_1(0), \\ u(x), v(x) \geq 0, & x \in B_1(0), \\ u(x), v(x) \equiv 0, & x \notin B_1(0), \end{cases} \tag{1.10}$$

where $\alpha, \beta \in (0,2)$

Theorem 1.3. *(Narrow Region principle for elliptic Fractional System) Let* $\Omega_\lambda = \Sigma_\lambda \cap B_1(0)$*,* Ω_λ *is a bounded narrow region in* Σ_λ*, assume that* $U_\lambda(x) \in \big(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\big)$*,* $V_\lambda(x) \in \big(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\big)$ *if*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} U_\lambda(x) \geq f_v(\xi(x)) V_\lambda(x), & x \in \Omega_\lambda, \\ (-\Delta)^{\frac{\beta}{2}} V_\lambda(x) \geq g_u(\eta(x)) U_\lambda(x), & x \in \Omega_\lambda, \end{cases} \tag{1.11}$$

then for λ *sufficiently close to* -1*, we have*

$$\begin{cases} U_\lambda(x) \geq 0, & x \in \Omega_\lambda, \\ V_\lambda(x) \geq 0, & x \in \Omega_\lambda, \end{cases} \tag{1.12}$$

We also introduce Narrow region principle for parabolic fractional systems:

Narrow Region Principle for fractional parabolic systems

Consider

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} u(x,t) = f(v(x,t)), & (x,t) \in B_1(0) \times (-\infty, \infty), \\ \frac{\partial v}{\partial t} + (-\Delta)^{\frac{\beta}{2}} v(x,t) = g(u(x,t)), & (x,t) \in B_1(0) \times (-\infty, \infty), \\ u(x,t), v(x,t) \geq 0, & (x,t) \in B_1(0) \times (-\infty, \infty), \\ u(x,t), v(x,t) \equiv 0, & x \notin B_1(0), \end{cases} \tag{1.13}$$

where $\alpha, \beta \in (0,2)$

Theorem 1.4. *(Narrow region principle on a parabolic cylinder) Let $\Omega_\lambda = \Sigma_\lambda \cap B_1(0)$, Ω_λ is a bounded narrow region in Σ_λ, assume that $U_\lambda(x,t), V_\lambda(x,t) \in [C^{1,1}_{loc}(\Omega_\lambda) \cap C(\bar{\Omega_\lambda}) \cap \mathcal{L}_{2s}] \times C^1([\underline{t}, T])$, and $U_\lambda(x,t)$, $V_\lambda(x,t)$ are lower semi-continuous on $\bar{\Omega}$. If*

$$\begin{cases} \frac{\partial U_\lambda}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} U_\lambda(x,t) \geq f_v(\xi(x,t)) V_\lambda(x,t), & (x,t) \in \Omega_\lambda \times [\underline{t}, T], \\ \frac{\partial V_\lambda}{\partial t} + (-\Delta)^{\frac{\beta}{2}} V_\lambda(x,t) \geq g_u(\eta(x,t)) U_\lambda(x,t), & (x,t) \in \Omega_\lambda \times [\underline{t}, T], \end{cases} \tag{1.14}$$

Then for λ sufficiently close to -1, we have

$$U_\lambda(x,t) \geq \min\{0, \inf_{\Omega_\lambda \times [\underline{t}, T]} U_\lambda(x, \underline{t})\}, \ (x,t) \in \Omega_\lambda \times [\underline{t}, T] \tag{1.15}$$

and

$$V_\lambda(x,t) \geq \min\{0, \inf_{\Omega_\lambda} V_\lambda(x, \underline{t})\}, \ (x,t) \in \Omega_\lambda \times [\underline{t}, T] \tag{1.16}$$

1.3.3 Sliding Method

The method of moving planes has been extensively utilized in studying qualitative properties of solutions, particularly to prove monotonicity, symmetry, and non-existence of solutions. In contrast, sliding methods, though less popular, are mainly employed to derive monotonicity and uniqueness of solutions.

When applying these methods to equations involving non-local operators defined by singular integrals, such as the fractional Laplacian defined by

$$(-\Delta)^s u(x) \;=\; C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy$$

it is necessary for the kernel of the operator to be monotone when using the method of moving planes, as we will explain below. In contrast, sliding methods do not have such a requirement.

For example, given a fractional elliptic equation:

$$\begin{cases} (-\Delta)^s u(x) = f(u(x)), & x \in B_1(0) \subset \mathbb{R}^n, \\ u(x) \equiv 0, & x \in B_1^c(0). \end{cases} \tag{1.17}$$

One wants to use the method of moving planes to show that u are radially symmetric about the origin. One common approach is to select a direction arbitrarily, such that choose the direction to be the x_1-direction and let

$$T_\lambda = \{x \in B_1(0) \mid x_1 = \lambda \text{ for } \lambda \in \mathbb{R}\}$$

be the moving planes,

$$\Omega_\lambda := \{x \in B_1(0) \mid x_1 < \lambda\}$$

be the region to the left of the hyperplane T_λ, and

$$x^\lambda := (2\lambda - x_1, x_2, ..., x_n)$$

be the reflection of x with respect to T_λ. Consider

$$w_\lambda(x) := u_\lambda(x) - u(x),$$

It is obvious that w_λ is an anti-symmetric function, i.e. $w_\lambda(x^\lambda) = -w_\lambda(x)$. The proof will be carried out in the following two steps.

Step 1 We first show that for $\lambda > -1$ and sufficiently close to -1,

$$w_\lambda(x) \geq 0, \;\; x \in \Omega_\lambda \tag{1.18}$$

This provides a starting point to move the plane T_λ.

Step 2 We continuously move the plane T_λ to the right along the x_1-axis as long as (1.18) holds to its limiting position T_{λ_0} with

$$\lambda_0 := \sup\{\lambda \leq 0 \mid w_\mu(x) \geq 0, \forall x \in \Omega_\mu, \mu \leq \lambda\}$$

We prove that $\lambda_0 = 0$, which implies that u must be radially symmetric and monotone decreasing about the origin due to the x_1 direction can be chosen arbitrarily.

Based on the above, it is evident that the crucial step is to establish (1.18), which is commonly done by utilizing the maximum principle for anti-symmetric functions w_λ. The simplest form of this principle is presented below.

Theorem 1.5. *(Maximum principles) Let Ω be a subset of Σ_λ, suppose*

$$\begin{cases} (-\Delta)^s w_\lambda(x) \geq 0, & x \in \Omega, \\ w_\lambda(x) \geq 0, & x \in \Sigma_\lambda \backslash \Omega \end{cases} \tag{1.19}$$

then

$$w_\lambda(x) \geq 0, \;\; x \in \Omega \tag{1.20}$$

Proof. We argue by contradiction, if (1.20) does not hold, there exists a point $x^o \in \Omega$ such that

$$w_\lambda(x^o) = \min_{\Omega} w_\lambda(x) < 0$$

By the definition of the fractional Laplacian, we have

$$\begin{aligned}
& (-\Delta)^s w_\lambda(x^o) \\
= \;& C_{n,s} PV\{\int_{\Sigma} \frac{w_\lambda(x^o) - w_\lambda(y)}{|x^o - y|^{n+2s}} dy + \int_{\mathbb{R}^n \setminus \Sigma_\lambda} \frac{w_\lambda(x^o) - w_\lambda(y)}{|x^o - y|^{n+2s}} dy\} \\
= \;& C_{n,s} PV\{\int_{\Sigma_\lambda} \frac{w_\lambda(x^o) - w_\lambda(y)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_\lambda} \frac{w_\lambda(x^o) + w_\lambda(y)}{|x^o - y^\lambda|^{n+2s}} dy\}
\end{aligned}$$

It is easy to verify that

$$|x^o - y| < |x^o - y^\lambda| \tag{1.21}$$

for any $y \in \Sigma_\lambda$.

Consequently

$$(-\Delta)^s w_\lambda(x^o) \leq C_{n,s} \int_{\Sigma_{\lambda_0}} \frac{2w_\lambda(x^o)}{|x^o - y^\lambda|^{n+2s}} < 0$$

which contradicts (1.19). Therefore, we must have $w_\lambda \geq 0$ in Ω. □

The arguments presented above demonstrate that establishing the monotonicity (1.21) of the fractional Laplacian's kernel is necessary for proving the maximum principles for anti-symmetric functions. However, in practice, there are numerous non-local operators that lack such monotonicity. Below are some examples:

- Uniformly elliptic fractional operator

$$(-\Delta)_a^s u(x) = C_{n,s} P.V \int_{\mathbb{R}^n} \frac{a(x-y)(u(x)-u(y))}{|x-y^{\lambda}|^{n+2s}} dy$$

where the function $a(\cdot)$ is uniformly bounded from above and away from 0. Note that the kernel $\frac{a(x-y)}{|x-y|^{n+2s}}$ is generally not monotone, unless some additional conditions are imposed on the weight function $a(\cdot)$.

More generally,

$$\mathcal{L}u(x) = C_{n,s} P.V. \int_{\mathbb{R}^n} (u(x)-u(y))K(x,y)dy$$

where

$$\frac{\lambda}{|x-y|^{n+2s}} \leq K(x,y) \leq \frac{\Lambda}{|x-y|^{n+2s}}$$

for some $0 < \lambda \leq \Lambda$

- Nonlocal Monge-Ampere operator

$$D_s u(x) = \inf\{P.V \int_{\mathbb{R}^n} \frac{u(y)-u(x)}{|A^{-1}(y-x)|^{n+2s}} dy \mid A > 0, \det A = 1\}$$

where A are positively definite matrixes.

In particular, in order these D_s to obey the maximum principle, we require the minimum eigenvalue of the above family of matrixes be bounded away from 0

$$\lambda_{min}(A) \geq \theta > 0$$

and we call the resulting operator D_s^{θ} is kind of uniformly elliptic.

Since the required monotonicity is absent, applying the method of moving planes to derive qualitative properties of solutions for nonlocal problems involving the aforementioned operators is not feasible. Instead, one can utilize the sliding method. The key advantage of the sliding method is that general maximum principles for solutions are established, rather than for anti-symmetric functions. Apart from establishing one-dimensional symmetry of solutions in the entire space, the sliding method can also be used to prove monotonicity of solutions on bounded domains, obtain uniform lower bounds for solutions in unbounded domains, and prove non-existence of solutions for certain fractional inequalities

1.3.4 Monotonicity in bounded domains

This subsection presents evidence that the sliding method can be employed to derive the monotonicity of solutions for nonlocal equations with uniformly elliptic fractional operators in bounded domains. The primary concept is based on comparing values of the solution of the equation at two distinct points, where one point is obtained by sliding the domain in a given direction, and the domain is then slid back to its initial position. The general uniformly elliptic operator $\mathcal{L}$ mentioned earlier will be used as an example.

Theorem 1.6. *(Monotonicity of solution of uniformly elliptic equation)*

Let Ω be a bounded domain in $\mathbb{R}^n$ which is convex in the x_n-direction. Assume that

$u(x) \in C^{1,1}_{loc}(\Omega) \cap \mathcal{L}_{2s}$ is a solution of

$$\begin{cases} \mathcal{L}u(x) = f(u(x)) & x \in \Omega, \\ u(x) = \varphi(x), & x \in \Omega^c, \end{cases} \tag{1.22}$$

where the nonhomogeneous term f is supposed to be Lipschitz continuous and the exterior function φ satisfies H:

For any three points $x = (x', x_n)$, $y = (x', y_n)$ and $z = (z', z_n)$ lying on a segment parallel to the x_n axis, $y_n < x_n < z_n$ with $y, z \in \Omega^c$, we have

$$\varphi(y) < u(x) < \varphi(z), \ \ x \in \Omega \tag{1.23}$$

and

$$\varphi(y) \leq \varphi(x) \leq \varphi(z), \ \ x \in \Omega^c \tag{1.24}$$

Then u is monotone increasing with respect to x_n in Ω. That is:

$u((x', x_n + \tau), t) > u((x', x_n), t)$ for $(x', x_n), (x', x_n + \tau) \in \Omega$ and $t \in \mathbb{R}$, where $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$

Proof. Let $\tau > 0$, it suffices to show that $u(x^\tau) > u(x)$ for any points

$$x = (x', x_n) \; and \; x^\tau = (x', x_n + \tau) \in \Omega$$

Let Ω be a bounded domain in $\mathbb{R}^n$, which is convex in the x_n-direction. By sliding Ω

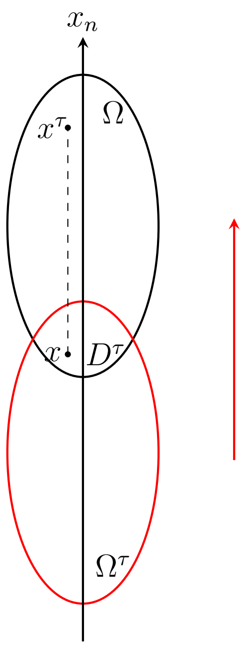


Figure 2: sliding method

downward τ units, we obtain Ω^τ:

$$\Omega^\tau = \Omega - \tau e_n, e_n = (0, 0, \cdots, 1)$$

Define

$$D^\tau = \Omega^\tau \cap \Omega$$

and

$$\widetilde{\tau} = \sup\{\tau \mid \tau > 0, D^\tau \neq \emptyset\}$$

and

$$W^\tau(x) = u^\tau(x) - u(x) \; in \; D^\tau$$

then W^τ satisifes

$$\mathcal{L}W^\tau(x) = f(u^\tau(x)) - f(u(x)) \tag{1.25}$$

Now we proceed in two steps.

Step 1 Using the assumption (H) and the continuity of u, it is obvious that

$$W_\lambda(x) > 0, \;\; x \in D^\tau \tag{1.26}$$

for τ sufficiently close to $\widetilde{\tau}$, which provides a starting point for sliding the domain along the x_n-axis.

Step 2 Continue to decrease τ as long as (1.26) holds, we claim that the limiting position is $\tau = 0$, If not, then there exists $\tau_0 > 0$ and a point $x^o \in D^{\tau_0}$ such that $W^{\tau_0}(x^o) = 0$, then

$$\mathcal{L}W^{\tau_0}(x^o) = f(u^{\tau_0}(x^o)) - f(u(x^o)) = 0 \tag{1.27}$$

and x^o is the minimum point in $\mathbb{R}^n$, which is ensured by the assumption (H). A combination of the definition of the uniformly elliptic fractional operator $\mathcal{L}$, equation (1.27), and the fact $W^{\tau_0} \geq 0$ yields that $W^{\tau_0}(x) \equiv 0$ in $\mathbb{R}^n$. This is a contradiction with the assumption (H).

From the above argument, one may see the essence of sliding. In general, equation (1.28) is not good, and from which one cannot use any maximum principle to derive $W^\tau(x) > 0$. However, if one can show that at beginning it holds $W^\tau(x) > 0$, then this must be true during the entire process of the sliding. Because once it is violated, we would arrive at a good equation (1.27) which would enable us to apply a maximum principle to derive a contradiction.

□

1.3.5 Monotonicity in unbounded domains

This subsection presents evidence that the sliding method can be employed to derive the monotonicity of solutions for nonlocal equations with uniformly elliptic operators in unbounded domains. The primary concept is based on comparing values of the solution of the equation at two distinct points, where one point is obtained by sliding the domain in a given direction, and the domain is then slid back to its initial position. The general uniformly elliptic operator $\mathcal{L}$ mentioned earlier will be used as an example.

Theorem 1.7. *(Monotonicity of solution of uniformly elliptic equation)*

Let Ω be a bounded domain in $\mathbb{R}^n$ which is convex in the x_n-direction. Assume that $u(x) \in C^{1,1}_{loc}(\Omega) \cap \mathcal{L}_{2s}$ is a solution of

$$\mathcal{L}u(x) = f(u(x)),\ x \in \mathbb{R}^n,$$

with condition

$$|u(x)| \leq 1$$

and

$$u((x', x_n)) \to \pm 1$$

uniformly in $x' = (x_1, \cdots, x_{n-1})$. Also, $f(|x|, u)$ is non-increasing near $u(x) = \pm 1$. Then u must be strictly increasing with respect to x_n, and it depends on x_n only.

where the nonhomogeneous term f is supposed to be Lipschitz continuous.

Then u is monotone increasing with respect to x_n in Ω. That is:

$u((x', x_n + \tau), t) > u((x', x_n), t)$ for $(x', x_n), (x', x_n + \tau) \in \Omega$ and $t \in \mathbb{R}$, where

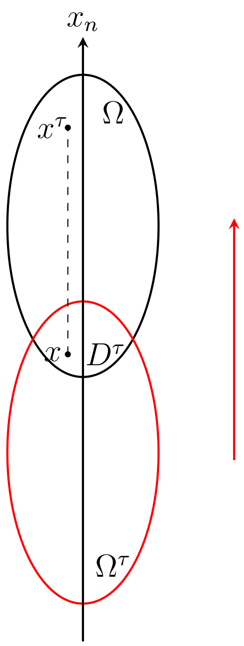


Figure 3: sliding method

$x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$

Proof. Let $\tau > 0$, it suffices to show that $u(x^\tau) > u(x)$ for any points

$$x = (x', x_n) \textit{ and } x^\tau = (x', x_n + \tau) \in \Omega$$

Let Ω be a bounded domain in $\mathbb{R}^n$, which is convex in the x_n-direction. By sliding Ω downward τ units, we obtain Ω^τ:

$$\Omega^\tau = \Omega - \tau e_n, e_n = (0, 0, \cdots, 1)$$

Define

$$D^\tau = \Omega^\tau \cap \Omega$$

and

$$\widetilde{\tau} = \sup\{\tau \mid \tau > 0, D^\tau \neq \emptyset\}$$

and

$$W^{\tau}(x) = u^{\tau}(x) - u(x) \; in \; D^{\tau}$$

then W^{τ} satisifes

$$\mathcal{L}W^{\tau}(x) = f(u^{\tau}(x)) - f(u(x)) \tag{1.28}$$

Now we proceed in two steps.

Step 1: *Begin sliding Ω^{τ} downward τ units along the x_n axis*

So then

$$|x| < |x^{\tau}|$$

We will show that for τ sufficiently close to $\widetilde{\tau}$, that is, when τ is sufficiently large, D^{τ} is narrow, we have

$$W^{\tau}(x) \leq 0, \;\; x \in D^{\tau}$$

Step 2: *Decrease τ as long as $W^{\tau}(x) \leq 0$ holds to its limiting position*

We would show the limit position is $\tau = 0$. In second step, we would divide the proof into two cases, one is $|x_n| \leq M$, the other is $|x_n| \geq M$. For $|x_n| \leq M$, we want to show $\sup_{-M \leq x_n \leq M} W^{\tau_0}(x) < 0$. Otherwise,

$$\sup_{-M \leq x_n \leq M} W^{\tau_0}(x) = 0$$

then there exists a sequence $\{x^k\} \subset \mathbb{R}^{n-1} \times [-M, M]$ such that

$$W^{\tau_0}(x^k) \to 0 \tag{1.29}$$

as $k \to \infty$

Then there exists $\tau_0 > 0$ and a point $x^o \in D^{\tau_0}$ such that $W^{\tau_0}(x^o) = 0$, then

$$\mathcal{L}W^{\tau_0}(x^o) = f(u^{\tau_0}(x^o)) - f(u(x^o)) = 0$$

and x^o is the maximum point in $\mathbb{R}^n$. A combination of the definition of the uniformly elliptic fractional operator $\mathcal{L}$, and the fact $W^{\tau_0} \leq 0$ yields that $W^{\tau_0}(x) \equiv 0$ in $\mathbb{R}^n$. So that $W_k^{\tau_0}$ converges uniformly to 0.

For all $m \in \mathbb{N}$, we have

$$u_\infty(x', x_n) = u_\infty(x', x_n + \tau_0) = u_\infty(x', x_n + 2\tau_0) = \cdots = u_\infty(x', x_n + m\tau_0)$$

If x_n is sufficiently negative and m is sufficiently large, then

$$u_\infty(x', x_n) \to -1$$

and

$$u_\infty(x', x_n + m\tau_0) \to 1$$

This is a contradiction, therefore, $\sup_{-M \leq x_n \leq M} W^{\tau_0}(x) < 0$ must be true.

Now we only need to prove when $|x_n| \geq M$, $\tau_0 > 0$

$$W^\tau(x) \leq 0, \, \forall \tau \in (\tau_0 - \delta, \, \tau_0], \tag{1.30}$$

Otherwise,

$$\sup_{\mathbb{R}^n \setminus (\mathbb{R}^{n-1} \times [-M,M])} W^\tau(x) = A > 0,\, \forall \tau \in (\tau_0 - \delta, \tau_0]$$

then there exists a sequence $\{x^k\}$ such that

$$W^\tau(x^k) \to A > 0$$

as $k \to \infty$

Then there exists a sequence $\{\varepsilon_k\} \to 0$ such that

$$U_k^\tau(x^k) = W^\tau(x^k) + \varepsilon_k \psi_k(x^k) = A$$

Then there exists $\bar{x}^k$ such that

$$U_k^\tau(\bar{x}^k) = \max_{\mathbb{R}^n} U_k^\tau(x^k) = A$$

By the definition of uniformly elliptic operator, we have

$$\mathcal{L} U_k^\tau(\bar{x}^k) \approx (-\triangle)^s (U_k^\tau)(\bar{x}^k) \geq c_0$$

We also have

$$\mathcal{L} U_k^\tau(\bar{x}^k) = f(W^\tau(\bar{x}^k)) - f(W(\bar{x}^k)) = 0$$

This is a contradiction, therefore, $\sup_{\mathbb{R}^n \setminus (\mathbb{R}^{n-1} \times [-M,M])} W^\tau(x) = A < 0,\, \forall \tau \in (\tau_0 - \delta, \tau_0]$ must be true.

In both cases we show the limiting position is $\tau = 0$. After we have completed the second step, we would prove $\forall\tau > 0$, $W^\tau(x) < 0$.

Thus we have completed proof of monotonicity of solution of uniformly elliptic equation in the whole space. In the last section, we would show $u(x)$ depends on x_n only, that is, $u(x) = u(x_n)$.

If we replace $u^\tau(x)$ by $u(x+\tau\nu)$, the argument still holds according to the above process, where $\nu = (\nu_1, \nu_2, \nu_3, \cdots, \nu_n)$ with $\nu_n > 0$ is an arbitrary vector that points upward. With the similar arguments as in Step 1 and Step 2, we can obtain that, for each of such ν,

$$u(x+\tau\nu) > u(x)$$

$\forall\tau > 0$ Let $\nu_n \to 0$, by continuity of u, we have that for arbitrary ν with $\nu_n = 0$

$$u(x+\tau\nu) \geq u(x)$$

By replacing ν by $-\nu$, we also have

$$u(x) \geq u(x+\tau\nu)$$

for arbitrary ν with $\nu_n = 0$, So we have

$$u(x+\tau\nu) = u(x) \tag{1.31}$$

(1.31) means that u is independent of $x' = (x_1, x_2, \cdots, x_{n-1})$. Therefore, $u(x) = u(x_n)$.

$\square$

2 Direct Method of the Moving Planes for Single Equations

In this chapter, I will prove the principles shown in section 1.

2.1 Narrow Region Principle

2.1.1 Narrow Region Principle for fractional elliptic equations

We first provide a simpler proof for a well-known maximum principle for s-super harmonic functions.

Theorem 2.1. *(Narrow region principle for an elliptic problem) Let* $\Omega_\lambda = \Sigma_\lambda \cap B_1(0)$*,* Ω_λ *is a bounded narrow region in* Σ_λ*, assume that* $u(x) \in \left(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\right)$*, if*

$$\begin{cases} (-\Delta)^s u(x) = f(u(x)), & u(x) > 0, \ x \in B_1(0), \\ u(x) \equiv 0, & x \notin B_1(0), \\ (-\Delta)^s w_\lambda(x) = c_\lambda w_\lambda(x), & x \in \Omega_\lambda \end{cases} \tag{2.1}$$

then for λ *sufficiently close to* -1*, we have*

$$w_\lambda(x) \geq 0, \ x \in \Omega_\lambda \tag{2.2}$$

Proof. Suppose otherwise, (2.2) does not hold, then w_λ is negative somewhere, hence there exists an $x^o \in \Omega_\lambda$ such that

$$w_\lambda(x^o) = \min_{\Omega_\lambda} w_\lambda(x) < 0$$

It follows that

$$
\begin{aligned}
& (-\Delta)^s w_\lambda(x^o) \\
= \;& C_{n,s} PV \int_{\mathbb{R}^n} \frac{w_\lambda(x^o) - w_\lambda(y)}{|x^o - y|^{n+2s}} dy \\
\leq \;& C_{n,s} \{ \int_{\Sigma_{\lambda_0}} \frac{w_\lambda(x^o) - w_\lambda(y)}{|x^o - y^\lambda|^{n+2s}} dy + \frac{w_\lambda(x^o) + w_\lambda(y)}{|x^o - y^\lambda|^{n+2s}} dy \} \\
= \;& C_{n,s} \int_{\Sigma_{\lambda_0}} \frac{2 w_\lambda(x^o)}{|x^o - y^\lambda|^{n+2s}}
\end{aligned}
$$

Denote

$$d = dis[x^0, T_\lambda] \leq width(\Omega_\lambda)$$

We also have

$$\int_{\Sigma_{\lambda_0}} \frac{1}{|x^o - y^\lambda|} dy \geq \frac{c}{d^{2s}}$$

So we deduce

$$(-\Delta)^s w_\lambda(x^o) \leq \frac{c}{d^{2s}} w_\lambda(x^o) \tag{2.3}$$

We also have

$$c_\lambda(x^o) w_\lambda(x^o) \leq \frac{c}{d^{2s}} w_\lambda(x^0)$$

Then we derive

$$\frac{c}{d^{2s}} \leq c_\lambda(x^o)$$

Which is a contradiction for d sufficiently small, since $c_\lambda(x)$ is bounded.

Therefore, (2.2) must be valid.

So far, we have proved the theorem 2.1.

□

Then we introduce a maximum principle for fractional parabolic equations.

2.2 Maximum Principle

2.2.1 Maximum principle for fractional parabolic equations

Theorem 2.2. *(Maximum principle on a parabolic cylinder) Assume that*

$$\begin{cases} \frac{\partial w}{\partial t} + (-\Delta)^s w = c(x,t)w(x,t), & x \in \Omega_\lambda \times [\underline{t}, T], \\ w(x^\lambda, t) = -w(x,t), & x \in \Omega_\lambda \times [\underline{t}, T], \\ w(x,t) \geq 0, & x \in \Sigma_\lambda \backslash \Omega_\lambda \times [\underline{t}, T], \end{cases} \tag{2.4}$$

Then for λ sufficiently close to -1, we have

$$w(x,t) \geq \min\{0, \inf_{\Omega_\lambda \times [\underline{t}, T]} w(x, \underline{t})\}, \; w \in [C^{1,1}_{loc}(\Omega_\lambda) \cap C(\bar{\Omega_\lambda}) \cap \mathcal{L}_{2s}] \times C^1([\underline{t}, T]) \tag{2.5}$$

Proof. If (2.5) does not hold, then the lower semi-continuity of $w(x,t)$ on $\bar{\Omega_\lambda} \times [\underline{t}, T]$ guarantees that there exists an $(x^o, t^o) \in \Omega_\lambda \times [\underline{t}, T]$ such that

$$w(x^o, t^o) = \min_{\Omega_\lambda \times (\underline{t}, T]} w < 0$$

And one can further deduce from condition (2.4) that (x^o, t^o) is in the interior of $\Omega_\lambda \times [\underline{t}, T]$

We have

$$
\begin{aligned}
& (-\Delta)^s w(x^o,t^o) \\
= & \; C_{n,s} PV \int_{\mathbb{R}^n} \frac{w(x^o,t^o)-w(y,t^o)}{|x^o-y|^{n+2s}} dy \\
\leq & \; C_{n,s} \{ \int_{\Sigma_{\lambda_0}} \frac{w(x^o,t^o)-w(y,t^o)}{|x^o-y^\lambda|^{n+2s}} dy + \frac{w(x^o,t^o)+w(y,t^o)}{|x^o-y^\lambda|^{n+2s}} dy \} \\
= & \; C_{n,s} \int_{\Sigma_{\lambda_0}} \frac{2w(x^o,t^o)}{|x^o-y^\lambda|^{n+2s}}
\end{aligned}
$$

Also we have

$$\int_{\Sigma_{\lambda_0}} \frac{1}{|x^o-y^\lambda|^{n+2s}} dy \geq \frac{c}{d^{2s}}$$

Thus,

$$(-\Delta)^{2s} w(x^o,t^o) \leq \frac{cw(x^o,t^o)}{d^{2s}} < 0 \tag{2.6}$$

We deduce

$$
\begin{aligned}
& c_\lambda(x^o,t^o)w(x^o,t^o) \leq \frac{cw(x^o,t^o)}{d^{2s}} + \frac{\partial w(x^o,t^o)}{\partial t} \\
= & \; c_\lambda(x^o,t^o)w(x^o,t^o) \leq \frac{cw(x^o,t^o)}{d^{2s}}
\end{aligned}
$$

Then we derive

$$\frac{c}{d^{2s}} \leq c_\lambda(x^o,t^o)$$

for λ sufficiently close to -1, d would be sufficiently small, since c_λ is bounded, we derive a contradiction. Therefore, (2.5) must be valid. So far, we have proved the Theorem 2.2.

□

3 Direct Method of the Moving Planes for Systems

Having witnessed the effectiveness and efficiency of the direct method of the moving planes for solving single equations that include fractional Laplacians, we naturally considered the possibility of extending this method to a broader range of problems, such as fractional systems and higher-order fractional equations.

3.1 Narrow Region Principle for A Fractional Systems

3.1.1 Narrow Region Principle for elliptic Fractional Systems

Theorem 3.1. *(Narrow Region principle for elliptic Fractional System) Let* $\Omega_\lambda = \Sigma_\lambda \cap B_1(0)$*,* Ω_λ *is a bounded narrow region in* Σ_λ*, assume that* $U_\lambda(x) \in \big(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\big)$*,* $V_\lambda(x) \in \big(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\big)$ *if*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} U_\lambda(x) \geq f_v(\xi(x)) V_\lambda(x), & x \in \Omega_\lambda, \\ (-\Delta)^{\frac{\beta}{2}} V_\lambda(x) \geq g_u(\eta(x)) U_\lambda(x), & x \in \Omega_\lambda, \end{cases} \tag{3.1}$$

then for λ *sufficiently close to* -1*, we have*

$$\begin{cases} U_\lambda(x) \geq 0, & x \in \Omega_\lambda, \\ V_\lambda(x) \geq 0, & x \in \Omega_\lambda, \end{cases} \tag{3.2}$$

Proof. Suppose otherwise, (3.2) does not hold, then U_λ is negative somewhere, hence there exists an $x^o \in \Omega_\lambda$ such that such that

$$U_\lambda(x^o) = \min_{\Omega_\lambda} U_\lambda(x) < 0$$

By the defining integral of the fractional Laplacian, we have

$$
\begin{aligned}
&(-\Delta)^{\frac{\alpha}{2}} U_\lambda(x^o) \\
\leq\ & C_{n,\alpha} \int_{\Sigma_\lambda} \frac{2U_\lambda(x^o)}{|x^o - y^\lambda|^{n+\alpha}}
\end{aligned}
$$

Denote

$$d = dis[x^0, T_\lambda] \leq width(\Omega_\lambda)$$

We have

$$\int_\Sigma \frac{1}{|x^o - y^\lambda|^{n+\alpha}} dy \geq \frac{c}{d^\alpha}$$

Hence

$$(-\Delta)^{\frac{\alpha}{2}} U_\lambda(x^o) \leq \frac{c}{d^\alpha} U_\lambda(x^o) < 0$$

On the other hand, we have

$$(-\Delta)^{\frac{\alpha}{2}} U_\lambda(x^o) = f(v_\lambda(x^o)) - f(v(x^o)) < 0$$

. Therefore, by the monotonicity of f, we have

$$V_\lambda(x^o) < 0$$

This implies that there exists some $\bar{x} \in \Omega_\lambda$ such that

$$V_\lambda(\bar{x}) = \min_{\Omega_{V_\lambda}} V_\lambda(x) < 0$$

Following the same argument, we can derive that

$$(-\Delta)^{\frac{\beta}{2}}V_\lambda(\bar{x}) \leq \frac{cV_\lambda(\bar{x})}{d^\beta} < 0$$

By assumption, We have

$$f_v(\xi(x))V_\lambda(x^o) \leq (-\Delta)^{\frac{\alpha}{2}}U_\lambda(x^o) \leq \frac{c}{d^\alpha}U_\lambda(x^o)$$

So we derive

$$\frac{d^\alpha}{c}f_v(\xi(x))V_\lambda(x^o) \leq U_\lambda(x^o)$$

By assumption, we have

$$(-\Delta)^{\frac{\beta}{2}}V_\lambda(\bar{x}) - g_u(\eta(\bar{x}))U_\lambda(\bar{x}) \geq 0$$

We derive

$$\begin{aligned}
0 &\leq (-\Delta)^{\frac{\beta}{2}}V_\lambda(\bar{x}) - g_u(\eta(\bar{x}))U_\lambda(\bar{x}) \\
&\leq \frac{cV_\lambda(\bar{x})}{d^\beta} - g_u(\eta(\bar{x}))U_\lambda(\bar{x})) \\
&\leq \frac{cV_\lambda(\bar{x})}{d^\beta} - g_u(\eta(\bar{x}))U_\lambda(x^o) \\
&\leq \frac{cV_\lambda(\bar{x})}{d^\beta} - g_u(\eta(\bar{x}))(f_v(\xi(x^o))V_\lambda(x^o)\frac{d^\alpha}{c}) \\
&\leq \frac{cV_\lambda(\bar{x})}{d^\beta} - g_u(\eta(\bar{x}))(f_v(\xi(x^o))V_\lambda(\bar{x})\frac{d^\alpha}{c}) \\
&\leq \frac{cV_\lambda(\bar{x})}{d^\beta}(1 - g_u(\eta(\bar{x}))f_v(\xi(x^o))\frac{d^{\alpha+\beta}}{c^2})
\end{aligned}$$

If λ is sufficiently close to -1, d would be sufficiently small,

$$g_u(\eta(\bar{x}))f_v(\xi(x^o))\frac{d^{\alpha+\beta}}{c^2} << 1$$

and

$$V_\lambda(\bar{x}) < 0$$

So we derive

$$\frac{cV_\lambda(\bar{x})}{d^\beta}(1 - g_u(\eta(\bar{x}))f_v(\xi(x^o))\frac{d^{\alpha+\beta}}{c^2}) < 0$$

This contradiction shows that (3.2) must be true. So far, we have proved the theorem 3.1. □

3.1.2 Narrow region principle for parabolic Fractional Systems

Theorem 3.2. *(Narrow region principle on a parabolic cylinder) Let $\Omega_\lambda = \Sigma_\lambda \cap B_1(0)$, Ω_λ is a bounded narrow region in Σ_λ, assume that $U_\lambda(x,t), V_\lambda(x,t) \in [C^{1,1}_{loc}(\Omega_\lambda) \cap C(\bar{\Omega_\lambda}) \cap \mathcal{L}_{2s}] \times C^1([\underline{t}, T])$, and $U_\lambda(x,t)$, $V_\lambda(x,t)$ are lower semi-continuous on $\bar{\Omega}$. If*

$$\begin{cases} \frac{\partial U_\lambda}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} U_\lambda(x,t) \geq f_v(\xi(x,t))V_\lambda(x,t), & (x,t) \in \Omega_\lambda \times [\underline{t}, T], \\ \frac{\partial V_\lambda}{\partial t} + (-\Delta)^{\frac{\beta}{2}} V_\lambda(x,t) \geq g_u(\eta(x,t))U_\lambda(x,t), & (x,t) \in \Omega_\lambda \times [\underline{t}, T], \end{cases} \tag{3.3}$$

Then for λ sufficiently close to -1, we have

$$U_\lambda(x,t) \geq \min\{0, \inf_{\Omega_\lambda \times [\underline{t}, T]} U_\lambda(x, \underline{t})\},\ (x,t) \in \Omega_\lambda \times [\underline{t}, T] \tag{3.4}$$

and

$$V_\lambda(x,t) \geq \min\{0, \inf_{\Omega_\lambda} V_\lambda(x,\underline{t})\}, \ (x,t) \in \Omega_\lambda \times [\underline{t}, T] \tag{3.5}$$

Proof. If (3.5) does not hold, then the lower semi-continuity of $U_\lambda(x,t)$ on $\bar{\Omega_\lambda} \times [\underline{t}, T]$ guarantees that there exists an $(x^o, t^o) \in \Omega_\lambda \times (\underline{t}, T]$ such that

$$U_\lambda(x^o, t^o) = \min_{\Omega_\lambda \times (\underline{t}, T]} U_\lambda < 0$$

By the defining integral of the fractional Laplacian, we have

$$\begin{aligned} & (-\Delta)^{\frac{\alpha}{2}} U_\lambda(x^o, t^o) \\ \leq \ & C_{n,\alpha} \int_{\Sigma_\lambda} \frac{2U_\lambda(x^o, t^o)}{|x^o - y^\lambda|^{n+\alpha}} \end{aligned}$$

Similar to the argument before, we have

$$\int_{\Sigma_\lambda} \frac{1}{|x^o - y^\lambda|^{n+\alpha}} dy \geq \frac{c}{d^\alpha}$$

Also, since (x^o, t^o) is the minimum,

If

$$t^o < T$$

$$\frac{\partial U_\lambda}{\partial t}(x^o, t^o) = 0$$

If

$$t^o = T$$

$$\frac{\partial U_\lambda}{\partial t}(x^o, t^o) \leq 0$$

Following from

$$(-\triangle)^{\frac{\alpha}{2}} U_\lambda(x^o, t^o) \leq C_{n,\alpha} \int_{\Sigma_\lambda} \frac{2U_\lambda(x^o, t^o)}{|x^o - y^\lambda|^{n+\alpha}}$$

We have

$$(-\triangle)^{\frac{\alpha}{2}} U_\lambda(x^o, t^o) \leq \frac{cU_\lambda(x^o, t^o)}{d^\alpha}$$

We deduce

$$\begin{aligned} & \frac{\partial U_\lambda(x^o, t^o)}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} U_\lambda(x^o, t^o) \\ \leq \;& \frac{cU_\lambda(x^o, t^o)}{d^\alpha} + \frac{\partial U_\lambda(x^o, t^o)}{\partial t}, \\ < \;& 0, \end{aligned}$$

On the other hand, we have

$$\frac{\partial U_\lambda(x^o, t^o)}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} U_\lambda(x^o, t^o) = f(v_\lambda(x^o, t^o)) - f(v(x^o, t^o)) < 0$$

Therefore, by the monotonicity of f, we have

$$V_\lambda(x^o, t^o) < 0$$

This implies that there exists some $(\bar{x}, \bar{t}) \in \Omega_{\lambda_0} \times (\underline{t}, T]$ such that

$$V_\lambda(\bar{x}, \bar{t}) = \min_{\Omega_{\lambda_0} \times (\underline{t}, T]} V_\lambda < 0$$

So that

$$\frac{\partial V_\lambda}{\partial t}(\bar{x}, \bar{t}) = 0$$

Following the same argument, we can derive that

$$(-\Delta)^{\frac{\beta}{2}} V_\lambda(\bar{x}, \bar{t}) \leq \frac{c V_\lambda(\bar{x}, \bar{t})}{d^\beta} < 0$$

We have

$$f_v(\xi(x,t)) V_\lambda(x^o, t^o) \leq \frac{c U_\lambda(x^o, t^o)}{d^\alpha} + \frac{\partial U_\lambda(x^o, t^o)}{\partial t}$$

we derive

$$f_v(\xi(x^o, t^o)) V_\lambda(x^o, t^o) \leq \frac{c U_\lambda(x^o, t^o)}{d^\alpha} < 0$$

so we derive

$$\frac{d^\alpha}{c} f_v(\xi(x^o, t^o)) V_\lambda(x^o, t^o) \leq U_\lambda(x^o, t^o)$$

So we have

$$\begin{aligned} & \frac{\partial V_\lambda}{\partial t}(\bar{x}, \bar{t}) + (-\Delta)^{\frac{\beta}{2}} V_\lambda(\bar{x}, \bar{t}) - g_u(\eta(\bar{x})) U_\lambda(\bar{x}) \geq 0 \\ := \quad & (-\Delta)^{\frac{\beta}{2}} V_\lambda(\bar{x}, \bar{t}) - g_u(\eta(\bar{x})) U_\lambda(\bar{x}) \geq 0 \end{aligned}$$

We derive

$$
\begin{aligned}
0 &\leq (-\Delta)^{\frac{\beta}{2}} V_\lambda(\bar{x},\bar{t}) - g_u(\eta(\bar{x},\bar{t}))U_\lambda(\bar{x},\bar{t}) \\
&\leq \frac{cV_\lambda(\bar{x},\bar{t})}{d^\beta} - g_u(\eta(\bar{x},\bar{t}))U_\lambda(\bar{x},\bar{t})) \\
&\leq \frac{cV_\lambda(\bar{x},\bar{t})}{d^\beta} - g_u(\eta(\bar{x},\bar{t}))U_\lambda(x^o,t^o) \\
&\leq \frac{cV_\lambda(\bar{x},\bar{t})}{d^\beta} - g_u(\eta(\bar{x},\bar{t}))(f_v(\xi(x^o,t^o))V_\lambda(x^o,t^o)\frac{d^\alpha}{c}) \\
&\leq \frac{cV_\lambda(\bar{x},\bar{t})}{d^\beta} - g_u(\eta(\bar{x},\bar{t}))(f_v(\xi(x^o,t^o))V_\lambda(\bar{x},\bar{t})\frac{d^\alpha}{c}) \\
&\leq \frac{cV_\lambda(\bar{x},\bar{t})}{d^\beta}(1 - g_u(\eta(\bar{x},\bar{t}))f_v(\xi(x^o,t^o))\frac{d^{\alpha+\beta}}{c^2})
\end{aligned}
$$

If λ is sufficiently close to -1, d would be sufficiently small,

$$
g_u(\eta(\bar{x},\bar{t}))f_v(\xi(x^o,t^o))\frac{d^{\alpha+\beta}}{c^2} << 1
$$

and

$$
V_\lambda(\bar{x},\bar{t}) < 0
$$

So we derive

$$
\frac{cV_\lambda(\bar{x},\bar{t})}{d^\beta}(1 - g_u(\eta(\bar{x},\bar{t}))f_v(\xi(x^o,t^o))\frac{d^{\alpha+\beta}}{c^2}) < 0
$$

This contradiction shows that (3.5) must be true. So far, we have proved theorem 3.2. $\square$

4 Our main results

4.1 Part 1: Method of Moving Planes and Its Applications: Radial symmetry and monotonicity of solutions for fractional elliptic and parabolic equations and systems

4.1.1 Fractional elliptic single equations

In section 5, we want to show radial symmetry and monotonicity of solution of fractional equations, where the equations here are represented by fractional elliptic equations and fractional parabolic equations. Section 5 is divided into two subsections, in the first subsection, we use moving of plane method to prove radial symmetry and monotonicity of fractional elliptic equations, where the elliptic equations are given by:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(u(x)), & x \in B_1(0), \\ u(x) \geq 0, & x \in B_1(0), \\ u(x) \equiv 0, & x \notin B_1(0), \end{cases} \tag{4.1}$$

We want to prove the following theorem:

Theorem 4.1. *(Radial Symmetry of solution of elliptic fractional equation)*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(u(x)), & x \in B_1(0), \\ u(x) \geq 0, & x \in B_1(0), \\ u(x) \equiv 0, & x \notin B_1(0), \end{cases} \tag{4.2}$$

where B_1 is a unit ball.

Let $0 < \alpha, \beta < 2$, and suppose that $u(x) \in \left(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\right)$ is positive bounded classical solutions of equation (5.1)*, and assume $f(|x|, u)$ satisfies the following assumptions:*

(X1) $f(|x|, u)$ are decreasing in $|x|$.

(X2) Assume that f is uniformly Lipschitz continuous in u. i.e:

$$|f(|x|, u_1) - f(|x|, u_2)| \leq c|u_1 - u_2|, \ \forall x \in B_1(0),$$

then $u(x)$ is radially symmetric and monotone decreasing about the origin, i.e.

$$u(x) = u(|x|)$$

.

$$u(x_1) > u(x_2), |x_1| < |x_2|$$

.

In the process to show $u(x)$ here is monotone and radial symmetric about the origin using moving of plane method, we should give the plane an initial position to start, where in this initial position $w_\lambda(x) = u_\lambda(x) - u(x)$ is non-negative, so the Narrow region theorem in the unit ball is avoidable to be an ingredient here, we first prove:

Theorem 4.2. *(Narrow region principle for an elliptic problem) Let $\Omega_\lambda = \Sigma_\lambda \cap B_1(0)$, Ω_λ*

is a bounded narrow region in Σ_λ, assume that $u(x) \in \left(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\right)$, if

$$\begin{cases} (-\Delta)^s u(x) = f(u(x)), & u(x) > 0,\ x \in B_1(0), \\ u(x) \equiv 0, & x \notin B_1(0), \\ (-\Delta)^s w_\lambda(x) = c_\lambda w_\lambda(x), & x \in \Omega_\lambda \end{cases} \tag{4.3}$$

then for λ sufficiently close to -1, we have

$$w_\lambda(x) \geq 0,\ x \in \Omega_\lambda \tag{4.4}$$

After we have an initial position to handle moving plane, we move the plane continuously to the right until its limiting position as long as w_λ holds, define

$$\lambda_0 = \sup\{\lambda \leq 0 \mid w_\mu(x) \geq 0, \forall x \in \Omega_\mu, \mu \leq \lambda\}$$

We show $\lambda_0 = 0$ by contradiction. Suppose $\lambda_0 < 0$, we show that the plane T_{λ_0} can be moved further to the right. First of all, we show $w_{\lambda_0}(x) > 0$ for $x \in \Omega_{\lambda_0}$, then we let the moving plane go back a little bit, show $w_{\lambda_0}(x) \geq c_o > 0$ for $x \in \Omega_{\lambda_0 - \delta}$.

To prove

$$w_{\lambda_0}(x) \geq c_o > 0,\ x \in \Omega_{\lambda_0 - \delta}$$

by contradiction, suppose there exists some sequences $x_k \in \Omega_{\lambda_0 - \delta}$ such that $w_{\lambda_0}(x_k) \to 0$ for $x_k \to x^o \in \Omega_{\lambda_o - \delta}$, by regularity theory, $w_{\lambda_0}(x_k)$ converges uniformly to $\bar{w}(x^o) = 0$.

We have

$$(-\Delta)^s \bar{w}(x^o) = c_\lambda(x^o)\bar{w}(x^o) = 0$$

We also have

$$(-\Delta)^s \bar{w}(x^o) = C_{n,s} PV \int_{\mathbb{R}^n} \frac{-\bar{w}(y)}{|x^o - y|^{n+2s}} dy \leq 0$$

That forces

$$\bar{w}(y) \equiv 0, \ \forall y \in \mathbb{R}^n$$

Same routine, by regularity theory, there exists some sequences x_k such that $u(x_k)$ converges uniformly to $\bar{u}(x)$, $f(u)$ converges uniformly to $\bar{f}(u)$ for $x \in \Omega_{\lambda_o}$.

By a Strong Maximum principle:

Lemma 4.3. *(Strong Maximum Principle for $(-\Delta)^s \bar{u} = \bar{f}(\bar{u})$.*

Assume that $\bar{u}(x) \in [C^{1,1}_{loc}(\Omega_\lambda) \cap C(\bar{\Omega_\lambda}) \cap \mathcal{L}_{2s}]$

$$\begin{cases} (-\Delta)^s \bar{u}(x) = \bar{f}(\bar{u}), & x \in \Omega_\lambda, \\ \bar{u}(x) \geq 0, & x \in \Omega_\lambda \end{cases} \tag{4.5}$$

we have either

$$\bar{u}(x) > 0, x \in B_1(0)$$

or

$$\bar{u}(x) \equiv 0, x \in \mathbb{R}^n$$

If $\bar{u}(x) > 0$, $x \in B_1(0)$, $\bar{w}(x) > 0$ somewhere, but we already derive $\bar{w}(x) \equiv 0$, hence we must have $\bar{u}(x) \equiv 0$, $x \in \mathbb{R}^n$. Thus, we know $u(x_k)$ converges to 0 uniformly.

In order to derive a contradiction for large k, Let

$$w_k(x) \equiv w_{\lambda_0}(x_k) = m_k \tag{4.6}$$

Let

$$v_k(x) = w_k(x) - 2m_k \tag{4.7}$$

with a minimum point $\bar{x_k}$, now for sufficiently large k,

$$(-\Delta)^s v_k(\bar{x}_k) = c_{\lambda_o}(\bar{x}_k) w_k(\bar{x}_k)$$

and

$$(-\Delta)^s v_k(\bar{x}_k) \leq \frac{c}{[d(\bar{x}_k, T_{\lambda_o})]^{2s}} v_k(\bar{x}_k) \leq -c_1 m_k$$

which is a contradiction, so we have proved

$$w_{\lambda_0}(x) \geq c_o > 0, \ \ x \in \Omega_{\lambda_0 - \delta}$$

Since w_λ depends on λ continuously, there exists $\epsilon > 0$ and $\epsilon < \delta$, such that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, we have

$$w_\lambda \geq 0, \ \ x \in \Omega_{\lambda_0 - \delta}$$

By Narrow region theorem, we derive

$$w_\lambda \geq 0, \ \ x \in \Omega_\lambda^- \backslash \Omega_{\lambda_0 - \delta}$$

We conclude that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$

$$w_\lambda(x) \geq 0, \;\; x \in \Omega_\lambda$$

This contradicts the definition of λ_0. Therefore, we must have

$$\lambda_0 = 0$$

and

$$w_{\lambda_0} \geq 0, \;\; \forall x \in \Omega_{\lambda_0}$$

Similarly, one can move the plane T_λ from $\lambda = 1$ to the left and show that

$$w_{\lambda_0} \leq 0, \;\; \forall x \in \Omega_{\lambda_0}$$

Now we have shown that

$$\lambda_0 = 0$$

and

$$w_{\lambda_0} \equiv 0, \;\; x \in \Omega_{\lambda_0}$$

This completes the setp 2.

So far, we have proved that u is symmetric about the plane T_0. Since the x_1 direction can be chosen arbitrarily, we have actually shown that u is radially symmetric about origin.

Since $w_\lambda(x) \not\equiv 0$, $x \in T_\lambda$, $\forall 0 < \lambda < \lambda_0$, if there exists x^o such that x^o is the minimum point, from the above process, on one hand,

$$(-\Delta)^s w_\lambda(x^o) \leq 0$$

On the other hand,

$$(-\Delta)^s w_\lambda(x^o) = 0$$

This forces

$$w_\lambda \equiv 0$$

which is a contradiction. Therefore, u is monotone decreasing about the origin.

4.1.2 Fractional parabolic single equations

From subsection 5.5, we use moving of plane method to prove radial symmetry and monotonicity of fractional parabolic equation, where the parabolic equation is given by:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = f(t,|x|,u), & (x,t) \in B_1(0) \times (-\infty,\infty), \\ u(x,t) > 0, & (x,t) \in B_1(0) \times (-\infty,\infty), \\ u(x,t) \equiv 0, & x \notin B_1(0), \end{cases} \tag{4.8}$$

We want to prove the following theorem:

Theorem 4.4. *(Radial Symmetry of solution of fractional parabolic equation)*

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = f(t,|x|,u), & (x,t) \in B_1(0) \times (-\infty,\infty), \\ u(x,t) > 0, & (x,t) \in B_1(0) \times (-\infty,\infty), \\ u(x,t) \equiv 0, & x \notin B_1(0), \end{cases} \tag{4.9}$$

where $B_1(0)$ is a unit ball.

Let $0 < s < 1$*, and suppose that* $u(x,t) \in \left(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\right) \times (-\infty, \infty)$ *is a positive bounded classical solution of equation* (5.17)*, and assume* $f(t, |x|, u)$ *satisfies the following assumptions:*

(f1) $f(t, |x|, u)$ *are decreasing in* $|x|$*.*

(f2) Assume that f *is uniformly Lipschitz continuous in* u*. i.e:*

$$f(t, |x|, u_1) - f(t, |x|, u_2) \leq c|u_1 - u_2|, \ \forall x \in B_1(0),$$

then $u(x,t)$ *is radially symmetric and monotone decreasing about the origin. i.e.*

$$u(x,t) = u(|x|, t)$$

$$u(x_1, t) > u(x_2, t), |x_1| < |x_2|.$$

.

In the process to show $u(x,t)$ here is monotone and radial symmetric about the origin using moving of plane method, we should give the plane an initial position to start, where in this initial position $w_\lambda(x,t) = u_\lambda(x,t) - u(x,t)$ is non-negative, so the Maximum principle in the unit ball is avoidable to be an ingredient here, we first prove:

Lemma 4.5. *(Maximum principle on a parabolic cylinder) Assume that*

$$\begin{cases} \frac{\partial w}{\partial t} + (-\Delta)^s w = c(x,t) w(x,t), & x \in \Omega_\lambda \times [\underline{t}, T], \\ w(x^\lambda, t) = -w(x,t), & x \in \Omega_\lambda \times [\underline{t}, T], \\ w(x,t) \geq 0, & x \in \Sigma_\lambda \backslash \Omega_\lambda \times [\underline{t}, T], \end{cases} \tag{4.10}$$

Then for λ sufficiently close to -1, we have

$$w(x,t) \geq \min\{0, \inf_{\Omega_\lambda \times [\underline{t},T]} w(x,\underline{t})\}, \ w \in [C_{loc}^{1,1}(\Omega_\lambda) \cap C(\bar{\Omega_\lambda}) \cap \mathcal{L}_{2s}] \times C^1([\underline{t},T]) \quad (4.11)$$

Let

$$\bar{w} = e^{m(t-\underline{t})} w_\lambda(x,t), \ m > 0$$

$$\frac{\partial \bar{w}}{\partial t} + (-\triangle)^s \bar{w} = \bar{c}\bar{w}$$

By Lemma, we have

$$\bar{w}(x,t) \geq \min\{0, \inf_{x \in \Omega_\lambda} \bar{w}(x,\underline{t})\}, \ \forall (x,t) \in \Omega_\lambda \times (\underline{t},T)$$

Thus

$$e^{m(t-\underline{t})} w_\lambda(x,t) \geq \min\{0, \inf_{x \in \Omega_\lambda} w_\lambda(x,\underline{t})\}$$

So

$$w_\lambda(x,t) \geq e^{-m(t-\underline{t})} \min\{0, \inf_{x \in \Omega_\lambda} w_\lambda(x,\underline{t})\}$$

$w_\lambda(x,t)$ is bounded from below. Let $\underline{t} \to -\infty$, $w_\lambda(x,t) \to \geq 0$.

Therefore,

$$w_\lambda(x,t) \geq 0$$

After we have an initial position to handle moving plane, we move the plane continuously to the right until its limiting position as long as w_λ holds, define

$$\lambda_0 = sup\{\lambda \leq 0 \mid w_\mu(x,t) \geq 0, \forall (x,t) \in \Omega_\mu \times \mathbb{R}, \mu \leq \lambda\}$$

We show $\lambda_0 = 0$ by contradiction. Suppose $\lambda_0 < 0$, we show that the plane T_{λ_0} can be moved further to the right. First of all, we show $w_{\lambda_0}(x,t) > 0$ for $(x,t) \in \Omega_{\lambda_0} \times \mathbb{R}$, then we let the moving plane go back a little bit, show $w_{\lambda_0}(x,t) \geq c_o > 0$ for $(x,t) \in \Omega_{\lambda_0 - \delta} \times \mathbb{R}$.

To prove

$$w_{\lambda_0}(x,t) \geq c_o > 0, \;\; x \in \Omega_{\lambda_0 - \delta}$$

by contradiction, suppose there exists some sequences $(x_k, t_k) \in \Omega_{\lambda_0 - \delta} \times \mathbb{R}$ such that $w_{\lambda_0}(x_k, t_k) \to 0$ for $x_k \to x^o \in \Omega_{\lambda_o - \delta}$.

Let

$$w_k(x,t) = w_{\lambda_0}(x, t + t_k)$$

by regularity theory, $w_k(x,t)$ converges uniformly to $\bar{w}(x,t)$.

Since

$$w_k(x_k, 0) = w_{\lambda_0}(x_k, t_k) \to 0$$

$$\bar{w}(x^o, 0) = 0$$

We have

$$(-\Delta)^s \bar{w}(x^o, 0) = c_\lambda(x^o, 0)\bar{w}(x^o, 0) = 0$$

We also have

$$(-\Delta)^s \bar{w}(x^o, 0) = C_{n,s} PV \int_{\mathbb{R}^n} \frac{-\bar{w}(y,0)}{|x^o - y|^{n+2s}} dy \leq 0$$

That forces

$$\bar{w}(y, 0) \equiv 0, \;\; \forall y \in \mathbb{R}^n$$

Same routine, Let $u_k(x,t) = u(x, t + t_k)$, by regularity theory, there exists some sequences x_k such that $u_k(x,t)$ converges uniformly to $\bar{u}(x,t)$, $f(0,u)$ converges uniformly to $\bar{f}(0,u)$ for $x \in \Omega_{\lambda_o}$.

By a Strong Maximum principle:

Lemma 4.6. *(Strong Maximum Principle for $\frac{\partial \bar{u}}{\partial t} + (-\Delta)^s \bar{u} = \bar{f}(t,\bar{u})$).*

Assume that $\bar{u}(x,t) \in [C^{1,1}_{loc}(\Omega_\lambda) \cap C(\bar{\Omega_\lambda}) \cap \mathcal{L}_{2s}] \times C^1([\underline{t}, T])$

$$\begin{cases} \frac{\partial \bar{u}(x,t)}{\partial t} + (-\Delta)^s \bar{u}(x,t) = \bar{f}(t,\bar{u}), & (x,t) \in \Omega_\lambda \times [\underline{t}, T], \\ \bar{u}(x,t) \geq 0, & (x,t) \in \Omega_\lambda \times [\underline{t}, T] \end{cases} \tag{4.12}$$

we have either

$$\bar{u}(x,0) > 0, x \in B_1(0)$$

or

$$\bar{u}(x,0) \equiv 0, x \in \mathbb{R}^n$$

If $\bar{u}(x,0) > 0$, $x \in B_1(0)$, $\bar{w}(x,0) > 0$ somewhere, but we already derive $\bar{w}(x,0) \equiv 0$, hence we must have $\bar{u}(x,0) \equiv 0$, $x \in \mathbb{R}^n$. Thus, we know $u_k(x,t)$ converges to 0 uniformly.

In order to derive a contradiction for large k, Let

$$w_k(x_k, 0) \equiv w_{\lambda_0}(x_k, t_k) = m_k \tag{4.13}$$

Let

$$v_k(x,t) = w_k(x,t) - 2m_k \eta(\epsilon_k(t - t_k)) \tag{4.14}$$

with a minimum point $(\bar{x_k}, \bar{t_k})$, now for sufficiently large k,

$$(-\Delta)^s v_k(\bar{x}_k, \bar{t}_k) = -\frac{\partial w_k}{\partial t}(\bar{x}_k, \bar{t}_k) + c_{\lambda_o}(\bar{x}_k, \bar{t}_k + t_k) w_k(\bar{x}_k, \bar{t}_k)$$

with

$$-\frac{\partial w_k}{\partial t}(\bar{x}_k, \bar{t}_k) \sim \epsilon_k m_k$$

and

$$(-\Delta)^s v_k(\bar{x}_k, \bar{t}_k) \leq \frac{c}{[d(\bar{x}_k, T_{\lambda_o})]^{2s}} v_k(\bar{x}_k, \bar{t}_k) \leq -c_1 m_k$$

which is a contradiction, so we have proved $\inf w_{\lambda_0}(x,t) > c_o > 0,\ (x,t) \in \Omega_{\lambda_0-\delta} \times \mathbb{R}$

Since w_λ depends on λ continuously, there exists $\epsilon > 0$ and $\epsilon < \delta$, such that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, we have

$$w_\lambda(x,t) \geq 0,\ (x,t) \in \Omega_{\lambda_0-\delta} \times \mathbb{R}$$

By Narrow region theorem, we derive

$$w_\lambda(x,t) \geq 0,\ (x,t) \in \Omega_\lambda^- \backslash \Omega_{\lambda_0-\delta} \times \mathbb{R}$$

We conclude that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$

$$w_\lambda(x,t) \geq 0,\ x \in \Omega_\lambda \times \mathbb{R}$$

This contradicts the definition of λ_0. Therefore, we must have

$$\lambda_0 = 0$$

and

$$w_{\lambda_0}(x,t) \geq 0,\ \forall (x,t) \in \Omega_{\lambda_0} \times \mathbb{R}$$

Similarly, one can move the plane T_λ from $\lambda = 1$ to the left and show that

$$w_{\lambda_0}(x,t) \leq 0,\ \forall (x,t) \in \Omega_{\lambda_0} \times \mathbb{R}$$

Now we have shown that

$$\lambda_0 = 0$$

and

$$w_{\lambda_0} \equiv 0,\ (x,t) \in \Omega_{\lambda_0} \times \mathbb{R}$$

This completes the setp 2.

So far, we have proved that u is symmetric about the plane T_0. Since the x_1 direction can be chosen arbitrarily, we have actually shown that u is radially symmetric about origin.

Since $w_\lambda(x,t) \not\equiv 0$, $x \in T_\lambda$, $\forall 0 < \lambda < \lambda_0$, if there exists (x^o, t^o) such that (x^o, t^o) is the minimum point, from the above process, on one hand,

$$(-\Delta)^s w_\lambda(x^o, t^o) \leq 0$$

On the other hand,

$$(-\Delta)^s w_\lambda(x^o, t^o) = 0$$

This forces

$$w_\lambda \equiv 0$$

which is a contradiction. Therefore, u is monotone decreasing about the origin.

4.1.3 Fractional elliptic systems

In section 6, we want to show radial symmetry and monotonicity of solution of fractional systems, where the systems here are represented by fractional elliptic systems and fractional parabolic systems. Section 6 is divided into two subsections, in the first subsection, we use moving of plane method to prove radial symmetry and monotonicity of fractional elliptic systems, where the fractional elliptic systems are given by:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(v(x)), & x \in B_1(0), \\ (-\Delta)^{\frac{\beta}{2}} v(x) = g(u(x)), & x \in B_1(0), \\ u(x), v(x) \geq 0, & x \in B_1(0), \\ u(x), v(x) \equiv 0, & x \notin B_1(0), \end{cases} \tag{4.15}$$

From the fractional systems given, it is easy to have

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} U_\lambda(x) \geq f_v(\xi(x)) V_\lambda(x), \\ (-\Delta)^{\frac{\beta}{2}} V_\lambda(x) \geq g_u(\eta(x)) U_\lambda(x), \end{cases} \tag{4.16}$$

We want to prove the following theorem:

Theorem 4.7. *(Radial Symmetry of solution of elliptic fractional system)*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(v(x)), & x \in B_1(0), \\ (-\Delta)^{\frac{\beta}{2}} v(x) = g(u(x)), & x \in B_1(0), \\ u(x), v(x) \geq 0, & x \in B_1(0), \\ u(x), v(x) \equiv 0, & x \notin B_1(0), \end{cases} \tag{4.17}$$

where B_1 is a unit ball.

Let $0 < \alpha, \beta < 2$, and suppose that $u(x), v(x) \in \left(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\right)$ are positive bounded classical solutions of equation (6.1)*, and assume $f(v(x)), g(u(x))$ satisfies the following assumptions:*

(X1) $f(\cdot)$ is non-decreasing in $v(\cdot)$, $g(\cdot)$ is non-decreasing in $u(\cdot)$.

(X2) Assume that f, g are uniformly Lipschitz continuous in u, v. i.e:

$$|f(v_1) - f(v_2)| \leq c|v_1 - v_2|,$$

$$|g(u_1) - f(u_2)| \leq c|u_1 - u_2|,$$

then $u(x), v(x)$ are radially symmetric about the origin and monotone decreasing about the origin, i.e.

$$u(x) = u(|x|), v(x) = v(|x|)$$

$$u(x_1) > u(x_2), v(x_1) > v(x_2), |x_1| < |x_2|.$$

.

In the process to show $u(x), v(x)$ here is monotone and radial symmetric about the origin using moving of plane method, we should give the plane an initial position to start, where in this initial position $U_\lambda(x) = u_\lambda(x) - u(x)$ and $V_\lambda(x) = v_\lambda(x) - v(x)$ are non-negative, so the Narrow region theorem in the unit ball is avoidable to be an ingredient here, we first prove:

Theorem 4.8. *(Narrow Region principle for elliptic Fractional System) Let $\Omega_\lambda = \Sigma_\lambda \cap B_1(0)$, Ω_λ is a bounded narrow region in Σ_λ, assume that $U_\lambda(x) \in \big(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\big)$, $V_\lambda(x) \in \big(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\big)$ if*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} U_\lambda(x) \geq f_v(\xi(x)) V_\lambda(x), & x \in \Omega_\lambda, \\ (-\Delta)^{\frac{\beta}{2}} V_\lambda(x) \geq g_u(\eta(x)) U_\lambda(x), & x \in \Omega_\lambda, \end{cases} \tag{4.18}$$

then for λ sufficiently close to -1, we have

$$\begin{cases} U_\lambda(x) \geq 0, & x \in \Omega_\lambda, \\ V_\lambda(x) \geq 0, & x \in \Omega_\lambda, \end{cases} \tag{4.19}$$

After we have an initial position to handle moving plane, we move the plane continuously to the right until its limiting position as long as U_λ, V_λ holds, define

$$\lambda_0 = sup\{\lambda \leq 0 \mid U_\mu(x) \geq 0, V_\mu(x) \geq 0, \forall x \in \Omega_\mu, \mu \leq \lambda\}$$

We show $\lambda_0 = 0$ by contradiction. Suppose $\lambda_0 < 0$, we show that the plane T_{λ_0} can be moved further to the right. First of all, we show $U_{\lambda_0}(x) > 0$, $V_{\lambda_0}(x) > 0$ for $x \in \Omega_{\lambda_0}$, then we let the moving plane go back a little bit, show $U_{\lambda_0}(x) \geq c_o > 0$, $V_{\lambda_0}(x) \geq c_o > 0$ for $x \in \Omega_{\lambda_0 - \delta}$.

To prove

$$\begin{cases} U_{\lambda_0}(x) > c_o > 0, \ x \in \Omega_{\lambda_0-\delta}, \\ V_{\lambda_0}(x) > c_o > 0, \ x \in \Omega_{\lambda_0-\delta}, \end{cases} \tag{4.20}$$

by contradiction, suppose there exists some sequences $x_k \in \Omega_{\lambda_0-\delta}$ such that $U_{\lambda_0}(x_k) \to 0$ for $x_k \to x^o \in \Omega_{\lambda_o-\delta}$, by regularity theory, $U_{\lambda_0}(x_k)$ converges uniformly to $\bar{U}(x)$.

We have

$$(-\Delta)^{\frac{\alpha}{2}}\bar{U}(x^o) \geq \bar{c}(x^o)\bar{V}(x^o) = 0$$

We also have

$$(-\Delta)^{\frac{\alpha}{2}}\bar{U}(x^o) = C_{n,\alpha}PV\int_{\mathbb{R}^n}\frac{-\bar{U}(y)}{|x^o-y|^{n+\alpha}}dy \leq 0$$

That forces

$$\bar{U}(y) \equiv 0, \ \forall y \in \mathbb{R}^n$$

Same routine, by regularity theory, there exists some sequences x_k such that $u(x_k)$ converges uniformly to $\bar{u}(x)$, $f(u)$ converges uniformly to $\bar{f}(u)$ for $x \in \Omega_{\lambda_o}$.

By a Strong Maximum principle:

Lemma 4.9. *(Strong Maximum Principle for $(-\Delta)^{\frac{\alpha}{2}}\bar{u} = f(\bar{v}(x))$).*

Assume that $\bar{u}(x) \in [C^{1,1}_{loc}(\Omega_\lambda) \cap C(\bar{\Omega_\lambda}) \cap \mathcal{L}_{2s}]$

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}}\bar{u}(x) = \bar{f}, & x \in \Omega_\lambda, \\ \bar{u}(x) \geq 0, & x \in \Omega_\lambda \end{cases} \tag{4.21}$$

we have either

$$\bar{u}(x) > 0, x \in B_1(0)$$

or

$$\bar{u}(x) \equiv 0, x \in \mathbb{R}^n$$

If $\bar{u}(x) > 0, x \in B_1(0)$, we know $\lambda_0 < 0, \bar{U}(x) > 0$ somewhere, but we already derive $\bar{U}(x) \equiv 0$, hence we must have $\bar{u}(x) \equiv 0$, $x \in \mathbb{R}^n$.

In order to derive a contradiction for large k, Let

$$U_k(x_k) \equiv U_{\lambda_0}(x_k) = m_k \tag{4.22}$$

Let

$$a_k(x) = U_k(x) - 2m_k \tag{4.23}$$

with a minimum point $\bar{x_k}$, now for sufficiently large k,

$$(-\Delta)^{\frac{\alpha}{2}} a_k(\bar{x}_k) = (-\Delta)^{\frac{\alpha}{2}} U_k \geq f_v(\xi(\bar{x}_k))V_k(\bar{x}_k)$$

and

$$(-\Delta)^{\frac{\alpha}{2}} a_k(\bar{x}_k) \leq \frac{c}{[d(\bar{x}_k, T_{\lambda_o})]^{\alpha}} a_k(\bar{x}_k) \leq -c_1 m_k$$

which is a contradiction, so that we have proved (4.20). Since U_λ, V_λ depends on λ continuously, there exists $\epsilon > 0$ and $\epsilon < \delta$, such that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, we have

$$\begin{cases} U_\lambda(x) \geq 0, \;\; x \in \Omega_{\lambda_0 - \delta}, \\ V_\lambda(x) \geq 0, \;\; x \in \Omega_{\lambda_0 - \delta}, \end{cases}$$

By Narrow region theorem, we derive

$$\begin{cases} U_\lambda(x) \geq 0, \ x \in \Omega_\lambda^- \backslash \Omega_{\lambda_0 - \delta}, \\ V_\lambda(x) \geq 0, \ x \in \Omega_\lambda^- \backslash \Omega_{\lambda_0 - \delta}, \end{cases}$$

We conclude that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$

$$\begin{cases} U_\lambda(x) \geq 0, \ x \in \Omega_{\lambda_0}, \\ V_\lambda(x) \geq 0, \ x \in \Omega_{\lambda_0}, \end{cases}$$

This contradicts the definition of λ_0. Therefore, we must have

$$\lambda_0 = 0$$

and

$$\begin{cases} U_{\lambda_0}(x) \geq 0, \ \forall x \in \Omega_{\lambda_0}, \\ V_{\lambda_0}(x) \geq 0, \ \forall x \in \Omega_{\lambda_0}, \end{cases}$$

Similarly, one can move the plane T_λ from $\lambda = 1$ to the left and show that

$$\begin{cases} U_{\lambda_0}(x) \leq 0, \ \forall x \in \Omega_{\lambda_0}, \\ V_{\lambda_0}(x) \leq 0, \ \forall x \in \Omega_{\lambda_0}, \end{cases}$$

Now we have shown that

$$\lambda_0 = 0$$

and

$$\begin{cases} U_{\lambda_0} \equiv 0, \ x \in \Omega_{\lambda_0}, \\ V_{\lambda_0} \equiv 0, \ x \in \Omega_{\lambda_0}, \end{cases}$$

This completes the step 2.

So far, we have proved that u, v are symmetric about the plane T_0. Since the x_1 direction can be chosen arbitrarily, we have actually shown that u, v are radially symmetric about origin.

Since $U_\lambda(x) \not\equiv 0$, $x \in T_\lambda$, $\forall 0 < \lambda < \lambda_0$, if there exists x^o such that x^o is the minimum point, from the above process, on one hand,

$$(-\Delta)^{\frac{\alpha}{2}} U_\lambda(x^o) \leq 0$$

On the other hand,

$$(-\Delta)^{\frac{\alpha}{2}} U_\lambda(x^o) = 0$$

This forces

$$U_\lambda \equiv 0$$

which is a contradiction. Therefore, u is monotone decreasing about the origin. Same reason for v.

4.1.4 Fractional parabolic systems

In the second subsection, we use moving of plane method to prove radial symmetry and monotonicity of fractional parabolic systems, where the fractional parabolic systems are given by:

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} u(x,t) = f(v(x,t)), & (x,t) \in B_1(0) \times (-\infty, \infty), \\ \frac{\partial v}{\partial t} + (-\Delta)^{\frac{\beta}{2}} v(x,t) = g(u(x,t)), & (x,t) \in B_1(0) \times (-\infty, \infty), \\ u(x,t), v(x,t) \geq 0, & (x,t) \in B_1(0) \times (-\infty, \infty), \\ u(x,t), v(x,t) \equiv 0, & x \notin B_1(0), \end{cases} \tag{4.24}$$

From the fractional systems given, it is easy to have

$$\begin{cases} \frac{\partial U_\lambda}{\partial t}(x,t) + (-\Delta)^{\frac{\alpha}{2}} U_\lambda(x,t) \geq f_v(\xi(x,t)) V_\lambda(x,t), \\ \frac{\partial V_\lambda}{\partial t}(x,t) + (-\Delta)^{\frac{\beta}{2}} V_\lambda(x,t) \geq g_u(\eta(x,t)) U_\lambda(x,t), \end{cases} \tag{4.25}$$

We want to prove the following theorem:

Theorem 4.10. *(Radial Symmetry of solution of parabolic fractional system)*

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} u(x,t) = f(v(x,t)), & (x,t) \in B_1(0) \times (-\infty, \infty), \\ \frac{\partial v}{\partial t} + (-\Delta)^{\frac{\beta}{2}} v(x,t) = g(u(x,t)), & (x,t) \in B_1(0) \times (-\infty, \infty), \\ u(x,t), v(x,t) \geq 0, & (x,t) \in B_1(0) \times (-\infty, \infty), \\ u(x,t), v(x,t) \equiv 0, & x \notin B_1(0), \end{cases} \tag{4.26}$$

where B_1 is a unit ball.

Let $0 < \alpha, \beta < 2$, and suppose that $u(x,t), v(x,t) \in \left(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\right) \times (-\infty, \infty)$ are positive bounded classical solutions of equation (6.22)*, and assume $f(v(x,t)), g(u(x,t))$ satisfies the following assumptions:*

(M1) $f(\cdot)$ is non-decreasing in $v(\cdot)$, $g(\cdot)$ is non-decreasing in $u(\cdot)$.

(M2) Assume that f, g are uniformly Lipschitz continuous in u, v. i.e:

$$|f(v_1) - f(v_2)| \leq c|v_1 - v_2|,$$

$$|g(u_1) - f(u_2)| \leq c|u_1 - u_2|,$$

then $u(x,t), v(x,t)$ are radially symmetric about the origin and monotone decreasing about the origin, i.e.

$$u(x,t) = u(|x|,t), v(x,t) = v(|x|,t)$$

$$u(x_1,t) > u(x_2,t), v(x_1,t) > v(x_2,t), |x_1| < |x_2|$$

.

In the process to show $u(x), v(x)$ here is monotone and radial symmetric about the origin using moving of plane method, we should give the plane an initial position to start, where in this initial position $U_\lambda(x) = u_\lambda(x) - u(x)$ and $V_\lambda(x) = v_\lambda(x) - v(x)$ are non-negative, so the Narrow region theorem in the unit ball is avoidable to be an ingredient here, we first prove:

Theorem 4.11. *(Narrow region principle on a parabolic cylinder) Let $\Omega_\lambda = \Sigma_\lambda \cap B_1(0)$, Ω_λ is a bounded narrow region in Σ_λ, assume that $U_\lambda(x,t), V_\lambda(x,t) \in [C^{1,1}_{loc}(\Omega_\lambda) \cap C(\bar{\Omega_\lambda}) \cap \mathcal{L}_{2s}] \times C^1([\underline{t}, T])$, and $U_\lambda(x,t)$, $V_\lambda(x,t)$ are lower semi-continuous on $\bar{\Omega}$. If*

$$\begin{cases} \frac{\partial U_\lambda}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} U_\lambda(x,t) \geq f_v(\xi(x,t)) V_\lambda(x,t), \quad (x,t) \in \Omega_\lambda \times [\underline{t}, T], \\ \frac{\partial V_\lambda}{\partial t} + (-\Delta)^{\frac{\beta}{2}} V_\lambda(x,t) \geq g_u(\eta(x,t)) U_\lambda(x,t), \quad (x,t) \in \Omega_\lambda \times [\underline{t}, T], \end{cases} \tag{4.27}$$

Then for λ sufficiently close to -1, we have

$$U_\lambda(x,t) \geq \min\{0, \inf_{\Omega_\lambda \times [\underline{t},T]} U_\lambda(x,\underline{t})\}, \ (x,t) \in \Omega_\lambda \times [\underline{t},T] \tag{4.28}$$

and

$$V_\lambda(x,t) \geq \min\{0, \inf_{\Omega_\lambda} V_\lambda(x,\underline{t})\}, \ (x,t) \in \Omega_\lambda \times [\underline{t},T] \tag{4.29}$$

After we have an initial position to handle moving plane, we move the plane continuously to the right until its limiting position as long as U_λ, V_λ holds, define

$$\lambda_0 = sup\{\lambda \leq 0 \mid U_\mu(x,t) \geq 0, V_\mu(x,t) \geq 0, \forall (x,t) \in \Omega_\mu \times \mathbb{R}, \mu \leq \lambda\}$$

We show $\lambda_0 = 0$ by contradiction. Suppose $\lambda_0 < 0$, we show that the plane T_{λ_0} can be moved further to the right. First of all, we show $U_{\lambda_0}(x,t) > 0$, $V_{\lambda_0}(x,t) > 0$ for $(x,t) \in \Omega_{\lambda_0} \times \mathbb{R}$, then we let the moving plane go back a little bit, show $U_{\lambda_0}(x,t) \geq c_o > 0$, $V_{\lambda_0}(x,t) \geq c_o > 0$ for $(x,t) \in \Omega_{\lambda_0 - \delta} \times \mathbb{R}$.

To prove

$$\begin{cases} \inf U_{\lambda_0}(x,t) > c_o > 0, \ (x,t) \in \Omega_{\lambda_0 - \delta} \times \mathbb{R}, \\ \inf V_{\lambda_0}(x,t) > c_o > 0, \ (x,t) \in \Omega_{\lambda_0 - \delta} \times \mathbb{R}, \end{cases} \tag{4.30}$$

Let

$$U_k(x,t) = U_{\lambda_0}(x, t + t_k)$$

by contradiction, suppose there exists some sequences $x_k \in \Omega_{\lambda_0 - \delta}$ such that $U_{\lambda_0}(x_k, t_k) \to 0$ for $x_k \to x^o \in \Omega_{\lambda_o - \delta}$, by regularity theory, $U_k(x,t)$ converges uniformly to $\bar{U}(x,t)$.

We have

$$\frac{\partial \bar{U}(x^o,0)}{\partial t} + (-\Delta)^{\frac{\alpha}{2}}\bar{U}(x^o,0) = (-\Delta)^{\frac{\alpha}{2}}\bar{U}(x^o,0) \geq \bar{c}(x^o,0)\bar{V}(x^o,0) = 0$$

We also have

$$\frac{\partial \bar{U}(x^o,0)}{\partial t} + (-\Delta)^{\frac{\alpha}{2}}\bar{U}(x^o,0) = (-\Delta)^{\frac{\alpha}{2}}\bar{U}(x^o,0) \leq 0$$

That forces

$$\bar{U}(y,0) \equiv 0,\ \forall y \in \mathbb{R}^n$$

Let $u_k(x,t) = u(x, t+t_k)$, same routine, by regularity theory, there exists some sequences x_k such that $u_k(x,t)$ converges uniformly to $\bar{u}(x,t)$, f converges uniformly to $\bar{f}$ for $x \in \Omega_{\lambda_o}$.

By a Strong Maximum principle:

Lemma 4.12. *(Strong Maximum Principle for $\frac{\partial \bar{u}}{\partial t} + (-\Delta)^{\frac{\alpha}{2}}\bar{u} = f(\bar{v}(x,t))$).*

Assume that $\bar{u}(x,t) \in [C^{1,1}_{loc}(\Omega_\lambda) \cap C(\bar{\Omega_\lambda}) \cap \mathcal{L}_{2s}] \times C^1([\underline{t}, T])$

$$\begin{cases} \frac{\partial \bar{u}(x,t)}{\partial t} + (-\Delta)^{\frac{\alpha}{2}}\bar{u}(x,t) = f(\bar{v}(x,t)), & (x,t) \in \Omega_\lambda \times [\underline{t}, T], \\ \bar{u}(x,t) \geq 0, & (x,t) \in \Omega_\lambda \times [\underline{t}, T] \end{cases} \tag{4.31}$$

we have either

$$\bar{u}(x,0) > 0, x \in B_1(0)$$

or

$$\bar{u}(x,0) \equiv 0, x \in \mathbb{R}^n$$

If $\bar{u}(x,0) > 0$, $x \in B_1(0)$, we know $\lambda_0 < 0$, $\bar{U}(x,0) > 0$ somewhere, but we already derive $\bar{U}(x,0) \equiv 0$, hence we must have $\bar{u}(x,0) \equiv 0$, $x \in \mathbb{R}^n$.

In order to derive a contradiction for large k, Let

$$U_k(x_k,0) \equiv U_{\lambda_0}(x_k,t_k) = m_k \tag{4.32}$$

Let

$$a_k(x,t) = U_k(x,t) - 2m_k\eta(\epsilon_k(t-t_k)) \tag{4.33}$$

with a minimum point $(\bar{x_k}, \bar{t_k})$, now for sufficiently large k,

$$(-\Delta)^{\frac{\alpha}{2}} a_k(\bar{x}_k,\bar{t}_k) = (-\Delta)^{\frac{\alpha}{2}} U_k \geq -\frac{\partial U_k}{\partial t}(\bar{x}_k,\bar{t}_k) + f_v(\xi(x,t))V_k(\bar{x}_k,\bar{t}_k)$$

and

$$(-\Delta)^{\frac{\alpha}{2}} a_k(\bar{x}_k,\bar{t}_k) \leq \frac{c}{[d(\bar{x}_k,T_{\lambda_o})]^\alpha} a_k(\bar{x}_k,\bar{t}_k) \leq -c_1 m_k$$

which is a contradiction, so we have proved (4.32)

Since U_λ, V_λ depends on λ continuously, there exists $\epsilon > 0$ and $\epsilon < \delta$, such that for all $\lambda \in (\lambda_0, \lambda_0+\epsilon)$, we have

$$\begin{cases} U_\lambda(x) \geq 0, \ \ x \in \Omega_{\lambda_0-\delta}, \\ V_\lambda(x) \geq 0, \ \ x \in \Omega_{\lambda_0-\delta}, \end{cases}$$

By Narrow region theorem, we derive

$$\begin{cases} U_\lambda(x) \geq 0, \ x \in \Omega_\lambda^- \backslash \Omega_{\lambda_0-\delta}, \\ V_\lambda(x) \geq 0, \ x \in \Omega_\lambda^- \backslash \Omega_{\lambda_0-\delta}, \end{cases}$$

We conclude that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$

$$\begin{cases} U_\lambda(x) \geq 0, \ x \in \Omega_{\lambda_0}, \\ V_\lambda(x) \geq 0, \ x \in \Omega_{\lambda_0}, \end{cases}$$

This contradicts the definition of λ_0. Therefore, we must have

$$\lambda_0 = 0$$

and

$$\begin{cases} U_{\lambda_0}(x,t) \geq 0, \ \forall (x,t) \in \Omega_{\lambda_0} \times \mathbb{R}, \\ V_{\lambda_0}(x,t) \geq 0, \ \forall (x,t) \in \Omega_{\lambda_0} \times \mathbb{R}, \end{cases}$$

Similarly, one can move the plane T_λ from $\lambda = 1$ to the left and show that

$$\begin{cases} U_{\lambda_0}(x,t) \leq 0, \ \forall (x,t) \in \Omega_{\lambda_0} \times \mathbb{R}, \\ V_{\lambda_0}(x,t) \leq 0, \ \forall (x,t) \in \Omega_{\lambda_0} \times \mathbb{R}, \end{cases}$$

Now we have shown that

$$\lambda_0 = 0$$

and

$$\begin{cases} U_{\lambda_0} \equiv 0, \ (x,t) \in \Omega_{\lambda_0} \times \mathbb{R}, \\ V_{\lambda_0} \equiv 0, \ (x,t) \in \Omega_{\lambda_0} \times \mathbb{R}, \end{cases}$$

This completes the step 2.

So far, we have proved that u, v are symmetric about the plane T_0. Since the x_1 direction can be chosen arbitrarily, we have actually shown that u, v are radially symmetric about origin.

Since $U_\lambda(x,t) \not\equiv 0$, $(x,t) \in T_\lambda \times \mathbb{R}$, $\forall 0 < \lambda < \lambda_0$, if there exists $(x^o.t^o)$ such that (x^o, t^o) is the minimum point, from the above process, on one hand,

$$(-\Delta)^{\frac{\alpha}{2}} U_\lambda(x^o, t^o) \leq 0$$

On the other hand,

$$(-\Delta)^{\frac{\alpha}{2}} U_\lambda(x^o, t^o) = 0$$

This forces

$$U_\lambda \equiv 0$$

which is a contradiction. Therefore, u is monotone decreasing about the origin. Same reason for v.

4.2 Part 2: Method of Sliding and Its Applications: Monotonicity of solutions of fractional parabolic and Monge-Ampére equations

4.2.1 Fractional parabolic equations in Bounded domains

In section 9, we use sliding method to prove the monotonicity of solution of fractional parabolic equation in a bounded domain, where the fractional parabolic equation is given by:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = f(t,|x|,u), & (x,t) \in \Omega \times (-\infty,\infty), \\ u(x,t) = \varphi(x,t), & (x,t) \in \Omega^c \times (-\infty,\infty), \end{cases} \tag{4.34}$$

We want to prove the following theorem:

Theorem 4.13. *(Monotonicity of solution of fractional parabolic equation)*

Let Ω be a bounded domain in $\mathbb{R}^n$ which is convex in the x_n-direction. Let $0 < s < 1$, and suppose that $u(x,t) \in \left(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\right) \times (-\infty,\infty)$ is a positive bounded classical solution of

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = f(t,|x|,u), & (x,t) \in \Omega \times (-\infty,\infty), \\ u(x,t) = \varphi(x,t), & (x,t) \in \Omega^c \times (-\infty,\infty), \end{cases} \tag{4.35}$$

We impose some conditions on u. Let $u(x,t) = \varphi(x,t)$ in Ω^c, suppose H:

For any three points $x = (x', x_n)$, $y = (x', y_n)$ and $z = (x', z_n)$ lying on a segment parallel to the x_n axis, $y_n < x_n < z_n$ with $y, z \in \Omega^c$, we have

$$\varphi(y,t) < u(x,t) < \varphi(z,t), \ (x,t) \in \Omega \times \mathbb{R} \tag{4.36}$$

and

$$\varphi(y,t) \leq \varphi(x,t) \leq \varphi(z,t),\ (x,t) \in \Omega^c \times \mathbb{R} \tag{4.37}$$

Assume that f is non-increasing about u and is uniformly Lipschitz continuous in u. i.e:

$$f(t,|x|,u_1) - f(t,|x|,u_2) \leq c|u_1 - u_2|,\ \forall x \in \Omega,$$

Then $u(x,t)$ is monotone increasing with respect to x_n in Ω, i.e: for any $\tau > 0$, we have $u((x',x_n+\tau),t) > u((x',x_n),t)$ for $(x',x_n),(x',x_n+\tau) \in \Omega$ and $t \in \mathbb{R}$, where $x' = (x_1,\cdots,x_{n-1}) \in \mathbb{R}^{n-1}$

In the process to show $u(x)$ here is monotone using sliding method, we begin sliding Ω^τ downward τ units along the x_n axis, we should give the plane an initial position to start, where in this initial position $W^\tau(x) = u^\tau(x) - u(x)$ is non-negative, so the Narrow region theorem in the bounded region is avoidable to be an ingredient here, we first prove:

Lemma 4.14. *(Narrow Region principle on a parabolic cylinder) Let D be a bounded narrow region in $\mathbb{R}^n$. Assume that $u(x,t) \in \left(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\right) \cap \mathcal{L}_{2s} \times [\underline{t},T]$. $W^\tau(x,t) = u^\tau(x,t) - u(x,t)$ is lower semi-continuous on $\bar{D} \times [\underline{t},T]$, and satisfies*

$$\begin{cases} \frac{\partial W^\tau}{\partial t} + (-\triangle)^s W^\tau = c(x,t)W^\tau(x,t), & (x,t) \in D \times [\underline{t},T], \\ W^\tau(x,t) \geq 0, & (x,t) \in (\mathbb{R}^n \setminus D) \times [\underline{t},T], \end{cases} \tag{4.38}$$

where $c(x,t)$ is bounded from below in D. Let $dn(D)$ be the width of D in the x_n-direction. Then:

$$W^\tau(x,t) \geq 0,\ (x,t) \in D \times [\underline{t},T] \tag{4.39}$$

Moreover, we have either $W^\tau(x,t) > 0$ *in* $D \times \mathbb{R}$ *or* $W^\tau(x,t) \equiv 0$ *in* $\mathbb{R}^n \times \mathbb{R}$*:*

Let

$$\bar{W} = e^{m(t-\underline{t})} W^\tau(x,t), \;\; m > 0$$

So then

$$\frac{\partial \bar{W}}{\partial t} + (-\triangle)^s \bar{W} = \bar{c}\bar{W}$$

By Narrow region theorem,

$$\bar{W}(x,t) \geq \min\{0, \inf_{x\in D} \bar{W}(x,\underline{t})\}, \;\; \forall (x,t) \in D \times (\underline{t}, T)$$

Thus

$$e^{m(t-\underline{t})} W^\tau(x,t) \geq \min\{0, \inf_{x\in D} \bar{W}(x,\underline{t})\}$$

So

$$W^\tau(x,t) \geq e^{-m(t-\underline{t})} \min\{0, \inf_{x\in D} \bar{W}(x,\underline{t})\}$$

$W^\tau(x,t)$ is bounded from below. Let $\underline{t} \to -\infty$, $W^\tau(x,t) \to \geq 0$.

Therefore,

$$W^\tau(x,t) \geq 0$$

The inequality provides a starting point, from which we can carry out the sliding. Now we decrease τ as long as $W^\tau(x,t) \geq 0$ holds to its limiting position. Define

$$\tau_0 = \inf\{\tau \mid W^\tau(x,t) \geq 0, \forall (x,t) \in D^\tau \times \mathbb{R}, 0 < \tau < \widetilde{\tau}\}$$

We show $\tau_0 = 0$ by contradiction. Suppose $\tau_0 > 0$, we will show that Ω^τ can be slid upward a little bit more and we will have $W^\tau(x,t) \geq 0$. First of all, we show

$W^{\tau_0}(x,t) > 0$ for $(x,t) \in D^{\tau_0} \times \mathbb{R}$, then we let Ω^τ can be slid upward a little bit more, show $\inf W^{\tau_0}(x,t) > c_o > 0,\ (x,t) \in D^{\tau_0-\epsilon} \times \mathbb{R}$.

To prove

$$\inf W^{\tau_0}(x,t) > c_o > 0,\ (x,t) \in D^{\tau_0-\epsilon} \times \mathbb{R}$$

by contradiction, suppose there exists some sequences $(x_k, t_k) \in D^{\tau_0-\epsilon} \times \mathbb{R}$ such that $W^{\tau_0}(x_k, t_k) \to 0$ for $x_k \to x^o \in D^{\tau_0-\epsilon}$.

Let

$$W_k(x,t) = W^{\tau_0}(x, t + t_k)$$

by regularity theory, $W_k(x,t)$ converges uniformly to $\bar{W}(x,t)$.

Since

$$W_k(x_k, 0) = W^{\tau_0}(x_k, t_k) \to 0$$

$$\bar{W}(x^o, 0) = 0$$

We have

$$\frac{\partial \bar{W}}{\partial t}(x^o, 0) + (-\Delta)^s \bar{W}(x^o, 0) = c^\tau(x^o, t_k)\bar{W}(x^o, 0) = 0$$

We also have

$$(-\Delta)^s \bar{W}(x^o, 0) = C_{n,s} PV \int_{\mathbb{R}^n} \frac{-\bar{W}(y,0)}{|x^o - y|^{n+2s}} dy \leq 0$$

That forces

$$\bar{W}(y, 0) \equiv 0,\ \forall y \in \mathbb{R}^n$$

Same routine, Let $u_k(x,t) = u(x, t + t_k)$, by regularity theory, there exists some sequences x_k such that $u_k(x,t)$ converges uniformly to $\bar{u}(x,t)$, $f(0,u)$ converges uniformly to $\bar{f}(0,u)$ for $x \in D^{\tau_0}$.

By a Strong Maximum principle:

Lemma 4.15. *(Strong Maximum Principle for $\frac{\partial \bar{u}}{\partial t} + (-\Delta)^s \bar{u} = \bar{f}(t,\bar{u})$).*

Assume that $\bar{u}(x,t) \in [C^{1,1}_{loc}(\Omega_\lambda) \cap C(\bar{\Omega_\lambda}) \cap \mathcal{L}_{2s}] \times C^1([\underline{t},T])$

$$\begin{cases} \frac{\partial \bar{u}(x,t)}{\partial t} + (-\Delta)^s \bar{u}(x,t) = \bar{f}(t,\bar{u}), & (x,t) \in D^\tau \times [\underline{t},T], \\ \bar{u}(x,t) \geq 0, & (x,t) \in D^\tau \times [\underline{t},T] \end{cases} \tag{4.40}$$

we have either

$$\bar{u}(x,0) > 0, x \in D^\tau$$

or

$$\bar{u}(x,0) \equiv 0, x \in \mathbb{R}^n$$

If $\bar{u}(x,0) > 0$, $x \in D^\tau$, $\bar{W}(x,0) > 0$ somewhere, but we already derive $\bar{W}(x,0) \equiv 0$, hence we must have $\bar{u}(x,0) \equiv 0$, $x \in \mathbb{R}^n$.Thus, we know $u(x,t_k)$ converges to 0 uniformly.

In order to derive a contradiction for large k, Let

$$W_k(x_k,0) \equiv W^{\tau_0}(x_k,t_k) = m_k \tag{4.41}$$

Let

$$V_k(x,t) = W_k(x,t) - 2m_k\eta(\epsilon_k(t-t_k)) \tag{4.42}$$

with a minimum point $(\bar{x_k}, \bar{t_k})$, now for sufficiently large k,

$$(-\Delta)^s V_k(\bar{x}_k, \bar{t}_k) = -\frac{\partial W_k}{\partial t}(\bar{x}_k, \bar{t}_k) + c^{\tau_0}(\bar{x}_k, \bar{t}_k + t_k) W_k(\bar{x}_k, \bar{t}_k)$$

with

$$-\frac{\partial W_k}{\partial t}(\bar{x}_k, \bar{t}_k) \sim \epsilon_k m_k$$

and

$$(-\Delta)^s V_k(\bar{x}_k, \bar{t}_k) \leq \frac{c}{[d(\bar{x}_k, T_{\lambda_o})]^{2s}} V_k(\bar{x}_k, \bar{t}_k) \leq -c_1 m_k$$

which is a contradiction, so that we have proved $\inf W^{\tau_0}(x,t) > c_o > 0,\ (x,t) \in D^{\tau_0 - \epsilon} \times \mathbb{R}$

Now we can carve out from D^{τ_0} a closed set $K \subset D^{\tau_0}$ such that $D^{\tau_0} \backslash K$ is narrow.

Since

$$W^{\tau_0}(x,t) > 0,\ (x,t) \in D^{\tau_0} \times \mathbb{R}$$

We have

$$W^{\tau_0}(x,t) \geq C_0 > 0,\ in\ K \tag{4.43}$$

Since W^τ is continuous with respect to τ, for small $\epsilon > 0$, we have:

$$W^{\tau_0 - \epsilon}(x,t) \geq 0,\ in\ K \tag{4.44}$$

According to (H), we have

$$W^{\tau_0 - \epsilon}(x,t) \geq 0,\ in\ (D^{\tau_0 - \epsilon})^c$$

It is obvious that $(D^{\tau_0-\epsilon}\backslash K)^c = K \cup (D^{\tau_0-\epsilon})^c$, then

$$\begin{cases} \frac{\partial W^{\tau_0-\epsilon}}{\partial t} + (-\Delta)^s W^{\tau_0-\epsilon} = c(x,t)W^{\tau_0-\epsilon}(x,t), & (x,t) \in D^{\tau_0-\epsilon}\backslash K \times \mathbb{R}, \\ W^{\tau_0-\epsilon}(x,t) \geq 0, & (x,t) \in (D^{\tau_0-\epsilon}\backslash K)^c \times \mathbb{R}, \end{cases} \tag{4.45}$$

By Narrow Region theorem, we have

$$W^{\tau_0-\epsilon}(x_k,t_k) \geq 0,\ (x,t) \in D^{\tau_0-\epsilon}\backslash K \times \mathbb{R} \tag{4.46}$$

From this and (4.44), we obtain $W^\tau(x,t) \geq 0$ for $\tau \in (\tau_0 - \epsilon, \tau_0)$ which contradicts the definition of τ_0.

Since $W^\tau(x,t) \not\equiv 0$, $(x,t) \in D^\tau \times \mathbb{R}$, $\forall 0 < \tau < \widetilde{\tau}$, if there exists (x^o, t^o) such that (x^o, t^o) is the minimum point, from the above process, on one hand,

$$(-\Delta)^s W^\tau(x^o, t^o) \leq 0$$

On the other hand,

$$(-\Delta)^s W^\tau(x^o, t^o) = 0$$

This forces

$$W^\tau \equiv 0$$

which contradicts (H)

Thus we have proved the Theorem.

4.2.2 Fractional parabolic Monge-Ampére equations in Bounded domains

In section 9.4, we use sliding method to prove the monotonicity of solution of parabolic Monge Ampere equation in a bounded domain, where the parabolic Monge Ampere equation is given by:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - D_s^{\theta} u(x,t) = f(t,|x|,u), & (x,t) \in \Omega \times (-\infty,\infty), \\ u(x,t) = \varphi(x,t), & (x,t) \in \Omega^c \times (-\infty,\infty), \end{cases} \tag{4.47}$$

We want to prove the following theorem:

Theorem 4.16. *(Monotonicity of solution of parabolic Monge Ampére equation)*

Let Ω be a bounded domain in $\mathbb{R}^n$ which is convex in the x_n-direction. Let $0 < s < 1$, and suppose that $u(x,t) \in \left(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\right) \times (-\infty,\infty)$ is a positive bounded classical solution of

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - D_s^{\theta} u(x,t) = f(t,|x|,u), & (x,t) \in \Omega \times (-\infty,\infty), \\ u(x,t) = \varphi(x,t), & (x,t) \in \Omega^c \times (-\infty,\infty), \end{cases} \tag{4.48}$$

We impose some conditions on u. Let $u(x,t) = \varphi(x,t)$ in Ω^c, suppose H:

For any three points $x = (x', x_n)$, $y = (x', y_n)$ and $z = (x', z_n)$ lying on a segment parallel to the x_n axis, $y_n < x_n < z_n$ with $y, z \in \Omega^c$, we have

$$\varphi(y,t) < u(x,t) < \varphi(z,t), \ (x,t) \in \Omega \times \mathbb{R} \tag{4.49}$$

and

$$\varphi(y,t) \leq \varphi(x,t) \leq \varphi(z,t), \ (x,t) \in \Omega^c \times \mathbb{R} \tag{4.50}$$

Assume that f is uniformly Lipschitz continuous in u. i.e:

$$f(t,|x|,u_1) - f(t,|x|,u_2) \leq c|u_1 - u_2|, \ \forall x \in \Omega,$$

then $u(x,t)$ is monotone increasing with respect to x_n in Ω, i.e: for any $\tau > 0$, we have

$u((x', x_n + \tau), t) > u((x', x_n), t)$ for $(x', x_n), (x', x_n + \tau) \in \Omega$ and $t \in \mathbb{R}$,

where $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$

In the process to show $u(x)$ here is monotone using sliding method, we begin sliding Ω^τ downward τ units along the x_n axis, we should give the plane an initial position to start, where in this initial position $W^\tau(x) = u^\tau(x) - u(x)$ is non-negative, so the Narrow region theorem in the bounded region is avoidable to be an ingredient here, we first prove:

Lemma 4.17. *(Narrow Region principle on a parabolic cylinder) Let D be a bounded narrow region in $\mathbb{R}^n$. Assume that $u(x,t) \in \left(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\right) \cap \mathcal{L}_{2s} \times [\underline{t}, T]$. $W^\tau(x,t) = u^\tau(x,t) - u(x,t)$ is lower semi-continuous on $\bar{D} \times [\underline{t}, T]$, and satisfies*

$$\begin{cases} \frac{\partial W^\tau}{\partial t} - D_s^\theta u^\tau + D_s^\theta u = c(x,t) W^\tau(x,t), & (x,t) \in D \times [\underline{t}, T], \\ W^\tau(x,t) \geq 0, & (x,t) \in (\mathbb{R}^n \setminus D) \times [\underline{t}, T], \end{cases} \tag{4.51}$$

where $c(x,t)$ is bounded from below in D. Let $dn(D)$ be the width of D in the x_n-direction. Then:

$$W^\tau(x,t) \geq 0, \ (x,t) \in D \times [\underline{t}, T] \tag{4.52}$$

Moreover, we have either $W^\tau(x,t) > 0$ in $D \times \mathbb{R}$ or $W^\tau(x,t) \equiv 0$ in $\mathbb{R}^n \times \mathbb{R}$:

Let

$$\bar{W} = e^{m(t-\underline{t})} W^{\tau}(x,t),\ \ m > 0$$

So then

$$\frac{\partial \bar{W}(x,t)}{\partial t} - D_s^{\theta} \bar{u}^{\tau}(x,t) + D_s^{\theta} \bar{u}(x,t) = \bar{c}(x,t) \bar{W}(x,t)$$

By Narrow region theorem,

$$\bar{W}(x,t) \geq \min\{0, \inf_{x \in D} \bar{W}(x,\underline{t})\},\ \ \forall (x,t) \in D \times (\underline{t}, T)$$

Thus

$$e^{m(t-\underline{t})} W^{\tau}(x,t) \geq \min\{0, \inf_{x \in D} \bar{W}(x,\underline{t})\}$$

So

$$W^{\tau}(x,t) \geq e^{-m(t-\underline{t})} \min\{0, \inf_{x \in D} \bar{W}(x,\underline{t})\}$$

$W^{\tau}(x,t)$ is bounded from below. Let $\underline{t} \to -\infty$, $W^{\tau}(x,t) \to \geq 0$.

Therefore,

$$W^{\tau}(x,t) \geq 0$$

The inequality provides a starting point, from which we can carry out the sliding. Now we decrease τ as long as $W^{\tau}(x,t) \geq 0$ holds to its limiting position. Define

$$\tau_0 = \inf\{\tau \mid W^{\tau}(x,t) \geq 0, \forall (x,t) \in D^{\tau} \times \mathbb{R}, 0 < \tau < \widetilde{\tau}\}$$

We show $\tau_0 = 0$ by contradiction. Suppose $\tau_0 > 0$, we will show that Ω^{τ} can be slid upward a little bit more and we will have $W^{\tau}(x,t) \geq 0$. First of all, we show $W^{\tau_0}(x,t) > 0$ for $(x,t) \in D^{\tau_0} \times \mathbb{R}$, then we let Ω^{τ} can be slid upward a little bit more, show $\inf W^{\tau_0}(x,t) > c_o > 0,\ (x,t) \in D^{\tau_0-\epsilon} \times \mathbb{R}$.

To prove

$$\inf W^{\tau_0}(x,t) > c_o > 0,\ (x,t) \in D^{\tau_0-\epsilon} \times \mathbb{R}$$

by contradiction, suppose there exists some sequences $(x_k, t_k) \in D^{\tau_0-\epsilon} \times \mathbb{R}$ such that $W^{\tau_0}(x_k, t_k) \to 0$ for $x_k \to x^o \in D^{\tau_0-\epsilon}$.

Let

$$W_k(x,t) = W^{\tau_0}(x, t+t_k)$$

by regularity theory, $W_k(x,t)$ converges uniformly to $\bar{W}(x,t)$.

Since

$$W_k(x_k, 0) = W^{\tau_0}(x_k, t_k) \to 0$$

$$\bar{W}(x^o, 0) = 0$$

$$\bar{W} \geq 0$$

So $(x^o, 0)$ is the minimum.

We have

$$\frac{\partial \bar{W}}{\partial t}(x^o,0) - D_s^\theta \bar{u}^{\tau_0}(x^o,0) + D_s^\theta \bar{u}(x^o,0) = c^\tau(x^o,t_k)\bar{W}(x^o,0) = 0$$

so we derive

$$D_s^\theta \bar{u}(x^o,0) - D_s^\theta \bar{u}^{\tau_0}(x^o,0) = 0$$

We also have

$$D_s^\theta \bar{u}(x^o,0) - D_s^\theta \bar{u}^{\tau_0}(x^o,0) \leq -c_0 cC - \eta < 0$$

Which is a contradiction, therefore, we have proved

$$\inf W^{\tau_0}(x,t) > c_o > 0,\ (x,t) \in D^{\tau_0-\epsilon} \times \mathbb{R}$$

Now we can carve out from D^{τ_0} a closed set $K \subset D^{\tau_0}$ such that $D^{\tau_0}\backslash K$ is narrow. We have

$$W^{\tau_0}(x,t) \geq c_0 > 0,\ in\ K \tag{4.53}$$

Since W^{τ} is continuous with respect to τ, for small $\epsilon > 0$, we have:

$$W^{\tau_0-\epsilon}(x,t) \geq 0,\ in\ K \tag{4.54}$$

According to (H), we have

$$W^{\tau_0-\epsilon}(x,t) \geq 0,\ in\ (D^{\tau_0-\epsilon})^c \times \mathbb{R}$$

It is obvious that $(D^{\tau_0-\epsilon}\backslash K)^c = K \cup (D^{\tau_0-\epsilon})^c$, then by a Narrow region theorem

$$\begin{cases} \frac{\partial W^{\tau_0-\epsilon}}{\partial t} - D_s^{\theta}u^{\tau_0-\epsilon} + D_s^{\theta}u^{\tau_0-\epsilon} = c(x,t)W^{\tau_0-\epsilon}(x,t), & (x,t) \in D^{\tau_0-\epsilon}\backslash K \times \mathbb{R}, \\ W^{\tau_0-\epsilon}(x,t) \geq 0, & (x,t) \in (D^{\tau_0-\epsilon}\backslash K)^c \times \mathbb{R}, \end{cases} \tag{4.55}$$

we have $W^{\tau_0-\epsilon}(x,t) \geq 0$, we obtain $W^{\tau}(x,t) \geq 0$ for $\tau \in (\tau_0 - \epsilon, \tau_0)$ which contradicts the definition of τ_0, so that $\tau_0 = 0$.

4.2.3 Fractional parabolic Monge-Ampére equations in the whole space

In section 9.7, we want to show the monotonicity of Monge-Ampere operator in whole space and depends on x_n only.

where the parabolic Monge Ampere equation is given by:

$$\frac{\partial u}{\partial t}(x,t) - D_s^{\theta} u(x,t) = f(t,|x|,u), \quad (x,t) \in \mathbb{R}^n \times (-\infty,\infty) \tag{4.56}$$

We want to prove the following theorem:

Theorem 4.18. *Let* $u(x,t) \in \left(C_{loc}^{1,1}(\Omega) \cap C(\bar{\Omega})\right) \cap \mathcal{L}_{2s} \times [\underline{t}, T]$ *be a solution of*

$$\frac{\partial u}{\partial t}(x,t) - D_s^{\theta} u(x,t) = f(t,|x|,u), \quad (x,t) \in \mathbb{R}^n \times (-\infty,\infty) \tag{4.57}$$

with condition

$$|u(x,t)| \leq 1$$

and

$$u((x', x_n), t) \to \pm 1 \tag{4.58}$$

uniformly in $x' = (x_1, \cdots, x_{n-1})$*. Also,* $f(t,|x|,u)$ *is non-increasing near* $u(x,t) = \pm 1$*. Then* u *must be strictly increasing with respect to* x_n*, and it depends on* x_n *only.*

Step 1: *Begin sliding* Ω^{τ} *downward* τ *units along the* x_n *axis*

So then

$$|x| < |x^{\tau}|$$

We will show that for τ sufficiently close to $\widetilde{\tau}$, that is, when τ is sufficiently large, D^{τ} is narrow, we have

$$W^{\tau}(x,t) \leq 0, \;\; (x,t) \in D^{\tau} \times \mathbb{R}$$

In the process to show $u(x)$ here is monotone using sliding method, we begin sliding

Ω^τ downward τ units along the x_n axis, we will show that for τ sufficiently close to $\widetilde{\tau}$, that is, when τ is sufficiently large, D^τ is narrow, we have $W^\tau(x) = u(x) - u^\tau(x) \le 0$. Then we give the plane an initial position to start sliding.

To show that for τ sufficiently large,

$$W^\tau(x,t) \le 0,\ \ (x,t) \in \mathbb{R}^n \times \mathbb{R} \tag{4.59}$$

Otherwise,

$$\sup_{\mathbb{R}^n \times \mathbb{R}} W^\tau(x,t) = A > 0$$

then there exists a sequence $\{x^k, t^k\} \subset \mathbb{R}^n \times \mathbb{R}$ such that

$$W^\tau(x^k, t^k) \to A > 0$$

as $k \to \infty$.

Denote $x^k = (x_1^k, x_2^k, \cdots, x_n^k)$. Let $\eta \in C_0^\infty$:

$$\eta(x,t) = \begin{cases} 1, & if\ |x|, |t| < 1, \\ 0, & if\ |x|, |t| \ge 2 \end{cases}$$

So $\max_{\mathbb{R}^n \times \mathbb{R}} \eta(x,t) = 1$. Set

$$\psi_k(x,t) = \eta(x - x^k, t - t^k)$$

There exists a sequence $\{\varepsilon_k\} \to 0$ such that

$$W^\tau(x^k, t^k) + \varepsilon_k \psi_k(x^k, t^k) = A$$

Set

$$U_k^\tau(x^k,t^k)=W^\tau(x^k,t^k)+\varepsilon_k\psi_k(x^k,t^k)$$

Then there exists $(\bar{x}^k,\bar{t}^k)\in B_1(x^k)\times B_1(t^k)$ such that

$$U_k^\tau(\bar{x}^k,\bar{t}^k)=\max_{\mathbb{R}^n\times\mathbb{R}}U_k^\tau(x^k,t^k)=A$$

Therefore

$$\frac{\partial U_k^\tau}{\partial t}(\bar{x}^k,\bar{t}^k)=0$$

We have

$$\varepsilon_k=A-W^\tau(x^k,t^k)$$

Therefore

$$\frac{\partial W^\tau}{\partial t}(\bar{x}^k,\bar{t}^k)\sim\varepsilon_k$$

By the definition of D_s^θ, we have

$$D_s^\theta(W^\tau+\varepsilon_k\psi_k)(\bar{x}^k,\bar{t}^k)=D_s^\theta(U_k^\tau)(\bar{x}^k,\bar{t}^k)\approx-(-\triangle)^s(U_k^\tau)(\bar{x}^k,\bar{t}^k)\leq -c_0$$

We also have

$$\begin{aligned}
& D_s^\theta(W^\tau+\varepsilon_k\psi_k)(\bar{x}^k,\bar{t}^k)\\
\geq\ & \inf\{P.V\int_{\mathbb{R}^n}\frac{W^\tau(y,\bar{t}^k)-W^\tau(\bar{x}^k,\bar{t}^k)}{|A^{-1}(y-\bar{x}^k)|^{n+2s}}dy\}+\inf\{P.V\int_{\mathbb{R}^n}\frac{\varepsilon_k\psi_k(y,\bar{t}^k)-\varepsilon_k\psi_k(\bar{x}^k,\bar{t}^k)}{|A^{-1}(y-\bar{x}^k)|^{n+2s}}dy\}\\
=\ & D_s^\theta W^\tau(\bar{x}^k,\bar{t}^k)-c\varepsilon_k
\end{aligned}$$

We also have

$$
\begin{aligned}
& D_s^\theta W^\tau(\bar{x}^k,\bar{t}^k) - c\varepsilon_k \\
= \;& \inf\{P.V\int_{\mathbb{R}^n} \frac{W^\tau(y,\bar{t}^k) - W^\tau(\bar{x}^k,\bar{t}^k)}{|A^{-1}(y-\bar{x}^k)|^{n+2s}}dy\} - c\varepsilon_k \\
= \;& P.V\int_{\mathbb{R}^n} \frac{W^\tau(y,\bar{t}^k) - W^\tau(\bar{x}^k,\bar{t}^k)}{|A^{-1}(y-\bar{x}^k)|^{n+2s}}dy - \varepsilon_A - c\varepsilon_k \\
= \;& P.V\int_{\mathbb{R}^n} \frac{u(y,\bar{t}^k) - u(\bar{x}^k,\bar{t}^k)}{|A^{-1}(y-\bar{x}^k)|^{n+2s}}dy - P.V\int_{\mathbb{R}^n} \frac{u^\tau(y,\bar{t}^k) - u^\tau(\bar{x}^k,\bar{t}^k)}{|A^{-1}(y-\bar{x}^k)|^{n+2s}}dy - \varepsilon_A - c\varepsilon_k \\
\geq \;& \inf\{P.V\int_{\mathbb{R}^n} \frac{u(y,\bar{t}^k) - u(\bar{x}^k,\bar{t}^k)}{|A^{-1}(y-\bar{x}^k)|^{n+2s}}dy\} - \inf\{P.V\int_{\mathbb{R}^n} \frac{u^\tau(y,\bar{t}^k) - u^\tau(\bar{x}^k,\bar{t}^k)}{|A^{-1}(y-\bar{x}^k)|^{n+2s}}dy\} - 2\varepsilon_A - c\varepsilon_k \\
= \;& D_s^\theta u(\bar{x}^k,\bar{t}^k) - D_s^\theta u^\tau(\bar{x}^k,\bar{t}^k) - 2\varepsilon_A - c\varepsilon_k \\
= \;& \frac{\partial u}{\partial t} - f(t,|\bar{x}^k|,u) - \frac{\partial u^\tau}{\partial t} + f(t,|\bar{x}^k|,u^\tau) - 2\varepsilon_A - c\varepsilon_k \\
= \;& \frac{\partial W^\tau}{\partial t} + f(t,|\bar{x}^k|,u^\tau) - f(t,|\bar{x}^k|,u) - 2\varepsilon_A - c\varepsilon_k \\
= \;& f(t,|\bar{x}^k|,u^\tau) - f(t,|\bar{x}^k|,u) - 2\varepsilon_A - c\varepsilon_k
\end{aligned}
$$

When τ is sufficiently large, we have either

1. $u^\tau(\bar{x}^k,\bar{t}^k)$ is close to 1 or

2. $u(\bar{x}^k,\bar{t}^k)$ is close to -1.

Since $u(\bar{x}^k,\bar{t}^k) > u^\tau(\bar{x}^k,\bar{t}^k)$, in case 1, both $u(\bar{x}^k,\bar{t}^k)$ and $u^\tau(\bar{x}^k,\bar{t}^k)$ are close to 1, while in case 2, both $u(\bar{x}^k,\bar{t}^k)$ and $u^\tau(\bar{x}^k,\bar{t}^k)$ are close to -1. Hence in any case, we can apply the monotonicity of f to derive that

$$f(t,|\bar{x}^k|,u^\tau) \geq f(t,|\bar{x}^k|,u)$$

Then we have

$$D_s^\theta W^\tau(\bar{x}^k,\bar{t}^k) - c\varepsilon_k \geq -2\varepsilon_A - c\varepsilon_k \to 0$$

Finally we derived

$$-c_0 \geq D_s^\theta(W^\tau + \varepsilon_k\psi_k)(\bar{x}^k, \bar{t}^k) \geq D_s^\theta W^\tau(\bar{x}^k, \bar{t}^k) - c\varepsilon_k \geq 0$$

which is a contradiction. So we verified (4.59)

The inequality provides a starting point, from which we can carry out the sliding. Now we decrease τ as long as $W^\tau(x,t) \leq 0$ holds to its limiting position. Define

$$\tau_0 = \inf\{\tau \mid W^\tau(x,t) \leq 0, \forall(x,t) \in \mathbb{R}^n \times \mathbb{R}\}$$

We show $\tau_0 = 0$ by contradiction, we would divide the proof into two cases, one is $|x_n| \leq M$, the other is $|x_n| \geq M$, in both cases we would show the limiting position is $\tau = 0$. After we have completed the second step, we would prove $\forall\tau > 0$, $W^\tau(x,t) < 0$, thus we have completed proof of monotonicity of solution of parabolic Monge-Ampere equation in the whole space. In the last section, we would show $u(x,t)$ depends on x_n only, that is, $u(x,t) = u(x_n,t)$.

Suppose $\tau_0 > 0$, we will show that Ω^τ can be slid upward a little bit more. First of all, we show $\sup_{-M \leq x_n \leq M} W^{\tau_0}(x,t) < 0$, if it does not hold, then $\sup_{-M \leq x_n \leq M} W^{\tau_0}(x,t) = 0$, then there exists a sequence $\{x^k, t^k\} \subset \mathbb{R}^{n-1} \times [-M, M] \times \mathbb{R}$ such that

$$W^{\tau_0}(x^k, t^k) \to 0 \tag{4.60}$$

as $k \to \infty$

Denote $x^k = (x_1^k, x_2^k, \cdots, x_n^k)$. Let $\eta \in C_0^\infty$:

$$\eta(x,t) = \begin{cases} 1, & if\ |x|, |t| < 1, \\ 0, & if\ |x|, |t| \geq 2 \end{cases}$$

So $\max_{\mathbb{R}^n \times \mathbb{R}} \eta(x,t) = 1$. Set

$$\psi_k(x,t) = \eta(x - x^k, t - t^k)$$

There exists a sequence $\{\varepsilon_k\} \to 0$ such that

$$W^{\tau_0}(x^k, t^k) + \varepsilon_k \psi_k(x^k, t^k) = 0$$

Set

$$U_k^{\tau_0}(x^k, t^k) = W^{\tau_0}(x^k, t^k) + \varepsilon_k \psi_k(x^k, t^k)$$

Since we have

$$U_k^{\tau_0}(x,t) = W^{\tau_0}(x,t) \leq 0, x \in \mathbb{R}^n \backslash B_2(x^k), t \in \mathbb{R} \backslash B_2(t^k)$$

and

$$W^{\tau_0}(x^k, t^k) + \varepsilon_k \psi_k(x^k, t^k) = 0$$

Then there exists $(\bar{x}^k, \bar{t}^k) \in B_1(x^k) \times B_1(t^k)$ such that

$$U_k^{\tau_0}(\bar{x}^k, \bar{t}^k) = \max_{\mathbb{R}^n \times \mathbb{R}} U_k^{\tau_0}(x^k, t^k) = 0$$

By the definition of D_s^θ, we have

$$D_s^\theta(W^{\tau_0} + \varepsilon_k\psi_k)(\bar{x}^k, \bar{t}^k) = D_s^\theta(U_k^{\tau_0})(\bar{x}^k, \bar{t}^k) \approx -(-\triangle)^s(U_k^{\tau_0})(\bar{x}^k, \bar{t}^k) \leq 0$$

On one hand, similar to the proof in Step 1, we have

$$D_s^\theta(W^{\tau_0} + \varepsilon_k\psi_k)(\bar{x}^k, t^o) \geq D_s^\theta W^{\tau_0}(\bar{x}^k, t^o) - c\varepsilon_k \tag{4.61}$$

We also have

$$\begin{aligned}
& D_s^\theta W^{\tau_0}(\bar{x}^k, \bar{t}^k) - c\varepsilon_k \\
= \; & \inf\{P.V \int_{\mathbb{R}^n} \frac{W^{\tau_0}(y, \bar{t}^k) - W^{\tau_0}(\bar{x}^k, \bar{t}^k)}{|A^{-1}(y - \bar{x}^k)|^{n+2s}} dy\} - c\varepsilon_k \\
\geq \; & \inf\{P.V \int_{\mathbb{R}^n} \frac{W^{\tau_0}(y, \bar{t}^k)}{|A^{-1}(y - \bar{x}^k)|^{n+2s}} dy\} - \inf\{P.V \int_{\mathbb{R}^n} \frac{W^{\tau_0}(\bar{x}^k, \bar{t}^k)}{|A_k^{-1}(y - \bar{x}^k)|^{n+2s}} dy\} - \varepsilon_k - c\varepsilon_k \\
\geq \; & -\varepsilon_k - c\varepsilon_k \to 0
\end{aligned} \tag{4.62}$$

Denote $u_k(x,t) = u(x^k, t^k)$, $W_k^{\tau_0}(x,t) = W^{\tau_0}(x^k, t^k)$

Since u is uniformly continuous, by the Arzela-Ascoli Theorem, we have

$$u_k(x,t) \to u_\infty(x,t) \; uniformly \; in \; \mathbb{R}^n \times \mathbb{R}, \; as \; k \to \infty$$

Let $k \to \infty$, by the continuity of f, and from (4.61) and (4.62), we have

$$W_k^{\tau_0}(x,t) \to 0, x \in (B_2(0))^c \; uniformly$$

Then

$$u_\infty(x,t) - u_\infty^{\tau_0}(x,t) \equiv 0, x \in (B_2(0))^c$$

For all $m \in \mathbb{N}$, we have

$$u_\infty(x', x_n) = u_\infty(x', x_n + \tau_0) = u_\infty(x', x_n + 2\tau_0) = \cdots = u_\infty(x', x_n + m\tau_0)$$

If x_n is sufficiently negative and m is sufficiently large, then

$$u_\infty(x', x_n) \to -1$$

and

$$u_\infty(x', x_n + m\tau_0) \to 1$$

This is a contradiction, therefore, $\sup_{-M \leq x_n \leq M} W^{\tau_0}(x,t) < 0$ must be true.

Since $\sup_{-M \leq x_n \leq M} W^{\tau_0}(x,t) < 0$, so there exists a $\delta > 0$ such that

$$\sup_{-M \leq x_n \leq M} W^{\tau}(x,t) \leq 0, \, \forall \tau \in (\tau_0 - \delta, \, \tau_0], \; |x_n| \leq M$$

which contradicts the definition of τ_0, therefore, we have $\tau_0 = 0$.

Now we only need to prove when $|x_n| \geq M$, $\tau_0 > 0$

$$W^{\tau}(x,t) \leq 0, \, \forall \tau \in (\tau_0 - \delta, \, \tau_0],$$

Since $\sup_{-M \leq x_n \leq M} W^{\tau_0}(x,t) < 0$, so there exists a $\delta > 0$ such that

$$\sup_{-M \leq x_n \leq M} W^{\tau}(x,t) \leq 0, \, \forall \tau \in (\tau_0 - \delta, \, \tau_0], \; |x_n| \leq M$$

which contradicts the definition of τ_0, therefore, we have $\tau_0 = 0$.

Now we only need to prove when $|x_n| \geq M$, $\tau_0 > 0$

$$W^\tau(x,t) \leq 0, \forall \tau \in (\tau_0 - \delta, \tau_0],$$

We use the same process of proof by contradiction to prove it, which is no difference as previous steps. Here I would not write it again. So we have proved $\tau_0 = 0$.

Then we will show that $u(x)$ depends on x_n only.

If we replace $u^\tau(x)$ by $u(x + \tau\nu)$, the argument still holds according to the above process, where $\nu = (\nu_1, \nu_2, \nu_3, \cdots, \nu_n)$ with $\nu_n > 0$ is an arbitrary vector that points upward. With the similar arguments as in Step 1 and Step 2, we can obtain that, for each of such ν,

$$u(x + \tau\nu) > u(x)$$

$\forall \tau > 0$ Let $\nu_n \to 0$, by continuity of u, we have that for arbitrary ν with $\nu_n = 0$

$$u(x + \tau\nu) \geq u(x)$$

By replacing ν by $-\nu$, we also have

$$u(x) \geq u(x + \tau\nu)$$

for arbitrary ν with $\nu_n = 0$, So we have

$$u(x + \tau\nu) = u(x) \tag{4.63}$$

(4.63) means that u is independent of $x' = (x_1, x_2, \cdots, x_{n-1})$. Therefore, $u(x) = u(x_n)$.

Part 1: Method of Moving Planes and Its Applications: Radial symmetry and monotonicity of solutions for fractional elliptic and parabolic equations and systems

5 Method of Moving Planes and Its Applications: Radial symmetry and monotonicity of solutions for fractional elliptic and parabolic equations

5.1 Radial Symmetry of solutions of fractional elliptic equations

In the following section, as a preliminary to proving the main equation, we will prove the solution of the elliptic fractional system is radially symmetric using the method of moving planes which I have explained in the previous sections.

Theorem 5.1. *(Radial Symmetry of solution of elliptic fractional equation)*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(u(x)), & x \in B_1(0), \\ u(x) \geq 0, & x \in B_1(0), \\ u(x) \equiv 0, & x \notin B_1(0), \end{cases} \tag{5.1}$$

where B_1 is a unit ball.

Let $0 < \alpha, \beta < 2$, and suppose that $u(x) \in \left(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\right)$ is positive bounded classical solutions of equation (5.1)*, and assume $f(|x|, u)$ satisfies the following assumptions:*

(X1) $f(|x|, u)$ are decreasing in $|x|$.

(X2) Assume that f is uniformly Lipschitz continuous in u. i.e:

$$|f(|x|, u_1) - f(|x|, u_2)| \leq c|u_1 - u_2|, \ \forall x \in B_1(0),$$

then $u(x)$ is radially symmetric and monotone decreasing about the origin, i.e.

$$u(x) = u(|x|)$$

.

$$u(x_1) > u(x_2), |x_1| < |x_2|$$

.

5.1.1 Basic set-up

For any given $\lambda \in \mathbb{R}$, let

$$T_\lambda = \{x \in \mathbb{R}^n \mid x_1 = \lambda \text{ for } \lambda \in \mathbb{R}\}$$

be the moving planes,

$$\Sigma_\lambda := \{x \in \mathbb{R}^n \mid x_1 < \lambda\}$$

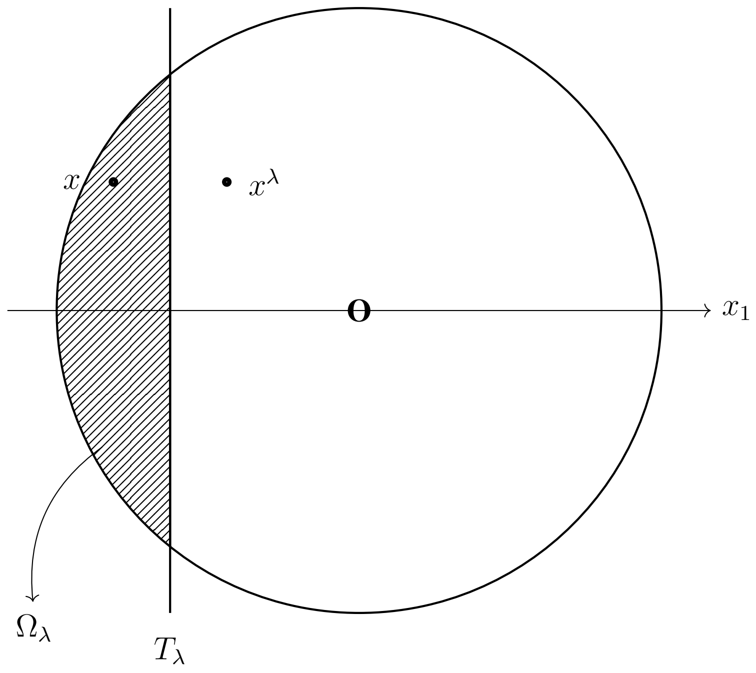


Figure 4: Moving Planes on the Unit Ball

be the region to the left of the plane, and

$$x^\lambda = (2\lambda - x_1, x_2, ..., x_n)$$

be the reflection of x about the plane T_λ. and

$$\Omega_\lambda = \Sigma_\lambda \cap B_1(0)$$

be the intersection of $B_1(0)$ and Σ_λ.

Assume that $u(x)$ is positive solution of equation (5.1). We compare the values of $u(x)$ with

$$u_\lambda(x) = u(x^\lambda)$$

Let

$$w_\lambda(x) = u_\lambda(x) - u(x),$$

Step 1: *Begin moving the plane from near the left end of $B_1(0)$ along the x_1 axis, but do not reach origin*, So then

$$|x^\lambda| < |x|$$

we derive

$$f(|x^\lambda|, u_\lambda) - f(|x|, u_\lambda) \geq 0$$

We deduce from the equation (5.1) and (X1), (X2) that w_λ satisfies

$$\begin{aligned} & (-\Delta)^s w_\lambda(x) \\ = \ & \frac{f(|x|, u_\lambda) - f(|x|, u)}{u_\lambda(x) - u(x)} w_\lambda(x) \\ := \ & c_\lambda(x) w_\lambda(x), \end{aligned} \tag{5.2}$$

where $c_\lambda(x) = \frac{f(|x|,u_\lambda)-f(t,|x|,u)}{u_\lambda(x)-u(x)}$ is bounded.

Apparently, Ω_λ is a narrow region in the x_1 direction for λ very close to -1. For further application of Narrow Region Principle, We will prove the Narrow region theorems in Elliptic problem.

5.2 *Step 1: show* $w_\lambda(x) \geq 0$

To show $w_\lambda(x) \geq 0$, we would prove a Narrow region theorem for an elliptic problem:

5.2.1 Narrow region theorem for an elliptic problem

Now, in order to prove the radial symmetry of the solution of fractional elliptic equation (5.3), in step 1, we first show $w_\lambda(x) \geq 0$. In step 2, move the plane continuously to the right until its limiting position as long as $w_\lambda(x) \geq 0$ holds.

Theorem 5.2. *(Narrow region principle for an elliptic problem) Let $\Omega_\lambda = \Sigma_\lambda \cap B_1(0)$, Ω_λ is a bounded narrow region in Σ_λ, assume that $u(x) \in \left(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\right)$, if*

$$\begin{cases} (-\Delta)^s u(x) = f(u(x)), & u(x) > 0,\ x \in B_1(0), \\ u(x) \equiv 0, & x \notin B_1(0), \\ (-\Delta)^s w_\lambda(x) = c_\lambda w_\lambda(x), & x \in \Omega_\lambda \end{cases} \tag{5.3}$$

then for λ sufficiently close to -1, we have

$$w_\lambda(x) \geq 0,\ x \in \Omega_\lambda \tag{5.4}$$

Suppose otherwise, (5.4) does not hold, then w_λ is negative somewhere, hence there exists an $x^o \in \Omega_\lambda$ such that

$$w_\lambda(x^o) = \min_{\Omega_\lambda} w_\lambda(x) < 0$$

It follows that

$$\begin{aligned} & (-\Delta)^s w_\lambda(x^o) \\ = \ & C_{n,s} PV \int_{\mathbb{R}^n} \frac{w_\lambda(x^o) - w_\lambda(y)}{|x^o - y|^{n+2s}} dy \\ = \ & C_{n,s} PV \{ \int_{\Sigma_\lambda} \frac{w_\lambda(x^o) - w_\lambda(y)}{|x^o - y|^{n+2s}} dy + \int_{\widetilde{\Sigma_\lambda}} \frac{w_\lambda(x^o) - w_\lambda(y)}{|x^o - y|^{n+2s}} dy \} \\ = \ & C_{n,s} PV \{ \int_{\Sigma_\lambda} \frac{w_\lambda(x^o) - w_\lambda(y)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_\lambda} \frac{w_\lambda(x^o) - w_\lambda(y^\lambda)}{|x^o - y^\lambda|^{n+2s}} dy \} \\ = \ & C_{n,s} PV \{ \int_{\Sigma_\lambda} \frac{w_\lambda(x^o) - w_\lambda(y)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_\lambda} \frac{w_\lambda(x^o) + w_\lambda(y)}{|x^o - y^\lambda|^{n+2s}} dy \} \\ \leq \ & C_{n,s} \{ \int_{\Sigma_\lambda} \frac{w_\lambda(x^o) - w_\lambda(y)}{|x^o - y^\lambda|^{n+2s}} dy + \frac{w_\lambda(x^o) + w_\lambda(y)}{|x^o - y^\lambda|^{n+2s}} dy \} \\ = \ & C_{n,s} \int_{\Sigma_\lambda} \frac{2 w_\lambda(x^o)}{|x^o - y^\lambda|^{n+2s}} \end{aligned}$$

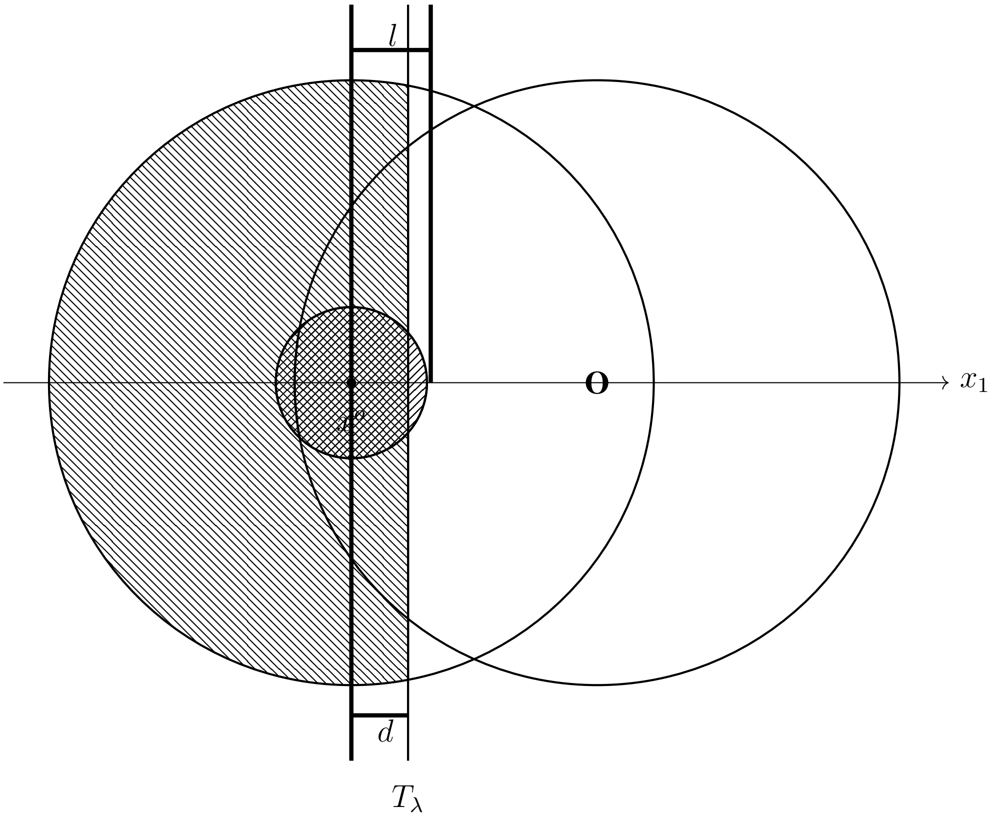


Figure 5: Choose l

To estimate the integral above, we may first consider the extreme case where $x_1^o = \lambda$. Then it is easy to see that

$$\int_{\Sigma_{\lambda_0}} \frac{1}{|x^o - y^\lambda|} dy = \infty$$

This suggests that we may obtain values arbitrarily large by integrating on a domain that is sufficiently close to the hyperplane $P := \{y \in \mathbb{R}^n \mid y_1 = \lambda\}$.

Denote

$$d = dis[x^0, T_\lambda] \leq width(\Omega_\lambda)$$

Choose a ball centered at x^o with radius l, also choose a unit ball centered at x^o, as Figure 5 shows, it is easy to see $d \leq l$.

Since

$$d \leq |x^o - y^\lambda| \leq 2d$$

Then

$$\frac{1}{|x^o - y^\lambda|} \geq \frac{1}{2d}$$

Lemma 5.3. *Here we want to prove a lemma that the integral* $\int_{\Sigma_{\lambda_0}} \frac{1}{|x^o-y^\lambda|}dy$ *is greater than or equal to* $\frac{c}{d^{2s}}$*, so as to use this results in subsequent sections:*

$$\begin{aligned} & \int_{\Sigma_{\lambda_0}} \frac{1}{|x^o - y^\lambda|}dy \\ \geq\ & \frac{c}{d^{2s}} \end{aligned}$$

Proof.

$$\begin{aligned} & \int_{\Sigma_{\lambda_0}} \frac{1}{|x^o - y^\lambda|}dy \\ \geq\ & \int_{\Sigma_{\lambda_0}\cap(B_1(x^o))\setminus(B_l(x^o))} \frac{1}{|x^o - y^\lambda|}dy \\ \geq\ & \int_D \frac{1}{|x^o - y^\lambda|^{n+2s}} \\ \geq\ & vol(D) \cdot \frac{1}{(2d)^{n+2s}} \\ \geq\ & c(2d)^n \cdot \frac{1}{(2d)^{n+2s}} \\ =\ & \frac{c}{d^{2s}} \end{aligned}$$

With the region D shown in Figure 6. □

Following from

$$(-\triangle)^s w_\lambda(x^o) \leq C_{n,s} \int_{\Sigma_{\lambda_0}} \frac{2w_\lambda(x^o)}{|x^o - y^\lambda|^{n+2s}}$$

we deduce

$$(-\Delta)^s w_\lambda(x^o) \leq \frac{c}{d^{2s}} w_\lambda(x^o) \tag{5.5}$$

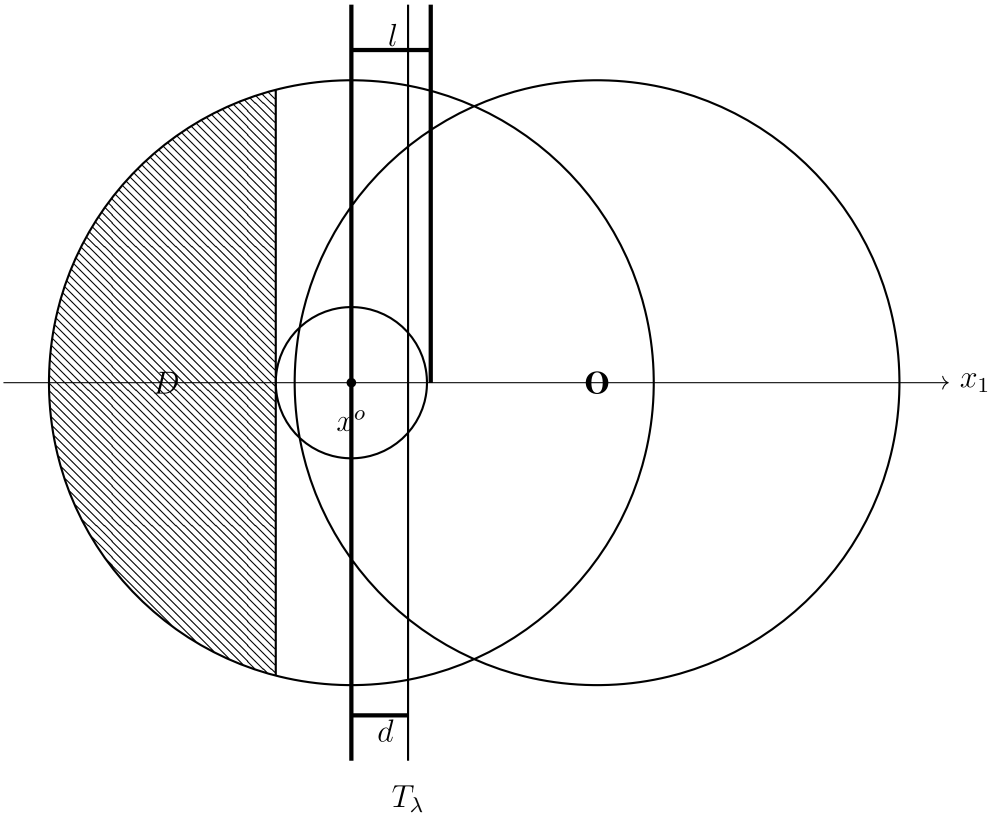


Figure 6: The region of D

From (5.3), we deduce

$$c_\lambda(x^o)w_\lambda(x^o) \leq \frac{c}{d^{2s}}w_\lambda(x^0)$$

Then we derive

$$\frac{c}{d^{2s}} \leq c_\lambda(x^o)$$

Which is a contradiction for d sufficiently small, since $c_\lambda(x)$ is bounded.

Therefore, (5.4) must be valid.

So far, we have proved the theorem 5.2.

(In Figure 5, the region of north-west pattern is $\Sigma_{\lambda_0} \cap B_1(x^o) \setminus B_l(x^o)$. In Figure 6, the region of north-west pattern is D.)

5.3 *Step 2*: *Move the plane continuously to the right until its limiting position as long as* $w_\lambda(x) \geq 0$ *holds*.

Define

$$\lambda_0 = \sup\{\lambda \leq 0 \mid w_\mu(x) \geq 0, \forall x \in \Omega_\mu, \mu \leq \lambda\}$$

In this part, we show that

$$\lambda_0 = 0$$

Suppose

$$\lambda_0 < 0$$

we show that the plane T_{λ_0} can be moved further to the right. To be more rigorous, there exists some $\epsilon > 0$, such that for any $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, we have $w_\lambda(x) \geq 0$, $x \in \Sigma_{\lambda_0}$

This is a contradiction with the definition of λ_0. Hence we must have

$$\lambda_0 = 0$$

5.3.1 Show $w_{\lambda_0}(x) > 0$ for $x \in \Omega_{\lambda_0}$

Suppose that $\lambda_0 < 0$, then the reflection of the curved part of $\partial\Omega_{\lambda_0}$ falls inside $B_1(0)$ and $w_{\lambda_0}(x) \geq 0$ for $x \in \partial\Omega_{\lambda_0}$. (See Figure 7)

We want to show $w_{\lambda_0}(x) > 0$ for $x \in \Omega_{\lambda_0}$:

Otherwise, $\exists x^o \in \Omega_{\lambda_0}$ such that $w_{\lambda_o}(x^o) = 0$

Since $w_{\lambda_0}(x) \geq 0$ inside Ω_{λ_o}, so for $w_{\lambda_0}(x^o) = 0$, we know x^o is the minimum.

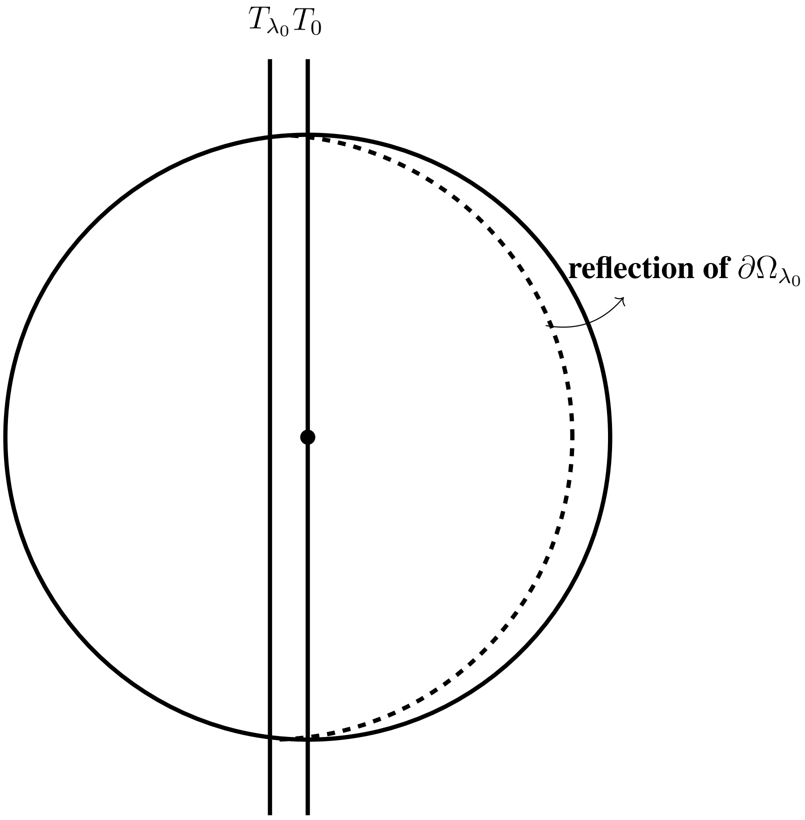


Figure 7: reflection of the curved part of $\partial\Omega_{\lambda_0}$

Following from (5.3), we then derive

$$\begin{aligned} & (-\Delta)^s w_{\lambda_0}(x^o) \\ := \;& c_{\lambda_0}(x^o) w_{\lambda_0}(x^o) \\ := \;& 0 \end{aligned}$$

It follows that

$$\begin{aligned} & 0 = (-\Delta)^s w_{\lambda_0}(x^o) \\ = \;& C_{n,s} PV \int_{\mathbb{R}^n} \frac{-w_{\lambda_0}(y)}{|x^o - y|^{n+2s}} dy \\ = \;& C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-w_{\lambda_0}(y)}{|x^o - y|^{n+2s}} dy + \int_{\widetilde{\Sigma_{\lambda_0}}} \frac{-w_{\lambda_0}(y)}{|x^o - y|^{n+2s}} dy \} \\ = \;& C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-w_{\lambda_0}(y)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_{\lambda_0}} \frac{-w_{\lambda_0}(y^\lambda)}{|x^o - y^\lambda|^{n+2s}} dy \} \\ = \;& C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-w_{\lambda_0}(y)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_{\lambda_0}} \frac{w_{\lambda_0}(y)}{|x^o - y^\lambda|^{n+2s}} dy \} \\ = \;& C_{n,s} PV \int_{\Sigma_{\lambda_0}} w_{\lambda_0}(y) \{ \frac{1}{|x^o - y^\lambda|^{n+2s}} - \frac{1}{|x^o - y|^{n+2s}} \} dy \\ \leq \;& 0 \end{aligned} \tag{5.6}$$

since

$$\frac{1}{|x^o - y^\lambda|^{n+2s}} - \frac{1}{|x^o - y|^{n+2s}} < 0$$

and

$$w_{\lambda_0}(y) \geq 0$$

This implies that

$$w_{\lambda_0}(y) \equiv 0, \;\; y \in \Omega_{\lambda_0}. \tag{5.7}$$

We derive a contradiction since the plane T_{λ_0} did not reach the origin. If we take a point $\bar{x}$ on the curved part $\partial\Omega_{\lambda_0}$, then it's reflection point $\bar{x}^{\lambda_0}$ is in the interior of the ball, and hence $u(\bar{x}^{\lambda_0}) > 0$, therefore, $w_{\lambda_0}(\bar{x}) = u(\bar{x}^{\lambda_0}) - u(\bar{x}) > 0$, which contradicts (5.7).

We conclude $w_{\lambda_0}(x) > 0$ for every $x \in \Omega_{\lambda_o}$.

Next, we want to further derive

$$w_{\lambda_0}(x) \geq c_o > 0, \;\; x \in \Omega_{\lambda_0 - \delta} \tag{5.8}$$

I will prove (5.8) by contradiction.

5.3.2 Show $\bar{w}(y) \equiv 0, \;\; \forall y \in \Omega_{\lambda_o - \delta}$

Proof: If (5.8) is violated, then $\exists x_k \in \Omega_{\lambda_0 - \delta}$ such that $w_{\lambda_0}(x_k) \to 0$

Without loss of generality, by Bolzano-Weierstrass theorem, $\exists$ subsequences of x_k, here we still denote this subsequence by x_k, such that $x_k \to x^o \in \Omega_{\lambda_o - \delta}$ Let

$$w_k(x) = w_{\lambda_0}(x_k)$$

From (5.2), we derive:

$$\begin{aligned}
& (-\Delta)^s w_k(x) \\
= \;& f(|x^{\lambda_0}|, u_{\lambda_0}) - f(|x|, u) \\
= \;& f(|x^{\lambda_0}|, u_{\lambda_0}) - f(|x|, u_{\lambda_0}) + f(|x|, u_{\lambda_0}) - f(|x|, u) \\
\geq \;& f(|x|, u_{\lambda_0}) - f(|x|, u) \\
= \;& \frac{f(|x|, u_{\lambda_0}) - f(|x|, u)}{u_{\lambda_0}(x) - u(x)} w_k(x, t) \\
:= \;& c_{\lambda_0}(x) w_k(x),
\end{aligned}$$

So w_k satisfies

$$(-\triangle)^s w_k(x) = c_{\lambda_0}(x) w_k(x) \tag{5.9}$$

By regularity theory for parabolic equations [45], there exists some functions $\bar{w}(x)$ and $\bar{c}(x)$ such that $k \to \infty$, $w_k(x)$ converges uniformly to $\bar{w}(x)$ for $x \in \Omega_{\lambda_o}$,

and $\bar{w}$ satisfies:

$$(-\triangle)^s \bar{w} = \bar{c}(x)\bar{w}(x)$$

Since

$$w_k(x) = w_{\lambda_0}(x_k) \to 0$$

$$\bar{w}(x^o) = 0$$

and

$$\bar{w} \geq 0$$

So x^o is the minimum. Also from (5.18), we derive,

$$\begin{aligned}
& (-\Delta)^s \bar{w}(x^o) \\
\geq \ & f(|x^o|, u_{\lambda_0}) - f(|x^o|, u) \\
= \ & \frac{f(|x^o|, u_{\lambda_0}) - f(|x^o|, u)}{\bar{u}_{\lambda_0}(x^o) - \bar{u}(x^o)} \bar{w}(x^o) \\
:= \ & c_\lambda(x^o)\bar{w}(x^o),
\end{aligned}$$

we derive

$$(-\triangle)^s \bar{w}(x^o) = 0$$

It follows that

$$
\begin{aligned}
& 0 = (-\Delta)^s \bar{w}(x^o) \\
= \;& C_{n,s} PV \int_{\mathbb{R}^n} \frac{-\bar{w}(y)}{|x^o - y|^{n+2s}} dy \\
= \;& C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{w}(y)}{|x^o - y|^{n+2s}} dy + \int_{\widetilde{\Sigma_{\lambda_0}}} \frac{-\bar{w}(y)}{|x^o - y|^{n+2s}} dy \} \\
= \;& C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{w}(y)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_{\lambda_0}} \frac{-\bar{w}(y^\lambda)}{|x^o - y^\lambda|^{n+2s}} dy \} \\
= \;& C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{w}(y)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_{\lambda_0}} \frac{\bar{w}(y)}{|x^o - y^\lambda|^{n+2s}} dy \} \\
= \;& C_{n,s} PV \int_{\Sigma_{\lambda_0}} \bar{w}(y) \{ \frac{1}{|x^o - y^\lambda|^{n+2s}} - \frac{1}{|x^o - y|^{n+2s}} \} dy \\
\leq \;& 0
\end{aligned}
\tag{5.10}
$$

since

$$\frac{1}{|x^o - y^\lambda|^{n+2s}} - \frac{1}{|x^o - y|^{n+2s}} < 0$$

and

$$\bar{w}(y) \geq 0$$

This implies that

$$\bar{w}(y) \equiv 0, \;\; \forall y \in \mathbb{R}^n$$

5.3.3 Show $\bar{u}(x) \equiv 0$

Let $u_k(x) = u(x_k)$, then by (5.18), we have

$$(-\Delta)^s u_k(x) = f(u_k(x))$$

By regularity theory for parabolic equations [45], there exists some functions $\bar{u}(x)$ such that as $k \to \infty$, $u_k(x)$ converges uniformly to $\bar{u}(x)$ for $x \in B_1(0)$, $f(u)$ converges uniformly to $\bar{f}(u)$ for $x \in \Omega_{\lambda_o}$

and

$$(-\Delta)^s \bar{u}(x) = \bar{f}(\bar{u}(x))$$

Since

$$f(0) \geq 0$$

Thus

$$\bar{f}(0) \geq 0$$

In order to show that

$$\bar{u}(x) \equiv 0, x \in \mathbb{R}^n, \tag{5.11}$$

we apply the following:

Lemma 5.4. *(Strong Maximum Principle for* $(-\Delta)^s \bar{u} = \bar{f}(\bar{u})$*.*

Assume that $\bar{u}(x) \in [C^{1,1}_{loc}(\Omega_\lambda) \cap C(\bar{\Omega_\lambda}) \cap \mathcal{L}_{2s}]$

$$\begin{cases} (-\Delta)^s \bar{u}(x) = \bar{f}(\bar{u}), & x \in \Omega_\lambda, \\ \bar{u}(x) \geq 0, & x \in \Omega_\lambda \end{cases} \tag{5.12}$$

we have either

$$\bar{u}(x) > 0, x \in B_1(0)$$

or

$$\bar{u}(x) \equiv 0, x \in \mathbb{R}^n$$

Proof. First, if $\bar{u}(x) \geq 0$ and $\bar{u}(x^o) = 0$, x^o then is a minimum, if $\bar{u}(x) \not\equiv 0$, then

$$\begin{aligned}
& (-\Delta)^s \bar{u}(x^o) \\
= \; & C_{n,s} PV \int_{\mathbb{R}^n} \frac{-\bar{u}(y)}{|x^o - y|^{n+2s}} dy \\
= \; & C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{u}(y)}{|x^o - y|^{n+2s}} dy + \int_{\widetilde{\Sigma_{\lambda_0}}} \frac{-\bar{u}(y)}{|x^o - y|^{n+2s}} dy \} \\
= \; & C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{u}(y)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_{\lambda_0}} \frac{-\bar{u}(y^\lambda)}{|x^o - y^\lambda|^{n+2s}} dy \} \\
= \; & C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{u}(y)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_{\lambda_0}} \frac{\bar{u}(y)}{|x^o - y^\lambda|^{n+2s}} dy \} \\
= \; & C_{n,s} PV \int_{\Sigma_{\lambda_0}} \bar{u}(y) \{ \frac{1}{|x^o - y^\lambda|^{n+2s}} - \frac{1}{|x^o - y|^{n+2s}} \} dy \\
< \; & 0
\end{aligned}$$

since

$$\frac{1}{|x^o - y^\lambda|^{n+2s}} - \frac{1}{|x^o - y|^{n+2s}} < 0$$

and

$$\bar{u}(y) \geq 0$$

and

$$\bar{u}(y) \not\equiv 0$$

which, by (5.12), is a contradiction with $\bar{f}(0) \geq 0$.

Therefore, we have either $\bar{u}(x) > 0$, $x \in B_1(0)$ or $\bar{u}(x) \equiv 0$, $x \in \mathbb{R}^n$.

□

If $\bar{u}(x) > 0$, $x \in B_1(0)$, we know $\lambda_0 < 0$, if we take a point $\bar{x}$ on the curved part $\partial\Omega_{\lambda_0}$, then it's reflection point $\bar{x}^{\lambda_0}$ is in the interior of the ball (see Figure 11) and hence $\bar{u}(\bar{x}^{\lambda_0}) > 0$, therefore, $\bar{w}(\bar{x}) = \bar{u}(\bar{x}^{\lambda_0}) - u(\bar{x}) > 0$. $\bar{w}(x) > 0$ somewhere, but we already derive $\bar{w}(x) \equiv 0$, hence we must have $\bar{u}(x) \equiv 0$, $x \in \mathbb{R}^n$.

Thus, we know $u(x_k)$ converges to 0 uniformly.

5.3.4 Derive a contradiction for large k

In order to derive a contradiction for large k, Let

$$w_k(x) \equiv w_{\lambda_0}(x_k) = m_k \tag{5.13}$$

which converges to zero.

Let

$$v_k(x) = w_k(x) - 2m_k \tag{5.14}$$

So

$$\begin{aligned} & v_k(x_k) \\ = \; & w_k(x_k) - 2m_k \end{aligned}$$

so $v_k(x)$ attains its minimum at some point, say $(\bar{x}_k)$ in $\Omega_{\lambda_0 - \delta}$.

Combining (5.13) and (5.14), it is easy to deduce

$$\begin{aligned} & v_k(x_k) \\ = \;& w_k(x_k) - 2m_k \\ = \;& m_k - 2m_k \\ = \;& -m_k \end{aligned}$$

Thus

$$v_k(\bar{x}_k) \leq -m_k$$

Then

$$(-\triangle)^s v_k = c_\lambda v_k + 2m_k$$

By regularity theory for parabolic equations [45], there exists some functions $\bar{v}(x)$ such that $k \to \infty$, $v_k(x) \to \bar{v}(x)$ converges uniformly for $x \in \Omega_{\lambda_o}$,

Moreover

$$(-\Delta)^s \bar{v} = c_\lambda \bar{v}$$

Passing to a subsequence, $(\bar{x}_k) \to (x^o) \in \Omega_{\lambda_o - \delta}$

$w_k \to \bar{w}$ uniformly, and

$$(-\Delta)^s \bar{w} = \bar{c}\bar{w}$$

As we have already derived

$$\bar{w}(x^o) = 0$$

Also following from (5.2)

$$
\begin{aligned}
& (-\Delta)^s \bar{w}(x^o) \\
\geq \;& f(|x^o|, u_{\lambda_0}) - f(|x^o|, u) \\
= \;& \frac{f(|x^o|, u_{\lambda_0}) - f(|x^o|, u)}{\bar{u}_{\lambda_0}(x^o) - \bar{u}(x^o)} \bar{w}(x^o) \\
:= \;& c_\lambda(x^o) \bar{w}(x^o),
\end{aligned}
$$

It is easy to deduct

$$(-\Delta)^s \bar{w}(x^o) = 0$$

It follows

$$
\begin{aligned}
& (-\Delta)^s \bar{w}(x^o) \\
= \;& C_{n,s} PV \int_{\mathbb{R}^n} \frac{-\bar{w}(y)}{|x^o - y|^{n+2s}} dy \\
= \;& C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{w}(y)}{|x^o - y|^{n+2s}} dy + \int_{\widetilde{\Sigma_{\lambda_0}}} \frac{-\bar{w}(y)}{|x^o - y|^{n+2s}} dy \} \\
= \;& C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{w}(y)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_{\lambda_0}} \frac{-\bar{w}(y^\lambda)}{|x^o - y^\lambda|^{n+2s}} dy \} \\
= \;& C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{w}(y)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_{\lambda_0}} \frac{\bar{w}(y)}{|x^o - y^\lambda|^{n+2s}} dy \} \\
= \;& C_{n,s} PV \int_{\Sigma_{\lambda_0}} \bar{w}(y) \{ \frac{1}{|x^o - y^\lambda|^{n+2s}} - \frac{1}{|x^o - y|^{n+2s}} \} dy \\
\leq \;& 0
\end{aligned}
$$

since

$$\frac{1}{|x^o - y^\lambda|^{n+2s}} - \frac{1}{|x^o - y|^{n+2s}} < 0$$

and

$$\bar{w}(y) \geq 0$$

This implies

$$\bar{w}(x) \equiv 0, \forall x \in \mathbb{R}^n$$

Similar with above, assume $f(0) \geq 0$, for

$$u_k(x) = u(x_k)$$

$$u_k(x) \to \bar{u}(x)$$

$$(-\Delta)^s \bar{u} = f(\bar{u})$$

We have

$$\bar{u}(x) \equiv 0$$

Now for sufficiently large k,

$$\begin{aligned} & (-\Delta)^s v_k(\bar{x}_k) \\ = \; & (-\Delta)^s w_k \\ = \; & c_{\lambda_o}(\bar{x}_k) w_k(\bar{x}_k) \end{aligned}$$

Since we know

$$\begin{aligned} & (-\Delta)^s v_k(\bar{x}_k) \\ \leq \; & \frac{c}{[d(\bar{x}_k, T_{\lambda_o})]^{2s}} v_k(\bar{x}_k) \\ \leq \; & -c_1 m_k \end{aligned}$$

where $c_1 > 0$.

$$c_{\lambda_o}(\bar{x}_k) = o(1) \to 0$$

Finally,

$$-c_1 m_k \geq o(1) m_k$$

which is a contradiction with $-c_1 m_k \geq o(1) m_k$ as $k \to \infty$.

Hence, we have proved (5.8).

Since w_λ depends on λ continuously, there exists $\epsilon > 0$ and $\epsilon < \delta$, such that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, we have

$$w_\lambda \geq 0, \; x \in \Omega_{\lambda_0 - \delta} \tag{5.15}$$

Now apply the Narrow region theorem 1.1 and in our case the narrow region is

$$\Omega_\lambda^- \backslash \Omega_{\lambda_0 - \delta}$$

By Narrow region theorem, we derive

$$w_\lambda \geq 0, \; x \in \Omega_\lambda^- \backslash \Omega_{\lambda_0 - \delta} \tag{5.16}$$

Combining (5.15) and (5.16), we conclude that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$

$$w_\lambda(x) \geq 0, \; x \in \Omega_\lambda$$

This contradicts the definition of λ_0. Therefore, we must have

$$\lambda_0 = 0$$

and

$$w_{\lambda_0} \geq 0, \ \forall x \in \Omega_{\lambda_0}$$

Similarly, one can move the plane T_λ from $\lambda = 1$ to the left and show that

$$w_{\lambda_0} \leq 0, \ \forall x \in \Omega_{\lambda_0}$$

Now we have shown that

$$\lambda_0 = 0$$

and

$$w_{\lambda_0} \equiv 0, \ x \in \Omega_{\lambda_0}$$

This completes the setp 2.

5.3.5 Conclude the solution is radially symmetric and monotone decreasing

So far, we have proved that u is symmetric about the plane T_0. Since the x_1 direction can be chosen arbitrarily, we have actually shown that u is radially symmetric about origin.

Since $w_\lambda(x) \not\equiv 0$, $x \in T_\lambda$, $\forall 0 < \lambda < \lambda_0$, if there exists x^o such that x^o is the minimum point, from the above process, on one hand,

$$(-\Delta)^s w_\lambda(x^o) \leq 0$$

On the other hand,

$$(-\Delta)^s w_\lambda(x^o) = 0$$

This forces

$$w_\lambda \equiv 0$$

which is a contradiction. Therefore, u is monotone decreasing about the origin.

5.4 Radial Symmetry of solutions of fractional parabolic equations

In the following section, we will try to prove the solution of the parabolic equation with assumption in section 2.2 is radially symmetric using the method of moving planes.

Theorem 5.5. *(Radial Symmetry of solution of fractional parabolic equation)*

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = f(t,|x|,u), & (x,t) \in B_1(0) \times (-\infty,\infty), \\ u(x,t) > 0, & (x,t) \in B_1(0) \times (-\infty,\infty), \\ u(x,t) \equiv 0, & x \notin B_1(0), \end{cases} \tag{5.17}$$

where $B_1(0)$ is a unit ball.

Let $0 < s < 1$, and suppose that $u(x,t) \in \left(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\right) \times (-\infty,\infty)$ is a positive bounded classical solution of equation (5.17)*, and assume $f(t,|x|,u)$ satisfies the following assumptions:*

(f1) $f(t,|x|,u)$ are decreasing in $|x|$.

(f2) Assume that f is uniformly Lipschitz continuous in u. i.e:

$$f(t,|x|,u_1) - f(t,|x|,u_2) \leq c|u_1 - u_2|, \ \forall x \in B_1(0),$$

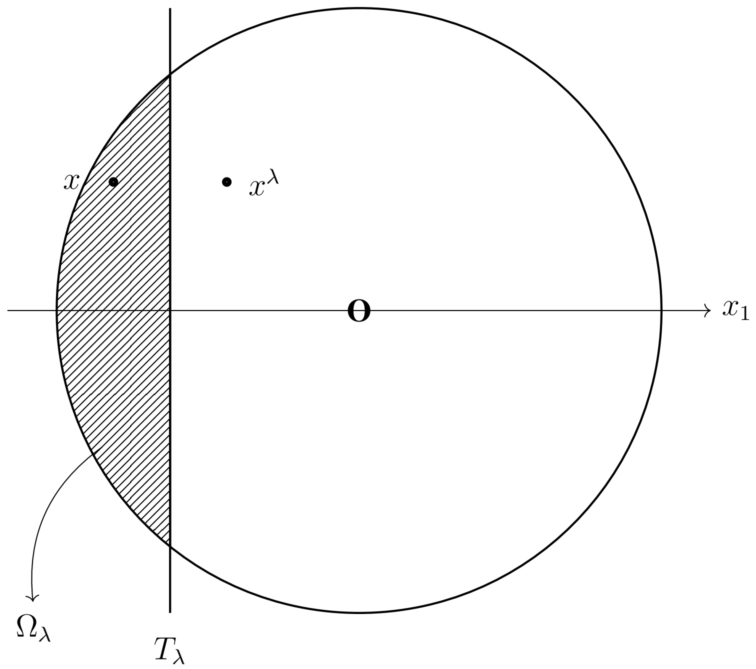


Figure 8: Moving Planes on the Unit Ball

then $u(x,t)$ is radially symmetric and monotone decreasing about the origin. i.e.

$$u(x,t) = u(|x|,t)$$

$$u(x_1,t) > u(x_2,t), |x_1| < |x_2|.$$

.

5.4.1 Basic set-up

For any given $\lambda \in \mathbb{R}$, let

$$T_\lambda = \{x \in \mathbb{R}^n \mid x_1 = \lambda \text{ for } \lambda \in \mathbb{R}\}$$

be the moving planes,

$$\Sigma_\lambda := \{x \in \mathbb{R}^n \mid x_1 < \lambda\}$$

be the region to the left of the plane, and

$$x^\lambda = (2\lambda - x_1, x_2, ..., x_n)$$

be the reflection of x about the plane T_λ. and

$$\Omega_\lambda = \Sigma_\lambda \cap B_1(0)$$

be the intersection of $B_1(0)$ and Σ_λ.

Assume that $u(x,t)$ is a positive solution of equation (5.17). We compare the values of $u(x,t)$ with

$$u_\lambda(x,t) = u(x^\lambda, t).$$

Let

$$w_\lambda(x,t) = u_\lambda(x,t) - u(x,t).$$

Step 1: *Begin moving the plane from near the left end of* $B_1(0)$ *along the* x_1 *axis, but do not reach origin*,

So then

$$|x^\lambda| < |x|$$

we derive

$$f(t, |x^\lambda|, u_\lambda) - f(t, |x|, u_\lambda) \geq 0$$

We deduce from the equation (1.8) and (f1), (f2) that w_λ satisfies

$$\begin{aligned} & \frac{\partial w_\lambda}{\partial t}(x,t) + (-\Delta)^s w_\lambda(x,t) \\ = \ & f(t,|x^\lambda|,u_\lambda) - f(t,|x|,u) \\ = \ & f(t,|x^\lambda|,u_\lambda) - f(t,|x|,u_\lambda) + f(t,|x|,u_\lambda) - f(t,|x|,u) \\ \geq \ & f(t,|x|,u_\lambda) - f(t,|x|,u) \\ = \ & \frac{f(t,|x|,u_\lambda) - f(t,|x|,u)}{u_\lambda(x,t) - u(x,t)} w_\lambda(x,t) \\ := \ & c_\lambda(x,t) w_\lambda(x,t), \end{aligned} \tag{5.18}$$

where $c_\lambda(x,t) = \frac{f(t,|x|,u_\lambda)-f(t,|x|,u)}{u_\lambda(x,t)-u(x,t)}$ is bounded.

Apparently, Ω_λ is a narrow region in the x_1 direction for λ very close to -1. For further application of Narrow Region Principle, We will prove the Narrow region theorem in parabolic problem in the following sections.

5.5 A parabolic problem

In Parabolic case, we add a time dimension on this unit ball with lower edge $t = \underline{t}$ and higher edge $t = T$, denotes this thin cylinder as $\Omega_\lambda \times (\underline{t}, T]$ to be the narrow region we want to use(See Figure 9).

Now, in order to prove the radial symmetry of the solution of parabolic equation (5.17), we consider

$$\bar{w} = e^{m(t-\underline{t})} w_\lambda(x,t), \ m > 0$$

and $w_\lambda \in [C^{1,1}_{loc}(\Omega_\lambda) \cap C(\bar{\Omega_\lambda}) \cap \mathcal{L}_{2s}] \times C^1([\underline{t}, T])$

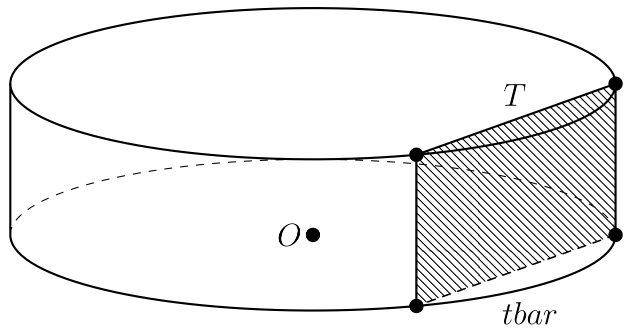


Figure 9: Thin cylinder in Unit Ball

5.6 *Step 1: show* $w_\lambda(x,t) \geq 0$

To show $w_\lambda(x,t) \geq 0$, we first show that $\bar{w}$ can not attain its negative minimum in $\Omega_\lambda \times (\underline{t}, T]$, to attain this goal, we will first prove a maximum principle in $\Omega_\lambda \times (\underline{t}, T]$.

5.6.1 Maximum principle on a parabolic cylinder

Lemma 5.6. *(Maximum principle on a parabolic cylinder) Assume that*

$$\begin{cases} \frac{\partial w}{\partial t} + (-\Delta)^s w = c(x,t)w(x,t), & x \in \Omega_\lambda \times [\underline{t}, T], \\ w(x^\lambda, t) = -w(x,t), & x \in \Omega_\lambda \times [\underline{t}, T], \\ w(x,t) \geq 0, & x \in \Sigma_\lambda \backslash \Omega_\lambda \times [\underline{t}, T], \end{cases} \tag{5.19}$$

Then for λ sufficiently close to -1, we have

$$w(x,t) \geq \min\{0, \inf_{\Omega_\lambda \times [\underline{t}, T]} w(x, \underline{t})\}, \;\; w \in [C^{1,1}_{loc}(\Omega_\lambda) \cap C(\bar{\Omega_\lambda}) \cap \mathcal{L}_{2s}] \times C^1([\underline{t}, T]) \tag{5.20}$$

Proof. If (5.20) does not hold, then the lower semi-continuity of $w(x,t)$ on $\bar{\Omega_\lambda} \times [\underline{t}, T]$ guarantees that there exists an $(x^o, t^o) \in \Omega_\lambda \times [\underline{t}, T]$ such that

$$w(x^o, t^o) = \min_{\Omega_\lambda \times (\underline{t}, T]} w < 0$$

And one can further deduce from condition (5.19) that (x^o, t^o) is in the interior of $\Omega_\lambda \times [\underline{t}, T]$

Similar to the argument in Section 4.3.1, we have

$$
\begin{aligned}
& (-\Delta)^s w(x^o, t^o) \\
= \; & C_{n,s} PV \int_{\mathbb{R}^n} \frac{w(x^o, t^o) - w(y, t^o)}{|x^o - y|^{n+2s}} dy \\
= \; & C_{n,s} PV \{ \int_{\Sigma_\lambda} \frac{w(x^o, t^o) - w(y, t^o)}{|x^o - y|^{n+2s}} dy + \int_{\widetilde{\Sigma_\lambda}} \frac{w(x^o, t^o) - w(y, t^o)}{|x^o - y|^{n+2s}} dy \} \\
= \; & C_{n,s} PV \{ \int_{\Sigma_\lambda} \frac{w(x^o, t^o) - w(y, t^o)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_\lambda} \frac{w(x^o, t^o) - w(y^\lambda, t^o)}{|x^o - y^\lambda|^{n+2s}} dy \} \\
= \; & C_{n,s} PV \{ \int_{\Sigma_\lambda} \frac{w(x^o, t^o) - w(y, t^o)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_\lambda} \frac{w(x^o, t^o) + w(y, t^o)}{|x^o - y^\lambda|^{n+2s}} dy \} \\
\leq \; & C_{n,s} \{ \int_{\Sigma_\lambda} \frac{w(x^o, t^o) - w(y, t^o)}{|x^o - y^\lambda|^{n+2s}} dy + \frac{w(x^o, t^o) + w(y, t^o)}{|x^o - y^\lambda|^{n+2s}} dy \} \\
= \; & C_{n,s} \int_{\Sigma_\lambda} \frac{2w(x^o, t^o)}{|x^o - y^\lambda|^{n+2s}}
\end{aligned}
$$

Lemma 5.7. *Here we want to prove a lemma that the integral of $\frac{1}{|x^o - y^\lambda|^{n+2s}}$ is greater than or equal to $\frac{c}{d^{2s}}$*

$$
\begin{aligned}
& \int_{\Sigma_{\lambda_0}} \frac{1}{|x^o - y^\lambda|^{n+2s}} dy \\
\geq \; & \frac{c}{d^{2s}}
\end{aligned}
$$

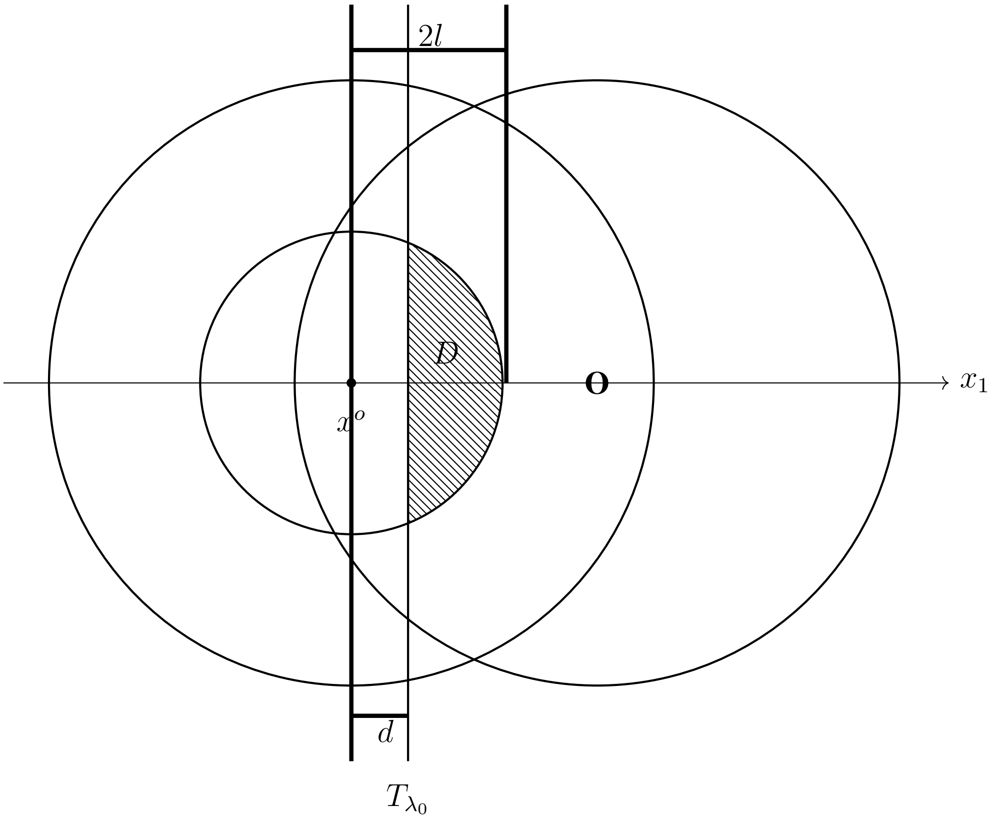


Figure 10: The region of $D = B_{2l} \cap \widetilde{\Sigma_{\lambda_0}}$

Proof.

$$
\begin{aligned}
& \int_{\Sigma_{\lambda_0}} \frac{1}{|x^o - y^\lambda|^{n+2s}} dy \\
\geq & \int_D \frac{1}{|x^o - y|^{n+2s}} dy \\
\geq & \; vol(D) \cdot \frac{1}{(2d)^{n+2s}} \\
\geq & \; c(2d)^n \cdot \frac{1}{(2d)^{n+2s}} \\
= & \; \frac{c}{d^{2s}}
\end{aligned}
$$

Let $D = B_{2l}(x^o, t^o) \cap \widetilde{\Sigma_{\lambda_0}}$. (See Figure 10, the shaded region is region $D = B_{2l}(x^o, t^o) \cap \widetilde{\Sigma_{\lambda_0}}$) $\square$

Thus,

$$(-\Delta)^{2s} w(x^o, t^o) \leq \frac{cw(x^o, t^o)}{d^{2s}} < 0 \tag{5.21}$$

Combining (5.18) and (5.21), we deduce

$$\begin{aligned} & c_\lambda(x^o, t^o) w(x^o, t^o) \leq \frac{cw(x^o, t^o)}{d^{2s}} + \frac{\partial w(x^o, t^o)}{\partial t} \\ = \quad & c_\lambda(x^o, t^o) w(x^o, t^o) \leq \frac{cw(x^o, t^o)}{d^{2s}} \end{aligned}$$

Then we derive

$$\frac{c}{d^{2s}} \leq c_\lambda(x^o, t^o)$$

for λ sufficiently close to -1, d would be sufficiently small, since c_λ is bounded, we derive a contradiction. Therefore, (5.20) must be valid. So far, we have proved the Lemma 5.6.

□

Let

$$\bar{w} = e^{m(t-\underline{t})} w_\lambda(x, t), \;\; m > 0$$

From (5.18) we derive

$$\frac{\partial \bar{w}}{\partial t} + (-\triangle)^s \bar{w} = \bar{c}\bar{w}$$

with $\bar{c}$ is still bounded.

This time, we want to show

$$w_\lambda(x, t) \geq 0 \tag{5.22}$$

Suppose otherwise, (5.22) does not hold, then $\bar{w}(x,t)$ is negative somewhere, hence there exists an $x^o \in \Omega_\lambda$ and $t^o \in [\underline{t}, T]$ such that

$$\bar{w}(x^o, t^o) = \min_{\Omega_\lambda \times (\underline{t}, T]} \bar{w} < 0$$

If

$$t^o < T$$

$$\frac{\partial \bar{w}}{\partial t}(x^o, t^o) = 0$$

If

$$t^o = T$$

$$\frac{\partial \bar{w}}{\partial t}(x^o, t^o) \leq 0$$

From (5.18), we derive

$$(-\Delta)^s \bar{w}(x^o, t^o) \geq \bar{c}(x^o, t^o) \bar{w}(x^o, t^o)$$

We also have

$$(-\Delta)^s \bar{w}(x^o, t^o) \leq \frac{c}{d^{2s}} \bar{w}(x^o, t^o) \tag{5.23}$$

We deduce

$$\frac{c}{d^{2s}} \bar{w}(x^o, t^o) \geq \bar{c}(x^o, t^o) \bar{w}(x^o, t^o)$$

Then we derive

$$\frac{c}{d^{2s}} \leq \bar{c}(x^o, t^o)$$

Which is a contradiction for d sufficiently small. Thus,

$$\bar{w}(x,t) \geq \min\{0, \inf_{x\in\Omega_\lambda} \bar{w}(x,\underline{t})\}, \; \forall (x,t) \in \Omega_\lambda \times (\underline{t}, T)$$

Thus

$$e^{m(t-\underline{t})} w_\lambda(x,t) \geq \min\{0, \inf_{x\in\Omega_\lambda} w_\lambda(x,\underline{t})\}$$

So

$$w_\lambda(x,t) \geq e^{-m(t-\underline{t})} \min\{0, \inf_{x\in\Omega_\lambda} w_\lambda(x,\underline{t})\}$$

$w_\lambda(x,t)$ is bounded from below. Let $\underline{t} \to -\infty$, $w_\lambda(x,t) \to \geq 0$.

Therefore,

$$w_\lambda(x,t) \geq 0$$

if Ω_λ is narrow.

5.7 *Step 2*: *Move the plane continuously to the right until its limiting position as long as* $w_\lambda(x,t) \geq 0$ *holds*.

Define

$$\lambda_0 = sup\{\lambda \leq 0 \mid w_\mu(x,t) \geq 0, \forall (x,t) \in \Omega_\mu \times \mathbb{R}, \mu \leq \lambda\}$$

In this part, we show that

$$\lambda_0 = 0$$

Suppose

$$\lambda_0 < 0$$

we show that the plane T_{λ_0} can be moved further to the right. To be more rigorous, there exists some $\epsilon > 0$, such that for any $\lambda \in (\lambda_0, \lambda_0+\epsilon)$, we have $w_\lambda(x,t) \geq 0$, $(x,t) \in \Sigma_{\lambda_0} \times \mathbb{R}$

This is a contradiction with the definition of λ_0. Hence we must have

$$\lambda_0 = 0$$

5.7.1 Show $w_{\lambda_0}(x,t) > 0$ for $(x,t) \in \Omega_{\lambda_o} \times \mathbb{R}$

Suppose that $\lambda_0 < 0$, then the reflection of the curved part of $\partial\Omega_{\lambda_0}$ falls inside $B_1(0)$ and $w_{\lambda_0}(x,t) \geq 0$ for $x \in \partial\Omega_{\lambda_0}$. (See Figure 11)

We want to show $w_{\lambda_0}(x,t) > 0$ for $(x,t) \in \Omega_{\lambda_o} \times \mathbb{R}$

Otherwise, $\exists (x^o, t^o) \in \Omega_{\lambda_o} \times \mathbb{R}$ such that $w_{\lambda_0}(x^o,t^o) = 0$

Since $w_{\lambda_0}(x,t) \geq 0$ inside $\Omega_{\lambda_o} \times \mathbb{R}$, so for $w_{\lambda_0}(x^o,t^o) = 0$, we know (x^o,t^o) is the minimum.

Then

$$\frac{\partial w_{\lambda_0}}{\partial t}(x^o,t^o) = 0$$

Following from (5.18), we then derive

$$\begin{aligned}
& \frac{\partial w_{\lambda_0}}{\partial t}(x^o,t^o) + (-\Delta)^s w_{\lambda_0}(x^o,t^o) \\
\geq \quad & f(t^o, |x^o|, u_{\lambda_0}) - f(t^o, |x^o|, u) \\
= \quad & \frac{f(t^o, |x^o|, u_{\lambda_0}) - f(t^o, |x^o|, u)}{u_{\lambda_0}(x^o,t^o) - u(x^o,t^o)} w_{\lambda_0}(x^o,t^o) \\
:= \quad & c_{\lambda_0}(x^o,t^o) w_{\lambda_0}(x^o,t^o),
\end{aligned}$$

So that

$$(-\triangle)^s w_{\lambda_0}(x^o,t^o) = 0$$

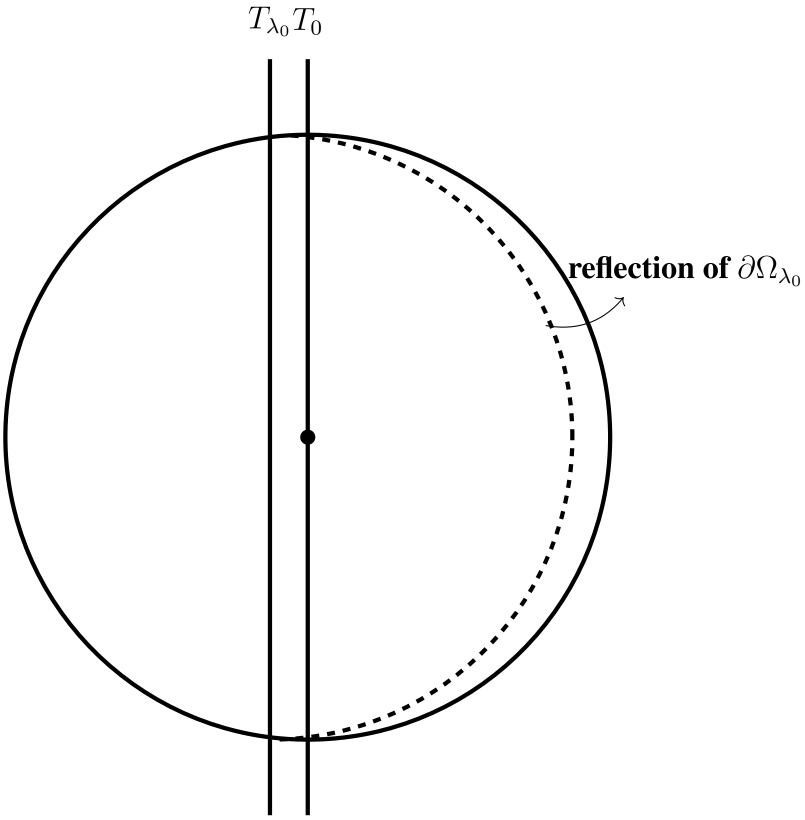


Figure 11: reflection of the curved part of $\partial\Omega_{\lambda_0}$

It follows that

$$
\begin{aligned}
& 0 = (-\Delta)^s w_{\lambda_0}(x^o, t^o) \\
= \;& C_{n,s} PV \int_{\mathbb{R}^n} \frac{-w_{\lambda_0}(y, t^o)}{|x^o - y|^{n+2s}} dy \\
= \;& C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-w_{\lambda_0}(y, t^o)}{|x^o - y|^{n+2s}} dy + \int_{\widetilde{\Sigma_{\lambda_0}}} \frac{-w_{\lambda_0}(y, t^o)}{|x^o - y|^{n+2s}} dy \} \\
= \;& C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-w_{\lambda_0}(y, t^o)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_{\lambda_0}} \frac{-w_{\lambda_0}(y^\lambda, t^o)}{|x^o - y^\lambda|^{n+2s}} dy \} \\
= \;& C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-w_{\lambda_0}(y, t^o)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_{\lambda_0}} \frac{w_{\lambda_0}(y, t^o)}{|x^o - y^\lambda|^{n+2s}} dy \} \\
= \;& C_{n,s} PV \int_{\Sigma_{\lambda_0}} w_{\lambda_0}(y, t^o) \{ \frac{1}{|x^o - y^\lambda|^{n+2s}} - \frac{1}{|x^o - y|^{n+2s}} \} dy \\
\leq \;& 0 \qquad (5.24)
\end{aligned}
$$

since

$$\frac{1}{|x^o - y^\lambda|^{n+2s}} - \frac{1}{|x^o - y|^{n+2s}} < 0$$

and

$$w_{\lambda_0}(y, t^o) \geq 0$$

This implies that

$$w_{\lambda_0}(y, t^o) \equiv 0, \;\; (y, t) \in \Omega_{\lambda_0} \times \mathbb{R}. \qquad (5.25)$$

We derive a contradiction since the plane T_{λ_0} did not reach the origin. If we take a point $\bar{x}$ on the curved part $\partial \Omega_{\lambda_0}$, then it's reflection point $\bar{x}^{\lambda_0}$ is in the interior of the ball, and hence $u(\bar{x}^{\lambda_0}, t) > 0$, therefore, $w_{\lambda_0}(\bar{x}, t) = u(\bar{x}^{\lambda_0}, t) - u(\bar{x}, t) > 0$, which contradicts (5.25).

We conclude $w_{\lambda_0}(x, t) > 0$ for every $(x, t) \in \Omega_{\lambda_0} \times \mathbb{R}$.

However, since $t \in (-\infty, \infty)$, $w_{\lambda_0}(x, t)$ may not be bounded away from 0.

We want to further derive

$$\inf w_{\lambda_0}(x,t) > c_o > 0,\ (x,t) \in \Omega_{\lambda_0-\delta} \times \mathbb{R} \tag{5.26}$$

I will prove (5.26) by contradiction.

5.7.2 Prove by contradiction: Show $\bar{w}(y,0) \equiv 0,\ \forall (y,0) \in \Omega_{\lambda_o-\delta} \times \mathbb{R}$

Proof: If (5.26) is violated, then $\exists (x_k, t_k) \in \Omega_{\lambda_0-\delta} \times \mathbb{R}$ such that $w_{\lambda_0}(x_k, t_k) \to 0$

Without loss of generality, by Bolzano-Weierstrass theorem, $\exists$ subsequences of x_k, here we still denote this subsequence by x_k, such that $x_k \to x^o \in \Omega_{\lambda_o-\delta}$

Now for each $t_k (k \geq k_0)$, Let

$$w_k(x,t) = w_{\lambda_0}(x, t+t_k)$$

so

$$w_k(x_k, 0) = w_{\lambda_0}(x_k, t_k) \to 0$$

From (5.18), we derive:

$$\begin{aligned}
& \frac{\partial w_k}{\partial t}(x,t) + (-\Delta)^s w_k(x,t) \\
= \; & f(t+t_k, |x^{\lambda_0}|, u_{\lambda_0}) - f(t+t_k, |x|, u) \\
= \; & f(t+t_k, |x^{\lambda_0}|, u_{\lambda_0}) - f(t+t_k, |x|, u_{\lambda_0}) + f(t+t_k, |x|, u_{\lambda_0}) - f(t+t_k, |x|, u) \\
\geq \; & f(t+t_k, |x|, u_{\lambda_0}) - f(t+t_k, |x|, u) \\
= \; & \frac{f(t+t_k, |x|, u_{\lambda_0}) - f(t+t_k, |x|, u)}{u_{\lambda_0}(x, t+t_k) - u(x, t+t_k)} w_k(x,t) \\
:= \; & c_{\lambda_0}(x, t+t_k) w_k(x,t),
\end{aligned}$$

So w_k satisfies

$$\frac{\partial w_k}{\partial t}(x,t) + (-\triangle)^s w_k(x,t) = c_{\lambda_0}(x, t+t_k) w_k(x,t) \tag{5.27}$$

By regularity theory for parabolic equations [45], there exists some functions $\bar{w}(x,t)$ and $\bar{c}(x,t)$ such that $k \to \infty$, $w_k(x,t)$ converges uniformly to $\bar{w}(x,t)$ for $(x,t) \in \Omega_{\lambda_o} \times \mathbb{R}$, and $\bar{w}$ satisfies:

$$\frac{\partial \bar{w}}{\partial t} + (-\triangle)^s \bar{w} = \bar{c}(x,t)\bar{w}(x,t)$$

Since

$$w_k(x_k, 0) = w_{\lambda_0}(x_k, t_k) \to 0$$

$$\bar{w}(x^o, 0) = 0$$

and

$$\bar{w} \geq 0$$

So $(x^o, 0)$ is the minimum.

$$\frac{\partial \bar{w}}{\partial t}(x^o, 0) = 0$$

Also from (5.18), we derive,

$$\begin{aligned}
& \frac{\partial \bar{w}}{\partial t}(x^o, 0) + (-\Delta)^s \bar{w}(x^o, 0) \\
\geq \quad & f(0, |x^o|, u_{\lambda_0}) - f(0, |x^o|, u) \\
= \quad & \frac{f(0, |x^o|, u_{\lambda_0}) - f(0, |x^o|, u)}{\bar{u}_{\lambda_0}(x^o, t_k) - \bar{u}(x^o, t_k)} \bar{w}(x^o, 0) \\
:= \quad & c_\lambda(x^o, t)\bar{w}(x^o, 0),
\end{aligned}$$

we derive

$$(-\triangle)^s \bar{w}(x^o, 0) = 0$$

It follows that

$$\begin{aligned}
& 0 = (-\Delta)^s \bar{w}(x^o, 0) \\
= \;& C_{n,s} PV \int_{\mathbb{R}^n} \frac{-\bar{w}(y,0)}{|x^o - y|^{n+2s}} dy \\
= \;& C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{w}(y,0)}{|x^o - y|^{n+2s}} dy + \int_{\widetilde{\Sigma_{\lambda_0}}} \frac{\bar{w}(y,0)}{|x^o - y|^{n+2s}} dy \} \\
= \;& C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{w}(y,0)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_{\lambda_0}} \frac{-\bar{w}(y^\lambda,0)}{|x^o - y^\lambda|^{n+2s}} dy \} \\
= \;& C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{w}(y,0)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_{\lambda_0}} \frac{\bar{w}(y,0)}{|x^o - y^\lambda|^{n+2s}} dy \} \\
= \;& C_{n,s} PV \int_{\Sigma_{\lambda_0}} \bar{w}(y,0) \{ \frac{1}{|x^o - y^\lambda|^{n+2s}} - \frac{1}{|x^o - y|^{n+2s}} \} dy \\
\leq \;& 0
\end{aligned} \tag{5.28}$$

since

$$\frac{1}{|x^o - y^\lambda|^{n+2s}} - \frac{1}{|x^o - y|^{n+2s}} < 0$$

and

$$\bar{w}(y, 0) \geq 0$$

This implies that

$$\bar{w}(y, 0) \equiv 0, \;\; \forall y \in \mathbb{R}^n$$

5.7.3 Show $\bar{u}(x, 0) \equiv 0$

Let $u_k(x,t) = u(x, t + t_k)$, then by (5.18), we have

$$\frac{\partial u_k(x,t)}{\partial t} + (-\Delta)^s u_k(x,t) = f(t + t_k, u_k(x,t))$$

By regularity theory for parabolic equations [45], there exists some functions $\bar{u}(x,t)$ such that as $k \to \infty$, $u_k(x,t)$ converges uniformly to $\bar{u}(x,t)$ for $(x,t) \in B_1(0) \times \mathbb{R}$, $f(0,u)$ converges uniformly to $\bar{f}(0,u)$ for $x \in \Omega_{\lambda_o}$

and

$$\frac{\partial \bar{u}(x,t)}{\partial t} + (-\Delta)^s \bar{u}(x,t) = \bar{f}(t, \bar{u}(x,t))$$

Since

$$f(0,u) \geq 0$$

Thus

$$\bar{f}(0,\bar{u}) \geq 0$$

In order to show that

$$\bar{u}(x,0) \equiv 0, x \in \mathbb{R}^n, \tag{5.29}$$

we apply the following:

Lemma 5.8. *(Strong Maximum Principle for $\frac{\partial \bar{u}}{\partial t} + (-\Delta)^s \bar{u} = \bar{f}(t,\bar{u})$).*

Assume that $\bar{u}(x,t) \in [C^{1,1}_{loc}(\Omega_\lambda) \cap C(\bar{\Omega_\lambda}) \cap \mathcal{L}_{2s}] \times C^1([\underline{t}, T])$

$$\begin{cases} \frac{\partial \bar{u}(x,t)}{\partial t} + (-\Delta)^s \bar{u}(x,t) = \bar{f}(t,\bar{u}), & (x,t) \in \Omega_\lambda \times [\underline{t}, T], \\ \bar{u}(x,t) \geq 0, & (x,t) \in \Omega_\lambda \times [\underline{t}, T] \end{cases} \tag{5.30}$$

we have either

$$\bar{u}(x,0) > 0, x \in B_1(0)$$

or

$$\bar{u}(x,0) \equiv 0, x \in \mathbb{R}^n$$

Proof. First, if $\bar{u}(x,t) \geq 0$ and $\bar{u}(x^o,0) = 0$, $(x^o,0)$ then is a minimum, thus we have $\frac{\partial \bar{u}}{\partial t}(x^o,0) = 0$.

If $\bar{u}(x,0) \not\equiv 0$, then

$$\begin{aligned}
& (-\Delta)^s \bar{u}(x^o,0) \\
= \; & C_{n,s} PV \int_{\mathbb{R}^n} \frac{-\bar{u}(y,0)}{|x^o - y|^{n+2s}} dy \\
= \; & C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{u}(y,0)}{|x^o - y|^{n+2s}} dy + \int_{\widetilde{\Sigma_{\lambda_0}}} \frac{-\bar{u}(y,0)}{|x^o - y|^{n+2s}} dy \} \\
= \; & C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{u}(y,0)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_{\lambda_0}} \frac{-\bar{u}(y^\lambda,0)}{|x^o - y^\lambda|^{n+2s}} dy \} \\
= \; & C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{u}(y,0)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_{\lambda_0}} \frac{\bar{u}(y,0)}{|x^o - y^\lambda|^{n+2s}} dy \} \\
= \; & C_{n,s} PV \int_{\Sigma_{\lambda_0}} \bar{u}(y,0) \{ \frac{1}{|x^o - y^\lambda|^{n+2s}} - \frac{1}{|x^o - y|^{n+2s}} \} dy \\
< \; & 0
\end{aligned}$$

since

$$\frac{1}{|x^o - y^\lambda|^{n+2s}} - \frac{1}{|x^o - y|^{n+2s}} < 0$$

and

$$\bar{u}(y,0) \geq 0$$

and

$$\bar{u}(y,0) \not\equiv 0$$

which, by 5.30, is a contradiction with $\bar{f}(0,\bar{u}) \geq 0$.

Therefore, we have either $\bar{u}(x,0) > 0$, $x \in B_1(0)$ or $\bar{u}(x,0) \equiv 0$, $x \in \mathbb{R}^n$.

$\square$

If $\bar{u}(x,0) > 0$, $x \in B_1(0)$, we know $\lambda_0 < 0$, if we take a point $\bar{x}$ on the curved part $\partial\Omega_{\lambda_0}$, then it's reflection point $\bar{x}^{\lambda_0}$ is in the interior of the ball (see Figure 11) and hence $\bar{u}(\bar{x}^{\lambda_0},0) > 0$, therefore, $\bar{w}(\bar{x},0) = \bar{u}(\bar{x}^{\lambda_0},0) - u(\bar{x},0) > 0$. $\bar{w}(x,0) > 0$ somewhere, but we already derive $\bar{w}(x,0) \equiv 0$, hence we must have $\bar{u}(x,0) \equiv 0$, $x \in \mathbb{R}^n$.

Thus, we know $u(x,t_k)$ converges to 0 uniformly.

5.7.4 Derive a contradiction for large k

In order to derive a contradiction for large k, we modify t_k a bit.

We still denote $w_k(x,t)$ by $w_{\lambda_0}(x,t+t_k)$, Let

$$w_k(x_k,0) \equiv w_{\lambda_0}(x_k,t_k) = m_k \tag{5.31}$$

which converges to zero.

Let

$$v_k(x,t) = w_k(x,t) - 2m_k\eta(\epsilon_k(t-t_k)) \tag{5.32}$$

where $\eta(t) \in C_0^\infty$ is a cut-off function such that $|\eta'(t)| \leq c$ and

$$\eta(t) = \begin{cases} 1, & |t| \leq 1, \\ 0, & |t| \geq 2. \end{cases}$$

When (x, t) is outside $\Omega_{\lambda_0-\delta} \times (t_k - 2, t_k + 2)$,

$$\begin{aligned} & v_k(x,t) \\ = \; & w_k(x,t), \end{aligned}$$

When (x, t) is inside $\Omega_{\lambda_0-\delta} \times (t_k - 2, t_k + 2)$, such that at (x, t_k)

$$\begin{aligned} & v_k(x,t_k) \\ = \; & w_k(x,t_k) - 2m_k \end{aligned}$$

The value of v_k outside $\Omega_{\lambda_0-\delta} \times (t_k - 2, t_k + 2)$ is greater than the value of v_k inside $\Omega_{\lambda_0-\delta} \times (t_k - 2, t_k + 2)$, so $v_k(x, t)$ attains its minimum at some point, say $(\bar{x}_k, \bar{t}_k)$ in $\Omega_{\lambda_0-\delta} \times (t_k - 2, t_k + 2)$.

This implies,

$$\frac{\partial v_k}{\partial t}(\bar{x}_k, \bar{t}_k) = 0$$

Combining (5.31) and (5.32), it is easy to deduce

$$\begin{aligned} & v_k(x_k, 0) \\ = \; & w_k(x_k, 0) - 2m_k \\ = \; & m_k - 2m_k \\ = \; & -m_k \end{aligned}$$

Thus

$$v_k(\bar{x}_k, \bar{t}_k) \leq -m_k$$

Let

$$\widetilde{v_k}(x,t) = v_k(x, t + \bar{t}_k)$$

Then

$$\widetilde{v_k}(\bar{x}_k, 0) = v_k(\bar{x}_k, \bar{t}_k)$$

Then

$$\frac{\partial \widetilde{v_k}}{\partial t} + (-\triangle)^s \widetilde{v_k} = c_\lambda \widetilde{v_k} + 2m_k \eta(\epsilon_k(t - t_k))$$

By regularity theory for parabolic equations [45], there exists some functions $\bar{v}(x,t)$ such that $k \to \infty$, $\widetilde{v_k}(x,t) \to \bar{v}(x,t)$ converges uniformly for $x \in \Omega_{\lambda_o}$,

Moreover

$$\frac{\partial \bar{v}}{\partial t} + (-\Delta)^s \bar{v} = c_\lambda \bar{v}$$

We know

$$\frac{\partial v_k}{\partial t} \sim \frac{\partial w_k}{\partial t} - 2m_k \epsilon_k c$$

Therefore we conclude

$$\frac{\partial w_k}{\partial t} \sim m_k \epsilon_k$$

Passing to a subsequence, $(\bar{x}_k, \bar{t}_k) \to (x^o, t^o) \in \Omega_{\lambda_o - \delta} \times [-2, 2]$

$w_k \to \bar{w}$ uniformly, and

$$\frac{\partial \bar{w}}{\partial t} + (-\Delta)^s \bar{w} = \bar{c}\bar{w}$$

As we have already derived

$$\bar{w}(x^o, t^o) = 0, \frac{\partial \bar{w}}{\partial t}(x^o, t^o) = 0$$

Also following from (5.18)

$$\begin{aligned}
& \frac{\partial \bar{w}}{\partial t}(x^o,t^o)+(-\Delta)^s\bar{w}(x^o,t^o) \\
\geq\ & f(t^o,|x^o|,u_{\lambda_0})-f(t^o,|x^o|,u) \\
=\ & \frac{f(t^o,|x^o|,u_{\lambda_0})-f(t^o,|x^o|,u)}{\bar{u}_{\lambda_0}(x^o,t^o)-\bar{u}(x^o,t^o)}\bar{w}(x^o,t^o) \\
:=\ & c_\lambda(x^o,t^o)\bar{w}(x^o,t^o),
\end{aligned}$$

It is easy to deduct

$$(-\Delta)^s\bar{w}(x^o,t^o)=0$$

It follows

$$\begin{aligned}
& (-\Delta)^s\bar{w}(x^o,t^o) \\
=\ & C_{n,s}PV\int_{\mathbb{R}^n}\frac{-\bar{w}(y,t^o)}{|x^o-y|^{n+2s}}dy \\
=\ & C_{n,s}PV\{\int_{\Sigma_{\lambda_0}}\frac{-\bar{w}(y,t^o)}{|x^o-y|^{n+2s}}dy+\int_{\widetilde{\Sigma_{\lambda_0}}}\frac{-\bar{w}(y,t^o)}{|x^o-y|^{n+2s}}dy\} \\
=\ & C_{n,s}PV\{\int_{\Sigma_{\lambda_0}}\frac{-\bar{w}(y,t^o)}{|x^o-y|^{n+2s}}dy+\int_{\Sigma_{\lambda_0}}\frac{-\bar{w}(y^\lambda,t^o)}{|x^o-y^\lambda|^{n+2s}}dy\} \\
=\ & C_{n,s}PV\{\int_{\Sigma_{\lambda_0}}\frac{-\bar{w}(y,t^o)}{|x^o-y|^{n+2s}}dy+\int_{\Sigma_{\lambda_0}}\frac{\bar{w}(y,t^o)}{|x^o-y^\lambda|^{n+2s}}dy\} \\
=\ & C_{n,s}PV\int_{\Sigma_{\lambda_0}}\bar{w}(y,t^o)\{\frac{1}{|x^o-y^\lambda|^{n+2s}}-\frac{1}{|x^o-y|^{n+2s}}\}dy \\
\leq\ & 0
\end{aligned}$$

since

$$\frac{1}{|x^o-y^\lambda|^{n+2s}}-\frac{1}{|x^o-y|^{n+2s}}<0$$

and

$$\bar{w}(y,t^o)\geq 0$$

This implies

$$\bar{w}(x, t^o) \equiv 0, \forall x \in \mathbb{R}^n$$

Similar with above, assume $f(t, 0) \geq 0$, for

$$u_k(x, t) = u(x, t + t_k)$$

$$u_k(x, \bar{t}_k) = u(x, \bar{t}_k + t_k)$$

$$u_k(x, t) \to \bar{u}(x, t)$$

$$\frac{\partial \bar{u}}{\partial t} + (-\Delta)^s \bar{u} = f(t, \bar{u})$$

We have

$$\bar{u}(x, t) \equiv 0$$

Now for sufficiently large k,

$$\begin{aligned}
& (-\Delta)^s v_k(\bar{x}_k, \bar{t}_k) \\
= \ & (-\Delta)^s w_k \\
= \ & -\frac{\partial w_k}{\partial t}(\bar{x}_k, \bar{t}_k) + c_{\lambda_o}(\bar{x}_k, \bar{t}_k + t_k) w_k(\bar{x}_k, \bar{t}_k)
\end{aligned}$$

Since we know

$$\begin{aligned}
& (-\Delta)^s v_k(\bar{x}_k, \bar{t}_k) \\
\leq \ & \frac{c}{[d(\bar{x}_k, T_{\lambda_o})]^{2s}} v_k(\bar{x}_k, \bar{t}_k) \\
\leq \ & -c_1 m_k
\end{aligned}$$

where $c_1 > 0$

$$-\frac{\partial w_k}{\partial t}(\bar{x}_k, \bar{t}_k) \sim \epsilon_k m_k$$

If we assume $\frac{\partial f}{\partial u}(t, 0) = 0$, as $u_k \to 0$ uniformly,

$$c_{\lambda_o}(\bar{x}_k, \bar{t}_k + t_k) = o(1) \to 0$$

Finally,

$$-c_1 m_k \geq o(1) m_k$$

or

$$c_1 \leq -o(1)$$

Since $o(1) \to 0$ as $k \to \infty$, which is a contradiction with $-c_1 m_k \geq o(1) m_k$ as $k \to \infty$.

Hence, we have proved (5.26).

Since w_λ depends on λ continuously, there exists $\epsilon > 0$ and $\epsilon < \delta$, such that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, we have

$$w_\lambda(x, t) \geq 0, \;\; (x, t) \in \Omega_{\lambda_0 - \delta} \times \mathbb{R} \tag{5.33}$$

Now apply the Narrow region theorem 1.1 and in our case the narrow region is

$$\Omega_\lambda^- \backslash \Omega_{\lambda_0 - \delta} \times \mathbb{R}$$

By Narrow region theorem, we derive

$$w_\lambda(x,t) \geq 0, \; (x,t) \in \Omega_\lambda^- \backslash \Omega_{\lambda_0-\delta} \times \mathbb{R} \tag{5.34}$$

Combining (5.33) and (5.34), we conclude that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$

$$w_\lambda(x,t) \geq 0, \; x \in \Omega_\lambda \times \mathbb{R}$$

This contradicts the definition of λ_0. Therefore, we must have

$$\lambda_0 = 0$$

and

$$w_{\lambda_0}(x,t) \geq 0, \; \forall (x.t) \in \Omega_{\lambda_0} \times \mathbb{R}$$

Similarly, one can move the plane T_λ from $\lambda = 1$ to the left and show that

$$w_{\lambda_0}(x,t) \leq 0, \; \forall (x,t) \in \Omega_{\lambda_0} \times \mathbb{R}$$

Now we have shown that

$$\lambda_0 = 0$$

and

$$w_{\lambda_0}(x,t) \equiv 0, \; (x,t) \in \Omega_{\lambda_0} \times \mathbb{R}$$

This completes the step 2.

5.7.5 Conclude the solution is radially symmetric and monotone decreasing

So far, we have proved that u is symmetric about the plane T_0. Since the x_1 direction can be chosen arbitrarily, we have actually shown that u is radially symmetric about origin.

Since $w_\lambda(x,t) \not\equiv 0$, $(x,t) \in T_\lambda \times \mathbb{R}$, $\forall 0 < \lambda < \lambda_0$, if there exists (x^o, t^o) such that (x^o, t^o) is the minimum point, from the above process, on one hand,

$$(-\Delta)^s w_\lambda(x^o, t^o) \leq 0$$

On the other hand,

$$(-\Delta)^s w_\lambda(x^o, t^o) = 0$$

This forces

$$w_\lambda \equiv 0$$

which is a contradiction. Therefore, u is monotone decreasing about the origin.

6 Method of Moving Planes and Its Applications: Radial symmetry and monotonicity of solutions for fractional elliptic and parabolic systems

While our previous work focused on proving properties of individual equations, it is also crucial to investigate systems of equations. Expanding our focus to systems of equations, we can further explore how direct method of moving planes are efficiently to be used to prove monotonicity of solutions for fractional elliptic and parabolic systems. As always, as the preliminary for proving monotonicity of solutions for fractional parabolic system, we would prove the monotonicity of solutions for fractional elliptic system first.

6.1 Radial Symmetry of solutions of elliptic fractional systems

In the following section, we will prove the solution of the elliptic fractional system is radially symmetric about the origin and monotone decreasing about the origin using the method of moving planes.

Theorem 6.1. *(Radial Symmetry of solution of elliptic fractional system)*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = f(v(x)), & x \in B_1(0), \\ (-\Delta)^{\frac{\beta}{2}} v(x) = g(u(x)), & x \in B_1(0), \\ u(x), v(x) \geq 0, & x \in B_1(0), \\ u(x), v(x) \equiv 0, & x \notin B_1(0), \end{cases} \tag{6.1}$$

where B_1 is a unit ball.

Let $0 < \alpha, \beta < 2$, and suppose that $u(x), v(x) \in \left(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\right)$ are positive bounded classical solutions of equation (6.1)*, and assume $f(v(x)), g(u(x))$ satisfies the following assumptions:*

(X1) $f(\cdot)$ is non-decreasing in $v(\cdot)$, $g(\cdot)$ is non-decreasing in $u(\cdot)$.

(X2) Assume that f, g are uniformly Lipschitz continuous in u, v. i.e:

$$|f(v_1) - f(v_2)| \leq c|v_1 - v_2|,$$

$$|g(u_1) - f(u_2)| \leq c|u_1 - u_2|,$$

then $u(x), v(x)$ are radially symmetric about the origin and monotone decreasing about the

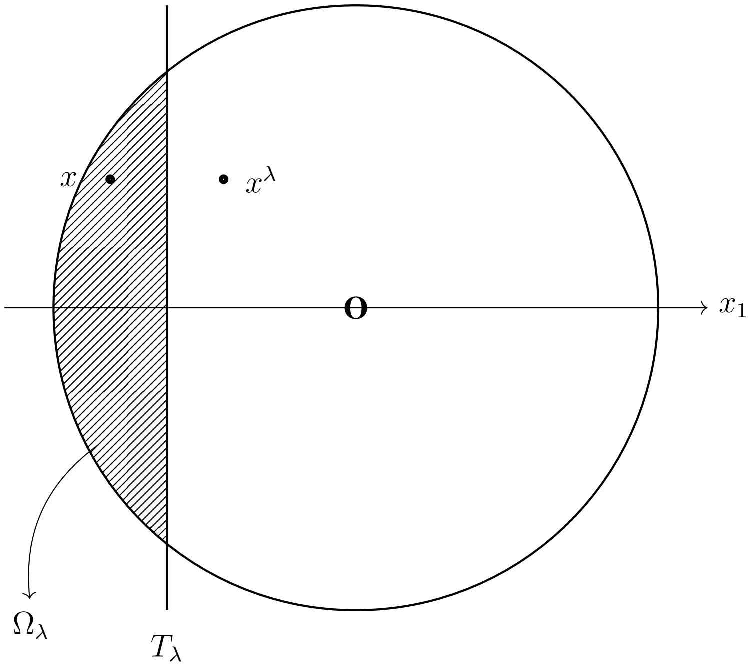


Figure 12: Moving Planes on the Unit Ball

origin, i.e.

$$u(x) = u(|x|), v(x) = v(|x|)$$

$$u(x_1) > u(x_2), v(x_1) > v(x_2), |x_1| < |x_2|.$$

.

6.1.1 Basic set-up

For any given $\lambda \in \mathbb{R}$, let

$$T_\lambda = \{x \in \mathbb{R}^n \mid x_1 = \lambda \text{ for } \lambda \in \mathbb{R}\}$$

be the moving planes,

$$\Sigma_\lambda := \{x \in \mathbb{R}^n \mid x_1 < \lambda\}$$

be the region to the left of the plane, and

$$x^\lambda = (2\lambda - x_1, x_2, ..., x_n)$$

be the reflection of x about the plane T_λ. and

$$\Omega_\lambda = \Sigma_\lambda \cap B_1(0)$$

be the intersection of $B_1(0)$ and Σ_λ.

Assume that $u(x), v(x)$ are positive solutions of equation (6.1). We compare the values of $u(x), v(x)$ with

$$u_\lambda(x) = u(x^\lambda), v_\lambda(x) = v(x^\lambda)$$

Let

$$U_\lambda(x) = u_\lambda(x) - u(x), V_\lambda(x) = v_\lambda(x) - v(x)$$

Step 1: *Begin moving the plane from near the left end of $B_1(0)$ along the x_1 axis, but do not reach origin*, So then

$$|x^\lambda| < |x|$$

We deduce from the equation (6.1) and (X1), (X2) and mean value theorem that U_λ, V_λ

satisfies

$$
\begin{aligned}
& (-\Delta)^{\frac{\alpha}{2}} U_\lambda(x) \\
= \quad & f(v_\lambda(x)) - f(v(x)) \\
= \quad & \frac{f(v_\lambda(x))}{v_\lambda^p(x)} v_\lambda^p(x) - \frac{f(v(x))}{v^p(x)} v^p(x) \\
\geq \quad & \frac{f(v(x))}{v^p(x)} [v_\lambda^p(x) - v^p(x)] \\
= \quad & \frac{f(v(x))}{v^p(x)} p\xi^{p-1}(x) V_\lambda(x) \\
:= \quad & f_v(\xi(x)) V_\lambda(x),
\end{aligned}
\tag{6.2}
$$

$$
\begin{aligned}
& (-\Delta)^{\frac{\beta}{2}} V_\lambda(x) \\
= \quad & g(u_\lambda(x^\lambda)) - g(u(x)) \\
= \quad & \frac{g(u_\lambda(x))}{u_\lambda^p(x)} u_\lambda^p(x) - \frac{g(u(x))}{u^p(x)} u^p(x) \\
\geq \quad & \frac{g(u(x))}{u^p(x)} [u_\lambda^p(x) - u^p(x)] \\
= \quad & \frac{g(u(x))}{u^p(x)} p\eta^{p-1}(x) U_\lambda(x) \\
:= \quad & g_u(\eta(x)) U_\lambda(x),
\end{aligned}
\tag{6.3}
$$

where

$$f_v(\xi(x)), g_u(\eta(x))$$

are bounded and positive.

Now, in order to prove the radial symmetry of the solution of elliptic fractional system (6.1), in step 1, we first show $U_\lambda(x) \geq 0, V_\lambda(x) \geq 0$, in step 2, move the plane continuously to the right until its limiting position as long as $U_\lambda(x) \geq 0, V_\lambda(x) \geq 0$ holds.

6.2 *Step 1: show* $U_\lambda(x) \geq 0, V_\lambda(x) \geq 0$

To show $U_\lambda(x) \geq 0, V_\lambda(x) \geq 0$, we would prove a Narrow region theorem for the elliptic fractional system:

6.2.1 Narrow Region Principle for Elliptic Fractional System

Theorem 6.2. *(Narrow Region principle for elliptic Fractional System) Let* $\Omega_\lambda = \Sigma_\lambda \cap B_1(0)$*,* Ω_λ *is a bounded narrow region in* Σ_λ*, assume that* $U_\lambda(x) \in \big(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\big)$*,* $V_\lambda(x) \in \big(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\big)$ *if*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} U_\lambda(x) \geq f_v(\xi(x)) V_\lambda(x), & x \in \Omega_\lambda, \\ (-\Delta)^{\frac{\beta}{2}} V_\lambda(x) \geq g_u(\eta(x)) U_\lambda(x), & x \in \Omega_\lambda, \end{cases} \tag{6.4}$$

then for λ *sufficiently close to* -1*, we have*

$$\begin{cases} U_\lambda(x) \geq 0, & x \in \Omega_\lambda, \\ V_\lambda(x) \geq 0, & x \in \Omega_\lambda, \end{cases} \tag{6.5}$$

Proof. Suppose otherwise, (6.5) does not hold, then U_λ is negative somewhere, hence there exists an $x^o \in \Omega_\lambda$ such that such that

$$U_\lambda(x^o) = \min_{\Omega_\lambda} U_\lambda(x) < 0$$

By the defining integral of the fractional Laplacian, we have

$$\begin{aligned} & (-\Delta)^{\frac{\alpha}{2}} U_\lambda(x^o) \\ \leq \;& C_{n,\alpha} \int_{\Sigma_\lambda} \frac{2U_\lambda(x^o)}{|x^o - y^\lambda|^{n+\alpha}} \end{aligned}$$

To estimate the integral above, we may first consider the extreme case where $x_1^o = \lambda$. Then it is easy to see that

$$\int_{\Sigma_\lambda} \frac{1}{|x^o - y^\lambda|} dy = \infty$$

This suggests that we may obtain values arbitrarily large by integrating on a domain that is sufficiently close to the hyperplane $P := \{y \in \mathbb{R}^n \mid y_1 = \lambda\}$.

Denote

$$d = dis[x^0, T_\lambda] \leq width(\Omega_\lambda)$$

Choose a ball centered at x^o with radius l, also choose a unit ball centered at x^o, as Figure 2 shows, it is easy to see $d \leq l$.

Since

$$d \leq |x^o - y^\lambda| \leq 2d$$

Then

$$\frac{1}{|x^o - y^\lambda|} \geq \frac{1}{2d}$$

By Lemma 5.3, we have

$$\int_{\Sigma} \frac{1}{|x^o - y^\lambda|^{n+\alpha}} dy \geq \frac{c}{d^\alpha}$$

Hence

$$(-\Delta)^{\frac{\alpha}{2}} U_\lambda(x^o) \leq \frac{c}{d^\alpha} U_\lambda(x^o) < 0 \tag{6.6}$$

On the other hand, by assumption 6.2, we have

$$(-\Delta)^{\frac{\alpha}{2}}U_\lambda(x^o) = f(v_\lambda(x^o)) - f(v(x^o)) < 0$$

. Therefore, by the monotonicity of f, we have

$$V_\lambda(x^o) < 0$$

This implies that there exists some $\bar{x} \in \Omega_\lambda$ such that

$$V_\lambda(\bar{x}) = \min_{\Omega_{V_\lambda}} V_\lambda(x) < 0$$

Following the same argument, we can derive that

$$(-\Delta)^{\frac{\beta}{2}}V_\lambda(\bar{x}) \leq \frac{cV_\lambda(\bar{x})}{d^\beta} < 0$$

By assumption 6.2, we have

$$f_v(\xi(x))V_\lambda(x^o) \leq (-\Delta)^{\frac{\alpha}{2}}U_\lambda(x^o) \leq \frac{c}{d^\alpha}U_\lambda(x^o)$$

so we derive

$$\frac{d^\alpha}{c}f_v(\xi(x))V_\lambda(x^o) \leq U_\lambda(x^o)$$

By assumption 6.3, we have

$$(-\Delta)^{\frac{\beta}{2}}V_\lambda(\bar{x}) - g_u(\eta(\bar{x}))U_\lambda(\bar{x}) \geq 0$$

We derive

$$
\begin{aligned}
0 &\leq (-\Delta)^{\frac{\beta}{2}} V_\lambda(\bar{x}) - g_u(\eta(\bar{x}))U_\lambda(\bar{x}) \\
&\leq \frac{cV_\lambda(\bar{x})}{d^\beta} - g_u(\eta(\bar{x}))U_\lambda(\bar{x})) \\
&\leq \frac{cV_\lambda(\bar{x})}{d^\beta} - g_u(\eta(\bar{x}))U_\lambda(x^o) \\
&\leq \frac{cV_\lambda(\bar{x})}{d^\beta} - g_u(\eta(\bar{x}))(f_v(\xi(x^o))V_\lambda(x^o)\frac{d^\alpha}{c}) \\
&\leq \frac{cV_\lambda(\bar{x})}{d^\beta} - g_u(\eta(\bar{x}))(f_v(\xi(x^o))V_\lambda(\bar{x})\frac{d^\alpha}{c}) \\
&\leq \frac{cV_\lambda(\bar{x})}{d^\beta}(1 - g_u(\eta(\bar{x}))f_v(\xi(x^o))\frac{d^{\alpha+\beta}}{c^2})
\end{aligned}
\tag{6.7}
$$

If λ is sufficiently close to -1, d would be sufficiently small,

$$g_u(\eta(\bar{x}))f_v(\xi(x^o))\frac{d^{\alpha+\beta}}{c^2} << 1$$

and

$$V_\lambda(\bar{x}) < 0$$

So we derive

$$\frac{cV_\lambda(\bar{x})}{d^\beta}(1 - g_u(\eta(\bar{x}))f_v(\xi(x^o))\frac{d^{\alpha+\beta}}{c^2}) < 0$$

This contradiction shows that (6.5) must be true. So far, we have proved the theorem 6.2. □

This completes step 1.

6.3 *Step 2*: *Move the plane continuously to the right until its limiting position as long as* $U_\lambda(x) \geq 0$ *,* $V_\lambda(x) \geq 0$ *holds.*

Define

$$\lambda_0 = sup\{\lambda \leq 0 \mid U_\mu(x) \geq 0, V_\mu(x) \geq 0, \forall x \in \Omega_\mu, \mu \leq \lambda\}$$

In this part, we show that

$$\lambda_0 = 0$$

Suppose

$$\lambda_0 < 0$$

we show that the plane T_{λ_0} can be moved further to the right. To be precise, for some $\epsilon > 0$ small such $\lambda_0 + \epsilon < 0$, it holds that for any $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, $U_\lambda(x), V_\lambda(x) \geq 0$, $x \in \Sigma_{\lambda_0}$.

This is a contradiction with the definition of λ_0. Hence we must have

$$\lambda_0 = 0$$

6.3.1 Show $U_{\lambda_0}(x), V_{\lambda_0}(x) > 0$ for $x \in \Omega_{\lambda_o}$

Suppose that $\lambda_0 < 0$, then the reflection of the curved part of $\partial\Omega_{\lambda_0}$ falls inside $B_1(0)$ and $U_{\lambda_0}(x), V_{\lambda_0}(x) \geq 0$ for $x \in \partial\Omega_{\lambda_0}$. (See Figure 7)

We want to show

$$\begin{cases} U_{\lambda_0}(x) > 0, & x \in \Omega_{\lambda_0}, \\ V_{\lambda_0}(x) > 0, & x \in \Omega_{\lambda_0}, \end{cases} \tag{6.8}$$

Suppose otherwise, (6.8) does not hold, then there exists $x^o \in \Omega_0$ such that $U_{\lambda_0}(x^o) = 0$.

Since $U_{\lambda_0}(x) \geq 0$ inside Ω_{λ_0}, so for $U_{\lambda_0}(x^o) = 0$, we know x^o is the minimum.

Following from the argument before, we have

$$(-\Delta)^{\frac{\alpha}{2}} U_{\lambda_0}(x^o) \leq \frac{c}{d^\alpha} U_{\lambda_0}(x^o) = 0 \tag{6.9}$$

On the other hand, by assumption 6.2, we have

$$(-\Delta)^{\frac{\alpha}{2}} U_{\lambda_0}(x^o) = f(v_{\lambda_0}(x^o)) - f(v(x^o)) \leq 0$$

Therefore, by the monotonicity of f, we have

$$V_{\lambda_0}(x^o) \leq 0$$

Since $V_{\lambda_0}(x) \geq 0$ inside Ω_{λ_0}, this implies that there exists some $\bar{x} \in \Omega_{\lambda_0}$ such that

$$V_{\lambda_0}(\bar{x}) = \min_{\Omega_{\lambda_0}} V_{\lambda_0}(x) = 0$$

Following the same argument, we can derive that

$$(-\Delta)^{\frac{\beta}{2}} V_{\lambda_0}(\bar{x}) \leq \frac{c V_{\lambda_0}(\bar{x})}{d^\beta} = 0$$

By 6.2, we have

$$f_v(\xi(x)) V_{\lambda_0}(x^o) \leq (-\Delta)^{\frac{\alpha}{2}} U_{\lambda_0}(x^o) \leq \frac{c}{d^\alpha} U_{\lambda_0}(x^o)$$

so we derive

$$\frac{d^\alpha}{c} f_v(\xi(x)) V_{\lambda_0}(x^o) \leq U_{\lambda_0}(x^o)$$

By 6.3, we have

$$(-\Delta)^{\frac{\beta}{2}} V_{\lambda_0}(\bar{x}) - g_u(\eta(\bar{x})) U_{\lambda_0}(\bar{x}) \geq 0$$

We derive

$$\begin{aligned}
0 &\leq (-\Delta)^{\frac{\beta}{2}} V_{\lambda_0}(\bar{x}) - g_u(\eta(\bar{x})) U_{\lambda_0}(\bar{x}) \\
&\leq \frac{c V_{\lambda_0}(\bar{x})}{d^\beta} - g_u(\eta(\bar{x})) U_{\lambda_0}(\bar{x})) \\
&\leq \frac{c V_{\lambda_0}(\bar{x})}{d^\beta} - g_u(\eta(\bar{x})) U_{\lambda_0}(x^o) \\
&\leq \frac{c V_{\lambda_0}(\bar{x})}{d^\beta} - g_u(\eta(\bar{x}))(f_v(\xi(x^o)) V_{\lambda_0}(x^o) \frac{d^\alpha}{c}) \\
&\leq \frac{c V_{\lambda_0}(\bar{x})}{d^\beta} - g_u(\eta(\bar{x}))(f_v(\xi(x^o)) V_{\lambda_0}(\bar{x}) \frac{d^\alpha}{c}) \\
&\leq \frac{c V_{\lambda_0}(\bar{x})}{d^\beta} (1 - g_u(\eta(\bar{x})) f_v(\xi(x^o)) \frac{d^{\alpha+\beta}}{c^2})
\end{aligned}$$

If λ is sufficiently close to -1, d would be sufficiently small,

$$g_u(\eta(\bar{x})) f_v(\xi(x^o)) \frac{d^{\alpha+\beta}}{c^2} << 1$$

and

$$V_{\lambda_0}(\bar{x}) = 0$$

So we derive

$$\frac{c V_{\lambda_0}(\bar{x})}{d^\beta} (1 - g_u(\eta(\bar{x})) f_v(\xi(x^o)) \frac{d^{\alpha+\beta}}{c^2}) = 0$$

This shows that

$$(-\Delta)^{\frac{\beta}{2}} V_{\lambda_0}(\bar{x}) - g_u(\eta(\bar{x})) U_{\lambda_0}(\bar{x}) = 0$$

This implies

$$g_u(\eta(\bar{x}))U_{\lambda_0}(\bar{x}) \le \frac{cV_{\lambda_0}(\bar{x})}{d^\beta} = 0$$

and since $g_u(\eta(\bar{x}))$ is positive and bounded, that means

$$U_{\lambda_0}(\bar{x}) \le \frac{cV_{\lambda_0}(\bar{x})}{d^\beta} = 0$$

but we know

$$U_{\lambda_0}(\bar{x}) \ge 0$$

So we conclude

$$U_{\lambda_0}(\bar{x}) \equiv 0 \tag{6.10}$$

We derive a contradiction since the plane T_{λ_0} did not reach the origin. If we take a point x_* on the curved part $\partial\Omega_{\lambda_0}$, then it's reflection point $x_*^{\lambda_0}$ is in the interior of the ball, and hence $u(x_*^{\lambda_0}) > 0$, therefore, $U_{\lambda_0}(x_*) = u(x_*^{\lambda_0}) - u(x_*) > 0$, which contradicts (6.10).

We conclude $U_{\lambda_0}(x) > 0, V_{\lambda_0}(x) > 0$ for every $x \in \Omega_{\lambda_0}$.

Next, We want to further derive

$$\begin{cases} \inf U_{\lambda_0}(x) > c_o > 0, \ x \in \Omega_{\lambda_0 - \delta}, \\ \inf V_{\lambda_0}(x) > c_o > 0, \ x \in \Omega_{\lambda_0 - \delta}, \end{cases} \tag{6.11}$$

I will prove (6.11) by contradiction.

6.3.2 Show $U_{\lambda_0}(y) \equiv 0, \ \forall y \in \Omega_{\lambda_o - \delta}$

Proof: If (6.11) is violated, then $\exists x_k \in \Omega_{\lambda_0 - \delta}$ such that $U_{\lambda_0}(x_k) \to 0$, Without loss of generality, by Bolzano-Weierstrass theorem, $\exists$ subsequences of x_k, here we still denote this subsequence by x_k, such that $x_k \to x^o \in \Omega_{\lambda_0-\delta}$,

so

$$U_k(x) = U_{\lambda_0}(x_k) \to 0$$

Following from (6.25), we have

$$\begin{aligned}
& (-\Delta)^{\frac{\alpha}{2}} U_k(x) \\
= \; & f(v_{\lambda_0}(x_k)) - f(v(x_k)) \\
= \; & \frac{f(v_{\lambda_0}(x_k))}{v^p_{\lambda_0}(x_k)} v^p_{\lambda_0}(x_k) - \frac{f(v(x_k))}{v^p(x_k)} v^p(x_k) \\
\geq \; & \frac{f(v(x_k))}{v^p(x_k)} [v^p_{\lambda_0}(x_k) - v^p(x_k)] \\
= \; & \frac{f(v(x_k))}{v^p(x_k)} p\xi^{p-1}(x_k) V_{\lambda_0}(x_k) \\
:= \; & f_v(\xi(x_k)) V_{\lambda_0}(x_k) \\
:= \; & f_v(\xi(x_k)) V_k(x)
\end{aligned}$$

So U_k satisfies

$$(-\triangle)^{\frac{\alpha}{2}} U_k(x) \geq f_v(\xi(x_k)) V_k(x) \tag{6.12}$$

By regularity theory for parabolic equations [45], there exists some functions $\bar{U}(x)$ and $\bar{c}(x)$ such that $k \to \infty$, $U_k(x)$ converges uniformly to $\bar{U}(x)$, and $f_v(\xi(x_k))$ converges uniformly to $\bar{c}(x)$ with $\bar{c}(x) \geq 0$.

Since

$$U_k(x_k) = U_{\lambda_0}(x_k) \to 0$$

$$\bar{U}(x^o) = 0$$

and

$$\bar{U} \geq 0$$

So x^o is the minimum. Following from the argument before, we have

$$(-\Delta)^{\frac{\alpha}{2}} \bar{U}(x^o) \leq \frac{c}{d^\alpha} \bar{U}(x^o) = 0 \tag{6.13}$$

On the other hand, we have

$$\begin{aligned} & (-\Delta)^{\frac{\alpha}{2}} \bar{U}(x^o) \\ \geq \; & \bar{c}(x^o) \bar{V}(x^o) \\ \geq \; & 0 \end{aligned}$$

but we know

$$\begin{aligned} & (-\Delta)^{\frac{\alpha}{2}} \bar{U}(x^o) \\ \leq \; & \frac{c}{d^\alpha} \bar{U}(x^o) \\ = \; & 0 \end{aligned}$$

So we conclude

$$(-\Delta)^{\frac{\alpha}{2}} \bar{U}(x^o) = 0 \tag{6.14}$$

It follows that

$$
\begin{aligned}
&0=(-\Delta)^{\frac{\alpha}{2}}\bar{U}(x^o)\\
=\ & C_{n,\alpha}PV\int_{\mathbb{R}^n}\frac{-\bar{U}(y)}{|x^o-y|^{n+2s}}dy\\
=\ & C_{n,\alpha}PV\{\int_{\Sigma_{\lambda_0}}\frac{-\bar{U}(y)}{|x^o-y|^{n+2s}}dy+\int_{\widetilde{\Sigma_{\lambda_0}}}\frac{-\bar{U}(y)}{|x^o-y|^{n+2s}}dy\}\\
=\ & C_{n,\alpha}PV\{\int_{\Sigma_{\lambda_0}}\frac{-\bar{U}(y)}{|x^o-y|^{n+2s}}dy+\int_{\Sigma_{\lambda_0}}\frac{-\bar{U}(y^\lambda)}{|x^o-y^\lambda|^{n+2s}}dy\}\\
=\ & C_{n,\alpha}PV\{\int_{\Sigma_{\lambda_0}}\frac{-\bar{U}(y)}{|x^o-y|^{n+2s}}dy+\int_{\Sigma_{\lambda_0}}\frac{\bar{U}(y)}{|x^o-y^\lambda|^{n+2s}}dy\}\\
=\ & C_{n,\alpha}PV\int_{\Sigma_{\lambda_0}}\bar{U}(y)\{\frac{1}{|x^o-y^\lambda|^{n+2s}}-\frac{1}{|x^o-y|^{n+2s}}\}dy\\
\le\ & 0
\end{aligned}
\tag{6.15}
$$

This implies

$$\bar{U}(y)\equiv 0,\ \ y\in\mathbb{R}^n$$

6.3.3 Show $\bar{u}(x)\equiv 0$ for $x\in\mathbb{R}^n$

Let $u_k(x)=u(x_k)$, then by (6.2), we have

$$(-\Delta)^{\frac{\alpha}{2}}u_k(x)=f(v_k(x))$$

By regularity theory for parabolic equations [45], there exists some functions $\bar{u}(x)$ such that as $k\to\infty$, $u_k(x)$ converges uniformly to $\bar{u}(x)$, and f converges uniformly to $\bar{f}$, here we assume:

$$f\ge 0$$

so

$$(-\Delta)^{\frac{\alpha}{2}}\bar{u}(x)=\bar{f}\ge 0$$

In order to show that

$$\bar{u}(x) \equiv 0, x \in \mathbb{R}^n \tag{6.16}$$

we apply the following:

Lemma 6.3. *(Strong Maximum Principle for $(-\Delta)^{\frac{\alpha}{2}}\bar{u} = f(\bar{v}(x))$).*

Assume that $\bar{u}(x) \in [C^{1,1}_{loc}(\Omega_\lambda) \cap C(\bar{\Omega_\lambda}) \cap \mathcal{L}_{2s}]$

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}}\bar{u}(x) = \bar{f}, & x \in \Omega_\lambda, \\ \bar{u}(x) \geq 0, & x \in \Omega_\lambda \end{cases} \tag{6.17}$$

we have either

$$\bar{u}(x) > 0, x \in B_1(0)$$

or

$$\bar{u}(x) \equiv 0, x \in \mathbb{R}^n$$

Proof. First, if $\bar{u}(x) \geq 0$ and $\bar{u}(x^o) = 0$, x^o then is a minimum, If $\bar{u}(x) \not\equiv 0$, then

$$\begin{aligned}
& (-\Delta)^{\frac{\alpha}{2}}\bar{u}(x^o) \\
= \;& C_{n,\alpha}PV\int_{\mathbb{R}^n}\frac{-\bar{u}(y)}{|x^o-y|^{n+\alpha}}dy \\
= \;& C_{n,\alpha}PV\{\int_{\Sigma_{\lambda_0}}\frac{-\bar{u}(y)}{|x^o-y|^{n+\alpha}}dy+\int_{\widetilde{\Sigma_{\lambda_0}}}\frac{-\bar{u}(y)}{|x^o-y|^{n+\alpha}}dy\} \\
= \;& C_{n,\alpha}PV\{\int_{\Sigma_{\lambda_0}}\frac{-\bar{u}(y)}{|x^o-y|^{n+\alpha}}dy+\int_{\Sigma_{\lambda_0}}\frac{-\bar{u}(y^\lambda)}{|x^o-y^\lambda|^{n+\alpha}}dy\} \\
= \;& C_{n,\alpha}PV\{\int_{\Sigma_{\lambda_0}}\frac{-\bar{u}(y)}{|x^o-y|^{n+\alpha}}dy+\int_{\Sigma_{\lambda_0}}\frac{\bar{u}(y)}{|x^o-y^\lambda|^{n+\alpha}}dy\} \\
= \;& C_{n,\alpha}PV\int_{\Sigma_{\lambda_0}}\bar{u}(y)\{\frac{1}{|x^o-y^\lambda|^{n+\alpha}}-\frac{1}{|x^o-y|^{n+\alpha}}\}dy \\
< \;& 0
\end{aligned}$$

since

$$\frac{1}{|x^o-y^\lambda|^{n+\alpha}}-\frac{1}{|x^o-y|^{n+\alpha}}<0$$

and

$$\bar{u}(y) \geq 0$$

and

$$\bar{u}(y) \not\equiv 0$$

However, we have

$$(-\Delta)^{\frac{\alpha}{2}}\bar{u}(x^o)=\bar{f}\geq 0$$

which is a contradiction. Therefore, we have either $\bar{u}(x) > 0$, $x \in B_1(0)$ or $\bar{u}(x) \equiv 0$, $x \in \mathbb{R}^n$.

□

If $\bar{u}(x) > 0$, $x \in B_1(0)$, we know $\lambda_0 < 0$, if we take a point $\bar{x}$ on the curved part

$\partial\Omega_{\lambda_0}$, then it's reflection point $\bar{x}^{\lambda_0}$ is in the interior of the ball (see Figure 11)and hence $u(\bar{x}^{\lambda_0}) > 0$, therefore, $\bar{U}(\bar{x}) = \bar{u}(\bar{x}^{\lambda_0}) - \bar{u}(\bar{x}) > 0$. $\bar{U}(x) > 0$ somewhere, but we already derive $\bar{U}(x) \equiv 0$, hence we must have $\bar{u}(x) \equiv 0$, $x \in \mathbb{R}^n$.

Thus, we know $u(x_k)$ converges to 0 uniformly.

6.3.4 Derive a contradiction for large k for U_k

In order to derive a contradiction for large k, Let

$$U_k(x_k) \equiv U_{\lambda_0}(x_k) = m_k \tag{6.18}$$

which converges to zero.

Let

$$a_k(x) = U_k(x) - 2m_k \tag{6.19}$$

So

$$\begin{aligned} & a_k(x_k) \\ = \ & U_k(x_k) - 2m_k \\ = \ & U_k - 2m_k \\ = \ & -m_k \end{aligned}$$

Thus there must be a minimum point $\bar{x_k}$ such that

$$a_k(\bar{x}_k) \leq -m_k$$

Then

$$(-\triangle)^s a_k = c_\lambda a_k + 2m_k$$

By regularity theory for parabolic equations [45], there exists some functions $\bar{a}(x)$ such that $k \to \infty$, $a_k(x) \to \bar{a}(x)$ converges uniformly for $x \in \Omega_{\lambda_o}$,

Moreover

$$(-\Delta)^s \bar{a} = c_\lambda \bar{a}$$

Passing to a subsequence, $(\bar{x}_k) \to (x^o) \in \Omega_{\lambda_o - \delta}$

$U_k \to \bar{U}$ uniformly, and

$$(-\Delta)^s \bar{U} = \bar{c}\bar{U}$$

As we have already derived

$$\bar{U}(x^o) = 0$$

Also following from (6.25)

$$\begin{aligned}
& (-\Delta)^{\frac{\alpha}{2}} \bar{U}(x^o) \\
= \quad & f(\bar{v}_{\lambda_0}(x^o)) - f(\bar{v}(x^o)) \\
= \quad & \frac{f(\bar{v}_{\lambda_0}(x^o))}{\bar{v}^p_{\lambda_0}(x^o)} \bar{v}^p_{\lambda_0}(x^o) - \frac{f(\bar{v}(x, t^o + t_k))}{\bar{v}^p(x^o)} \bar{v}^p(x^o) \\
\geq \quad & \frac{f(\bar{v}(x^o))}{\bar{v}^p(x^o)} [\bar{v}^p_{\lambda_0}(x^o) - \bar{v}^p(x^o)] \\
= \quad & \frac{f(\bar{v}(x^o))}{\bar{v}^p(x^o)} p\xi^{p-1}(x^o) V_{\lambda_0}(x^o) \\
:= \quad & f_{\bar{v}}(\xi(x^o)) V_{\lambda_0}(x^o)
\end{aligned}$$

This implies

$$(-\Delta)^{\frac{\alpha}{2}} \bar{U}(x^o) \geq f_{\bar{v}}(\xi(x^o)) V_{\lambda_0}(x^o)$$

and

$$f_{\bar{v}}(\xi(x^o))V_{\lambda_0}(x^o) \geq 0$$

It follows

$$\begin{aligned}
& 0 \leq (-\Delta)^{\frac{\alpha}{2}}\bar{U}(x^o) \\
= \;& C_{n,\alpha}PV\int_{\mathbb{R}^n}\frac{-\bar{U}(y)}{|x^o-y|^{n+\alpha}}dy \\
= \;& C_{n,\alpha}PV\{\int_{\Sigma_{\lambda_0}}\frac{-\bar{U}(y)}{|x^o-y|^{n+\alpha}}dy+\int_{\widetilde{\Sigma_{\lambda_0}}}\frac{-\bar{U}(y)}{|x^o-y|^{n+\alpha}}dy\} \\
= \;& C_{n,\alpha}PV\{\int_{\Sigma_{\lambda_0}}\frac{-\bar{U}(y)}{|x^o-y|^{n+\alpha}}dy+\int_{\Sigma_{\lambda_0}}\frac{-\bar{U}(y^\lambda)}{|x^o-y^\lambda|^{n+\alpha}}dy\} \\
= \;& C_{n,\alpha}PV\{\int_{\Sigma_{\lambda_0}}\frac{-\bar{U}(y)}{|x^o-y|^{n+\alpha}}dy+\int_{\Sigma_{\lambda_0}}\frac{\bar{U}(y)}{|x^o-y^\lambda|^{n+\alpha}}dy\} \\
= \;& C_{n,\alpha}PV\int_{\Sigma_{\lambda_0}}\bar{U}(y)\{\frac{1}{|x^o-y^\lambda|^{n+\alpha}}-\frac{1}{|x^o-y|^{n+\alpha}}\}dy \\
\leq \;& 0
\end{aligned}$$

since

$$\frac{1}{|x^o-y^\lambda|^{n+\alpha}}-\frac{1}{|x^o-y|^{n+\alpha}}<0$$

and

$$\bar{U}(y) \geq 0$$

This implies

$$\bar{U}(y) \equiv 0, \forall y \in \mathbb{R}^n$$

Similar with above, for

$$u_k(x) = u(x_k)$$

$$u_k(x) \to \bar{u}(x)$$

$$(-\Delta)^{\frac{\alpha}{2}}\bar{u} \geq \bar{f}_v$$

We have

$$\bar{u}(x) \equiv 0$$

Now for sufficiently large k,

$$\begin{aligned} & (-\Delta)^{\frac{\alpha}{2}}a_k(\bar{x}_k) \\ = \; & (-\Delta)^{\frac{\alpha}{2}}U_k \\ \geq \; & f_v(\xi(\bar{x}_k))V_k(\bar{x}_k) \\ > \; & 0 \end{aligned}$$

Since we know

$$\begin{aligned} & (-\Delta)^{\frac{\alpha}{2}}a_k(\bar{x}_k) \\ \leq \; & \frac{c}{[d(\bar{x}_k, T_{\lambda_o})]^{\alpha}}a_k(\bar{x}_k) \\ \leq \; & -c_1 m_k \end{aligned}$$

Finally,

$$-c_1 m_k \geq 0$$

which is a contradiction.

Hence, we have proved (6.11).

Since U_λ, V_λ depends on λ continuously, there exists $\epsilon > 0$ and $\epsilon < \delta$, such that for all

$\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, we have

$$\begin{cases} U_\lambda(x) \geq 0, \ \ x \in \Omega_{\lambda_0 - \delta}, \\ V_\lambda(x) \geq 0, \ \ x \in \Omega_{\lambda_0 - \delta}, \end{cases} \tag{6.20}$$

Now apply the Narrow region theorem **??** and in our case the narrow region is

$$\Omega_\lambda^- \backslash \Omega_{\lambda_0 - \delta}$$

By Narrow region theorem, we derive

$$\begin{cases} U_\lambda(x) \geq 0, \ \ x \in \Omega_\lambda^- \backslash \Omega_{\lambda_0 - \delta}, \\ V_\lambda(x) \geq 0, \ \ x \in \Omega_\lambda^- \backslash \Omega_{\lambda_0 - \delta}, \end{cases} \tag{6.21}$$

Combining (6.20) and (6.21), we conclude that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$

$$\begin{cases} U_\lambda(x) \geq 0, \ \ x \in \Omega_{\lambda_0}, \\ V_\lambda(x) \geq 0, \ \ x \in \Omega_{\lambda_0}, \end{cases}$$

This contradicts the definition of λ_0. Therefore, we must have

$$\lambda_0 = 0$$

and

$$\begin{cases} U_{\lambda_0}(x) \geq 0, \ \ \forall x \in \Omega_{\lambda_0}, \\ V_{\lambda_0}(x) \geq 0, \ \ \forall x \in \Omega_{\lambda_0}, \end{cases}$$

Similarly, one can move the plane T_λ from $\lambda = 1$ to the left and show that

$$\begin{cases} U_{\lambda_0}(x) \leq 0, \ \forall x \in \Omega_{\lambda_0}, \\ V_{\lambda_0}(x) \leq 0, \ \forall x \in \Omega_{\lambda_0}, \end{cases}$$

Now we have shown that

$$\lambda_0 = 0$$

and

$$\begin{cases} U_{\lambda_0} \equiv 0, \ x \in \Omega_{\lambda_0}, \\ V_{\lambda_0} \equiv 0, \ x \in \Omega_{\lambda_0}, \end{cases}$$

This completes the step 2.

6.3.5 Conclude the solution is radially symmetric and monotone decreasing

So far, we have proved that u, v are symmetric about the plane T_0. Since the x_1 direction can be chosen arbitrarily, we have actually shown that u, v are radially symmetric about origin.

Since $U_\lambda(x) \not\equiv 0$, $x \in T_\lambda$, $\forall 0 < \lambda < \lambda_0$, if there exists x^o such that x^o is the minimum point, from the above process, on one hand,

$$(-\Delta)^{\frac{\alpha}{2}} U_\lambda(x^o) \leq 0$$

On the other hand,

$$(-\Delta)^{\frac{\alpha}{2}} U_\lambda(x^o) = 0$$

This forces

$$U_\lambda \equiv 0$$

which is a contradiction. Therefore, u is monotone decreasing about the origin. Same reason for v.

6.4 Radial Symmetry of solutions of parabolic fractional systems

In the following section, we will prove the solution of the parabolic fractional system is radially symmetric using the method of moving planes.

Theorem 6.4. *(Radial Symmetry of solution of parabolic fractional system)*

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} u(x,t) = f(v(x,t)), & (x,t) \in B_1(0) \times (-\infty, \infty), \\ \frac{\partial v}{\partial t} + (-\Delta)^{\frac{\beta}{2}} v(x,t) = g(u(x,t)), & (x,t) \in B_1(0) \times (-\infty, \infty), \\ u(x,t), v(x,t) \geq 0, & (x,t) \in B_1(0) \times (-\infty, \infty), \\ u(x,t), v(x,t) \equiv 0, & x \notin B_1(0), \end{cases} \tag{6.22}$$

where B_1 is a unit ball.

Let $0 < \alpha, \beta < 2$, and suppose that $u(x,t), v(x,t) \in \left(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\right) \times (-\infty, \infty)$ are positive bounded classical solutions of equation (6.22)*, and assume $f(v(x,t)), g(u(x,t))$ satisfies the following assumptions:*

(M1) $f(\cdot)$ is non-decreasing in $v(\cdot)$, $g(\cdot)$ is non-decreasing in $u(\cdot)$.

(M2) Assume that f, g are uniformly Lipschitz continuous in u, v. i.e:

$$|f(v_1) - f(v_2)| \leq c|v_1 - v_2|,$$

$$|g(u_1) - f(u_2)| \leq c|u_1 - u_2|,$$

then $u(x,t), v(x,t)$ are radially symmetric about the origin and monotone decreasing about the origin, i.e.

$$u(x,t) = u(|x|,t), v(x,t) = v(|x|,t)$$

$$u(x_1,t) > u(x_2,t), v(x_1,t) > v(x_2,t), |x_1| < |x_2|$$

.

6.4.1 Basic set-up

For any given $\lambda \in \mathbb{R}$, let

$$T_\lambda = \{x \in \mathbb{R}^n \mid x_1 = \lambda \text{ for } \lambda \in \mathbb{R}\}$$

be the moving planes,

$$\Sigma_\lambda := \{x \in \mathbb{R}^n \mid x_1 < \lambda\}$$

be the region to the left of the plane, and

$$x^\lambda = (2\lambda - x_1, x_2, ..., x_n)$$

be the reflection of x about the plane T_λ. and

$$\Omega_\lambda = \Sigma_\lambda \cap B_1(0)$$

be the intersection of $B_1(0)$ and Σ_λ.

Assume that $u(x,t), v(x,t)$ are positive solutions of equation (6.22). We compare the values of $u(x,t), v(x,t)$ with

$$u_\lambda(x,t) = u(x^\lambda, t), v_\lambda(x,t) = v(x^\lambda, t)$$

Let

$$U_\lambda(x,t) = u_\lambda(x,t) - u(x,t), V_\lambda(x,t) = v_\lambda(x,t) - v(x,t)$$

Step 1: *Begin moving the plane from near the left end of $B_1(0)$ along the x_1 axis, but do not reach origin*,

So then

$$|x^\lambda| < |x|$$

We deduce from the equation (6.22) and (M1), (M2) that U_λ, V_λ satisfies

$$\begin{aligned}
& \frac{\partial U_\lambda}{\partial t}(x,t) + (-\Delta)^{\frac{\alpha}{2}} U_\lambda(x,t) \\
= \quad & f(v_\lambda(x,t)) - f(v(x,t)) \\
= \quad & \frac{f(v_\lambda(x,t))}{v_\lambda^p(x,t)} v_\lambda^p(x,t) - \frac{f(v(x,t))}{v^p(x,t)} v^p(x,t) \\
\geq \quad & \frac{f(v(x,t))}{v^p(x,t)} [v_\lambda^p(x,t) - v^p(x,t)] \\
= \quad & \frac{f(v(x,t))}{v^p(x,t)} p\xi^{p-1}(x,t) V_\lambda(x,t) \\
:= \quad & f_v(\xi(x,t)) V_\lambda(x,t),
\end{aligned} \tag{6.23}$$

$$
\begin{aligned}
& \frac{\partial V_\lambda}{\partial t}(x,t) + (-\Delta)^{\frac{\beta}{2}} V_\lambda(x,t) \\
= \quad & g(u_\lambda(x,t)) - g(u(x,t)) \\
= \quad & \frac{g(u_\lambda(x,t))}{u_\lambda^p(x,t)} u_\lambda^p(x,t) - \frac{g(u(x,t))}{u^p(x,t)} u^p(x,t) \\
\geq \quad & \frac{g(u(x,t))}{u^p(x,t)} [u_\lambda^p(x,t) - u^p(x,t)] \\
= \quad & \frac{g(u(x,t))}{u^p(x,t)} p\eta^{p-1}(x,t) U_\lambda(x,t) \\
:= \quad & g_u(\eta(x,t)) U_\lambda(x,t),
\end{aligned}
\tag{6.24}
$$

where

$$
f_v(\xi(x,t)), g_u(\eta(x,t))
$$

are bounded and positive.

Apparently, Ω_λ is a narrow region in the x_1 direction for λ very close to -1. For further application of Narrow Region Principle, We will prove the Narrow region theorem in parabolic fractional system in the following sections.

In Parabolic case, we add a time dimension on this unit ball with lower edge $t = \underline{t}$ and higher edge $t = T$, denotes this thin cylinder as $\Omega_\lambda \times (\underline{t}, T]$ to be the narrow region we want to use(See Figure 9).

6.5 *Step 1: show* $U_\lambda(x,t), V_\lambda(x,t) \geq 0$

To show $U_\lambda(x,t), V_\lambda(x,t) \geq 0$, we first show that $U_\lambda(x,t), V_\lambda(x,t)$ can not attain its negative minimum in $\Omega_\lambda \times (\underline{t}, T]$, to attain this goal, we will first prove a narrow region principle in $\Omega_\lambda \times (\underline{t}, T]$.

6.5.1 Narrow region principle on a parabolic cylinder

Theorem 6.5. *(Narrow region principle on a parabolic cylinder) Let $\Omega_\lambda = \Sigma_\lambda \cap B_1(0)$, Ω_λ is a bounded narrow region in Σ_λ, assume that $U_\lambda(x,t), V_\lambda(x,t) \in [C^{1,1}_{loc}(\Omega_\lambda) \cap C(\bar{\Omega_\lambda}) \cap \mathcal{L}_{2s}] \times C^1([\underline{t},T])$, and $U_\lambda(x,t)$, $V_\lambda(x,t)$ are lower semi-continuous on $\bar{\Omega}$. If*

$$\begin{cases} \frac{\partial U_\lambda}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} U_\lambda(x,t) \geq f_v(\xi(x,t)) V_\lambda(x,t), & (x,t) \in \Omega_\lambda \times [\underline{t},T], \\ \frac{\partial V_\lambda}{\partial t} + (-\Delta)^{\frac{\beta}{2}} V_\lambda(x,t) \geq g_u(\eta(x,t)) U_\lambda(x,t), & (x,t) \in \Omega_\lambda \times [\underline{t},T], \end{cases} \tag{6.25}$$

Then for λ sufficiently close to -1, we have

$$U_\lambda(x,t) \geq \min\{0, \inf_{\Omega_\lambda \times [\underline{t},T]} U_\lambda(x,\underline{t})\}, \ (x,t) \in \Omega_\lambda \times [\underline{t},T] \tag{6.26}$$

and

$$V_\lambda(x,t) \geq \min\{0, \inf_{\Omega_\lambda} V_\lambda(x,\underline{t})\}, \ (x,t) \in \Omega_\lambda \times [\underline{t},T] \tag{6.27}$$

Proof. If (6.26) does not hold, then the lower semi-continuity of $U_\lambda(x,t)$ on $\bar{\Omega_\lambda} \times [\underline{t},T]$ guarantees that there exists an $(x^o,t^o) \in \Omega_\lambda \times (\underline{t},T]$ such that

$$U_\lambda(x^o,t^o) = \min_{\Omega_\lambda \times (\underline{t},T]} U_\lambda < 0$$

By the defining integral of the fractional Laplacian, we have

$$\begin{aligned} & (-\Delta)^{\frac{\alpha}{2}} U_\lambda(x^o,t^o) \\ \leq \ & C_{n,\alpha} \int_{\Sigma_\lambda} \frac{2U_\lambda(x^o,t^o)}{|x^o - y^\lambda|^{n+\alpha}} \end{aligned}$$

Similar to the argument in Section 3, by Lemma 5.7, we have

$$\int_{\Sigma_\lambda} \frac{1}{|x^o - y^\lambda|^{n+\alpha}} dy \geq \frac{c}{d^\alpha}$$

Also, since (x^o, t^o) is the minimum,

If

$$t^o < T$$

$$\frac{\partial U_\lambda}{\partial t}(x^o, t^o) = 0$$

If

$$t^o = T$$

$$\frac{\partial U_\lambda}{\partial t}(x^o, t^o) \leq 0$$

Following from

$$(-\triangle)^{\frac{\alpha}{2}} U_\lambda(x^o, t^o) \leq C_{n,\alpha} \int_{\Sigma_\lambda} \frac{2U_\lambda(x^o, t^o)}{|x^o - y^\lambda|^{n+\alpha}}$$

We have

$$(-\triangle)^{\frac{\alpha}{2}} U_\lambda(x^o, t^o) \leq \frac{cU_\lambda(x^o, t^o)}{d^\alpha}$$

we deduce

$$\begin{aligned} & \frac{\partial U_\lambda(x^o, t^o)}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} U_\lambda(x^o, t^o) \\ \leq \quad & \frac{cU_\lambda(x^o, t^o)}{d^\alpha} + \frac{\partial U_\lambda(x^o, t^o)}{\partial t}, \\ < \quad & 0, \end{aligned} \tag{6.28}$$

On the other hand, by 6.25, we have

$$\frac{\partial U_\lambda(x^o,t^o)}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} U_\lambda(x^o,t^o) = f(v_\lambda(x^o,t^o)) - f(v(x^o,t^o)) < 0$$

Therefore, by the monotonicity of f, we have

$$V_\lambda(x^o,t^o) < 0$$

This implies that there exists some $(\bar{x},\bar{t}) \in \Omega_{\lambda_0} \times (\underline{t},T]$ such that

$$V_\lambda(\bar{x},\bar{t}) = \min_{\Omega_{\lambda_0}\times(\underline{t},T]} V_\lambda < 0$$

So that

$$\frac{\partial V_\lambda}{\partial t}(\bar{x},\bar{t}) = 0$$

Following the same argument, we can derive that

$$(-\Delta)^{\frac{\beta}{2}} V_\lambda(\bar{x},\bar{t}) \leq \frac{cV_\lambda(\bar{x},\bar{t})}{d^\beta} < 0$$

From (6.25), we have

$$f_v(\xi(x,t))V_\lambda(x^o,t^o) \leq \frac{cU_\lambda(x^o,t^o)}{d^\alpha} + \frac{\partial U_\lambda(x^o,t^o)}{\partial t}$$

we derive

$$f_v(\xi(x^o,t^o))V_\lambda(x^o,t^o) \leq \frac{cU_\lambda(x^o,t^o)}{d^\alpha} < 0$$

so we derive

$$\frac{d^\alpha}{c} f_v(\xi(x^o,t^o))V_\lambda(x^o,t^o) \leq U_\lambda(x^o,t^o)$$

By 6.25, we have

$$\begin{aligned} & \frac{\partial V_\lambda}{\partial t}(\bar{x},\bar{t}) + (-\Delta)^{\frac{\beta}{2}} V_\lambda(\bar{x},\bar{t}) - g_u(\eta(\bar{x}))U_\lambda(\bar{x}) \geq 0 \\ := \quad & (-\Delta)^{\frac{\beta}{2}} V_\lambda(\bar{x},\bar{t}) - g_u(\eta(\bar{x}))U_\lambda(\bar{x}) \geq 0 \end{aligned}$$

We derive

$$\begin{aligned} 0 &\leq (-\Delta)^{\frac{\beta}{2}} V_\lambda(\bar{x},\bar{t}) - g_u(\eta(\bar{x},\bar{t}))U_\lambda(\bar{x},\bar{t}) \\ &\leq \frac{cV_\lambda(\bar{x},\bar{t})}{d^\beta} - g_u(\eta(\bar{x},\bar{t}))U_\lambda(\bar{x},\bar{t})) \\ &\leq \frac{cV_\lambda(\bar{x},\bar{t})}{d^\beta} - g_u(\eta(\bar{x},\bar{t}))U_\lambda(x^o,t^o) \\ &\leq \frac{cV_\lambda(\bar{x},\bar{t})}{d^\beta} - g_u(\eta(\bar{x},\bar{t}))(f_v(\xi(x^o,t^o))V_\lambda(x^o,t^o)\frac{d^\alpha}{c}) \\ &\leq \frac{cV_\lambda(\bar{x},\bar{t})}{d^\beta} - g_u(\eta(\bar{x},\bar{t}))(f_v(\xi(x^o,t^o))V_\lambda(\bar{x},\bar{t})\frac{d^\alpha}{c}) \\ &\leq \frac{cV_\lambda(\bar{x},\bar{t})}{d^\beta}(1 - g_u(\eta(\bar{x},\bar{t}))f_v(\xi(x^o,t^o))\frac{d^{\alpha+\beta}}{c^2}) \end{aligned} \tag{6.29}$$

If λ is sufficiently close to -1, d would be sufficiently small,

$$g_u(\eta(\bar{x},\bar{t}))f_v(\xi(x^o,t^o))\frac{d^{\alpha+\beta}}{c^2} << 1$$

and

$$V_\lambda(\bar{x},\bar{t}) < 0$$

So we derive

$$\frac{cV_\lambda(\bar{x},\bar{t})}{d^\beta}(1 - g_u(\eta(\bar{x},\bar{t}))f_v(\xi(x^o,t^o))\frac{d^{\alpha+\beta}}{c^2}) < 0$$

This contradiction shows that (6.26) and (6.27) must be true. So far, we have proved theorem 6.5. □

Define

$$\begin{cases} \bar{U}(x,t) = e^{m(t-\underline{t})}U_\lambda(x,t), \;\; m>0, \\ \bar{V}(x,t) = e^{m(t-\underline{t})}V_\lambda(x,t), \;\; m>0, \end{cases} \tag{6.30}$$

Then we have

$$\begin{cases} U_\lambda(x,t) = e^{-m(t-\underline{t})}\bar{U}(x,t), \;\; m>0, \\ V_\lambda(x,t) = e^{-m(t-\underline{t})}\bar{V}(x,t), \;\; m>0, \end{cases}$$

Let

$$\bar{U}(x,t) = e^{m(t-\underline{t})}U_\lambda(x,t), \;\; m>0,$$

From (6.26), we derive:

$$\frac{\partial \bar{U}(x,t)}{\partial t} + (-\triangle)^{\frac{\alpha}{2}}\bar{U}(x,t) \geq \bar{f}_v(\xi(x,t))\bar{V}(x,t)$$

$$\frac{\partial \bar{V}(x,t)}{\partial t} + (-\triangle)^{\frac{\beta}{2}}\bar{V}(x,t) \geq \bar{g}_u(\eta(x,t))\bar{U}(x,t)$$

This time, we want to show

$$\begin{cases} U_\lambda(x,t) \geq 0, \\ V_\lambda(x,t) \geq 0, \end{cases} \tag{6.31}$$

Suppose otherwise, (6.31), does not hold, then U_λ is negative somewhere, by 6.30, hence $\bar{U}$ is negative somewhere, hence there exists an $x^o \in \Omega_\lambda$ and $t^o \in [\underline{t}, T]$ such that

$$\bar{U}(x^o, t^o) = \min_{\Omega_\lambda \times (\underline{t}, T]} \bar{U} < 0$$

If

$$t^o < T$$

$$\frac{\partial \bar{U}}{\partial t}(x^o, t^o) = 0$$

If

$$t^o = T$$

$$\frac{\partial \bar{U}}{\partial t}(x^o, t^o) \leq 0$$

Hence

$$\frac{\partial \bar{U}}{\partial t}(x^o, t^o) + (-\Delta)^{\frac{\alpha}{2}} \bar{U}(x^o, t^o) \leq \frac{c}{d^\alpha} \bar{U}(x^o, t^o) < 0$$

On the other hand, we have

$$\frac{\partial \bar{U}}{\partial t}(x^o, t^o) + (-\Delta)^{\frac{\alpha}{2}} \bar{U}(x^o, t^o) \geq \bar{f}_v(\xi(x^o, t^o)\bar{V}(x^o, t^o)$$

Therefore

$$\bar{f}_v(\xi(x^o, t^o))\bar{V}(x^o, t^o) \leq \frac{c\bar{U}(x^o, t^o)}{d^\alpha} < 0$$

so we derive

$$\frac{d^\alpha}{c} \bar{f}_v(\xi(x^o, t^o))\bar{V}(x^o, t^o) \leq \bar{U}(x^o, t^o) < 0$$

Therefore, we must have

$$\bar{V}(x^o, t^o) < 0$$

This implies that there exists some $(\bar{x}, \bar{t}) \in \Omega_{\lambda_0} \times (\underline{t}, T]$ such that

$$\bar{V}(\bar{x}, \bar{t}) = \min_{\Omega_{\lambda_0} \times (\underline{t}, T]} \bar{V} < 0$$

So that

$$\frac{\partial \bar{V}}{\partial t}(\bar{x}, \bar{t}) = 0$$

Following the same argument, we can derive that

$$(-\Delta)^{\frac{\beta}{2}} \bar{V}(\bar{x}, \bar{t}) \leq \frac{c\bar{V}(\bar{x}, \bar{t})}{d^\beta} < 0$$

We derive

$$\begin{aligned}
0 &\leq (-\triangle)^{\frac{\alpha}{2}} \bar{V}(\bar{x}, \bar{t}) - \bar{g}_u(\eta(\bar{x}, \bar{t})\bar{U}(\bar{x}, \bar{t}) \\
&\leq \frac{c\bar{V}(\bar{x}, \bar{t})}{d^\beta} - \bar{g}_u(\eta(\bar{x}, \bar{t}))\bar{U}(\bar{x}, \bar{t})) \\
&\leq \frac{c\bar{V}(\bar{x}, \bar{t})}{d^\beta} - \bar{g}_u(\eta(\bar{x}, \bar{t}))\bar{U}(x^o, t^o) \\
&\leq \frac{c\bar{V}(\bar{x}, \bar{t})}{d^\beta} - \bar{g}_u(\eta(\bar{x}, \bar{t}))(f_v(\xi(x^o, t^o))\bar{V}(x^o, t^o)\frac{d^\alpha}{c}) \\
&\leq \frac{c\bar{V}(\bar{x}, \bar{t})}{d^\beta} - \bar{g}_u(\eta(\bar{x}, \bar{t}))(\bar{f}_v(\xi(x^o, t^o))\bar{V}(\bar{x}, \bar{t})\frac{d^\alpha}{c}) \\
&\leq \frac{c\bar{V}(\bar{x}, \bar{t})}{d^\beta}(1 - \bar{g}_u(\eta(\bar{x}, \bar{t}))\bar{f}_v(\xi(x^o, t^o))\frac{d^{\alpha+\beta}}{c^2})
\end{aligned} \tag{6.32}$$

If λ is sufficiently close to -1, d would be sufficiently small,

$$\bar{g}_u(\eta(\bar{x}, \bar{t}))\bar{f}_v(\xi(x^o, t^o))\frac{d^{\alpha+\beta}}{c^2} << 1$$

Also

$$V_\lambda(\bar{x}, \bar{t}) < 0$$

So we derive

$$\frac{c\bar{V}(\bar{x}, \bar{t})}{d^\beta}(1 - \bar{g}_u(\eta(\bar{x}, \bar{t}))\bar{f}_v(\xi(x^o, t^o))\frac{d^{\alpha+\beta}}{c^2}) < 0$$

Which is a contradiction. Thus,

$$\bar{U}(x, t) \geq \min\{0, \inf_{x \in \Omega_\lambda} \bar{U}(x, \underline{t})\}, \ \forall (x, t) \in \Omega_\lambda \times (\underline{t}, T)$$

Thus

$$e^{m(t-\underline{t})} U_\lambda(x, t) \geq \min\{0, \inf_{x \in \Omega_\lambda} U_\lambda(x, \underline{t})\}$$

So

$$U_\lambda(x, t) \geq e^{-m(t-\underline{t})} \min\{0, \inf_{x \in \Omega_\lambda} U_\lambda(x, \underline{t})\}$$

$U_\lambda(x, t)$ is bounded from below. Let $\underline{t} \to -\infty$, $U_\lambda(x, t) \to \geq 0$.

Therefore,

$$U_\lambda(x, t) \geq 0$$

if Ω_λ is narrow.

Also,

$$\bar{V}(x, t) \geq \min\{0, \inf_{x \in \Omega_\lambda} \bar{V}(x, \underline{t})\}, \ \forall (x, t) \in \Omega_\lambda \times (\underline{t}, T)$$

Thus

$$e^{m(t-\underline{t})} V_\lambda(x, t) \geq \min\{0, \inf_{x \in \Omega_\lambda} V_\lambda(x, \underline{t})\}$$

So

$$V_\lambda(x,t) \geq e^{-m(t-\underline{t})} \min\{0, \inf_{x \in \Omega_\lambda} V_\lambda(x,\underline{t})\}$$

$V_\lambda(x,t)$ is bounded from below. Let $\underline{t} \to -\infty$, $V_\lambda(x,t) \to \geq 0$.

Therefore,

$$V_\lambda(x,t) \geq 0$$

if Ω_λ is narrow.

6.6 *Step 2*: *Move the plane continuously to the right until its limiting position as long as* $U_\lambda(x,t) \geq 0$ *,* $V_\lambda(x,t) \geq 0$ *holds.*

Define

$$\lambda_0 = sup\{\lambda \leq 0 \mid U_\mu(x,t) \geq 0, V_\mu(x,t) \geq 0, \forall (x,t) \in \Omega_\mu \times \mathbb{R}, \mu \leq \lambda\}$$

In this part, we show that

$$\lambda_0 = 0$$

Suppose

$$\lambda_0 < 0$$

we show that the plane T_{λ_0} can be moved further to the right. To be more rigorous, there exists some $\epsilon > 0$, such that for any $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, we have

$$\begin{cases} U_\lambda(x,t) \geq 0, \\ V_\lambda(x,t) \geq 0, \end{cases}$$

for $(x,t) \in \Sigma_{\lambda_0} \times \mathbb{R}$

This is a contradiction with the definition of λ_0. Hence we must have

$$\lambda_0 = 0$$

6.6.1 Show $U_{\lambda_0} > 0, V_{\lambda_0} > 0$ for $(x,t) \in \Omega_{\lambda_0} \times \mathbb{R}$

Suppose that $\lambda_0 < 0$, then the reflection of the curved part of $\partial\Omega_{\lambda_0}$ falls inside $B_1(0)$ and $U_{\lambda_0}(x,t) \geq 0, V_{\lambda_0}(x,t) \geq 0$ for $(x,t) \in \Omega_{\lambda_0} \times \mathbb{R}$. (See Figure 11)

We want to show

$$\begin{cases} U_{\lambda_0}(x,t) > 0, \quad (x,t) \in \Omega_{\lambda_0} \times \mathbb{R}, \\ V_{\lambda_0}(x,t) > 0, \quad (x,t) \in \Omega_{\lambda_0} \times \mathbb{R}, \end{cases} \tag{6.33}$$

Suppose otherwise, (6.33) does not hold, then there exists $(x^o, t^o) \in \Omega_{\lambda_0} \times \mathbb{R}$ such that $U_{\lambda_0}(x^o, t^o) = 0$.

Since $U_{\lambda_0}(x,t) \geq 0$ inside $\Omega_{\lambda_0} \times \mathbb{R}$, so for $U_{\lambda_0}(x^o, t^o) = 0$, we know (x^o, t^o) is the minimum.

Then

$$\frac{\partial U_{\lambda_0}(x^o, t^o)}{\partial t} = 0$$

Following from the argument before, we have

$$(-\Delta)^{\frac{\alpha}{2}} U_{\lambda_0}(x^o, t^o) \leq \frac{c}{d^\alpha} U_{\lambda_0}(x^o, t^o) = 0 \tag{6.34}$$

On the other hand, by 6.25, we have

$$\begin{aligned} & \frac{\partial U_{\lambda_0}(x^o,t^o)}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} U_{\lambda_0}(x^o,t^o) \\ := \; & (-\Delta)^{\frac{\alpha}{2}} U_{\lambda_0}(x^o,t^o) \\ \geq \; & f_v(\xi(x^o,t^o)) V_{\lambda_0}(x^o,t^o) \\ \geq \; & 0 \end{aligned}$$

but we know

$$\begin{aligned} & \frac{\partial U_{\lambda_0}(x^o,t^o)}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} U_{\lambda_0}(x^o,t^o) \\ := \; & (-\Delta)^{\frac{\alpha}{2}} U_{\lambda_0}(x^o,t^o) \\ \leq \; & \frac{c}{d^\alpha} U_{\lambda_0}(x^o,t^o) \\ \leq \; & 0 \end{aligned}$$

So we conclude

$$U_{\lambda_0}(x^o,t^o) \equiv 0 \tag{6.35}$$

We derive a contradiction since the plane T_{λ_0} did not reach the origin. If x^o is on the curved part $\partial\Omega_{\lambda_0}$, then it's reflection point $x^{o\lambda_0}$ is in the interior of the ball, and hence $U_{\lambda_0}(x^o,t^o) = u(x^{o\lambda_0},t^o) - u(x^o,t^o) > 0$, which contradicts (6.35).

We conclude $U_{\lambda_0}(x,t) > 0, V_{\lambda_0}(x,t) > 0$ for every $x \in \Omega_{\lambda_0} \times \mathbb{R}$.

However, since $t \in (-\infty,\infty)$, $U_{\lambda_0}(x,t)$, $V_{\lambda_0}(x,t)$ may not be bounded away from 0.

We want to further derive

$$\begin{cases} \inf U_{\lambda_0}(x,t) > c_o > 0,\ (x,t) \in \Omega_{\lambda_0-\delta} \times \mathbb{R}, \\ \inf V_{\lambda_0}(x,t) > c_o > 0,\ (x,t) \in \Omega_{\lambda_0-\delta} \times \mathbb{R}, \end{cases} \tag{6.36}$$

I will prove (6.36) by contradiction.

6.6.2 Prove by contradiction: Show $\bar{U}(y,0) \equiv 0, \bar{V}(y,0) \equiv 0,\ \forall (y,0) \in \Omega_{\lambda_o-\delta} \times \mathbb{R}$

Proof: If (6.36) is violated, then $\exists (x_k, t_k) \in \Omega_{\lambda_0-\delta} \times \mathbb{R}$ such that $U_{\lambda_0}(x_k, t_k) \to 0$,

Without loss of generality, by Bolzano-Weierstrass theorem, $\exists$ subsequences of x_k, here we still denote this subsequence by x_k, such that $x_k \to x^o \in \Omega_{\lambda_0-\delta}$,

Now for each $t_k (k \geq k_0)$, Let

$$U_k(x,t) = U_{\lambda_0}(x, t + t_k)$$

so

$$U_k(x_k, 0) = U_{\lambda_0}(x_k, t_k) \to 0$$

Following from (6.25), we have

$$
\begin{aligned}
& \frac{\partial U_k}{\partial t}(x,t) + (-\Delta)^{\frac{\alpha}{2}} U_k(x,t) \\
= \quad & f(v_{\lambda_0}(x,t+t_k)) - f(v(x,t+t_k)) \\
= \quad & \frac{f(v_{\lambda_0}(x,t+t_k))}{v^p_{\lambda_0}(x,t+t_k)} v^p_{\lambda_0}(x,t+t_k) - \frac{f(v(x,t+t_k))}{v^p(x,t+t_k)} v^p(x,t+t_k) \\
\geq \quad & \frac{f(v(x,t+t_k))}{v^p(x,t+t_k)} [v^p_{\lambda_0}(x,t+t_k) - v^p(x,t+t_k)] \\
= \quad & \frac{f(v(x,t+t_k))}{v^p(x,t+t_k)} p\xi^{p-1}(x,t+t_k) V_{\lambda_0}(x,t+t_k) \\
:= \quad & f_v(\xi(x,t+t_k)) V_{\lambda_0}(x,t+t_k) \\
:= \quad & f_v(\xi(x,t+t_k)) V_k(x,t)
\end{aligned}
$$

So U_k satisfies

$$
\frac{\partial U_k}{\partial t}(x,t) + (-\triangle)^{\frac{\alpha}{2}} U_k(x,t) \geq f_v(\xi(x,t+t_k)) V_k(x,t) \tag{6.37}
$$

By regularity theory for parabolic equations [45], there exists some functions $\bar{U}(x,t)$ and $\bar{c}(x,t)$ such that $k \to \infty$, $U_k(x,t)$ converges uniformly to $\bar{U}(x,t)$ for $(x,t) \in \Omega_{\lambda_0} \times \mathbb{R}$, and $f_v(\xi(x,t+t_k))$ converges uniformly to $\bar{c}(x,t)$ with $\bar{c}(x,t) \geq 0$.

and $\bar{U}$ satisfies:

$$
\frac{\partial \bar{U}}{\partial t} + (-\triangle)^{\frac{\alpha}{2}} \bar{U} \geq \bar{c}(x,t) \bar{V}(x,t)
$$

Since

$$
U_k(x_k,0) = U_{\lambda_0}(x_k,t_k) \to 0
$$

$$
\bar{U}(x^o,0) = 0
$$

and

$$\bar{U} \geq 0$$

So $(x^o, 0)$ is the minimum.

$$\frac{\partial \bar{U}}{\partial t}(x^o, 0) = 0$$

Following from the argument before, we have

$$(-\Delta)^{\frac{\alpha}{2}} \bar{U}(x^o, 0) \leq \frac{c}{d^\alpha} \bar{U}(x^o, 0) = 0 \tag{6.38}$$

On the other hand, by 6.25, we have

$$\begin{aligned} & \frac{\partial \bar{U}(x^o, 0)}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} \bar{U}(x^o, 0) \\ := \;& (-\Delta)^{\frac{\alpha}{2}} \bar{U}(x^o, 0) \\ \geq \;& \bar{c}(x^o, 0) \bar{V}(x^o, 0) \\ \geq \;& 0 \end{aligned}$$

but we know

$$\begin{aligned} & \frac{\partial \bar{U}(x^o, 0)}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} \bar{U}(x^o, 0) \\ := \;& (-\Delta)^{\frac{\alpha}{2}} \bar{U}(x^o, 0) \\ \leq \;& \frac{c}{d^\alpha} \bar{U}(x^o, 0) \\ = \;& 0 \end{aligned}$$

So we conclude

$$\bar{U}(x^o, 0) \equiv 0 \tag{6.39}$$

We derive a contradiction since the plane T_{λ_0} did not reach the origin. If x^o is on the curved part $\partial\Omega_{\lambda_0}$, then it's reflection point $x^{o\lambda_0}$ is in the interior of the ball, and hence $u(x^{o\lambda_0}, t_k) > 0$, therefore, $U_{\lambda_0}(x^o, t_k) = u(x^{o\lambda_0}, t_k) - u(x^o, t_k) > 0$, which contradicts (6.39).

So we proved 6.36

6.6.3 Show $\bar{u}(x, 0) \equiv 0, \bar{v}(x, 0) \equiv 0$ for $x \in \mathbb{R}^n$

Let $u_k(x, t) = u(x, t + t_k)$, then by (6.25), we have

$$\frac{\partial u_k(x,t)}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} u_k(x,t) = f(v_k(x,t))$$

By regularity theory for parabolic equations [45], there exists some functions $\bar{u}(x,t)$ such that as $k \to \infty$, $u_k(x,t)$ converges uniformly to $\bar{u}(x,t)$, $v_k(x,t)$ converges uniformly to $\bar{v}(x,t)$ for $(x,t) \in B_1(0) \times \mathbb{R}$ and f converges to $\bar{f}$.

$$\frac{\partial \bar{u}(x,t)}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} \bar{u}(x,t) = f(\bar{v}(x,t)) \geq 0$$

Since we assume

$$f(\bar{v}(x,0)) \geq 0$$

Thus

$$\bar{f}(\bar{v}(x,0)) \geq 0$$

In order to show that

$$\bar{u}(x,0) \equiv 0, \bar{v}(x,0) \equiv 0, x \in \mathbb{R}^n \tag{6.40}$$

we apply the following:

Lemma 6.6. *(Strong Maximum Principle for* $\frac{\partial \bar{u}}{\partial t} + (-\Delta)^{\frac{\alpha}{2}}\bar{u} = f(\bar{v}(x,t))$*).*

Assume that $\bar{u}(x,t) \in [C^{1,1}_{loc}(\Omega_\lambda) \cap C(\bar{\Omega_\lambda}) \cap \mathcal{L}_{2s}] \times C^1([\underline{t}, T])$

$$\begin{cases} \frac{\partial \bar{u}(x,t)}{\partial t} + (-\Delta)^{\frac{\alpha}{2}}\bar{u}(x,t) = f(\bar{v}(x,t)), & (x,t) \in \Omega_\lambda \times [\underline{t}, T], \\ \bar{u}(x,t) \geq 0, & (x,t) \in \Omega_\lambda \times [\underline{t}, T] \end{cases} \tag{6.41}$$

we have either

$$\bar{u}(x,0) > 0, x \in B_1(0)$$

or

$$\bar{u}(x,0) \equiv 0, x \in \mathbb{R}^n$$

Proof. First, if $\bar{u}(x,0) \geq 0$ and $\bar{u}(x^o,0) = 0$, $(x^o,0)$ then is a minimum, thus we have $\frac{\partial \bar{u}}{\partial t}(x^o,0) = 0$.

If $\bar{u}(x,0) \not\equiv 0$, then

$$
\begin{aligned}
& (-\Delta)^{\frac{\alpha}{2}} \bar{u}(x^o,0) \\
= \;& C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{-\bar{u}(y,0)}{|x^o - y|^{n+\alpha}} dy \\
= \;& C_{n,\alpha} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{u}(y,0)}{|x^o - y|^{n+\alpha}} dy + \int_{\widetilde{\Sigma_{\lambda_0}}} \frac{-\bar{u}(y,0)}{|x^o - y|^{n+\alpha}} dy \} \\
= \;& C_{n,\alpha} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{u}(y,0)}{|x^o - y|^{n+\alpha}} dy + \int_{\Sigma_{\lambda_0}} \frac{-\bar{u}(y^\lambda,0)}{|x^o - y^\lambda|^{n+\alpha}} dy \} \\
= \;& C_{n,\alpha} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{u}(y,0)}{|x^o - y|^{n+\alpha}} dy + \int_{\Sigma_{\lambda_0}} \frac{\bar{u}(y,0)}{|x^o - y^\lambda|^{n+\alpha}} dy \} \\
= \;& C_{n,\alpha} PV \int_{\Sigma_{\lambda_0}} \bar{u}(y,0) \{ \frac{1}{|x^o - y^\lambda|^{n+\alpha}} - \frac{1}{|x^o - y|^{n+\alpha}} \} dy \\
< \;& 0
\end{aligned}
$$

since

$$
\frac{1}{|x^o - y^\lambda|^{n+\alpha}} - \frac{1}{|x^o - y|^{n+\alpha}} < 0
$$

and

$$
\bar{u}(y,0) \geq 0
$$

and

$$
\bar{u}(y,0) \not\equiv 0
$$

However, we have

$$
\begin{aligned}
& \frac{\partial \bar{u}(x^o,0)}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} \bar{u}(x^o,0) \\
:= \;& (-\Delta)^{\frac{\alpha}{2}} \bar{u}(x^o,0) \\
= \;& f(\bar{v}(x^o,0)) \\
\geq \;& 0
\end{aligned}
$$

Which is a contradiction. Therefore, we have either $\bar{u}(x,0) > 0, x \in B_1(0)$ or $\bar{u}(x,0) \equiv 0, x \in \mathbb{R}^n$.

$\square$

If $\bar{u}(x,0) > 0$, $x \in B_1(0)$, we know $\lambda_0 < 0$, if we take a point $\bar{x}$ on the curved part $\partial\Omega_{\lambda_0}$, then it's reflection point $\bar{x}^{\lambda_0}$ is in the interior of the ball (see Figure 11)and hence $u(\bar{x}^{\lambda_0}, 0) > 0$, therefore, $\bar{U}(\bar{x}, 0) = \bar{u}(\bar{x}^{\lambda_0}, 0) - \bar{u}(\bar{x}, 0) > 0$. $\bar{U}(x,0) > 0$ somewhere, but we already derive $\bar{U}(x,0) \equiv 0$, hence we must have $\bar{U}(x,0) \equiv 0$, $x \in \mathbb{R}^n$.

Thus, we know $u(x, t_k)$ converges to 0 uniformly.

Following the same argument, we know $v(x, t_k)$ converges to 0 uniformly.

6.6.4 Derive a contradiction for large k for U_k

In order to derive a contradiction for large k, we modify t_k a bit.

We still denote $U_{\lambda_0}(x, t + t_k)$ by $U_k(x,t)$, Let

$$U_k(x_k, 0) \equiv U_{\lambda_0}(x_k, t_k) = m_k \tag{6.42}$$

which converges to zero.

Let

$$a_k(x,t) = U_k(x,t) - 2m_k \eta(\epsilon_k(t - t_k)) \tag{6.43}$$

where $\eta(t) \in C_0^\infty$ is a cut-off function such that $|\eta'(t)| \le c$ and

$$\eta(t) = \begin{cases} 1, & |t| \le 1, \\ 0, & |t| \ge 2. \end{cases}$$

When (x,t) is outside $\Omega_{\lambda_0-\delta} \times (t_k-2, t_k+2)$,

$$\begin{aligned} & a_k(x,t) \\ = \;& U_k(x,t), \end{aligned}$$

When (x,t) is inside $\Omega_{\lambda_0-\delta} \times (t_k-2, t_k+2)$, such that at (x,t_k)

$$\begin{aligned} & a_k(x,t_k) \\ = \;& U_k(x,t_k) - 2m_k \end{aligned}$$

The value of a_k outside $\Omega_{\lambda_0-\delta} \times (t_k-2, t_k+2)$ is greater than the value of a_k inside $\Omega_{\lambda_0-\delta} \times (t_k-2, t_k+2)$, so $a_k(x,t)$ attains its minimum at some point, say $(\bar{x}_k, \bar{t}_k)$ in $\Omega_{\lambda_0-\delta} \times (t_k-2, t_k+2)$.

This implies,

$$\frac{\partial a_k}{\partial t}(\bar{x}_k, \bar{t}_k) = 0$$

Combining (6.42) and (6.43), it is easy to deduce

$$\begin{aligned} & a_k(x_k,0) \\ = \;& U_k(x_k,0) - 2m_k \\ = \;& U_k - 2m_k \\ = \;& -m_k \end{aligned}$$

Thus

$$a_k(\bar{x}_k, \bar{t}_k) \le -m_k$$

Let

$$\widetilde{a_k}(x,t) = a_k(x, t + \bar{t}_k)$$

Then

$$\widetilde{a_k}(\bar{x}_k, 0) = a_k(\bar{x}_k, \bar{t}_k)$$

Then

$$\frac{\partial \widetilde{a}_k}{\partial t} + (-\triangle)^s \widetilde{a}_k = c_\lambda \widetilde{a}_k + 2m_k \eta(\epsilon_k(t - t_k))$$

By regularity theory for parabolic equations [45], there exists some functions $\bar{a}(x,t)$ such that $k \to \infty$, $\widetilde{a}_k(x,t) \to \bar{a}(x,t)$ converges uniformly for $x \in \Omega_{\lambda_o}$,

Moreover

$$\frac{\partial \bar{a}}{\partial t} + (-\Delta)^s \bar{a} = c_\lambda \bar{a}$$

We know

$$\frac{\partial a_k}{\partial t} \sim \frac{\partial U_k}{\partial t} - 2m_k \epsilon_k c$$

Therefore we conclude

$$\frac{\partial U_k}{\partial t} \sim m_k \epsilon_k$$

Passing to a subsequence, $(\bar{x}_k, \bar{t}_k) \to (x^o, t^o) \in \Omega_{\lambda_o - \delta} \times [-2, 2]$

$U_k \to \bar{U}$ uniformly, and

$$\frac{\partial \bar{U}}{\partial t} + (-\Delta)^s \bar{U} = \bar{c}\bar{U}$$

As we have already derived

$$\bar{U}(x^o, t^o) = 0, \frac{\partial \bar{U}}{\partial t}(x^o, t^o) = 0$$

Also following from (6.25)

$$
\begin{aligned}
& \frac{\partial \bar{U}}{\partial t}(x^o,t^o) + (-\Delta)^{\frac{\alpha}{2}} \bar{U}(x^o,t^o) \\
= \quad & f(\bar{v}_{\lambda_0}(x^o,t^o+t_k)) - f(\bar{v}(x^o,t^o+t_k)) \\
= \quad & \frac{f(\bar{v}_{\lambda_0}(x^o,t^o+t_k))}{\bar{v}^p_{\lambda_0}(x^o,t^o+t_k)} \bar{v}^p_{\lambda_0}(x^o,t^o+t_k) - \frac{f(\bar{v}(x,t^o+t_k))}{\bar{v}^p(x^o,t^o+t_k)} \bar{v}^p(x^o,t^o+t_k) \\
\geq \quad & \frac{f(\bar{v}(x^o,t^o+t_k))}{\bar{v}^p(x^o,t^o+t_k)} [\bar{v}^p_{\lambda_0}(x^o,t^o+t_k) - \bar{v}^p(x^o,t^o+t_k)] \\
= \quad & \frac{f(\bar{v}(x^o,t^o+t_k))}{\bar{v}^p(x^o,t^o+t_k)} p\xi^{p-1}(x^o,t^o+t_k) V_{\lambda_0}(x^o,t^o+t_k) \\
:= \quad & f_{\bar{v}}(\xi(x^o,t_k)) V_{\lambda_0}(x^o,t^o+t_k) \\
:= \quad & f_{\bar{v}}(\xi(x^o,t_k)) V_k(x^o,t^o) \\
:= \quad & f_{\bar{v}}(\xi(x^o,t_k)) \bar{V}(x^o,t^o)
\end{aligned}
$$

This implies

$$(-\Delta)^{\frac{\alpha}{2}} \bar{U}(x^o,t^o) \geq f_{\bar{v}}(\xi(x^o,t_k)) \bar{V}(x^o,t^o)$$

and

$$f_{\bar{v}}(\xi(x^o,t_k)) \bar{V}(x^o,t^o) \geq 0$$

It follows

$$
\begin{aligned}
& 0 \leq (-\Delta)^{\frac{\alpha}{2}} \bar{U}(x^o, t^o) \\
= \;& C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{-\bar{U}(y,t^o)}{|x^o - y|^{n+\alpha}} dy \\
= \;& C_{n,\alpha} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{U}(y,t^o)}{|x^o - y|^{n+\alpha}} dy + \int_{\widetilde{\Sigma_{\lambda_0}}} \frac{-\bar{U}(y,t^o)}{|x^o - y|^{n+\alpha}} dy \} \\
= \;& C_{n,\alpha} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{U}(y,t^o)}{|x^o - y|^{n+\alpha}} dy + \int_{\Sigma_{\lambda_0}} \frac{-\bar{U}(y^\lambda,t^o)}{|x^o - y^\lambda|^{n+\alpha}} dy \} \\
= \;& C_{n,\alpha} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{U}(y,t^o)}{|x^o - y|^{n+\alpha}} dy + \int_{\Sigma_{\lambda_0}} \frac{\bar{U}(y,t^o)}{|x^o - y^\lambda|^{n+\alpha}} dy \} \\
= \;& C_{n,\alpha} PV \int_{\Sigma_{\lambda_0}} \bar{U}(y,t^o) \{ \frac{1}{|x^o - y^\lambda|^{n+\alpha}} - \frac{1}{|x^o - y|^{n+\alpha}} \} dy \\
\leq \;& 0
\end{aligned}
$$

since

$$\frac{1}{|x^o - y^\lambda|^{n+\alpha}} - \frac{1}{|x^o - y|^{n+\alpha}} < 0$$

and

$$\bar{U}(y,t^o) \geq 0$$

This implies

$$\bar{U}(y,t^o) \equiv 0, \forall y \in \mathbb{R}^n$$

Similar with above, for

$$u_k(x,t) = u(x, t + t_k)$$

$$u_k(x, \bar{t}_k) = u(x, \bar{t}_k + t_k)$$

$$u_k(x,t) \to \bar{u}(x,t)$$

$$\frac{\partial \bar{u}}{\partial t} + (-\Delta)^{\frac{\alpha}{2}} \bar{u} = \bar{f}(\bar{v}(x,t))$$

We have

$$\bar{u}(x,t) \equiv 0$$

Now for sufficiently large k,

$$\begin{aligned}
& (-\Delta)^{\frac{\alpha}{2}} a_k(\bar{x}_k, \bar{t}_k) \\
= \ & (-\Delta)^{\frac{\alpha}{2}} U_k \\
\geq \ & -\frac{\partial U_k}{\partial t}(\bar{x}_k, \bar{t}_k) + f_v(\xi(x,t)) V_k(\bar{x}_k, \bar{t}_k)
\end{aligned}$$

Since we know

$$\begin{aligned}
& (-\Delta)^{\frac{\alpha}{2}} a_k(\bar{x}_k, \bar{t}_k) \\
\leq \ & \frac{c}{[d(\bar{x}_k, T_{\lambda_o})]^{\alpha}} a_k(\bar{x}_k, \bar{t}_k) \\
\leq \ & -c_1 m_k
\end{aligned}$$

where $c_1 > 0$

$$-\frac{\partial U_k}{\partial t}(\bar{x}_k, \bar{t}_k) \sim \epsilon_k m_k$$

If we assume $\frac{\partial f}{\partial V} = 0$, as $U_k \to 0$ uniformly,

$$f_v(\xi(x,t)) = o(1) \to 0$$

Finally,

$$-c_1 m_k \geq o(1) m_k$$

or

$$c_1 \leq -o(1)$$

Since $o(1) \to 0$ as $k \to \infty$, which is a contradiction with $-c_1 m_k \geq o(1) m_k$ as $k \to \infty$.

Hence, we have proved (6.36).

Since U_λ, V_λ depends on λ continuously, there exists $\epsilon > 0$ and $\epsilon < \delta$, such that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, we have

$$\begin{cases} U_\lambda(x,t) \geq 0, \ (x,t) \in \Omega_{\lambda_0 - \delta} \times \mathbb{R}, \\ V_\lambda(x,t) \geq 0, \ (x,t) \in \Omega_{\lambda_0 - \delta} \times \mathbb{R}, \end{cases} \tag{6.44}$$

Now apply the Narrow region theorem **??** and in our case the narrow region is

$$\Omega_\lambda^- \backslash \Omega_{\lambda_0 - \delta} \times \mathbb{R}$$

By Narrow region theorem, we derive

$$\begin{cases} U_\lambda(x,t) \geq 0, \ (x,t) \in \Omega_\lambda^- \backslash \Omega_{\lambda_0 - \delta} \times \mathbb{R}, \\ V_\lambda(x,t) \geq 0, \ (x,t) \in \Omega_\lambda^- \backslash \Omega_{\lambda_0 - \delta} \times \mathbb{R}, \end{cases} \tag{6.45}$$

Combining (6.44) and (6.45), we conclude that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$

$$\begin{cases} U_\lambda(x,t) \geq 0, \ (x,t) \in \Omega_{\lambda_0} \times \mathbb{R}, \\ V_\lambda(x,t) \geq 0, \ (x,t) \in \Omega_{\lambda_0} \times \mathbb{R}, \end{cases}$$

This contradicts the definition of λ_0. Therefore, we must have

$$\lambda_0 = 0$$

and

$$\begin{cases} U_{\lambda_0}(x,t) \geq 0, \ \forall (x,t) \in \Omega_{\lambda_0} \times \mathbb{R}, \\ V_{\lambda_0}(x,t) \geq 0, \ \forall (x,t) \in \Omega_{\lambda_0} \times \mathbb{R}, \end{cases}$$

Similarly, one can move the plane T_λ from $\lambda = 1$ to the left and show that

$$\begin{cases} U_{\lambda_0}(x,t) \leq 0, \ \forall (x,t) \in \Omega_{\lambda_0} \times \mathbb{R}, \\ V_{\lambda_0}(x,t) \leq 0, \ \forall (x,t) \in \Omega_{\lambda_0} \times \mathbb{R}, \end{cases}$$

Now we have shown that

$$\lambda_0 = 0$$

and

$$\begin{cases} U_{\lambda_0} \equiv 0, \ (x,t) \in \Omega_{\lambda_0} \times \mathbb{R}, \\ V_{\lambda_0} \equiv 0, \ (x,t) \in \Omega_{\lambda_0} \times \mathbb{R}, \end{cases}$$

This completes the step 2.

6.6.5 Conclude the solution is radially symmetric and monotone decreasing

So far, we have proved that u, v are symmetric about the plane T_0. Since the x_1 direction can be chosen arbitrarily, we have actually shown that u, v are radially symmetric about origin.

Since $U_\lambda(x,t) \not\equiv 0$, $(x,t) \in T_\lambda \times \mathbb{R}$, $\forall 0 < \lambda < \lambda_0$, if there exists $(x^o.t^o)$ such that (x^o, t^o) is the minimum point, from the above process, on one hand,

$$(-\Delta)^{\frac{\alpha}{2}} U_\lambda(x^o, t^o) \leq 0$$

On the other hand,

$$(-\Delta)^{\frac{\alpha}{2}} U_\lambda(x^o, t^o) = 0$$

This forces

$$U_\lambda \equiv 0$$

which is a contradiction. Therefore, u is monotone decreasing about the origin. Same reason for v.

Part 2: Method of Sliding and Its Applications: Monotonicity of solution of fractional parabolic and Monge-Ampére equations

7 Method of Sliding and Its Applications: Monotonicity of solutions of fractional parabolic and Monge-Ampére equations

7.1 Monotonicity of solution of fractional parabolic equation

In the following section, we will try to prove the solution of the fractional parabolic equation with assumption below is monotone increasing using the sliding method.

Theorem 7.1. *(Monotonicity of solution of fractional parabolic equation)*

Let Ω be a bounded domain in $\mathbb{R}^n$ which is convex in the x_n-direction. Let $0 < s < 1$, and suppose that $u(x,t) \in \left(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\right) \times (-\infty, \infty)$ is a positive bounded classical solution of

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = f(t,|x|,u), & (x,t) \in \Omega \times (-\infty,\infty), \\ u(x,t) = \varphi(x,t), & (x,t) \in \Omega^c \times (-\infty,\infty), \end{cases} \tag{7.1}$$

We impose some conditions on u. Let $u(x,t) = \varphi(x,t)$ in Ω^c, suppose H:

For any three points $x = (x', x_n)$, $y = (x', y_n)$ and $z = (x', z_n)$ lying on a segment parallel to the x_n axis, $y_n < x_n < z_n$ with $y, z \in \Omega^c$, we have

$$\varphi(y,t) < u(x,t) < \varphi(z,t),\ (x,t) \in \Omega \times \mathbb{R} \tag{7.2}$$

and

$$\varphi(y,t) \leq \varphi(x,t) \leq \varphi(z,t),\ (x,t) \in \Omega^c \times \mathbb{R} \tag{7.3}$$

Assume that f is non-increasing about u and is uniformly Lipschitz continuous in u. i.e:

$$f(t,|x|,u_1) - f(t,|x|,u_2) \leq c|u_1 - u_2|,\ \forall x \in \Omega,$$

Then $u(x,t)$ is monotone increasing with respect to x_n in Ω, i.e: for any $\tau > 0$, we have

$u((x', x_n + \tau), t) > u((x', x_n), t)$ for $(x', x_n), (x', x_n + \tau) \in \Omega$ and $t \in \mathbb{R}$,

where $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$

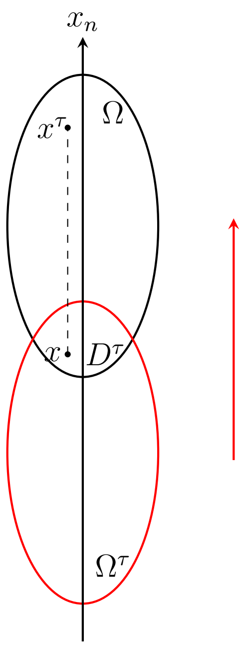


Figure 13: sliding method

7.1.1 Basic set-up

Write

$$x = (x', x_n)$$

For any $\tau \in \mathbb{R}$, define

$$u^\tau(x) = u(x', x_n + \tau)$$

Let Ω be a bounded domain in $\mathbb{R}^n$, which is convex in the x_n-direction. By sliding Ω downward τ units, we obtain Ω^τ:

$$\Omega^\tau = \Omega - \tau e_n, e_n = (0, 0, \cdots, 1)$$

Define

$$D^\tau = \Omega^\tau \cap \Omega$$

and

$$\widetilde{\tau} = \sup\{\tau \mid \tau > 0, D^\tau \neq \emptyset\}$$

Assume that $u(x,t)$ is a positive solution of equation (7.1). We compare the values of $u(x,t)$ with

$$u^\tau(x,t) = u((x', x_n + \tau), t).$$

Let

$$W^\tau(x,t) = u^\tau(x,t) - u(x,t)$$

$u^\tau(x,t)$ satisfies equation (7.1), from which, we have

$$\begin{aligned}
&\frac{\partial W^\tau}{\partial t}(x,t) + (-\triangle)^s W^\tau(x,t) \\
= \quad & f(t,|x|,u^\tau) - f(t,|x|,u) \\
= \quad & \frac{f(t,|x|,u^\tau) - f(t,|x|,u)}{u^\tau(x,t) - u(x,t)} W^\tau(x,t) \\
:= \quad & c^\tau(x,t) W^\tau(x,t),
\end{aligned} \tag{7.4}$$

where

$$c^\tau(x,t) = \frac{f(t,|x|,u^\tau) - f(t,|x|,u)}{u^\tau(x,t) - u(x,t)}$$

Since f is Lipschitz continuous, we have

$$c^\tau(x,t) \leq L, \ \forall x \in D^\tau$$

where L is the Lipschitz constant.

The main part of the proof is to show that

$$W^\tau(x,t) > 0,\ \forall (x,t) \in D^\tau \times \mathbb{R}$$

We divide our proof in two steps.

Step 1: *Begin sliding Ω^τ downward τ units along the x_n axis*

So then

$$|x| < |x^\tau|$$

We will show that for τ sufficiently close to $\widetilde{\tau}$ when D^τ is narrow, we have

$$W^\tau(x,t) \geq 0,\ (x,t) \in D^\tau \times \mathbb{R}$$

Apparently, D^τ is a narrow region in the x_n direction for τ sufficiently close to $\widetilde{\tau}$. We first establish a narrow region principle for the fractional parabolic operator, which is an important ingredient in applying the sliding method on bounded domains.

Step 2: *Decrease τ as long as $W^\tau(x,t) \geq 0$ holds to its limiting position*

We would show the limit position is $\tau = 0$. After we have completed the second step, we would prove $\forall \tau > 0$, $W^\tau(x,t) > 0$, thus we have completed proof of monotonicity of solution of fractional parabolic equation in the bounded domain.

7.2 *Step 1: show* $W^\tau(x,t) \geq 0$

7.2.1 Narrow Region principle on a parabolic cylinder

Lemma 7.2. *(Narrow Region principle on a parabolic cylinder) Let D be a bounded narrow region in $\mathbb{R}^n$. Assume that $u(x,t) \in \left(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\right) \cap \mathcal{L}_{2s} \times [\underline{t}, T]$. $W^\tau(x,t) =$*

$u^\tau(x,t)-u(x,t)$ is lower semi-continuous on $\bar{D}\times[\underline{t},T]$, and satisfies

$$\begin{cases} \frac{\partial W^\tau}{\partial t}+(-\triangle)^s W^\tau = c(x,t)W^\tau(x,t), & (x,t)\in D\times[\underline{t},T], \\ W^\tau(x,t)\geq 0, & (x,t)\in(\mathbb{R}^n\setminus D)\times[\underline{t},T], \end{cases} \tag{7.5}$$

where $c(x,t)$ is bounded from below in D. Let $dn(D)$ be the width of D in the x_n-direction. Then:

$$W^\tau(x,t)\geq 0,\ (x,t)\in D\times[\underline{t},T] \tag{7.6}$$

Moreover, we have either $W^\tau(x,t)>0$ in $D\times\mathbb{R}$ or $W^\tau(x,t)\equiv 0$ in $\mathbb{R}^n\times\mathbb{R}$:

Proof. First, we will prove

$$W^\tau(x,t)\geq\min\{0,\inf_{D\times[\underline{t},T]}W^\tau(x,\underline{t})\},\ (x,t)\in D\times[\underline{t},T] \tag{7.7}$$

If (7.7) does not hold, then the lower semi-continuity of $W^\tau(x,t)$ on $D\times[\underline{t},T]$ guarantees that there exists an $(x^o,t^o)\in D\times[\underline{t},T]$ such that

$$W^\tau(x^o,t^o)=\min_{D\times(\underline{t},T]}W^\tau<0$$

And one can further deduce from condition (7.5) that (x^o,t^o) is in the interior of $D^\tau\times[\underline{t},T]$

Since (x^o,t^o) is the minimum, thus

$$\frac{\partial W^\tau(x^o,t^o)}{\partial t}=0$$

Similar to the argument in previous section, we have

$$
\begin{aligned}
& (-\Delta)^s W^\tau(x^o,t^o) \\
= \;& C_{n,s} PV \int_{\mathbb{R}^n} \frac{W^\tau(x^o,t^o) - W^\tau(y,t^o)}{|x^o-y|^{n+2s}} dy \\
= \;& C_{n,s} PV \{ \int_D \frac{W^\tau(x^o,t^o) - W^\tau(y,t^o)}{|x^o-y|^{n+2s}} dy + \int_{\mathbb{R}^n \setminus D} \frac{W^\tau(x^o,t^o) - W^\tau(y,t^o)}{|x^o-y|^{n+2s}} dy \} \\
\leq \;& C_{n,s} PV \int_{\mathbb{R}^n \setminus D} \frac{W^\tau(x^o,t^o) - W^\tau(y,t^o)}{|x^o-y|^{n+2s}} dy \} \\
\leq \;& C_{n,s} PV \int_{\mathbb{R}^n \setminus D} \frac{W^\tau(x^o,t^o)}{|x^o-y|^{n+2s}} dy \\
\leq \;& c_0 W^\tau(x^o,t^o) \int_{\mathbb{R}^n \setminus D} \frac{1}{|x^o-y|^{n+2s}} dy
\end{aligned}
$$

By lemma 5.7, we have

$$
(-\Delta)^s W^\tau(x^o,t^o) \leq \frac{cW^\tau(x^o,t^o)}{dn(D)^{2s}} < 0 \tag{7.8}
$$

Combining (7.4) and (7.8), we deduce

$$
c^\tau(x^o,t^o) W^\tau(x^o,t^o) \leq \frac{cW^\tau(x^o,t^o)}{dn(D)^{2s}}
$$

Then we derive

$$
\frac{c}{dn(D)^{2s}} \leq c^\tau(x^o,t^o)
$$

for τ sufficiently close to $\widetilde{\tau}$, $dn(D)$ would be sufficiently small, since c^τ is bounded, we derive a contradiction. Therefore, (7.7) must be valid.

Let

$$
\bar{W} = e^{m(t-\underline{t})} W^\tau(x,t), \;\; m > 0
$$

So then

$$\frac{\partial \bar{W}}{\partial t} + (-\triangle)^s \bar{W} = \bar{c}\bar{W}$$

with

$$\bar{c}$$

is still bounded.

This time, we want to show

$$W^\tau(x,t) \geq 0 \tag{7.9}$$

Suppose otherwise, (7.9) does not hold, then $\bar{W}(x,t)$ is negative somewhere, hence there exists an $x^o \in D$ and $t^o \in [\underline{t}, T]$ such that

$$\bar{W}(x^o, t^o) = \min_{D\times(\underline{t},T]} \bar{W} < 0$$

If

$$t^o < T$$

$$\frac{\partial \bar{W}}{\partial t}(x^o, t^o) = 0$$

If

$$t^o = T$$

$$\frac{\partial \bar{W}}{\partial t}(x^o, t^o) \leq 0$$

From (7.4), we derive

$$(-\Delta)^s \bar{W}(x^o, t^o) \geq \bar{c}\bar{W}(x^o, t^o)$$

We also have

$$(-\Delta)^s \bar{W}(x^o, t^o) \leq \frac{c}{dn(D)^{2s}} \bar{W}(x^o, t^o) \tag{7.10}$$

We deduce

$$\frac{c}{dn(D)^{2s}} \bar{W}(x^o, t^o) \geq \bar{c} \bar{W}(x^o, t^o)$$

Then we derive

$$\frac{c}{dn(D)^{2s}} \leq \bar{c}(x^o, t^o)$$

Which is a contradiction for $dn(D)$ sufficiently small. Thus,

$$\bar{W}(x, t) \geq \min\{0, \inf_{x \in D} \bar{W}(x, \underline{t})\}, \ \forall (x, t) \in D \times (\underline{t}, T)$$

Thus

$$e^{m(t-\underline{t})} W^\tau(x, t) \geq \min\{0, \inf_{x \in D} \bar{W}(x, \underline{t})\}$$

So

$$W^\tau(x, t) \geq e^{-m(t-\underline{t})} \min\{0, \inf_{x \in D} \bar{W}(x, \underline{t})\}$$

$W^\tau(x, t)$ is bounded from below. Let $\underline{t} \to -\infty$, $W^\tau(x, t) \to \geq 0$.

We conclude that for τ sufficiently close to $\widetilde{\tau}$ when D^τ is narrow,

$$W^\tau(x, t) \geq 0$$

If $W^\tau(x^o, t^o) = 0$ at $(x^o, t^o) \in D^\tau \times \mathbb{R}$, then (x^o, t^o) is a minimum point of $W^\tau(x, t)$ in $D^\tau \times \mathbb{R}$.

So then

$$\frac{W^\tau(x^o,t^o)}{\partial t} = 0$$

If $W^\tau(x,t) \not\equiv 0$ in $\mathbb{R}^n \times \mathbb{R}$, then there exists a point (y^o,t^o) and one of the neighborhood $N(y^o,t^o)$ such that

$$W^\tau(y,t^o) \geq c > 0,$$

in $N(y^o,t^o)$

Then we have

$$\begin{aligned}
& (-\Delta)^s W^\tau(x^o,t^o) \\
= \ & C_{n,s} PV \int_{\mathbb{R}^n} \frac{W^\tau(x^o,t^o) - W^\tau(y,t^o)}{|x^o-y|^{n+2s}} dy \\
= \ & C_{n,s} PV \int_{\mathbb{R}^n} \frac{-W^\tau(y,t^o)}{|x^o-y|^{n+2s}} dy \\
\leq \ & C_{n,s} PV \int_{N(y^o,t^o)} \frac{-c}{|x^o-y|^{n+2s}} dy \\
< \ & 0
\end{aligned} \tag{7.11}$$

This is a contradiction with (7.5).

Then we have either

$$W^\tau(x,t) > 0$$

in $D^\tau \times \mathbb{R}$ or

$$W^\tau(x,t) \equiv 0$$

in $\mathbb{R}^n \times \mathbb{R}$.

So far, we have proved the Narrow Region Theorem 7.5. □

We first proved that for τ sufficiently close to $\widetilde{\tau}$ when D^τ is narrow, we have

$$W^\tau(x,t) \geq 0 \tag{7.12}$$

7.3 *Step 2: decrease τ as long as $W^\tau \geq 0$ holds to its limiting position*

The inequality (7.12) provides a starting point, from which we can carry out the sliding. Now we decrease τ as long as $W^\tau(x,t) \geq 0$ holds to its limiting position. Define

$$\tau_0 = \inf\{\tau \mid W^\tau(x,t) \geq 0, \forall (x,t) \in D^\tau \times \mathbb{R}, 0 < \tau < \widetilde{\tau}\}$$

In this part, we show that

$$\tau_0 = 0$$

Suppose

$$\tau_0 > 0$$

we will show that Ω^τ can be slid upward a little bit more and we will have

$$W^\tau(x,t) \geq 0$$

To be more rigorous, there exists some $\epsilon > 0$, such that for any $\tau \in (\tau_0 - \epsilon, \tau_0)$, we have $W^\tau(x,t) \geq 0$, $x \in D^\tau$

This is a contradiction with the definition of τ_0. Hence we must have

$$\tau_0 = 0$$

7.3.1 Show $W^{\tau_0} > 0$ for $(x,t) \in D^{\tau_0} \times \mathbb{R}$

Since $W^{\tau_0}(x,t) \geq 0$ for $(x,t) \in D^{\tau_0} \times \mathbb{R}$ and $W^{\tau_0}(x,t) > 0$ for $(x,t) \in \partial D^{\tau_0} \times \mathbb{R}$, we have

$$W^{\tau_0}(x,t) \not\equiv 0 \tag{7.13}$$

for $(x,t) \in D^{\tau_0} \times \mathbb{R}$

We want to show $W^{\tau_0}(x,t) > 0$ for $(x,t) \in D^{\tau_0} \times \mathbb{R}$

Otherwise, $\exists (x^o, t^o) \in D^{\tau_0} \times \mathbb{R}$ such that $W^{\tau_0}(x^o,t^o) = 0$

Since $W^{\tau_0}(x,t) \geq 0$ inside $D^{\tau_0} \times \mathbb{R}$, so for $W^{\tau_0}(x^o,t^o) = 0$, we know (x^o,t^o) is the minimum.

Then

$$\frac{\partial W^{\tau_0}(x^o,t^o)}{\partial t} = 0$$

Following from (7.4), we then derive

$$\begin{aligned} & \frac{\partial W^{\tau_0}}{\partial t}(x^o,t^o) + (-\Delta)^s W^{\tau_0}(x^o,t^o) \\ = \ & f(t^o,|x^o|,u^{\tau_0}) - f(t^o,|x^o|,u) \\ = \ & \frac{f(t^o,|x^o|,u^{\tau_0}) - f(t^o,|x^o|,u)}{u^{\tau_0}(x^o,t^o) - u(x^o,t^o)} W^{\tau_0}(x^o,t^o) \\ := \ & c^{\tau_0}(x^o,t^o) W^{\tau_0}(x^o,t^o), \end{aligned}$$

So that

$$(-\triangle)^s W^{\tau_0}(x^o,t^o) = 0$$

It follows that

$$
\begin{aligned}
& 0=(-\Delta)^s W^{\tau_0}(x^o,t^o) \\
= \; & C_{n,s} PV \int_{\mathbb{R}^n} \frac{-W^{\tau_0}(y,t^o)}{|x^o-y|^{n+2s}} dy \\
\leq \; & 0
\end{aligned}
\tag{7.14}
$$

since

$$\frac{1}{|x^o-y|^{n+2s}}>0$$

and

$$W^{\tau_0}(y,t^o)\geq 0$$

This implies that

$$W^{\tau_0}(y,t^o)\equiv 0,\ (y,t)\in\mathbb{R}^n\times\mathbb{R}. \tag{7.15}$$

which contradicts with (7.13).

We conclude

$$W^{\tau_0}(x,t)>0 \tag{7.16}$$

for every $(x,t)\in D^{\tau_0}\times\mathbb{R}$.

However, since $t\in(-\infty,\infty)$, $W^{\tau_0}(x,t)$ may not be bounded away from 0.

We want to further derive

$$\inf W^{\tau_0}(x,t)>c_o>0,\ (x,t)\in D^{\tau_0-\epsilon}\times\mathbb{R} \tag{7.17}$$

I will prove (7.17) by contradiction.

7.3.2 Prove by contradiction: Show $\bar{W}(y,0) \equiv 0,\ \forall (y,0) \in D^{\tau_0-\epsilon} \times \mathbb{R}$

Proof: If (7.17) is violated, then $\exists (x_k, t_k) \in D^{\tau_0-\epsilon} \times \mathbb{R}$ such that $W^{\tau_0}(x_k, t_k) \to 0$

Without loss of generality, by Bolzano-Weierstrass theorem, $\exists$ subsequences of x_k, here we still denote this subsequence by x_k, such that $x_k \to x^o \in D^{\tau_0-\epsilon}$

Now for each $t_k (k \geq k_0)$, Let

$$W_k(x,t) = W^{\tau_0}(x, t + t_k)$$

so

$$W_k(x_k, 0) = W^{\tau_0}(x_k, t_k) \to 0$$

From (7.4), we derive:

$$\begin{aligned}
& \frac{\partial W_k}{\partial t}(x,t) + (-\triangle)^s W_k(x,t) \\
= \quad & f(t+t_k, |x^{\tau_0}|, u^{\tau_0}) - f(t+t_k, |x|, u) \\
= \quad & \frac{f(t+t_k, |x^{\tau_0}|, u^{\tau_0}) - f(t+t_k, |x|, u)}{u^{\tau_0}(x, t+t_k) - u(x, t+t_k)} W_k(x,t) \\
:= \quad & c^{\tau_0}(x, t+t_k) W_k(x,t),
\end{aligned}$$

So W_k satisfies

$$\frac{\partial W_k}{\partial t}(x,t) + (-\triangle)^s W_k(x,t) = c^{\tau_0}(x, t+t_k) W_k(x,t) \tag{7.18}$$

By regularity theory for parabolic equations [45], there exists some functions $\bar{W}(x,t)$ and $\bar{c}(x,t)$ such that $k \to \infty$, $W_k(x,t)$ converges uniformly to $\bar{W}(x,t)$ for $(x,t) \in D^{\tau_0} \times \mathbb{R}$, and $\bar{W}$ satisfies:

$$\frac{\partial \bar{W}}{\partial t} + (-\triangle)^s \bar{W} = \bar{c}(x,t)\bar{W}(x,t)$$

Since

$$W_k(x_k, 0) = W^{\tau_0}(x_k, t_k) \to 0$$

$$\bar{W}(x^o, 0) = 0$$

and

$$\bar{W} \geq 0$$

So $(x^o, 0)$ is the minimum.

$$\frac{\partial \bar{W}}{\partial t}(x^o, 0) = 0$$

Also from (7.4), we derive,

$$\begin{aligned}
& \frac{\partial \bar{W}}{\partial t}(x^o, 0) + (-\Delta)^s \bar{W}(x^o, 0) \\
= \quad & f(t_k, |x^{o\tau_0}|, \bar{u}^{\tau_0}) - f(t_k, |x^o|, \bar{u}) \\
= \quad & \frac{f(t_k, |x^{o\tau_0}|, \bar{u}^{\tau_0}) - f(t_k, |x^o|, \bar{u})}{\bar{u}^{\tau_0}(x^o, t_k) - \bar{u}(x^o, t_k)} \bar{W}(x^o, 0) \\
:= \quad & c^{\tau}(x^o, t_k)\bar{W}(x^o, 0),
\end{aligned}$$

we derive

$$(-\triangle)^s \bar{W}(x^o, 0) = 0$$

It follows that

$$
\begin{aligned}
& 0=(-\Delta)^s \bar{W}(x^o,0) \\
= \;& C_{n,s}PV\int_{\mathbb{R}^n}\frac{-\bar{W}(y,0)}{|x^o-y|^{n+2s}}dy \\
= \;& C_{n,s}PV\{\int_{\Sigma_{\lambda_0}}\frac{-\bar{W}(y,0)}{|x^o-y|^{n+2s}}dy+\int_{\widetilde{\Sigma_{\lambda_0}}}\frac{\bar{W}(y,0)}{|x^o-y|^{n+2s}}dy\} \\
= \;& C_{n,s}PV\{\int_{\Sigma_{\lambda_0}}\frac{-\bar{W}(y,0)}{|x^o-y|^{n+2s}}dy+\int_{\Sigma_{\lambda_0}}\frac{-\bar{W}(y^\lambda,0)}{|x^o-y^\lambda|^{n+2s}}dy\} \\
= \;& C_{n,s}PV\{\int_{\Sigma_{\lambda_0}}\frac{-\bar{W}(y,0)}{|x^o-y|^{n+2s}}dy+\int_{\Sigma_{\lambda_0}}\frac{\bar{W}(y,0)}{|x^o-y^\lambda|^{n+2s}}dy\} \\
= \;& C_{n,s}PV\int_{\Sigma_{\lambda_0}}\bar{W}(y,0)\{\frac{1}{|x^o-y^\lambda|^{n+2s}}-\frac{1}{|x^o-y|^{n+2s}}\}dy \\
\leq \;& 0
\end{aligned}
\tag{7.19}
$$

since

$$\frac{1}{|x^o-y^\lambda|^{n+2s}}-\frac{1}{|x^o-y|^{n+2s}}<0$$

and

$$\bar{W}(y,0)\geq 0$$

This implies that

$$\bar{W}(y,0)\equiv 0,\ \forall y\in\mathbb{R}^n$$

7.3.3 Show $\bar{u}(x,0)\equiv 0$

Let $u_k(x,t)=u(x,t+t_k)$, then by (7.1), we have

$$\frac{\partial u_k(x,t)}{\partial t}+(-\Delta)^s u_k(x,t)=f(t+t_k,u_k(x,t))$$

By regularity theory for parabolic equations [45], there exists some functions $\bar{u}(x,t)$

such that as $k \to \infty$, $u_k(x,t)$ converges uniformly to $\bar{u}(x,t)$ for $(x,t) \in D^{\tau_0} \times \mathbb{R}$, $f(t,u)$ converges uniformly to $\bar{f}(0,\bar{u})$ for $x \in D^{\tau_0}$

and

$$\frac{\partial \bar{u}(x,t)}{\partial t} + (-\Delta)^s \bar{u}(x,t) = \bar{f}(t, \bar{u}(x,t))$$

Since

$$f(0,u) \geq 0$$

Thus

$$\bar{f}(0,\bar{u}) \geq 0$$

In order to show that

$$\bar{u}(x,0) \equiv 0, x \in \mathbb{R}^n, \tag{7.20}$$

we apply the following:

Lemma 7.3. *(Strong Maximum Principle for $\frac{\partial \bar{u}}{\partial t} + (-\Delta)^s \bar{u} = \bar{f}(t,\bar{u})$).*

Assume that $\bar{u}(x,t) \in [C^{1,1}_{loc}(\Omega_\lambda) \cap C(\bar{\Omega_\lambda}) \cap \mathcal{L}_{2s}] \times C^1([\underline{t}, T])$

$$\begin{cases} \frac{\partial \bar{u}(x,t)}{\partial t} + (-\Delta)^s \bar{u}(x,t) = \bar{f}(t,\bar{u}), & (x,t) \in D^\tau \times [\underline{t}, T], \\ \bar{u}(x,t) \geq 0, & (x,t) \in D^\tau \times [\underline{t}, T] \end{cases} \tag{7.21}$$

we have either

$$\bar{u}(x,0) > 0, x \in D^\tau$$

or

$$\bar{u}(x,0) \equiv 0, x \in \mathbb{R}^n$$

Proof. First, if $\bar{u}(x,0) \geq 0$ and $\bar{u}(x^o,0) = 0$, $(x^o,0)$ then is a minimum, thus we have $\frac{\partial \bar{u}}{\partial t}(x^o,0) = 0$.

If $\bar{u}(x,0) \not\equiv 0$, then

$$\begin{aligned}
& (-\Delta)^s \bar{u}(x^o,0) \\
= \;& C_{n,s} PV \int_{\mathbb{R}^n} \frac{-\bar{u}(y,0)}{|x^o - y|^{n+2s}} dy \\
= \;& C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{u}(y,0)}{|x^o - y|^{n+2s}} dy + \int_{\widetilde{\Sigma_{\lambda_0}}} \frac{-\bar{u}(y,0)}{|x^o - y|^{n+2s}} dy \} \\
= \;& C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{u}(y,0)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_{\lambda_0}} \frac{-\bar{u}(y^\lambda,0)}{|x^o - y^\lambda|^{n+2s}} dy \} \\
= \;& C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{u}(y,0)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_{\lambda_0}} \frac{\bar{u}(y,0)}{|x^o - y^\lambda|^{n+2s}} dy \} \\
= \;& C_{n,s} PV \int_{\Sigma_{\lambda_0}} \bar{u}(y,0) \{ \frac{1}{|x^o - y^\lambda|^{n+2s}} - \frac{1}{|x^o - y|^{n+2s}} \} dy \\
< \;& 0
\end{aligned}$$

since

$$\frac{1}{|x^o - y^\lambda|^{n+2s}} - \frac{1}{|x^o - y|^{n+2s}} < 0$$

and

$$\bar{u}(y,0) \geq 0$$

and

$$\bar{u}(y,0) \not\equiv 0$$

which, by 7.21, is a contradiction with $\bar{f}(0,\bar{u}) \geq 0$.

Therefore, we have either $\bar{u}(x,0) > 0$, $x \in D^{\tau}$ or $\bar{u}(x,0) \equiv 0$, $x \in \mathbb{R}^n$.

□

If $\bar{u}(x,0) > 0$, $x \in D^{\tau}$, then $\bar{u}(x^{\tau_0},0) > \bar{u}(x,0)$, therefore, $\bar{W}(\bar{x},0) = \bar{u}(\bar{x}^{\tau_0},0) - u(\bar{x},0) > 0$. $\bar{W}(x,0) > 0$ somewhere, but we already derive $\bar{W}(x,0) \equiv 0$, hence we must have $\bar{u}(x,0) \equiv 0$, $x \in \mathbb{R}^n$.

Thus, we know $u(x,t_k)$ converges to 0 uniformly.

7.3.4 Derive a contradiction for large k

In order to derive a contradiction for large k, we modify t_k a bit.

We still denote $W_k(x,t)$ by $W^{\tau_0}(x,t+t_k)$, Let

$$W_k(x_k,0) \equiv W^{\tau_0}(x_k,t_k) = m_k \tag{7.22}$$

which converges to zero.

Let

$$V_k(x,t) = W_k(x,t) - 2m_k\eta(\epsilon_k(t-t_k)) \tag{7.23}$$

where $\eta(t) \in C_0^{\infty}$ is a cut-off function such that $|\eta'(t)| \leq c$ and

$$\eta(t) = \begin{cases} 1, & |t| \leq 1, \\ 0, & |t| \geq 2. \end{cases}$$

When (x,t) is outside $D^{\tau_0-\epsilon} \times (t_k-2, t_k+2)$,

$$
\begin{aligned}
& V_k(x,t) \\
= \;& W_k(x,t),
\end{aligned}
$$

When (x,t) is inside $D^{\tau_0-\epsilon} \times (t_k-2, t_k+2)$, such that at (x,t_k)

$$
\begin{aligned}
& V_k(x,t_k) \\
= \;& W_k(x,t_k) - 2m_k
\end{aligned}
$$

The value of V_k outside $D^{\tau_0-\epsilon} \times (t_k-2, t_k+2)$ is greater than the value of V_k inside $D^{\tau_0-\epsilon} \times (t_k-2, t_k+2)$, so $V_k(x,t)$ attains its minimum at some point, say $(\bar{x}_k, \bar{t}_k)$ in $D^{\tau_0-\epsilon} \times (t_k-2, t_k+2)$.

This implies,

$$\frac{\partial V_k}{\partial t}(\bar{x}_k, \bar{t}_k) = 0$$

Combining (7.22) and (7.23), it is easy to deduce

$$
\begin{aligned}
& V_k(x_k,0) \\
= \;& W_k(x_k,0) - 2m_k \\
= \;& m_k - 2m_k \\
= \;& -m_k
\end{aligned}
$$

Thus

$$V_k(\bar{x}_k, \bar{t}_k) \le -m_k$$

Let

$$\widetilde{V}_k(x,t) = V_k(x, t+\bar{t}_k)$$

Then

$$\widetilde{V}_k(\bar{x}_k, 0) = V_k(\bar{x}_k, \bar{t}_k)$$

Then

$$\frac{\partial \widetilde{V}_k}{\partial t} + (-\triangle)^s \widetilde{V}_k = c_\lambda \widetilde{V}_k + 2m_k \eta(\epsilon_k(t - t_k))$$

By regularity theory for parabolic equations [45], there exists some functions $\bar{V}(x,t)$ such that $k \to \infty$, $\widetilde{V}_k(x,t) \to \bar{V}(x,t)$ converges uniformly for $x \in D^{\tau_0}$,

Moreover

$$\frac{\partial \bar{V}}{\partial t} + (-\Delta)^s \bar{V} = c_\lambda \bar{V}$$

We know

$$\frac{\partial V_k}{\partial t} \sim \frac{\partial W_k}{\partial t} - 2m_k \epsilon_k c$$

Therefore we conclude

$$\frac{\partial W_k}{\partial t} \sim m_k \epsilon_k$$

Passing to a subsequence, $(\bar{x}_k, \bar{t}_k) \to (x^o, t^o) \in D^{\tau_0 - \epsilon} \times [-2, 2]$

$W_k \to \bar{W}$ uniformly, and

$$\frac{\partial \bar{W}}{\partial t} + (-\Delta)^s \bar{W} = \bar{c} \bar{W}$$

As we have already derived

$$\bar{W}(x^o, t^o) = 0, \frac{\partial \bar{W}}{\partial t}(x^o, t^o) = 0$$

Also following from (7.4)

$$
\begin{aligned}
& \frac{\partial \bar{W}}{\partial t}(x^o, t^o) + (-\Delta)^s \bar{W}(x^o, t^o) \\
= \;& f(t^o, |x^{o\tau_0}|, u^{\tau_0}) - f(t^o, |x^o|, u) \\
= \;& \frac{f(t^o, |x^{o\tau_0}|, u^{\tau_0}) - f(t^o, |x^o|, u)}{\bar{u}^{\tau_0}(x^o, t^o) - \bar{u}(x^o, t^o)} \bar{W}(x^o, t^o) \\
:= \;& c^\tau(x^o, t^o) \bar{W}(x^o, t^o),
\end{aligned}
$$

It is easy to deduct

$$(-\Delta)^s \bar{W}(x^o, t^o) = 0$$

It follows

$$
\begin{aligned}
& (-\Delta)^s \bar{W}(x^o, t^o) \\
= \;& C_{n,s} PV \int_{\mathbb{R}^n} \frac{-\bar{W}(y, t^o)}{|x^o - y|^{n+2s}} dy \\
= \;& C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{W}(y, t^o)}{|x^o - y|^{n+2s}} dy + \int_{\widetilde{\Sigma_{\lambda_0}}} \frac{-\bar{W}(y, t^o)}{|x^o - y|^{n+2s}} dy \} \\
= \;& C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{W}(y, t^o)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_{\lambda_0}} \frac{-\bar{W}(y^\lambda, t^o)}{|x^o - y^\lambda|^{n+2s}} dy \} \\
= \;& C_{n,s} PV \{ \int_{\Sigma_{\lambda_0}} \frac{-\bar{W}(y, t^o)}{|x^o - y|^{n+2s}} dy + \int_{\Sigma_{\lambda_0}} \frac{\bar{W}(y, t^o)}{|x^o - y^\lambda|^{n+2s}} dy \} \\
= \;& C_{n,s} PV \int_{\Sigma_{\lambda_0}} \bar{W}(y, t^o) \{ \frac{1}{|x^o - y^\lambda|^{n+2s}} - \frac{1}{|x^o - y|^{n+2s}} \} dy \\
\leq \;& 0
\end{aligned}
$$

since

$$\frac{1}{|x^o - y^\lambda|^{n+2s}} - \frac{1}{|x^o - y|^{n+2s}} < 0$$

and

$$\bar{W}(y, t^o) \geq 0$$

This implies

$$\bar{W}(x,t^o) \equiv 0, \forall y \in \mathbb{R}^n$$

Similar with above, assume $f(t,0) \geq 0$, for

$$u_k(x,t) = u(x,t+t_k)$$

$$u_k(x,\bar{t}_k) = u(x,\bar{t}_k + t_k)$$

$$u_k(x,t) \to \bar{u}(x,t)$$

$$\frac{\partial \bar{u}}{\partial t} + (-\Delta)^s \bar{u} = f(t,\bar{u})$$

We have

$$\bar{u}(x,t) \equiv 0$$

Now for sufficiently large k,

$$\begin{aligned}
& (-\Delta)^s V_k(\bar{x}_k,\bar{t}_k) \\
= \;& (-\Delta)^s W_k \\
= \;& -\frac{\partial W_k}{\partial t}(\bar{x}_k,\bar{t}_k) + c^{\tau_0}(\bar{x}_k,\bar{t}_k + t_k) W_k(\bar{x}_k,\bar{t}_k)
\end{aligned}$$

Since we know

$$\begin{aligned}
& (-\Delta)^s V_k(\bar{x}_k,\bar{t}_k) \\
\leq \;& \frac{c}{[d(\bar{x}_k, T_{\lambda_o})]^{2s}} V_k(\bar{x}_k,\bar{t}_k) \\
\leq \;& -c_1 m_k
\end{aligned}$$

where $c_1 > 0$

$$-\frac{\partial W_k}{\partial t}(\bar{x}_k, \bar{t}_k) \sim \epsilon_k m_k$$

If we assume $\frac{\partial f}{\partial u}(t, 0) = 0$, as $u_k \to 0$ uniformly,

$$c^{\tau_0}(\bar{x}_k, \bar{t}_k + t_k) = o(1) \to 0$$

Finally,

$$-c_1 m_k \geq o(1) m_k$$

or

$$c_1 \leq -o(1)$$

Since $o(1) \to 0$ as $k \to \infty$, which is a contradiction with $-c_1 m_k \geq o(1) m_k$ as $k \to \infty$.

Now we can carve out from D^{τ_0} a closed set $K \subset D^{\tau_0}$ such that $D^{\tau_0} \backslash K$ is narrow.

According to (7.16), we have

$$W^{\tau_0}(x, t) \geq C_0 > 0, \ \ in\ K \tag{7.24}$$

Since W^{τ} is continuous with respect to τ, for small $\epsilon > 0$, we have:

$$W^{\tau_0 - \epsilon}(x, t) \geq 0, \ \ in\ K \tag{7.25}$$

According to (H), we have

$$W^{\tau_0 - \epsilon}(x, t) \geq 0, \ \ in\ (D^{\tau_0 - \epsilon})^c$$

It is obvious that $(D^{\tau_0-\epsilon}\backslash K)^c = K \cup (D^{\tau_0-\epsilon})^c$, then

$$\begin{cases} \frac{\partial W^{\tau_0-\epsilon}}{\partial t} + (-\Delta)^s W^{\tau_0-\epsilon} = c(x,t) W^{\tau_0-\epsilon}(x,t), & (x,t) \in D^{\tau_0-\epsilon}\backslash K \times \mathbb{R}, \\ W^{\tau_0-\epsilon}(x,t) \geq 0, & (x,t) \in (D^{\tau_0-\epsilon}\backslash K)^c \times \mathbb{R}, \end{cases} \tag{7.26}$$

By Lemma 7.5, we have

$$W^{\tau_0-\epsilon}(x_k, t_k) \geq 0, \ (x,t) \in D^{\tau_0-\epsilon}\backslash K \times \mathbb{R} \tag{7.27}$$

From this and (7.25), we obtain $W^\tau(x,t) \geq 0$ for $\tau \in (\tau_0 - \epsilon, \tau_0)$ which contradicts the definition of τ_0.

Since $W^\tau(x,t) \not\equiv 0$, $(x,t) \in D^\tau \times \mathbb{R}$, $\forall 0 < \tau < \widetilde{\tau}$, if there exists (x^o, t^o) such that (x^o, t^o) is the minimum point, from the above process, on one hand,

$$(-\Delta)^s W^\tau(x^o, t^o) \leq 0$$

On the other hand,

$$(-\Delta)^s W^\tau(x^o, t^o) = 0$$

This forces

$$W^\tau \equiv 0$$

which contradicts (H)

Thus we have proved the Theorem.

7.4 Monotonicity of solutions of parabolic Monge-Ampére equations

In the following section, we will try to prove the solution of the parabolic nonlocal Monge-Ampére equation with assumption below is monotone increasing using the sliding method.

Theorem 7.4. *(Monotonicity of solution of parabolic Monge Ampere equation)*

Let Ω be a bounded domain in $\mathbb{R}^n$ which is convex in the x_n-direction. Let $0 < s < 1$, and suppose that $u(x,t) \in \left(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\right) \times (-\infty, \infty)$ is a positive bounded classical solution of

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - D_s^\theta u(x,t) = f(t,|x|,u), & (x,t) \in \Omega \times (-\infty,\infty), \\ u(x,t) = \varphi(x,t), & (x,t) \in \Omega^c \times (-\infty,\infty), \end{cases} \tag{7.28}$$

We impose some conditions on u. Let $u(x,t) = \varphi(x,t)$ in Ω^c, suppose H:

For any three points $x = (x', x_n)$, $y = (x', y_n)$ and $z = (x', z_n)$ lying on a segment parallel to the x_n axis, $y_n < x_n < z_n$ with $y, z \in \Omega^c$, we have

$$\varphi(y,t) < u(x,t) < \varphi(z,t), \ (x,t) \in \Omega \times \mathbb{R} \tag{7.29}$$

and

$$\varphi(y,t) \leq \varphi(x,t) \leq \varphi(z,t), \ (x,t) \in \Omega^c \times \mathbb{R} \tag{7.30}$$

Assume that f is uniformly Lipschitz continuous in u. i.e:

$$f(t,|x|,u_1) - f(t,|x|,u_2) \leq c|u_1 - u_2|, \ \forall x \in \Omega,$$

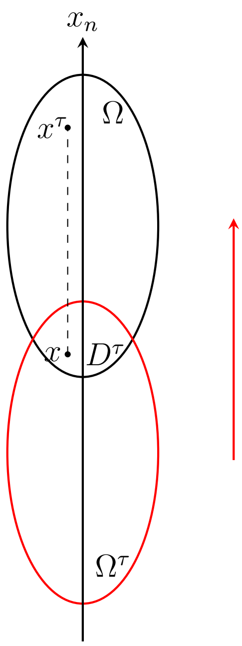


Figure 14: sliding method

then $u(x,t)$ *is monotone increasing with respect to* x_n *in* Ω*, i.e: for any* $\tau > 0$*, we have*

$u((x', x_n + \tau), t) > u((x', x_n), t)$ *for* $(x', x_n), (x', x_n + \tau) \in \Omega$ *and* $t \in \mathbb{R}$*,*

where $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$

7.4.1 Basic set-up

Write

$$x = (x', x_n)$$

For any $\tau \in \mathbb{R}$, define

$$u^\tau(x) = u(x', x_n + \tau)$$

Let Ω be a bounded domain in $\mathbb{R}^n$, which is convex in the x_n-direction. By sliding Ω downward τ units, we obtain Ω^τ:

$$\Omega^\tau = \Omega - \tau e_n, e_n = (0, 0, \cdots, 1)$$

Define

$$D^\tau = \Omega^\tau \cap \Omega$$

and

$$\widetilde{\tau} = \sup\{\tau \mid \tau > 0, D^\tau \neq \emptyset\}$$

Assume that $u(x,t)$ is a positive solution of equation (7.28). We compare the values of $u(x,t)$ with

$$u^\tau(x,t) = u((x', x_n + \tau), t).$$

Let

$$W^\tau(x,t) = u^\tau(x,t) - u(x,t)$$

$u^\tau(x,t)$ satisfies equation (7.28), from which, we have

$$\begin{aligned}
& \frac{\partial W^\tau}{\partial t}(x,t) - D_s^\theta u^\tau(x,t) + D_s^\theta u(x,t) \\
= \;& f(t,|x|,u^\tau) - f(t,|x|,u) \\
= \;& \frac{f(t,|x|,u^\tau) - f(t,|x|,u)}{u^\tau(x,t) - u(x,t)} W^\tau(x,t) \\
:= \;& c^\tau(x,t) W^\tau(x,t), \qquad (7.31)
\end{aligned}$$

where

$$c^\tau(x,t) = \frac{f(t,|x|,u^\tau) - f(t,|x|,u)}{u^\tau(x,t) - u(x,t)}$$

Since f is Lipschitz continuous, we have

$$c^\tau(x,t) \leq L, \;\; \forall x \in D^\tau$$

where L is the Lipschitz constant.

The main part of the proof is to show that

$$W^{\tau}(x,t) > 0,\ \forall (x,t) \in D^{\tau} \times \mathbb{R}$$

We divide our proof in two steps.

Step 1: *Begin sliding Ω^{τ} downward τ units along the x_n axis*

So then

$$|x| < |x^{\tau}|$$

We will show that for τ sufficiently close to $\widetilde{\tau}$ when D^{τ} is narrow, we have

$$W^{\tau}(x,t) \geq 0,\ (x,t) \in D^{\tau} \times \mathbb{R}$$

Apparently, D^{τ} is a narrow region in the x_n direction for τ sufficiently close to $\widetilde{\tau}$. We first establish a narrow region principle for the parabolic Monge-Ampere equation, which is an important ingredient in applying the sliding method on bounded domains.

Step 2: *Decrease τ as long as $W^{\tau}(x,t) \geq 0$ holds to its limiting position*

We would show the limit position is $\tau = 0$. After we have completed the second step, we would prove $\forall \tau > 0$, $W^{\tau}(x,t) > 0$, thus we have completed proof of monotonicity of solution of parabolic Monge-Ampere equation in the bounded domain.

7.5 *Step 1: show* $W^\tau(x,t) \geq 0$

7.5.1 Narrow Region principle on a parabolic cylinder

Lemma 7.5. *(Narrow Region principle on a parabolic cylinder) Let D be a bounded narrow region in $\mathbb{R}^n$. Assume that $u(x,t) \in \left(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\right) \cap \mathcal{L}_{2s} \times [\underline{t}, T]$. $W^\tau(x,t) = u^\tau(x,t) - u(x,t)$ is lower semi-continuous on $\bar{D} \times [\underline{t}, T]$, and satisfies*

$$\begin{cases} \frac{\partial W^\tau}{\partial t} - D_s^\theta u^\tau + D_s^\theta u = c(x,t) W^\tau(x,t), & (x,t) \in D \times [\underline{t}, T], \\ W^\tau(x,t) \geq 0, & (x,t) \in (\mathbb{R}^n \setminus D) \times [\underline{t}, T], \end{cases} \tag{7.32}$$

where $c(x,t)$ is bounded from below in D. Let $dn(D)$ be the width of D in the x_n-direction. Then:

$$W^\tau(x,t) \geq 0, \;\; (x,t) \in D \times [\underline{t}, T] \tag{7.33}$$

Moreover, we have either $W^\tau(x,t) > 0$ in $D \times \mathbb{R}$ or $W^\tau(x,t) \equiv 0$ in $\mathbb{R}^n \times \mathbb{R}$:

Proof. First, we will prove

$$W^\tau(x,t) \geq \min\{0, \inf_{D \times [\underline{t}, T]} W^\tau(x, \underline{t})\}, \;\; (x,t) \in D \times [\underline{t}, T] \tag{7.34}$$

If (7.34) does not hold, then the lower semi-continuity of $W^\tau(x,t)$ on $D \times [\underline{t}, T]$ guarantees that there exists an $(x^o, t^o) \in D \times [\underline{t}, T]$ such that

$$W^\tau(x^o, t^o) = \min_{D \times (\underline{t}, T]} W^\tau < 0$$

And one can further deduce from condition (7.32) that (x^o, t^o) is in the interior of $D \times$

$[\underline{t}, T]$

Since (x^o, t^o) is the minimum, thus

$$\frac{\partial W^\tau(x^o,t^o)}{\partial t} = 0$$

By definition of D_s^θ we have

$$\begin{aligned}
& D_s^\theta u(x^o,t^o) - D_s^\theta u^\tau(x^o,t^o) \\
= \;& \inf\{P.V\int_{\mathbb{R}^n} \frac{u(y,t^o)-u(x^o,t^o)}{|A^{-1}(y-x^o)|^{n+2s}}dy\} - \inf\{P.V\int_{\mathbb{R}^n} \frac{u^\tau(y,t^o)-u^\tau(x^o,t^o)}{|A^{-1}(y-x^o)|^{n+2s}}dy\}
\end{aligned}$$

For any $\eta > 0$, there exists an A_η, such that

$$\begin{aligned}
& D_s^\theta u(x^o,t^o) - D_s^\theta u^\tau(x^o,t^o) \\
= \;& \inf\{P.V\int_{\mathbb{R}^n} \frac{u(y,t^o)-u(x^o,t^o)}{|A^{-1}(y-x^o)|^{n+2s}}dy\} - \inf\{P.V\int_{\mathbb{R}^n} \frac{u^\tau(y,t^o)-u^\tau(x^o,t^o)}{|A^{-1}(y-x^o)|^{n+2s}}dy\} \\
\geq \;& P.V\int_{\mathbb{R}^n} \frac{u(y,t^o)-u(x^o,t^o)}{|A_\eta^{-1}(y-x^o)|^{n+2s}}dy - \eta - P.V\int_{\mathbb{R}^n} \frac{u^\tau(y,t^o)-u^\tau(x^o,t^o)}{|A_\eta^{-1}(y-x^o)|^{n+2s}}dy \\
= \;& P.V\int_{\mathbb{R}^n} \frac{W^\tau(x^o,t^o)-W^\tau(y,t^o)}{|A_\eta^{-1}(y-x^o)|^{n+2s}}dy - \eta \\
= \;& P.V\{\int_D \frac{W^\tau(x^o,t^o)-W^\tau(y,t^o)}{|A_\eta^{-1}(y-x^o)|^{n+2s}}dy + \int_{\mathbb{R}^n\setminus D} \frac{W^\tau(x^o,t^o)-W^\tau(y,t^o)}{|A_\eta^{-1}(y-x^o)|^{n+2s}}dy\} - \eta \\
\leq \;& P.V\int_{\mathbb{R}^n\setminus D} \frac{W^\tau(x^o,t^o)-W^\tau(y,t^o)}{|A_\eta^{-1}(y-x^o)|^{n+2s}}dy - \eta \\
\leq \;& P.V\int_{\mathbb{R}^n\setminus D} \frac{W^\tau(x^o,t^o)}{|A_\eta^{-1}(y-x^o)|^{n+2s}}dy - \eta \\
\leq \;& c_0 W^\tau(x^o,t^o)\int_{\mathbb{R}^n\setminus D} \frac{1}{|A_\eta^{-1}(y-x^o)|^{n+2s}}dy - \eta
\end{aligned}$$

By lemma 5.7, we have

$$D_s^\theta u(x^o,t^o) - D_s^\theta u^\tau(x^o,t^o) \leq \frac{cW^\tau(x^o,t^o)}{dn(D)^{2s}} < 0 \tag{7.35}$$

Combining (7.31) and (7.35), we deduce

$$c^{\tau}(x^o,t^o)W^{\tau}(x^o,t^o) \leq \frac{cW^{\tau}(x^o,t^o)}{dn(D)^{2s}}$$

Then we derive

$$\frac{c}{dn(D)^{2s}} \leq c^{\tau}(x^o,t^o)$$

for τ sufficiently close to $\widetilde{\tau}$, $dn(D)$ would be sufficiently small, since c^{τ} is bounded, we derive a contradiction. Therefore, (7.34) must be valid.

Let

$$\bar{W} = e^{m(t-\underline{t})}W^{\tau}(x,t), \;\; m > 0$$

From 7.31,

$$\begin{aligned} & \frac{\partial \bar{W}(x,t)}{\partial t} - D_s^{\theta}\bar{u}^{\tau}(x,t) + D_s^{\theta}\bar{u}(x,t) \\ = \;\; & \bar{c}(x,t)\bar{W}(x,t) \end{aligned}$$

with $\bar{c}(x,t)$ is still bounded.

This time, we want to show

$$W^{\tau}(x,t) \geq 0 \tag{7.36}$$

Suppose otherwise, (7.36) does not hold, then $\bar{W}(x,t)$ is negative somewhere, hence there exists an $x^o \in D$ and $t^o \in [\underline{t},T]$ such that

$$\bar{W}(x^o,t^o) = \min_{D\times(\underline{t},T]} \bar{W} < 0$$

If

$$t^o < T$$

$$\frac{\partial \bar{W}}{\partial t}(x^o, t^o) = 0$$

If

$$t^o = T$$

$$\frac{\partial \bar{W}}{\partial t}(x^o, t^o) \leq 0$$

From (7.31), we derive

$$(-\Delta)^s \bar{W}(x^o, t^o) \geq \bar{c}(x^o, t^o)\bar{W}(x^o, t^o)$$

We also have

$$(-\Delta)^s \bar{W}(x^o, t^o) \leq \frac{c}{dn(D)^{2s}} \bar{W}(x^o, t^o) \tag{7.37}$$

We deduce

$$\frac{c}{dn(D)^{2s}} \bar{W}(x^o, t^o) \geq \bar{c}(x^o, t^o)\bar{W}(x^o, t^o)$$

Then we derive

$$\frac{c}{dn(D)^{2s}} \leq \bar{c}(x^o, t^o)$$

Which is a contradiction for $dn(D)$ sufficiently small. Thus,

$$\bar{W}(x, t) \geq \min\{0, \inf_{x \in D} \bar{W}(x, \underline{t})\},\ \forall (x, t) \in D \times (\underline{t}, T)$$

Thus

$$e^{m(t-\underline{t})}W^{\tau}(x,t) \geq \min\{0, \inf_{x\in D} \bar{W}(x,\underline{t})\}$$

So

$$W^{\tau}(x,t) \geq e^{-m(t-\underline{t})} \min\{0, \inf_{x\in D} \bar{W}(x,\underline{t})\}$$

$W^{\tau}(x,t)$ is bounded from below. Let $\underline{t} \to -\infty$, $W^{\tau}(x,t) \to \geq 0$.

We conclude that for τ sufficiently close to $\widetilde{\tau}$ when D^{τ} is narrow,

$$W^{\tau}(x,t) \geq 0$$

If $W^{\tau}(x^o,t^o) = 0$ at $(x^o,t^o) \in D^{\tau} \times \mathbb{R}$, then (x^o,t^o) is a minimum point of $W^{\tau}(x,t)$ in $D^{\tau} \times \mathbb{R}$.

So then

$$\frac{W^{\tau}(x^o,t^o)}{\partial t} = 0$$

If $W^{\tau}(x,t) \not\equiv 0$ in $\mathbb{R}^n \times \mathbb{R}$, then there exists a point (y^o,t^o) and one of the neighborhood $N(y^o,t^o)$ such that

$$W^{\tau}(y,t^o) \geq c > 0,$$

in $N(y^o,t^o)$

Then we have

$$
\begin{aligned}
& D_s^\theta u(x^o,t^o) - D_s^\theta u^\tau(x^o,t^o) \\
= \; & \inf\{P.V \int_{\mathbb{R}^n} \frac{u(y,t^o)-u(x^o,t^o)}{|A^{-1}(y-x^o)|^{n+2s}} dy\} - \inf\{P.V \int_{\mathbb{R}^n} \frac{u^\tau(y,t^o)-u^\tau(x^o,t^o)}{|A^{-1}(y-x^o)|^{n+2s}} dy\} \\
\geq \; & P.V \int_{\mathbb{R}^n} \frac{u(y,t^o)-u(x^o,t^o)}{|A_\eta^{-1}(y-x^o)|^{n+2s}} dy - \eta - P.V \int_{\mathbb{R}^n} \frac{u^\tau(y,t^o)-u^\tau(x^o,t^o)}{|A_\eta^{-1}(y-x^o)|^{n+2s}} dy \\
= \; & P.V \int_{\mathbb{R}^n} \frac{W^\tau(x^o,t^o)-W^\tau(y,t^o)}{|A_\eta^{-1}(y-x^o)|^{n+2s}} dy - \eta \\
= \; & P.V \int_{\mathbb{R}^n} \frac{-W^\tau(y,t^o)}{|A_\eta^{-1}(y-x^o)|^{n+2s}} dy - \eta \\
\leq \; & c_0 \int_{\mathbb{R}^n} \frac{-W^\tau(y,t^o)}{|y-x^o|^{n+2s}} dy - \eta \\
\leq \; & c_0 c \int_{N(y^o,t^o)} \frac{-1}{|y-x^o|^{n+2s}} dy - \eta \\
\leq \; & -c_0 c C - \eta
\end{aligned}
$$

where c_0, c and C are positive constants. Let $\eta \to 0$, we have

$$
\begin{aligned}
& D_s^\theta u(x^o,t^o) - D_s^\theta u^\tau(x^o,t^o) \\
\leq \; & -c_0 c C \\
< \; & 0
\end{aligned}
$$

However, from (7.32), we have

$$D_s^\theta u(x^o,t^o) - D_s^\theta u^\tau(x^o,t^o) = 0$$

This is a contradiction. Then we have either

$$W^\tau(x,t) > 0$$

in $D^\tau \times \mathbb{R}$ or

$$W^\tau(x,t) \equiv 0$$

in $\mathbb{R}^n \times \mathbb{R}$.

So far, we have proved the Narrow Region Theorem 7.5. □

We first proved that for τ sufficiently close to $\widetilde{\tau}$ when D^τ is narrow, we have

$$W^\tau(x,t) \geq 0 \tag{7.38}$$

7.6 *Step 2: decrease τ as long as $W^\tau \geq 0$ holds to its limiting position*

The inequality (7.38) provides a starting point, from which we can carry out the sliding. Now we decrease τ as long as $W^\tau(x,t) \geq 0$ holds to its limiting position. Define

$$\tau_0 = \inf\{\tau \mid W^\tau(x,t) \geq 0, \forall (x,t) \in D^\tau \times \mathbb{R}, 0 < \tau < \widetilde{\tau}\}$$

In this part, we show that

$$\tau_0 = 0$$

Suppose

$$\tau_0 > 0$$

we will show that Ω^τ can be slid upward a little bit more and we will have

$$W^\tau(x,t) \geq 0$$

To be more rigorous, there exists some $\epsilon > 0$, such that for any $\tau \in (\tau_0 - \epsilon, \tau_0)$, we have

$W^\tau(x,t) \geq 0, x \in D^\tau$

This is a contradiction with the definition of τ_0. Hence we must have

$$\tau_0 = 0$$

7.6.1 Show $W^{\tau_0} > 0$ for $(x,t) \in D^{\tau_0} \times \mathbb{R}$

Since $W^{\tau_0}(x,t) \geq 0$ for $(x,t) \in D^{\tau_0} \times \mathbb{R}$ and $W^{\tau_0}(x,t) > 0$ for $(x,t) \in \Omega \cap \partial D^{\tau_0} \times \mathbb{R}$, we have

$$W^{\tau_0}(x,t) \not\equiv 0 \tag{7.39}$$

for $(x,t) \in D^{\tau_0} \times \mathbb{R}$

We want to show $W^{\tau_0}(x,t) > 0$ for $(x,t) \in D^{\tau_0} \times \mathbb{R}$

Otherwise, $\exists (x^o, t^o) \in D^{\tau_0} \times \mathbb{R}$ such that $W^{\tau_0}(x^o, t^o) = 0$

Since $W^{\tau_0}(x,t) \geq 0$ inside $D^{\tau_0} \times \mathbb{R}$, so for $W^{\tau_0}(x^o,t^o) = 0$, we know (x^o,t^o) is the minimum.

Then

$$\frac{\partial W^{\tau_0}(x^o,t^o)}{\partial t} = 0$$

Following from (7.31), we then derive

$$\begin{aligned}
& \frac{\partial W^{\tau_0}}{\partial t}(x^o,t^o) - D_s^\theta u^\tau(x^o,t^o) + D_s^\theta u(x^o,t^o) \\
= \; & f(t^o, |x^o|, u^{\tau_0}) - f(t^o, |x^o|, u) \\
= \; & \frac{f(t^o, |x^o|, u^{\tau_0}) - f(t^o, |x^o|, u)}{u^{\tau_0}(x^o,t^o) - u(x^o,t^o)} W^{\tau_0}(x^o,t^o) \\
:= \; & c^{\tau_0}(x^o,t^o) W^{\tau_0}(x^o,t^o),
\end{aligned}$$

So that

$$D_s^\theta u(x^o,t^o) - D_s^\theta u^\tau(x^o,t^o) = 0 \tag{7.40}$$

It follows that

$$\begin{aligned}
& 0 = D_s^\theta u(x^o,t^o) - D_s^\theta u^\tau(x^o,t^o) \\
= \; & \inf\{P.V\int_{\mathbb{R}^n} \frac{u(y,t^o)-u(x^o,t^o)}{|A^{-1}(y-x^o)|^{n+2s}}dy\} - \inf\{P.V\int_{\mathbb{R}^n} \frac{u^\tau(y,t^o)-u^\tau(x^o,t^o)}{|A^{-1}(y-x^o)|^{n+2s}}dy\} \\
\geq \; & P.V\int_{\mathbb{R}^n} \frac{u(y,t^o)-u(x^o,t^o)}{|A_\eta^{-1}(y-x^o)|^{n+2s}}dy - \eta - P.V\int_{\mathbb{R}^n} \frac{u^\tau(y,t^o)-u^\tau(x^o,t^o)}{|A_\eta^{-1}(y-x^o)|^{n+2s}}dy \\
= \; & P.V\int_{\mathbb{R}^n} \frac{W^\tau(x^o,t^o)-W^\tau(y,t^o)}{|A_\eta^{-1}(y-x^o)|^{n+2s}}dy - \eta \\
= \; & P.V\int_{\mathbb{R}^n} \frac{-W^\tau(y,t^o)}{|A_\eta^{-1}(y-x^o)|^{n+2s}}dy - \eta \\
\leq \; & c_0\int_{\mathbb{R}^n} \frac{-W^\tau(y,t^o)}{|y-x^o|^{n+2s}}dy - \eta \\
\leq \; & c_0 c\int_{N(y^o,t^o)} \frac{-1}{|y-x^o|^{n+2s}}dy - \eta \\
\leq \; & -c_0cC - \eta
\end{aligned} \tag{7.41}$$

where c_0, c and C are positive constants. Let $\eta \to 0$, we have

$$\begin{aligned}
& D_s^\theta u(x^o,t^o) - D_s^\theta u^\tau(x^o,t^o) \\
\leq \; & -c_0cC \\
< \; & 0
\end{aligned}$$

which contradicts with (7.40).

We conclude

$$W^{\tau_0}(x,t) > 0 \tag{7.42}$$

for every $(x,t) \in D^{\tau_0} \times \mathbb{R}$.

However, since $t \in (-\infty, \infty)$, $W^{\tau_0}(x,t)$ may not be bounded away from 0.

We want to further derive

$$\inf W^{\tau_0}(x,t) > c_o > 0, \ (x,t) \in D^{\tau_0-\epsilon} \times \mathbb{R} \tag{7.43}$$

I will prove (7.43) by contradiction.

Proof: If (7.43) is violated, then $\exists (x_k, t_k) \in D^{\tau_0-\epsilon} \times \mathbb{R}$ such that $W^{\tau_0}(x_k, t_k) \to 0$

Without loss of generality, by Bolzano-Weierstrass theorem, $\exists$ subsequences of x_k, here we still denote this subsequence by x_k, such that $x_k \to x^o \in D^{\tau_0-\epsilon}$

Now for each $t_k (k \geq k_0)$, Let

$$W_k(x,t) = W^{\tau_0}(x, t+t_k)$$

so

$$W_k(x_k, 0) = W^{\tau_0}(x_k, t_k) \to 0$$

From (7.31), we derive:

$$\begin{aligned}
& \frac{\partial W_k}{\partial t}(x,t) - D_s^{\theta} u_k^{\tau_0}(x, t+t_k) + D_s^{\theta} u_k(x, t+t_k) \\
= \ & f(t+t_k, |x^{\tau_0}|, u^{\tau_0}) - f(t+t_k, |x|, u) \\
= \ & \frac{f(t+t_k, |x^{\tau_0}|, u^{\tau_0}) - f(t+t_k, |x|, u)}{u^{\tau_0}(x, t+t_k) - u(x, t+t_k)} W_k(x,t) \\
:= \ & c^{\tau_0}(x, t+t_k) W_k(x,t), \qquad (7.44)
\end{aligned}$$

So W_k satisfies

$$\frac{\partial W_k}{\partial t}(x,t) - D_s^\theta u_k^{\tau_0}(x,t+t_k) + D_s^\theta u_k(x,t+t_k) = c^{\tau_0}(x,t+t_k)W_k(x,t) \tag{7.45}$$

By regularity theory for parabolic equations [45], there exists some functions $\bar{W}(x,t)$ and $\bar{c}(x,t)$ such that $k \to \infty$, $W_k(x,t)$ converges uniformly to $\bar{W}(x,t)$ for $(x,t) \in D^{\tau_0} \times \mathbb{R}$, there exists some functions $\bar{u}(x,t)$ such that as $k \to \infty$, $u_k(x,t)$ converges uniformly to $\bar{u}(x,t)$ for $(x,t) \in D^{\tau_0} \times \mathbb{R}$,

and $\bar{W}$ satisfies:

$$\frac{\partial \bar{W}}{\partial t} - D_s^\theta \bar{u}^{\tau_0} + D_s^\theta \bar{u} = \bar{c}(x,t)\bar{W}(x,t) \tag{7.46}$$

Since

$$W_k(x_k,0) = W^{\tau_0}(x_k,t_k) \to 0$$

$$\bar{W}(x^o,0) = 0$$

and

$$\bar{W} \geq 0$$

So $(x^o,0)$ is the minimum.

$$\frac{\partial \bar{W}}{\partial t}(x^o,0) = 0$$

Also from (7.31), we derive,

$$
\begin{aligned}
& \frac{\partial \bar{W}}{\partial t}(x^o,0) - D_s^\theta \bar{u}^{\tau_0}(x^o,0) + D_s^\theta \bar{u}(x^o,0) \\
= \;& f(t_k,|x^{o\tau_0}|,\bar{u}^{\tau_0}) - f(t_k,|x^o|,\bar{u}) \\
= \;& \frac{f(t_k,|x^{o\tau_0}|,\bar{u}^{\tau_0}) - f(t_k,|x^o|,\bar{u})}{\bar{u}^{\tau_0}(x^o,t_k) - \bar{u}(x^o,t_k)} \bar{W}(x^o,0) \\
:= \;& c^\tau(x^o,t_k)\bar{W}(x^o,0),
\end{aligned}
$$

we derive

$$
D_s^\theta \bar{u}(x^o,0) - D_s^\theta \bar{u}^{\tau_0}(x^o,0) = 0
$$

It follows that

$$
\begin{aligned}
& 0 = D_s^\theta \bar{u}(x^o,0) - D_s^\theta \bar{u}^{\tau_0}(x^o,0) \\
= \;& \inf\{P.V \int_{\mathbb{R}^n} \frac{\bar{u}(y,0)-\bar{u}(x^o,0)}{|A^{-1}(y-x^o)|^{n+2s}} dy\} - \inf\{P.V \int_{\mathbb{R}^n} \frac{\bar{u}^\tau(y,0)-\bar{u}^{\tau_0}(x^o,0)}{|A^{-1}(y-x^o)|^{n+2s}} dy\} \\
\geq \;& P.V \int_{\mathbb{R}^n} \frac{\bar{u}(y,0)-\bar{u}(x^o,0)}{|A_\eta^{-1}(y-x^o)|^{n+2s}} dy - \eta - P.V \int_{\mathbb{R}^n} \frac{\bar{u}^{\tau_0}(y,0)-\bar{u}^{\tau_0}(x^o,0)}{|A_\eta^{-1}(y-x^o)|^{n+2s}} dy \\
= \;& P.V \int_{\mathbb{R}^n} \frac{W^{\tau_0}(x^o,0)-W^{\tau_0}(y,0)}{|A_\eta^{-1}(y-x^o)|^{n+2s}} dy - \eta \\
= \;& P.V \int_{\mathbb{R}^n} \frac{-W^{\tau_0}(y,0)}{|A_\eta^{-1}(y-x^o)|^{n+2s}} dy - \eta \\
\leq \;& c_0 \int_{\mathbb{R}^n} \frac{-W^{\tau_0}(y,0)}{|y-x^o|^{n+2s}} dy - \eta \\
\leq \;& c_0 c \int_{N(y^o,0)} \frac{-1}{|y-x^o|^{n+2s}} dy - \eta \\
\leq \;& -c_0 c C - \eta
\end{aligned}
\tag{7.47}
$$

where c_0, c and C are positive constants. Let $\eta \to 0$, we have

$$\begin{aligned} & D_s^\theta \bar{u}(x^o, 0) - D_s^\theta \bar{u}^\tau(x^o, 0) \\ \leq \; & -c_0 c C \\ < \; & 0 \end{aligned}$$

which is a contradiction, therefore, we have proved (7.43).

Now we can carve out from D^{τ_0} a closed set $K \subset D^{\tau_0}$ such that $D^{\tau_0} \backslash K$ is narrow. According to (7.42), we have

$$W^{\tau_0}(x,t) \geq c_0 > 0, \; in \; K \tag{7.48}$$

Since W^τ is continuous with respect to τ, for small $\epsilon > 0$, we have:

$$W^{\tau_0 - \epsilon}(x,t) \geq 0, \; in \; K \tag{7.49}$$

According to (H), we have

$$W^{\tau_0 - \epsilon}(x,t) \geq 0, \; in \; (D^{\tau_0 - \epsilon})^c \times \mathbb{R}$$

It is obvious that $(D^{\tau_0 - \epsilon} \backslash K)^c = K \cup (D^{\tau_0 - \epsilon})^c$, then

$$\begin{cases} \frac{\partial W^{\tau_0 - \epsilon}}{\partial t} - D_s^\theta u^{\tau_0 - \epsilon} + D_s^\theta u^{\tau_0 - \epsilon} = c(x,t) W^{\tau_0 - \epsilon}(x,t), & (x,t) \in D^{\tau_0 - \epsilon} \backslash K \times \mathbb{R}, \\ W^{\tau_0 - \epsilon}(x,t) \geq 0, & (x,t) \in (D^{\tau_0 - \epsilon} \backslash K)^c \times \mathbb{R}, \end{cases} \tag{7.50}$$

By Lemma 7.5, we have

$$W^{\tau_0-\epsilon}(x,t) \geq 0 \tag{7.51}$$

in $D^{\tau_0-\epsilon}\backslash K$.

From this and (7.49), we obtain $W^{\tau}(x,t) \geq 0$ for $\tau \in (\tau_0-\epsilon, \tau_0)$ which contradicts the definition of τ_0.

Since $W^{\tau}(x,t) \not\equiv 0$, $(x,t) \in D^{\tau} \times \mathbb{R}$, $\forall 0 < \tau < \widetilde{\tau}$, if there exists (x^o, t^o) such that (x^o, t^o) is the minimum point, from the above process, on one hand,

$$D_s^{\theta}u(x^o,t^o) - D_s^{\theta}u^{\tau}(x^o,t^o) < 0$$

On the other hand,

$$D_s^{\theta}u(x^o,t^o) - D_s^{\theta}u^{\tau}(x^o,t^o) = 0$$

This forces

$$W^{\tau} \equiv 0$$

which contradicts (H)

Thus we have proved the Theorem.

7.7 u is strictly decreasing with respect to x_n in the whole space

In this section, we would prove u must be strictly increasing with respect to x_n, and it depends on x_n only.

Theorem 7.6. *Let* $u(x,t) \in \left(C^{1,1}_{loc}(\Omega) \cap C(\bar{\Omega})\right) \cap \mathcal{L}_{2s} \times [\underline{t}, T]$ *be a solution of*

$$\frac{\partial u}{\partial t}(x,t) - D_s^{\theta} u(x,t) = f(t,|x|,u), \quad (x,t) \in \mathbb{R}^n \times (-\infty, \infty) \tag{7.52}$$

with condition

$$|u(x,t)| \leq 1$$

and

$$u((x', x_n), t) \to \pm 1 \tag{7.53}$$

uniformly in $x' = (x_1, \cdots, x_{n-1})$*. Also,* $f(t,|x|,u)$ *is non-increasing near* $u(x,t) = \pm 1$*. Then* u *must be strictly increasing with respect to* x_n*, and it depends on* x_n *only.*

The basic set-up is same as before except we let

$$W^{\tau}(x,t) = u(x,t) - u^{\tau}(x,t)$$

. We divide our proof in two steps.

Step 1: *Begin sliding* Ω^{τ} *downward* τ *units along the* x_n *axis*

So then

$$|x| < |x^{\tau}|$$

We will show that for τ sufficiently close to $\widetilde{\tau}$, that is, when τ is sufficiently large, D^{τ} is narrow, we have

$$W^{\tau}(x,t) \leq 0, \ (x,t) \in D^{\tau} \times \mathbb{R}$$

Step 2: *Decrease* τ *as long as* $W^{\tau}(x,t) \leq 0$ *holds to its limiting position*

We would show the limit position is $\tau = 0$. In second step, we would divide the proof into two cases, one is $|x_n| \leq M$, the other is $|x_n| \geq M$, in both cases we would show the limiting position is $\tau = 0$. After we have completed the second step, we would prove $\forall \tau > 0$, $W^\tau(x,t) < 0$, thus we have completed proof of monotonicity of solution of parabolic Monge-Ampere equation in the whole space. In the last section, we would show $u(x,t)$ depends on x_n only, that is, $u(x,t) = u(x_n,t)$.

7.8 *Step 1: show* $W^\tau(x,t) \leq 0$

In step 1, we will show that for τ sufficiently large,

$$W^\tau(x,t) \leq 0,\ (x,t) \in \mathbb{R}^n \times \mathbb{R} \tag{7.54}$$

Otherwise,

$$\sup_{\mathbb{R}^n \times \mathbb{R}} W^\tau(x,t) = A > 0$$

then there exists a sequence $\{x^k, t^k\} \subset \mathbb{R}^n \times \mathbb{R}$ such that

$$W^\tau(x^k, t^k) \to A > 0 \tag{7.55}$$

as $k \to \infty$.

Denote $x^k = (x_1^k, x_2^k, \cdots, x_n^k)$. Let $\eta \in C_0^\infty$:

$$\eta(x,t) = \begin{cases} 1, & if\ |x|, |t| < 1, \\ 0, & if\ |x|, |t| \geq 2 \end{cases}$$

So $\max_{\mathbb{R}^n \times \mathbb{R}} \eta(x,t) = 1$. Set

$$\psi_k(x,t) = \eta(x - x^k, t - t^k)$$

According to (7.55), there exists a sequence $\{\varepsilon_k\} \to 0$ such that

$$W^\tau(x^k, t^k) + \varepsilon_k \psi_k(x^k, t^k) = A$$

Set

$$U_k^\tau(x^k, t^k) = W^\tau(x^k, t^k) + \varepsilon_k \psi_k(x^k, t^k)$$

Since we have

$$U_k^\tau(x,t) = W^\tau(x,t) \le A, x \in \mathbb{R}^n \backslash B_2(x^k), t \in \mathbb{R} \backslash B_2(t^k)$$

and

$$W^\tau(x^k, t^k) + \varepsilon_k \psi_k(x^k, t^k) = A$$

Then there exists $(\bar{x}^k, \bar{t}^k) \in B_1(x^k) \times B_1(t^k)$ such that

$$U_k^\tau(\bar{x}^k, \bar{t}^k) = \max_{\mathbb{R}^n \times \mathbb{R}} U_k^\tau(x^k, t^k) = A$$

Therefore

$$\frac{\partial U_k^\tau}{\partial t}(\bar{x}^k, \bar{t}^k) = 0$$

We have

$$\varepsilon_k = A - W^\tau(x^k, t^k)$$

Therefore

$$\frac{\partial W^\tau}{\partial t}(\bar{x}^k, \bar{t}^k) \sim \varepsilon_k$$

By the definition of D_s^θ, we have

$$D_s^\theta(W^\tau + \varepsilon_k\psi_k)(\bar{x}^k, \bar{t}^k) = D_s^\theta(U_k^\tau)(\bar{x}^k, \bar{t}^k) \approx -(-\triangle)^s(U_k^\tau)(\bar{x}^k, \bar{t}^k) \leq -c_0$$

We also have,

$$\begin{aligned} & D_s^\theta W^\tau(\bar{x}^k, \bar{t}^k) \\ = \ & \inf\{P.V \int_{\mathbb{R}^n} \frac{W^\tau(y, \bar{t}^k) - W^\tau(\bar{x}^k, \bar{t}^k)}{|A^{-1}(y - \bar{x}^k)|^{n+2s}} dy\} \end{aligned}$$

and

$$\begin{aligned} & D_s^\theta(W^\tau + \varepsilon_k\psi_k)(\bar{x}^k, \bar{t}^k) \\ = \ & \inf\{P.V \int_{\mathbb{R}^n} \frac{(W^\tau + \varepsilon_k\psi_k)(y, \bar{t}^k) - (W^\tau + \varepsilon_k\psi_k)(\bar{x}^k, \bar{t}^k)}{|A^{-1}(y - \bar{x}^k)|^{n+2s}} dy\} \\ = \ & \inf\{P.V \int_{\mathbb{R}^n} \frac{W^\tau(y, \bar{t}^k) - W^\tau(\bar{x}^k, \bar{t}^k) + \varepsilon_k\psi_k(y, \bar{t}^k) - \varepsilon_k\psi_k(\bar{x}^k, \bar{t}^k)}{|A^{-1}(y - \bar{x}^k)|^{n+2s}} dy\} \\ \geq \ & \inf\{P.V \int_{\mathbb{R}^n} \frac{W^\tau(y, \bar{t}^k) - W^\tau(\bar{x}^k, \bar{t}^k)}{|A^{-1}(y - \bar{x}^k)|^{n+2s}} dy\} + \inf\{P.V \int_{\mathbb{R}^n} \frac{\varepsilon_k\psi_k(y, \bar{t}^k) - \varepsilon_k\psi_k(\bar{x}^k, \bar{t}^k)}{|A^{-1}(y - \bar{x}^k)|^{n+2s}} dy\} \end{aligned}$$

and we also have

$$\begin{aligned} & \inf\{P.V \int_{\mathbb{R}^n} \frac{\varepsilon_k\psi_k(y, \bar{t}^k) - \varepsilon_k\psi_k(\bar{x}^k, \bar{t}^k)}{|A^{-1}(y - \bar{x}^k)|^{n+2s}} dy \\ = \ & \varepsilon_k P.V \int_{\mathbb{R}^n} \frac{\psi_k(y, \bar{t}^k) - \psi_k(\bar{x}^k, \bar{t}^k)}{|A_k^{-1}(y - \bar{x}^k)|^{n+2s}} dy \\ \geq \ & -c\varepsilon_k \end{aligned}$$

So we derive

$$
\begin{aligned}
& D_s^{\theta}(W^{\tau} + \varepsilon_k \psi_k)(\bar{x}^k, \bar{t}^k) \\
\geq \; & D_s^{\theta} W^{\tau}(\bar{x}^k, \bar{t}^k) - c\varepsilon_k
\end{aligned}
$$

We also have

$$
\begin{aligned}
& D_s^{\theta} W^{\tau}(\bar{x}^k, \bar{t}^k) - c\varepsilon_k \\
= \; & \inf\{P.V \int_{\mathbb{R}^n} \frac{W^{\tau}(y, \bar{t}^k) - W^{\tau}(\bar{x}^k, \bar{t}^k)}{|A^{-1}(y - \bar{x}^k)|^{n+2s}} dy\} - c\varepsilon_k \\
= \; & P.V \int_{\mathbb{R}^n} \frac{W^{\tau}(y, \bar{t}^k) - W^{\tau}(\bar{x}^k, \bar{t}^k)}{|A^{-1}(y - \bar{x}^k)|^{n+2s}} dy - \varepsilon_A - c\varepsilon_k \\
= \; & P.V \int_{\mathbb{R}^n} \frac{u(y, \bar{t}^k) - u(\bar{x}^k, \bar{t}^k)}{|A^{-1}(y - \bar{x}^k)|^{n+2s}} dy - P.V \int_{\mathbb{R}^n} \frac{u^{\tau}(y, \bar{t}^k) - u^{\tau}(\bar{x}^k, \bar{t}^k)}{|A^{-1}(y - \bar{x}^k)|^{n+2s}} dy - \varepsilon_A - c\varepsilon_k \\
\geq \; & \inf\{P.V \int_{\mathbb{R}^n} \frac{u(y, \bar{t}^k) - u(\bar{x}^k, \bar{t}^k)}{|A^{-1}(y - \bar{x}^k)|^{n+2s}} dy\} - \inf\{P.V \int_{\mathbb{R}^n} \frac{u^{\tau}(y, \bar{t}^k) - u^{\tau}(\bar{x}^k, \bar{t}^k)}{|A^{-1}(y - \bar{x}^k)|^{n+2s}} dy\} - 2\varepsilon_A - c\varepsilon_k \\
= \; & D_s^{\theta} u(\bar{x}^k, \bar{t}^k) - D_s^{\theta} u^{\tau}(\bar{x}^k, \bar{t}^k) - 2\varepsilon_A - c\varepsilon_k \\
= \; & \frac{\partial u}{\partial t} - f(t, |\bar{x}^k|, u) - \frac{\partial u^{\tau}}{\partial t} + f(t, |\bar{x}^k|, u^{\tau}) - 2\varepsilon_A - c\varepsilon_k \\
= \; & \frac{\partial W^{\tau}}{\partial t} + f(t, |\bar{x}^k|, u^{\tau}) - f(t, |\bar{x}^k|, u) - 2\varepsilon_A - c\varepsilon_k \\
= \; & f(t, |\bar{x}^k|, u^{\tau}) - f(t, |\bar{x}^k|, u) - 2\varepsilon_A - c\varepsilon_k
\end{aligned}
$$

When τ is sufficiently large, we have either

1. $u^{\tau}(\bar{x}^k, \bar{t}^k)$ is close to 1 or

2. $u(\bar{x}^k, \bar{t}^k)$ is close to -1.

Since $u(\bar{x}^k, \bar{t}^k) > u^{\tau}(\bar{x}^k, \bar{t}^k)$, in case 1, both $u(\bar{x}^k, \bar{t}^k)$ and $u^{\tau}(\bar{x}^k, \bar{t}^k)$ are close to 1, while in case 2, both $u(\bar{x}^k, \bar{t}^k)$ and $u^{\tau}(\bar{x}^k, \bar{t}^k)$ are close to -1. Hence in any case, we can apply the monotonicity of f to derive that

$$f(t, |\bar{x}^k|, u^\tau) \geq f(t, |\bar{x}^k|, u)$$

Then we have

$$D_s^\theta W^\tau(\bar{x}^k, \bar{t}^k) - c\varepsilon_k \geq -2\varepsilon_A - c\varepsilon_k \to 0$$

Thus we derive

$$-c_0 \geq D_s^\theta(W^\tau + \varepsilon_k \psi_k)(\bar{x}^k, \bar{t}^k) \geq D_s^\theta W^\tau(\bar{x}^k, \bar{t}^k) - c\varepsilon_k \geq 0 \tag{7.56}$$

which is a contradiction. This verifies (7.54).

7.9 *Step 2: decrease τ as long as $W^\tau \leq 0$ holds to its limiting position*

The inequality (7.54) provides a starting point, from which we can carry out the sliding. Now we decrease τ as long as $W^\tau(x, t) \leq 0$ holds to its limiting position. Define

$$\tau_0 = \inf\{\tau \mid W^\tau(x, t) \leq 0, \forall (x, t) \in \mathbb{R}^n \times \mathbb{R}\}$$

In this part, we show that

$$\tau_0 = 0$$

Suppose

$$\tau_0 > 0$$

we will show that Ω^τ can be slid upward a little bit more and we will have

$$W^\tau(x,t) \leq 0$$

To be more rigorous, there exists some $\epsilon > 0$, such that for any $\tau \in (\tau_0 - \epsilon, \tau_0)$, we have $W^\tau(x,t) \leq 0$, $x \in D^\tau$

This is a contradiction with the definition of τ_0. Hence we must have

$$\tau_0 = 0$$

We will divide this section into two parts, first we will show $\tau_0 = 0$ is the limiting position for $|x_n| \leq M$, second we will show $\tau_0 = 0$ is the limiting position for $|x_n| \geq M$.

7.9.1 Show $\sup_{-M \leq x_n < M} W^{\tau_0} < 0$

Otherwise, we have $\tau_0 > 0$ such that

$$\sup_{-M \leq x_n \leq M} W^{\tau_0}(x,t) < 0 \tag{7.57}$$

If (7.57) does not hold, then

$$\sup_{-M \leq x_n \leq M} W^{\tau_0}(x,t) = 0$$

then there exists a sequence $\{x^k, t^k\} \subset \mathbb{R}^{n-1} \times [-M, M] \times \mathbb{R}$ such that

$$W^{\tau_0}(x^k, t^k) \to 0 \tag{7.58}$$

as $k \to \infty$

Denote $x^k = (x_1^k, x_2^k, \cdots, x_n^k)$. Let $\eta \in C_0^\infty$:

$$\eta(x,t) = \begin{cases} 1, & if \ |x|, |t| < 1, \\ 0, & if \ |x|, |t| \geq 2 \end{cases}$$

So $\max_{\mathbb{R}^n \times \mathbb{R}} \eta(x,t) = 1$. Set

$$\psi_k(x,t) = \eta(x - x^k, t - t^k)$$

According to (7.55), there exists a sequence $\{\varepsilon_k\} \to 0$ such that

$$W^{\tau_0}(x^k, t^k) + \varepsilon_k \psi_k(x^k, t^k) = 0$$

Set

$$U_k^{\tau_0}(x^k, t^k) = W^{\tau_0}(x^k, t^k) + \varepsilon_k \psi_k(x^k, t^k)$$

Since we have

$$U_k^{\tau_0}(x,t) = W^{\tau_0}(x,t) \leq 0, x \in \mathbb{R}^n \backslash B_2(x^k), t \in \mathbb{R} \backslash B_2(t^k)$$

and

$$W^{\tau_0}(x^k, t^k) + \varepsilon_k \psi_k(x^k, t^k) = 0$$

Then there exists $(\bar{x}^k, \bar{t}^k) \in B_1(x^k) \times B_1(t^k)$ such that

$$U_k^{\tau_0}(\bar{x}^k, \bar{t}^k) = \max_{\mathbb{R}^n \times \mathbb{R}} U_k^{\tau_0}(x^k, t^k) = 0$$

By the definition of D_s^θ, we have

$$D_s^\theta(W^{\tau_0}+\varepsilon_k\psi_k)(\bar{x}^k,\bar{t}^k)=D_s^\theta(U_k^{\tau_0})(\bar{x}^k,\bar{t}^k)\approx -(-\triangle)^s(U_k^{\tau_0})(\bar{x}^k,\bar{t}^k)\leq 0$$

On one hand, similar to the proof in Step 1, we have

$$D_s^\theta(W^{\tau_0}+\varepsilon_k\psi_k)(\bar{x}^k,t^o)\geq D_s^\theta W^{\tau_0}(\bar{x}^k,t^o)-c\varepsilon_k \tag{7.59}$$

We also have

$$\begin{aligned}
& D_s^\theta W^{\tau_0}(\bar{x}^k,\bar{t}^k)-c\varepsilon_k \\
= \;& \inf\{P.V\int_{\mathbb{R}^n}\frac{W^{\tau_0}(y,\bar{t}^k)-W^{\tau_0}(\bar{x}^k,\bar{t}^k)}{|A^{-1}(y-\bar{x}^k)|^{n+2s}}dy\}-c\varepsilon_k \\
\geq \;& \inf\{P.V\int_{\mathbb{R}^n}\frac{W^{\tau_0}(y,\bar{t}^k)}{|A^{-1}(y-\bar{x}^k)|^{n+2s}}dy\}-\inf\{P.V\int_{\mathbb{R}^n}\frac{W^{\tau_0}(\bar{x}^k,\bar{t}^k)}{|A_k^{-1}(y-\bar{x}^k)|^{n+2s}}dy\}-\varepsilon_k-c\varepsilon_k \\
\geq \;& -\varepsilon_k-c\varepsilon_k\to 0
\end{aligned} \tag{7.60}$$

Denote $u_k(x,t)=u(x^k,t^k)$, $W_k^{\tau_0}(x,t)=W^{\tau_0}(x^k,t^k)$

Since u is uniformly continuous, by the Arzela-Ascoli Theorem, we have

$$u_k(x,t)\to u_\infty(x,t)\ uniformly\ in\ \mathbb{R}^n\times\mathbb{R},\ as\ k\to\infty$$

Let $k\to\infty$, by the continuity of f, and from (7.59) and (7.60), we have

$$W_k^{\tau_0}(x,t)\to 0, x\in(B_2(0))^c\ uniformly$$

Then

$$u_\infty(x,t)-u_\infty^{\tau_0}(x,t)\equiv 0, x\in(B_2(0))^c$$

For all $m \in \mathbb{N}$, we have

$$u_\infty(x', x_n) = u_\infty(x', x_n + \tau_0) = u_\infty(x', x_n + 2\tau_0) = \cdots = u_\infty(x', x_n + m\tau_0)$$

If x_n is sufficiently negative and m is sufficiently large, then

$$u_\infty(x', x_n) \to -1$$

and

$$u_\infty(x', x_n + m\tau_0) \to 1$$

This is a contradiction, therefore, (7.57) must be true.

Since $\sup_{-M \le x_n \le M} W^{\tau_0}(x,t) < 0$, so there exists a $\delta > 0$ such that

$$\sup_{-M \le x_n \le M} W^{\tau}(x,t) \le 0, \, \forall \tau \in (\tau_0 - \delta, \, \tau_0], \; |x_n| \le M \tag{7.61}$$

which contradicts the definition of τ_0, therefore, we have $\tau_0 = 0$.

Now we only need to prove when $|x_n| \ge M$, $\tau_0 > 0$

$$W^{\tau}(x,t) \le 0, \, \forall \tau \in (\tau_0 - \delta, \, \tau_0], \tag{7.62}$$

Otherwise,

$$\sup_{\mathbb{R}^n \setminus (\mathbb{R}^{n-1} \times [-M, M])} W^{\tau}(x,t) = A > 0, \, \forall \tau \in (\tau_0 - \delta, \tau_0]$$

then there exists a sequence $\{x^k, t^k\}$ such that

$$W^\tau(x^k, t^k) \to A > 0 \tag{7.63}$$

as $k \to \infty$

Denote $x^k = (x_1^k, x_2^k, \cdots, x_n^k)$. Let $\eta \in C_0^\infty$:

$$\eta(x,t) = \begin{cases} 1, & if\ |x|, |t| < 1, \\ 0, & if\ |x|, |t| \geq 2 \end{cases}$$

So $\max_{\mathbb{R}^n \times \mathbb{R}} \eta(x,t) = 1$. Set

$$\psi_k(x,t) = \eta(x - x^k, t - t^k)$$

According to (7.55), there exists a sequence $\{\varepsilon_k\} \to 0$ such that

$$W^\tau(x^k, t^k) + \varepsilon_k \psi_k(x^k, t^k) = A$$

Set

$$U_k^\tau(x^k, t^k) = W^\tau(x^k, t^k) + \varepsilon_k \psi_k(x^k, t^k)$$

Since we have

$$U_k^\tau(x,t) = W^\tau(x,t) \leq A, x \in \mathbb{R}^n \backslash B_2(x^k), t \in \mathbb{R} \backslash B_2(t^k)$$

and

$$W^\tau(x^k, t^k) + \varepsilon_k \psi_k(x^k, t^k) = A$$

Then there exists $(\bar{x}^k, \bar{t}^k) \in B_1(x^k) \times B_1(t^k)$ such that

$$U_k^\tau(\bar{x}^k, \bar{t}^k) = \max_{\mathbb{R}^n \times \mathbb{R}} U_k^\tau(x^k, t^k) = A$$

By the definition of D_s^θ, we have

$$D_s^\theta(W^\tau + \varepsilon_k \psi_k)(\bar{x}^k, \bar{t}^k) = D_s^\theta(U_k^\tau)(\bar{x}^k, \bar{t}^k) \approx -(-\triangle)^s(U_k^\tau)(\bar{x}^k, \bar{t}^k) \leq -c_0$$

Similarly to previous steps, we have

$$\begin{aligned} & D_s^\theta(W^\tau + \varepsilon_k \psi_k)(\bar{x}^k, \bar{t}^k) \\ \geq \;& D_s^\theta W^\tau(\bar{x}^k, \bar{t}^k) - c\varepsilon_k \end{aligned}$$

We also have

$$\begin{aligned} & D_s^\theta W^{\tau_0}(\bar{x}^k, \bar{t}^k) - c\varepsilon_k \\ = \;& \inf\{P.V \int_{\mathbb{R}^n} \frac{W^{\tau_0}(y, \bar{t}^k) - W^{\tau_0}(\bar{x}^k, \bar{t}^k)}{|A^{-1}(y - \bar{x}^k)|^{n+2s}} dy\} - c\varepsilon_k \\ \geq \;& \inf\{P.V \int_{\mathbb{R}^n} \frac{W^{\tau_0}(y, \bar{t}^k)}{|A^{-1}(y - \bar{x}^k)|^{n+2s}} dy\} - \inf\{P.V \int_{\mathbb{R}^n} \frac{W^{\tau_0}(\bar{x}^k, \bar{t}^k)}{|A_k^{-1}(y - \bar{x}^k)|^{n+2s}} dy\} - \varepsilon_k - c\varepsilon_k \\ \geq \;& -\varepsilon_k - c\varepsilon_k \to 0 \end{aligned} \tag{7.64}$$

Let $k \to \infty$, then $\varepsilon_k \to 0$. We get a contradiction and obtain (7.62), which contradicts the definition of τ_0. Therefore, we derive $\tau_0 = 0$.

Finally, we will prove that u is strictly increasing with respect to x_n and $u(x, t)$ depends on x_n only.

We already have

$$W^\tau(x,t) \leq 0 \; in \; \mathbb{R}^n \times \mathbb{R}, \forall \tau > 0$$

If there exists $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$ such that $W^\tau(x_0, t_0) = 0$, then (x_0, t_0) is the maximum point of $W^\tau(x,t)$ in $\mathbb{R}^n \times \mathbb{R}$

Therefore,

$$\frac{\partial W^\tau}{\partial t}(x_0, t_0) = 0$$

On one hand,

$$\begin{aligned}
& D_s^\theta u(x_0,t_0) - D_s^\theta u^\tau(x_0,t_0) \\
= \; & \frac{\partial u}{\partial t}(x_0,t_0) - f(t_0,|x_0|,u) - \frac{\partial u^\tau}{\partial t}(x_0,t_0) + f(t_0,|x_0|,u^\tau) \\
= \; & \frac{\partial W^\tau}{\partial t}(x_0,t_0) + f(t_0,|x_0|,u^\tau) - f(t_0,|x_0|,u) \\
= \; & f(t_0,|x_0|,u^\tau) - f(t_0,|x_0|,u) \\
= \; & \frac{f(t_0,|x_0|,u^\tau) - f(t_0,|x_0|,u)}{u^\tau(x_0,t_0) - u(x_0,t_0)} W^\tau(x_0,t_0) \\
= \; & 0
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
& D_s^\theta u(x_0,t_0) - D_s^\theta u^\tau(x_0,t_0) \\
= \; & \inf\{P.V \int_{\mathbb{R}^n} \frac{u(y,t_0)-u(x_0,t_0)}{|A^{-1}(y-x_0)|^{n+2s}}dy\} - \inf\{P.V \int_{\mathbb{R}^n} \frac{u^\tau(y,t_0)-u^\tau(x_0,t_0)}{|A^{-1}(y-x_0)|^{n+2s}}dy\} \\
\leq \; & P.V \int_{\mathbb{R}^n} \frac{u(y,t_0)-u(x_0,t_0)}{|A_\delta^{-1}(y-x_0)|^{n+2s}}dy - P.V \int_{\mathbb{R}^n} \frac{u^\tau(y,t_0)-u^\tau(x_0,t_0)}{|A_\delta^{-1}(y-x_0)|^{n+2s}}dy + \delta \\
= \; & P.V \int_{\mathbb{R}^n} \frac{W^\tau(y,t_0)-W^\tau(x_0,t_0)}{|A_\delta^{-1}(y-x_0)|^{n+2s}}dy + \delta \\
\leq \; & c_0 \int_{\mathbb{R}^n} \frac{W^\tau(y,t_0)}{|y-x_0|^{n+2s}}dy + \delta \\
\leq \; & c_0(-a) \int_{N_{(y_0,t_0)}} \frac{1}{|y-x^o|^{n+2s}}dy + \delta \\
\leq \; & c_0(-a)c_1 + \delta
\end{aligned}
$$

Let $\delta \to 0$, we have

$$0 = D_s^\theta u(x_0,t_0) - D_s^\theta u^\tau(x_0,t_0) \leq c_0(-a)c_1$$

This is a contradiction. Therefore, we have

$$W^\tau(x,t) < 0, \; (x,t) \in \mathbb{R}^n \times \mathbb{R}, \; \tau > 0$$

Then we will show that $u(x)$ depends on x_n only.

If we replace $u^\tau(x)$ by $u(x+\tau\nu)$, the argument still holds according to the above process, where $\nu = (\nu_1, \nu_2, \nu_3, \cdots, \nu_n)$ with $\nu_n > 0$ is an arbitrary vector that points upward. With the similar arguments as in Step 1 and Step 2, we can obtain that, for each of such ν,

$$u(x+\tau\nu) > u(x)$$

$\forall \tau > 0$ Let $\nu_n \to 0$, by continuity of u, we have that for arbitrary ν with $\nu_n = 0$

$$u(x + \tau\nu) \geq u(x)$$

By replacing ν by $-\nu$, we also have

$$u(x) \geq u(x + \tau\nu)$$

for arbitrary ν with $\nu_n = 0$, So we have

$$u(x + \tau\nu) = u(x) \tag{7.65}$$

(7.65) means that u is independent of $x' = (x_1, x_2, \cdots, x_{n-1})$. Therefore, $u(x) = u(x_n)$. This proves the theorem 7.6.

References

[1] A.V. Chechkin, R. Gorenflo, and I.M. Sokolov *Fractional diffusion in inhomogeneous media.* Journal of Physics A: Mathematical and General, 38(42):L679–L684, 2005.

[2] Brändle, C., Colorado, E., de Pablo, A., & Sánchez, U.*A concaveconvex elliptic problem involving the fractional Laplacian*. Proc. Roy. Soc. Edinburgh, 143(2013) 39-71.

[3] Caffarelli, L., & Silvestre, L. *Regularity theory for fully nonlinear integro-differential equations* C.P.A.M. 62(2009) 597-638.

[4] Caffarelli, L., & Silvestre, L.*An extension problem related to the fractional Laplacian*. Comm. PDE. 32(2007) 1245-1260.

[5] Chen,W., Li, C., Li, Y. *A direct method of moving planes for fractional Laplacian* Accepted by Adv. in Math. 2016.

[6] Chen,W., Li, C., Li, Y.*A direct blowing-up and rescaling argument on nonlocal elliptic equation* Int. J. Math. 2016.

[7] Chen, W., Li, C.*A necessary and sufficient condition for the Nirenberg problem.* Comm. Pure Appl. Math. 48(1995) 657-667.

[8] Chen,W., Li, C., & Li, Y.*A direct method of moving planes for fractional Laplacian* Accepted by Adv. in Math. 2016.

[9] Chen,W, Li, C., & Ou, B. *Classification of solutions for an integral equation.* C.P.A.M. 59(2006) 330-343.

[10] Chen,W., & Zhu, J. *Indefinite fractional elliptic problem and Liouville theorems.* J.D.E. 260(2016) 2758-2785.

[11] Chen,W., Li, C., & Ou, B.*Qualitative properties of solutions for an integral equation* Disc. Cont. Dyn. Sys. 12(2005) 347-354.

[12] Chen, W., Li, C.,*Regularity of solutions for a system of integral equations.* C.P.A.A. 4(2005) 1-8..

[13] Chen, W., Li, C., & Li, G.*Symmetry of solutions for nonlinear problems involving fully nonlinear nonlocal operators* 2015. ArXiv:1604.04806.

[14] Chen, W., Li, C., & Li, G.*Symmetry of solutions for nonlinear problems involving fully nonlinear nonlocal operators* 2015. ArXiv:1604.04806.

[15] Cheng Z, Huang G, Li C.*A Liouville theorem for subcritical Lane-Emden system.* 2014, arXiv:1412.7275

[16] D. Horstmann and G. W. Jones. *Anomalous diffusion in a heterogeneous reaction-diffusion system modeling population growth* Nonlinear Analysis: Real World Applications, vol. 10, no. 5, pp. 3135-3152, 2009.

[17] E. G. Kostadinova & R. Banka *Fractional Laplacian spectral approach to turbulence in a dusty plasma monolayer* Physics of Plasma, vol. 28, no. 7, 2021.

[18] Fernando Diaz-Diaz & Ernesto Estrada *Time and space generalized diffusion equation on graph/networks* Institute of Cross-Disciplinary Physics and Complex Systems, 2022.

[19] H. Berestycki, L. Nirenberg*On the method of moving planes and the sliding method* Bol. Soc. Bras. Mat, 22 (1991), 1-37.

[20] Juan Luis Vázquez,*Nonlinear Diffusion with Fractional Laplacian Operators* Nonlinear Partial Differential Equations, vol. 7, pp. 271-298.

[21] Li, Y., Ma, P.*Symmetry of solutions for a fractional system.* Accepted by Science China Mathematics. 2016.

[22] L. Wu, W. Chen. *The sliding methods for the fractional p-Laplacian* Adv. Math., 361 (2020), 106933

[23] L. Ma, Z. Zhang. *Monotonicity for fractional Laplacian systems in unbounded Lipschitz domains*Discrete Contin. Dyn. Syst., 41 (2021), 537-552.

[24] L. Ma, Z. Zhang. *Monotonicity of positive solutions for fractional p-systems in unbounded Lipschitz domains* Nonlinear Anal., 198 (2020), 111892.

[25] L. Caffarelli, L. Silvestre*An extension problem related to the fractional Laplacian*Comm. Partial Differential Equations, 32 (2007), 1245-1260.

[26] Nick Laskin *Fractional Quantum Mechanics* 2018.

[27] Mou C.*Nonlinear elliptic systems involving the fractional Laplacian in the unit ball and on a half space* C.P.A.A. 6(2015) 2335-2362.

[28] M. Raberto, E. Scalas, and F. Mainardi *Waiting-times and returns in high-frequency fnancial data: an empirical study* Physica A, 314 (2002), 749-755.

[29] M. Sun, B. Liu. *Sliding methods for a class of generalized fractional Laplacian equations* Bull. Malays. Math.Sci. Soc., 45 (2022), 2225-2247.

[30] M. Qu, J.Wu, and T. Zhang. *Sliding method for the semi-linear elliptic equations involving the uniformly elliptic nonlocal operators* Discrete Contin. Dyn. Syst., 41 (2021), 2285-2300.

[31] P.Le. *Rigidity of phase transitions for the fractional elliptic Gross-Pitaevskii system* Fract. Calc. Appl. Anal.,26 (2023), 237-252.

[32] Q. Yu, & F. Liu, & I. Turner, & K. Burrage & V. Vegh *The use of a Riesz fractional differential-based approach for texture enhancement in image processing* ANZIAM Journal, 2013.

[33] Tang, D., & Fang, Y. *Application of the fractional Laplacian equation of direct moving plane method* preprint, 2015.

[34] Wenxiong Chen, Pengyan Wang, Yahui Niu, Yunyun Hu.*Asymptotic Method Of Moving Planes For Fractional Parabolic Equations*

[35] Wenxiong Chen, Pengyan Wang.*Hopf lemma for fractional parabolic equation and parabolic p-equation*

[36] Wenxiong Chen, Leyun Wu *Liouville Theorems for Fractional Parabolic Equations*

[37] Wenxiong Chen,Leyun Wu, Pengyan Wang *Nonexistence of solutions for indefinite fractional parabolic equations*

[38] Wenxiong Chen, Wei Dai.*Some Recent Developments on Fractional Parabolic Equations*

[39] Wenxiong Chen, Yan Li, Pei Ma.*The Fractional Laplacian*

[40] W. Chen, Y. Hu, J. *Monotonicity of positive solutions for nonlocal problems in unbounded domains* Funct.Anal., 281 (2021), 109187.

[41] W. Chen, Y.Hu, and L.Ma. *Moving planes and sliding methods for fractional elliptic and parabolic equations*submitted to Adv. Nonlinear Stud., 2023.

[42] W. Chen, L. Wu. *Monotonicity and one-deimensional symmetry of solutions for fractional reaction-diffusion equations and various applications of sliding methods*preprint, 2023.

[43] W. Dai, G. Qin, and D. Wu. *Direct Methods for pseudo-relativistic Schrödinger operators* J. Geom. Anal., 31(2021), 5555-5618.

[44] X. Chen, G. Bao, and G. Li. *The sliding method for the nonlocal Monge Ampére operator* Nonlinear Anal., 196(2020), 111786.

[45] X. Fernandez-Real, X. Ros-Oton. *Regularity theory for general stable operators: parabolic equations, J. Funct. Anal., 272 (2017), 4165-4221.*

[46] Xueying Chen, Gejun Bao,Guanfeng Li. *The sliding method for the nonlocal Monge–Ampère operator* Nonlinear Analysis.

[47] Yu, X. *Liouville type theorems for integral equations and integral systems* Calc. Var.46(2013) 75-95.

[48] Z. Liu. *Maximum principles and monotonicity of solutions for fractional p-equations in unbounded domains* J.Differential Equations, 270 (2021), 1043-1078.

[49] Zhuo, R., Chen,W., Cui, X. & Yuan, Z. *A Liouville theorem for the fractional Laplacian* Accepted by Disc. Cont. Dyn. Sys. 2015.

Part 3: Topic in long flight in infinite horizon Lorentz Gas

8 Mathematical billiards: The periodic Lorentz gas

The background and history for Lorentz Gas was referenced from [10]. The Lorentz Gas, initially introduced by Lorentz [38] (1905), serves as a model for the movement of electrons within a metal. It characterizes a collection of unconnected point particles navigating through an infinite arrangement of spherical obstacles. Lorentz specifically focused on the stochastic behavior arising in the Boltzmann-Grad limit, wherein the scatterer size diminishes to zero, investigating the dynamic properties under such conditions.

The initial step in proving the existence of a limiting process for the periodic Lorentz gas involves understanding the distribution of free path lengths as the scatterer radius r approaches zero. This crucial result, presented by Jens Marklof and Andreas Strömbergsson in their paper [22], sets the foundation. The investigation of the free path length distribution in the periodic Lorentz gas was previously explored by Pólya, who reformulated the problem in terms of visibility within a periodic forest [16]. Jens Marklof and Andreas Strömbergsson further advance the analysis of the limiting process in their papers [[23], [24], and [25]]. They establish a Markov property, provide explicit formulas, and offer asymptotic estimates for the resulting distributions.

Two major physics problems that served as the primary impetus for the initial investigation of mathematical billiards were: **a**) the ergodic hypothesis, and **b**) the pursuit to comprehend Brownian motion based on microscopic principles. NS Krylov, who is a great Russian statistical physicist, brought hyperbolic billiards, to the attention of the community of mathematicians for further study of hyperbolic behaviors. In the field of mathematics, the 1960s witnessed the emergence and rapid advancement of the theory of smooth hyperbolic dynamical systems, in which Sinai played a prominent role as one of the leading contributors. Sinai's significant contribution to this theory came in his 1970 paper [35], where he introduced a new subject of study: hyperbolic billiards as hyperbolic dynamical systems characterized by singularities. Sinai aspired to develop a mathematical theory for Brownian motion, also known as the dynamical theory of Brownian motion (cf. [29]). The ultimate objective of this theory is to deduce the behavior of Brownian motion based on microscopic assumptions, specifically rooted in Newtonian dynamics. Sinai's contributions include statistical properties of hyperbolic billiards via Markov approximations. Roughly speaking, the following steps outline the main development of his contributions:

1. Markov partitions and Markov approximations for Anosov systems (and Axiom A systems) (cf. [[36], [34], [3], [37]]);

2. Markov partitions and Markov approximations for 2D Sinai billiards (cf. [[4], [5]]);

3. Markov sieves and Markov approximations for 2D Sinai billiards (cf. [[6], [7]]);

Except Sinai, there are a lot of other mathematicians who offered a concise historical summary regarding the role of Markov partitions in the field of dynamical systems theory. Chernov and Young wrote an excellent survey [8], [39] for hyperbolic systems with singularities, in particular for 2D Sinai billiards. Sinai's comprehensive development of Anosov maps [34], along with its far-reaching implications [36], unveiled the profound insights encompassed by the Markov partitions for Anosov maps. In 1980, Bunimovich and Sinai updated the diffusively-rescaled version of the Lorentz process [4], they also used Markov approximation to prove Lorentz process converges to Brownian motion [5], and they proved the diffusively-rescaled version of the Lorentz process weakly converges to planar Wiener process as time coefficient goes to infinity [5]. A decade later, Bunimovich, Chernov, and Sinai revisited the subject in a pair of companion papers. In addition to simplifying the initial constructions and proofs presented in [4] [5], the authors also provided clarification and substantially relaxed the conditions that were previously imposed.

The findings of Jens Marklof and Andreas Strömbergsson in their work [22] complement classical research in ergodic theory, which focuses on the stochastic behavior in the limit of long times while keeping the radius of each scatterer fixed. In a two-dimensional setting with a finite horizon, Bunimovich and Sinai [21] demonstrated that the dynamics become diffusive and adhere to a central limit theorem as time approaches infinity. The notion of a ”finite horizon” implies that the scatterers are sufficiently large to ensure that the distance traveled between consecutive collisions remains bounded. This assumption was recently relaxed by Szász and Varjú [12] following earlier work by Bleher [33]. Recent studies exploring the statistical properties of the periodic Lorentz gas in two dimensions also include references such as [9], [18], and [19]. It is important to note that, currently, there is no proof available for the central limit theorem in higher dimensions, even in the case of a finite horizon [30], [31].

Our paper investigates the long free-flight motion of particles in high-dimensional infinite horizons. Our results show that, with time T goes to infinity, the conditional dynamic invariant measure of the flying time converges to some probability measure of some stochastic processes.

9 Introduction on Homogeneous dynamics

Homogeneous dynamics is the study of actions of subgroups of Lie groups on their quotients called homogeneous spaces. This has been a very active area of research for several decades. One application of homogeneous dynamics is to describe the periodic Lorentz gas in the Boltzmann-Grad limit [22]. The Lorentz gas describes an ensemble of noninteracting point particles in an infinite array of spherical scatterers. Jens Marklof and Strömbergsson had developed a framework for proving, for a given deterministic scatterer configuration, the convergence of the particle dynamics to a limiting transport process.

We will start by giving an introduction to the setting and problems in homogeneous dynamics, and we will discuss various examples in detail. We will also describe the basic set-up for how results from homogeneous dynamics are applied in the proof of the theorems in our thesis.

9.1 Homogeneous space

Let G be a Lie group, and let Γ be a discrete subgroup of G. Then the set

$$X = \Gamma\backslash G = \{\Gamma g : g \in G\}$$

is our homogeneous space.

9.1.1 Example 1

Let $G = \mathbb{R}^d$, $L = \{c_1v_1 + c_2v_2 + \cdots + c_dv_d : c_1, c_2, \cdots, c_d \in \mathbb{Z}\}$ for some $v_1, v_2, \cdots, v_d \in \mathbb{R}^d$ which forms a basis of $\mathbb{R}^d$

Then

$$X = \Gamma\backslash G = L\backslash\mathbb{R}^d = \mathbb{R}^d/L$$

which is a torus, that is a homogeneous space.

9.1.2 Example 2

Let $G = SL_2(\mathbb{R})$ and Γ is a discrete subgroup of G with $-I \in \Gamma$. Recall that $SL_2(\mathbb{R})$ acts by isometries on the hyperbolic upper half space:

$$H = \{z = x + iy : x, y \in \mathbb{R}, y > 0\}$$

with metric

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y}$$

Now $\Gamma\backslash H$ is a hyperbolic surface of finite area

A fundamental domain for $\Gamma\backslash H$ is:

$$F = \{z \in H, |Rez| \leq \frac{1}{z}, |z| \geq 1\}$$

$H = \cup_{\delta\in\Gamma}\delta F$ and $\forall\delta_1 \neq \delta_2 \in \Gamma$ such that $\delta_1 F \cap \delta_2 F = \emptyset$

So

$$X = \Gamma\backslash G = SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R}) \approx T^1(\Gamma\backslash H)$$

where $\Gamma\backslash G$ can be identified with the unit tangent bundle of $\Gamma\backslash H$.

where $\Gamma\backslash H$ looks as follows:

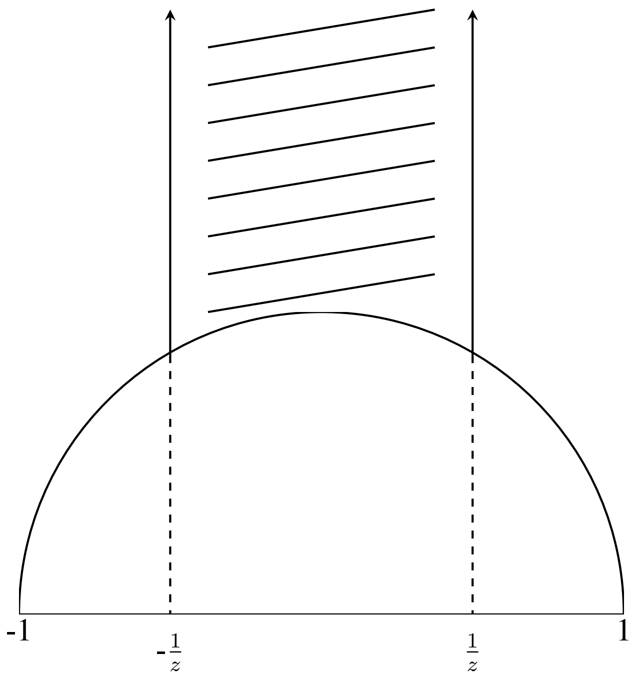


Figure 15: $\Gamma\backslash H$

9.1.3 Example 3

$G = SL_d(\mathbb{R})$ and $\Gamma = SL_d(\mathbb{Z})$, $d \geq 2$ Then

$$X = \Gamma\backslash G = SL_d(\mathbb{Z})\backslash SL_d(\mathbb{R}) = \textit{ the space of lattices in } \mathbb{R}^d \textit{ of covolume } 1$$

with identification map

$$\Gamma g \to \mathbb{Z}^d g$$

is a bijection onto the set of lattices of covolume 1.

where the definition of lattice is defined as follows:

A lattice in $\mathbb{R}^d$ is a set of the form

$$\mathcal{L} = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_d = \{m_1 v_1 + \cdots + m_d v_d \mid m_j \in \mathbb{Z}\}$$

with $v_1, \cdots, v_d$ being an R basis of $\mathbb{R}^d$.

Then

$$F = \{x_1 v_1 + \cdots + x_d v_d, x_1, \cdots, x_d \in [0, 1)\}$$

is a fundamental cell of $\mathbb{R}^d/\mathcal{L} = \mathcal{L}\backslash\mathbb{R}^d$, which means it is same to take left and right cosets.

And

$$covol(\mathcal{L}) = vol(\mathbb{R}^d/\mathcal{L}) = vol(F) = |\det(v_1, v_2, \cdots, v_d)|$$

Also, an affine lattice in $\mathbb{R}^d$ is a set $\mathcal{L}' \subset \mathbb{R}^d$ of the form

$$\mathcal{L}' = w + \mathcal{L}$$

with $w \in \mathbb{R}^d$ and $\mathcal{L}$ is a lattice.

$$covol(\mathcal{L}) = covol(\mathcal{L}')$$

Set

$$G' = ASL_d(\mathbb{R}) := SL_d(\mathbb{R}) \ltimes \mathbb{R}^d$$

G' acts on $\mathbb{R}^d$ as

$$(g_1, w_1)(g_2, w_2) = (g_1 g_2, w_1 g_2 + w_2)$$

based on the law that

$$v(g, w) = vg + w$$

9.2 What is homogeneous dynamics

Consider a homogeneous space of the form:

$$X = \Gamma \backslash G = \{\Gamma g : g \in G\}$$

where G is a Lie group and Γ is a lattice in G.

A lattice Γ in G is a discrete subgroup such that there is a fundamental domain $\mathcal{F}_\Gamma$ of the Γ action on G with finite left Haar measure (Derivation of Haar measure would be based on Ratner's theorem, which would be explained in section 9.3).

We can generate our dynamics on $X = \Gamma \backslash G$ by right multiplication by a fixed element: Let $(h_t)_{t \in \mathbb{R}}$ be a 1-parameter subgroup of G, and Φ_t is a flow on X, then

$$\Phi_t(\Gamma g) := \Gamma g h_t$$

and the flow preserves the measure.

Example 1: geodesic flow

Let $a_t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}$ then $\Phi_t(\Gamma g) := \Gamma g a_t$ is a geodesic flow on $T^1(\Gamma \backslash H)$.
More generally, We can define the geodesic flow as follows:

$$\Phi_t(I) := I a_t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}$$

$$\Phi_t(I) : PSL_2(\mathbb{R}) \to PSL_2(\mathbb{R})$$

$$g \to g\Phi_t$$

where

$$PSL_2(\mathbb{R}) = SL_2(\mathbb{R}) \backslash \{\pm I_2\}$$

Example 2: horocycle flow
Let $n_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ then $h_t(\Gamma g) := \Gamma g n_t$ is a horocycle flow on $T^1(\Gamma \backslash H)$.
Also, Let $n_t^- = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ then $h_t^-(\Gamma g) := \Gamma g n_t^-$ is a horocycle flow on $T^1(\Gamma \backslash H)$.
More generally, $G = SL_d(\mathbb{R})$ and any

$$n_t = \begin{pmatrix} 1 & * & * & \cdots & * \\ 0 & 1 & * & \cdots & * \\ 0 & 0 & 1 & & * \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$n_t^- = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & 1 & 0 & \cdots & 0 \\ * & * & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ * & * & \cdots & 0 & 1 \end{pmatrix}$$

is a horocycle map.

9.3 Ratner's theorem

We asserts that the orbit of any unipotent flow is dense in some homogeneous space of finite volume, based on Ratner's orbit closure theorem [cf [27],Theorem A]:

Theorem 9.1. *(Ratner's orbit closure theorem) Let $X = G/\Gamma$ be a homogeneous space of finite volume with a connected finite-dimensional Lie group G, and let U be a connected subgroup of G generated by unipotent elements. Let U_x be an orbit of U in X. Then the closure $\overline{U_x}$ is itself a homogeneous space of finite volume; In particular, there exists a closed subgroup $U \leq H \leq G$ such that $\overline{U_x} = H_x$.*

Ratner's orbit closure theorem was applied to prove the Oppenheim conjecture in Number theory. In certain scenarios, density is not enough; We also desire equidistribution. Fortunately, Ratner also have a theorem on that [cf [26],Theorem 1]:

Theorem 9.2. *(Ratner's equidistribution theorem) Let $X = G/\Gamma$ be a homogeneous space of finite volume with a connected finite-dimensional Lie group G, and let U be a connected subgroup of G generated by unipotent elements. Assume also that U is a one-parameter group, thus $U = \{g_t : t \in \mathbb{R}\}$ for some homomorphism $t \mapsto g_t$. Then U_x is equidistributed* *in H_x; Thus for any continuous function $f : H_x \to \mathbb{R}$ we have*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(g_t x)\, dt = \int_{H_x} f$$

where $\int_{H_x}$ represents integration on the normalised Haar measure on H_x.

By employing the equidistribution theorem alongside a dash of ergodic theory, a measure-theoretic corollary emerges, describing the ergodic measures of a group formed by unipotent elements [cf [27],Theorem B]:

Theorem 9.3. *(Ratner's equidistribution theorem) Let X be a finite volume homogeneous space for a connected Lie group G, and let U be a connected subgroup of G generated by unipotent elements. Let μ be a probability measure on X which is ergodic under the action of U. Then μ is the Haar measure of some closed finite volume orbit H_x for some $U \leq H \leq G$.*

Ratner's theorem and the Iwasawa decomposition are related in the context of homogeneous spaces and Lie groups.

Ratner's theorem, specifically the Ratner-Margulis theorem, is a significant result in the field of ergodic theory and homogeneous dynamics. It provides deep insights into the long-term behavior of orbits in homogeneous spaces.

On the other hand, the Iwasawa decomposition is a decomposition theorem that is particularly relevant for connected semisimple Lie groups. It states that any element in such a Lie group can be expressed uniquely as the product of three components: an element from a maximal compact subgroup, an element from a maximal abelian subgroup, and an element from a unipotent subgroup.

The connection between Ratner's theorem and the Iwasawa decomposition lies in their implications for the structure and dynamics of homogeneous spaces associated with semisimple Lie groups. In particular, Ratner's theorem can be employed to analyze the orbits of unipotent flows on homogeneous spaces, which aligns with the unipotent component in the Iwasawa decomposition. The theorem provides information about the distribution and equidistribution of these orbits, shedding light on their dynamical behavior. Let me simply introduce the Iwasawa decomposition in the following:

9.3.1 Iwasawa decomposition

Define

$$A = \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_d \end{bmatrix} : a_1, a_2, \cdots, a_d > 0 \right\}$$

and

$$N = \left\{ \begin{bmatrix} 1 & n_{12} & n_{13} & n_{14} & \cdots \\ 0 & 1 & n_{23} & \cdots & \cdots \\ 0 & 0 & 1 & n_{34} & \cdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} : n_{ij} \in \mathbb{R} \right\}$$

and

$$K = SO(d) = \{k \in SL_d(\mathbb{R}) : k^t k = 1_d\}$$

The theorem of Iwasawa decomposition states that, for every $g \in G$, where G is a real Lie group, there exists $n \in N$, $a \in A$, $k \in K$ such that $g = nak$

The sketch of the proof is as follows:

Proof.

$$g = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix}$$

Apply Gram-Schmidt process to $v_d, v_{d-1}, \cdots, v_1$, we get orthogonal vectors $w_d, w_{d-1}, \cdots, w_1$ with

$$v_j \mathbb{R}_{>0} \in w_j + Span\{w_j, \cdots, w_d\}$$

So

$$g = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} = \begin{pmatrix} a_1 & * & * & * & * \\ 0 & a_2 & * & * & * \\ 0 & 0 & a_3 & * & * \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{pmatrix}$$

with

$$\begin{pmatrix} a_1 & * & * & * & * \\ 0 & a_2 & * & * & * \\ 0 & 0 & a_3 & * & * \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_d \end{pmatrix} \in AN$$

and

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{pmatrix} \in K$$

For $g = nak$,

$$d\mu(g) = dn(\Pi_{i<j}\frac{a_j}{a_i})(\Pi_{j=1}^{d-1}\frac{da_j}{a_j})dk$$

where dn is the Haar measure on N, dk is the Haar measure on K.

and

$$\Pi_{j=1}^{d-1}\frac{da_j}{a_j}$$

is the Haar measure in A.

The computation on the Haar measure of g based on the law that

$$d\nu(g) = |detg|^{-d}dx_{11}dx_{12}\cdots dx_{dd}$$

for every

$$g = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ 0 & x_{22} & x_{23} & \cdots & \cdots \\ 0 & 0 & x_{33} & x_{34} & \cdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & x_{dd} \end{pmatrix}$$

$\square$

9.4 Review of Marklof-Strömbergsson's Methodology

9.4.1 Basic set-up

Fix a lattice $\mathcal{L} \subset \mathbb{R}^d$ of covolume one, also fix a choice of $M_0 \in SL(d, \mathbb{R})$ such that $\mathcal{L} = \mathbb{Z}^d M_0$, given $\alpha \in \mathbb{R}^d$ we then define the affine lattice

$$\mathcal{L}_\alpha := (\mathbb{Z}^d + \alpha)M_0 = \mathbb{Z}^d(1, \alpha)(M_0, 0)$$

Consider the set $\mathcal{P}_T$ of lattice points $y \in \mathcal{L}_\alpha$ inside the ball $\mathcal{B}_T^d$ of radius T

More generally, the spherical shell $\mathcal{B}_T^d$ is defined as:

$$\mathcal{B}_T^d(c) = \{x \in \mathbb{R}^d : cT \leq \|x\| < T\}, 0 \leq c < 1$$

For T large there are approximately $C \cdot vol(B_1^d)T^d$ such points. It is well known that as $T \to \infty$, these points become uniformly distributed on S_T^{d-1}.

In Marklof's paper, they study the corresponding directions such that

$$\|y\|^{-1}y \in S_1^{d-1}$$

for

$$y \in \mathcal{P}_T = \mathcal{L}_\alpha \cap \mathcal{B}_T^d(c) \backslash \{0\}$$

They define $\mathcal{D}_T(\sigma, v) \subset S_1^{d-1}$ to be an open disc with center v and set a counting function as

$$\mathcal{N}_{c,T}(\sigma, v) = \#\{y \in \mathcal{P}_T : \|y\|^{-1}y \in \mathcal{D}_T(\sigma, v)\}$$

Referenced from theorem 2.1, Let λ be a Borel probability measure on S_1^d absolutely continuous with respect to Lebesgue measure. Then, for every $\sigma \geq 0$ and $r \in \mathbb{Z}_{\geq 0}$, the limit

$$E_{c,\alpha}(r,\sigma) := lim_{T\to\infty}\lambda(\{v \in S_1^{d-1} : \mathcal{N}_{c,T}(\sigma, v) = r\})$$

exists, and for fixed c, α, r the convergence is uniform with respect to σ in any compact subset of $\mathbb{R}_{\geq 0}$. The limit function is given by

$$E_{c,\alpha}(r,\sigma) := \mu(\{(M.\xi) \in X : \#((\mathbb{Z}^d M + \xi) \cap \Theta(c,\sigma)) = r\})$$

for $\alpha \notin \mathbb{Q}^d$

where

$$\Theta(c,\sigma) = \{(x_1, \cdots, x_d) \in \mathbb{R}^d : c < x_1 < 1, \|(x_2, \cdots, x_d)\| \leq x_1 A(c,\sigma)\}$$

is a cone, and $A(c,\sigma)$ is an area function related with c, σ and volume of $\mathcal{B}_1^{d-1}$.

Also, $ASL(d,\mathbb{R}) = SL(d,\mathbb{R}) \times \mathbb{R}^d$ is the semi-direct product group with multiplication law

$$(M,\xi)(M^{'}\xi^{'}) = (MM^{'}, \xi M^{'} + \xi^{'})$$

and X is defined as

$$X = ASL(d,\mathbb{Z})\backslash ASL(d,\mathbb{R})$$

In our paper, we use the same methodology except that we change the cone to parabloid.

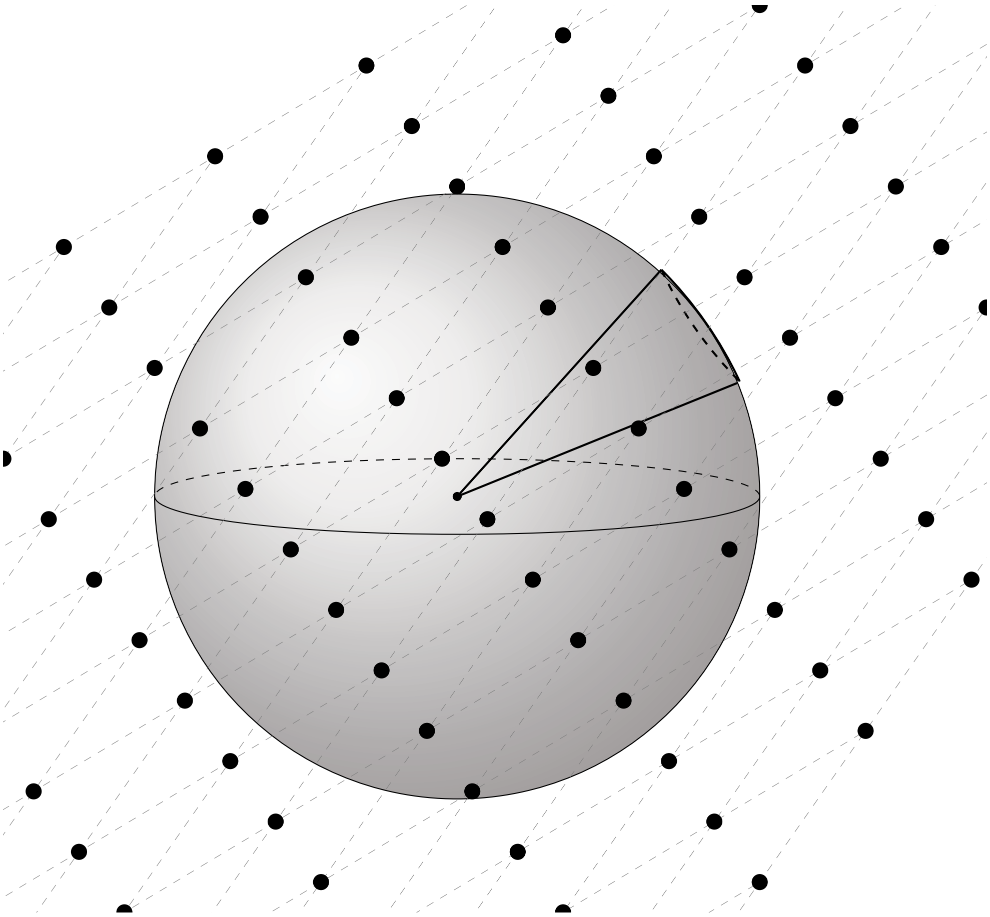


Figure 16: Cone

10 Our model: Lorentz gas on the billiard

Fix some dimension $D \geq 3$ and a finite collection of disjoint open convex subsets $B_1, ..., B_k$ (called scatterers in the sequel) of $\mathbb{T}^D$ with $\mathcal{C}^3$ smooth boundary.

Here, $\mathbb{T}^D$ is the unit D-torus, canonically identified with $[0,1]^D$.

We study the Lorentz gas, that is the dynamics of a point particle on the billiard table

$$\mathcal{D} = \mathbb{R}^D \setminus \cup_{i=1}^k \cup_{\ell \in \mathbb{Z}^D} (B_i + \ell),$$

where the particle flies freely in the interior of $\mathcal{D}$ and bounces back elastically from the set $(B_i + \ell)$ upon reaching $\partial\mathcal{D}$. The speed of the particle is constant 1 and its velocity is denoted by $v \in S^{D-1}$, where S^{D-1} is the unit sphere in $\mathbb{R}^D$.

The phase space of the dynamics is

$$\Omega := \mathcal{D} \times S^{D-1}$$

where on the boundary, pre- and post-collisional vectors are identified (that is, if $q \in \partial\mathcal{D}$, then (q, v) is identified with (q, v') if $v = v_\perp + v_\|$ and $v' = -v_\perp + v_2$ where $v_\perp$ is perpendicular to $\partial\mathcal{D}$ at q and $v_\|$ in the tangent space of $\partial\mathcal{D}$ at q).

Let us write

$$\mathcal{D}_0 = \mathcal{D} \cap \mathbb{T}^D$$

Let ν_0 be the normalized Lebesgue measure on $\Omega_0 := \mathcal{D}_0 \times S^{D-1}$. Then the Sinai billiard flow $\Phi_0^t : \Omega_0 \to \Omega_0$, defined for any $t \in \mathbb{R}$, preserves the measure ν_0

Similarly, the $\mathbb{Z}^d$ extension of the Sinai billarid flow, namely the flow

$$\Phi^t : \Omega \to \Omega$$

preserves the infnite measure

$$\nu = \nu_0 \times \text{counting}_{\mathbb{Z}^D}$$

Let $\tau : \Omega \to [0, \infty]$ be the first collision time. Let $\mathcal{M} = \partial\mathcal{D} \times S_1^{D-1}$ denote thus the boundary of the phase space, where $S_1^{D-1} = \{v \in S^{D-1} : v_1 > 0\}$ and $v = (1, 0, ..., 0)$ is identified with the unit normal vector of $\partial\mathcal{D}$ pointing into $\mathcal{D}$. With this common choice, we use the post-collisional velocity to represent both the pre- and the post-collisional velocity at the time of the collision.

Let $\mathcal{F} : \Omega \to \mathcal{M}$ be the first collision map, that is $\mathcal{F}(x) = \Phi^{\tau(x)}(x)$.

We note that $\mathcal{F}(x)$ may not be defined in case $\tau(x) = \infty$. The measure of such points x is zero and their geometry is the major topic of the present work.

We denote by

$$\tau_k : \Omega \to [0, \infty]; \quad \tau_k(x) = \tau \circ \mathcal{F}^k$$

the time of the kth collision. Note that in case $\tau_k = \infty$, τ_{k+1} is undefined, but this will only happen on a set of ν-measure zero. Since we only study sets of positive ν measure, we can disregard the fact that τ_k may be undefined.

Throughout this work, We assume that the billiard table $\mathcal{D}$ is such that there exists a hyperplane V and an interval $I \subset \mathbb{R}$ of non-empty interior so that

$$(V + V^{\perp}I) \subset \mathcal{D} \tag{10.1}$$

where $V^\perp$ is a unit vector orthogonal to V.

According to a common terminology, billiard tables allowing infinite free flight without collision are said to have called *inifnite horizon* (and those without infinife flight, in fact then necessarily only having bounded flight, are of *finite horizon*). We note that the condition 10.1 implies that the table is of infinite horizon but in case $D \geq 3$ is stronger than that because there are billiard tables where infinite unbounded flights are only possible along subspaces of codimension at least 2.

We will call the set $V + V^\perp I$ a *principal corridor*, where "principal" refers to the fact that V is of codimension 1.

The geometry of billiard tables with infinite horizon have been studied in the literature. We note that in dimension $D = 2$, since $dimV = 1$, the geometry is quite simple and it has been extensively studied in [1, 38, 32].

In dimension $D \geq 3$, the geometry can be substantially more complicated. The asymptotic measure of length flight has been studied in [11] and [28]. In particular, it is proven in [28] that for billiard tables with at least one principal corridor,

$$\nu_0(\tau > T) \sim \frac{C}{T}$$

as $T \to \infty$ with an explicit constant C.

In this work, we are interested in the length of a few consecutive long free flights. In dimension $D = 2$, it is well known that a flight of length $T \gg 1$ is typically followed by a flight of length $C\sqrt{T}$. Here, we extend this result to any dimension D and in particular we resolve Conjecture C from [28].

10.1 Geometry of corridors

Let $H = V + IV^{\perp}$ be any set as in (10.1) where $I = [a, b]$ is maximal for containment (recall that the scatterers B_i are open). We say that H is a principal corridor, associated with V and $I = [a, b]$. When proving Theorem 11.1, we can assume that (q, v) is such that q is in a small neighborhood of H and v is close to V. Indeed, by the main result of [28], the contribution of points not close to a principal horizon is negligible.

Let us assume there is a single spherical scatterer or radius $r < 1/2$ inside $\mathcal{D}_0$.

Let $\nu_{H,c}$ be the normalized Lebesgue measure on $U_{H,c} := (V + cV^{\perp})/\mathcal{L}_b$ for a fixed corridor H and side $c = a, b$. To simplify notation let us write $\hat{\nu} = \nu_{H,c}$. Let λ_T be the normalized Lebesgue measure on the set

$$\left\{ v \in S^{D-1} : v \cdot V^{\perp} = \frac{1}{T} \right\}$$

We need to keep in mind that a horizon H and an endpoint c has been fixed.

An important special case is when $\mathcal{D} = \mathbb{R}^D \setminus \cup_{\ell \in \mathbb{Z}^D} B(\ell, r)$, where $B(\ell, r)$ is the ball of radius r centered at ℓ. $r < 1/2$ clearly hold by assumption. In case $r < \sqrt{2}/4$, for every principal corridor H, V_H is a coordinate direction, i.e. $V_H^{\perp} = e_i$ for some $i = 1, ..., D$ (e_i denoting the ith coordinate direction) and I_H is an interval of length $1 - 2r$ and for both endpoints of this interval c, we have $\mathcal{L}_c = \mathbb{Z}^{D-1}$.

(See Appendix for proof of this part)

11 Theorems we want to prove

Here, we are interested in the length of a few consecutive long free flights. Namely, we want to prove the following theorems.

11.1 Theorem 1: Main theorem

Theorem 11.1. *Let $D \geq 2$ and assume there is a single spherical scatterer with radius $r < \frac{1}{2}$. There exists a stochastic process $\mathcal{X}_1, \mathcal{X}_2, ...$ so that for any finite n and for any sets $A_i \subset \mathbb{R}$ with $Leb(\partial A_i) = 0$,*

$$\lim_{T\to\infty} \frac{\nu_0(\tau_1 > T, \tau_i \in A_i T^{1/D^{i-1}}, i = 1, ..., n)}{\nu_0(\tau_1 > T)} = \mathbb{P}(\mathcal{X}_i \in A_i, i = 1, ..., n).$$

11.2 Theorem 2: Special case for theorem 1

Theorem 11.2. *Let $D \geq 2$ and assume there is a single spherical scatterer with radius $r < \frac{1}{2}$. For $n = 1$ and for the set $A \subset \mathbb{R}$ with $Leb(\partial A) = 0$, (In this case A is an interval)*

$$\lim_{T\to\infty} \frac{\nu_0(\tau_1 > T, \tau_1 \in AT, i = 1)}{\nu_0(\tau_1 > T)} = \mathbb{P}(\mathcal{X}_1 \in A) \sim \frac{1}{A}.$$

11.3 Theorem 3: Special case for theorem 1

Theorem 11.3. *Let $D = 2$, $\mathcal{D} = \mathbb{R}^2 \setminus \cup_{z\in\mathbb{Z}^2} B(z, r)$ with $\sqrt{2}/4 < r < 1/2$. This condition ensures that principal corridors exist and they are all parallel to coordinate hyperplanes. There exists a stochastic process $\mathcal{X}_1, \mathcal{X}_2, ...$ so that for any finite n and for any sets $A_i \subset \mathbb{R}$*

with $Leb(\partial A_i) = 0$*,*

$$\lim_{T\to\infty} \frac{\nu_0(\tau_1 > T, \tau_i \in A_i T^{1/2^{i-1}}, i = 1, ..., n)}{\nu_0(\tau_1 > T)} = \mathbb{P}(\mathcal{X}_i \in A_i, i = 1, ..., n).$$

We also generalize the theorem 3 to any n and any r as the following sections would prove.

11.4 Theorem 4: Marklof-Strömbergsson theory

Theorem 11.4. *There exists a continuous function* $\Psi : \mathbb{R}_+ \to \mathbb{R}$ *so that for all* ξ*,*

$$\lim_{T\to\infty} (\hat{\nu} \times \lambda_T)(\tau > \xi T^{\frac{D-2}{D}}) = \int_\xi^\infty \Psi(\xi')d\xi'.$$

A variant of Theorem 4 was recently established by Boca and Zaharescu [2] in dimension $d = 2$, utilizing techniques from analytic number theory. Their earlier collaboration with Gologan [15], as well as the work by Caglioti and Golse [13], are also relevant in this context. In Marklof-Strömbergsson's study, they employ dynamics and equidistribution of flows on homogeneous spaces and extend the results to arbitrary dimensions. Previous research in higher dimensions $(d > 2)$ includes the papers by Bourgain, Golse, and Wennberg [20], [14], which provide estimates on the tail behavior of potential limiting distributions of converging subsequences. In our theorem, we generalize the result to D dimensional.

11.5 Theorem 5

Theorem 11.5. *There exists a continuous density function Φ so that for all $\underline{\xi}, \bar{\xi}$, $[a,b] \subset [0,1]$*

$$\lim_{T\to\infty}(\nu \times \lambda_T)(\tau \in [\underline{\xi}T^{\frac{D-2}{D}}, \bar{\xi}T^{\frac{D-2}{D}}], w \in [a,b]) = \int_{\underline{\xi}}^{\bar{\xi}}\int_a^b \Phi(\xi, w)dwd\xi.$$

The function Φ is explicitly given by

$$\Phi(\xi, w) = \begin{cases} \nu_y(\{M \in X_q(y) : (\mathbb{Z}^d + \alpha)M \cap (\Upsilon(0,\xi,1) + z)\}) & \text{if } \alpha \in q^{-1}\mathbb{Z}^d \\ \nu_y(\{g \in X(y) : \mathbb{Z}^d g \cap \Upsilon(0,\xi,1) + z) = \emptyset\}) & \text{if } \alpha \notin \mathbb{Q}^d \end{cases}$$

where $y = \xi e_1 + \frac{\sqrt{2\xi}}{\sqrt{\kappa covol(\mathcal{L})^{1/d}}} we_2$, and

$$\Upsilon(\xi) = \left\{(x_1, ..., x_d) \in \mathbb{R}^d : 0 < x_1 < \frac{\xi}{[covol(\mathcal{L})]^{1/d}}, \|(x_2, ..., x_d)\| \leq \frac{\sqrt{2x_1}}{\sqrt{\kappa}[covol(\mathcal{L})]^{1/2d}}\right\}.$$

11.6 Theorem 6

Theorem 11.6. *There exists a continuous density function Φ, ψ so that for all ξ, ξ',*

$$\begin{aligned}\lim_{T\to\infty}(\nu \times \lambda_T)(\tau_1 \in [\underline{\xi}T^{\frac{D-2}{D}}, \bar{\xi}T^{\frac{D-2}{D}}], w_1 \in [a,b] \\ \tau_2 \in [\underline{\xi}'T^{\frac{D-2}{2D}}, \bar{\xi}'T^{\frac{D-2}{2D}}], w_2 \in [a',b']) \\ = \int_{\underline{\xi}}^{\bar{\xi}}\int_a^b \Phi(\xi, w)\int_{\underline{\xi}'}^{\bar{\xi}'}\int_{a'}^{b'} \psi(\xi', w', \xi, w)dwd\xi.\end{aligned}$$

where

$$\psi(\xi', w', \xi, w) = \Phi(\sqrt{\frac{\kappa}{8}}\frac{1}{\sqrt{\xi(1-w)}}\xi', \sqrt{\frac{8}{\kappa}}\sqrt{\xi(1-w)}w')$$

We extend Theorem 6 to cover cases with k collisions for particles, leading to the development of Theorem 7.

11.7 Theorem 7

Theorem 11.7. *There exists continuous density functions* $\Phi, \psi, \cdots$ *so that for all* $\xi_1, \xi_2, \cdots, \xi_k$,

$$\lim_{T\to\infty}(\nu\times\lambda_T)(\tau_1\in[\underline{\xi}T^{\frac{D-2}{D}},\bar{\xi}T^{\frac{D-2}{D}}],w_1\in[a,b],\cdots,$$
$$\tau_k\in[\underline{\xi_k}T^{\frac{D-2}{2D}},\bar{\xi_k}T^{\frac{D-2}{2D}}],w_k\in[a_k,b_k])$$
$$=\int_{\underline{\xi_1}}^{\bar{\xi_1}}\int_{a_1}^{b_1}\Phi(\xi_1,w_1)\int_{\underline{\xi_2}}^{\bar{\xi_2}}\int_{a_2}^{b_2}\psi(\xi_1,w_1,\xi_2,w_2)$$
$$\int_{\underline{\xi_3}}^{\bar{\xi_3}}\int_{a_3}^{b_3}\psi(\xi_2,w_2,\xi_3,w_3)\cdots\int_{\underline{\xi_k}}^{\bar{\xi_k}}\int_{a_k}^{b_k}\psi(\xi_{k-1},w_{k-1},\xi_k,w_k)$$
$$dw_k d\xi_k dw_{k-1} d\xi_{k-1}\cdots dw_1 d\xi_1$$

12 Proof of Theorem 1: Main Theorem

The proof of the theorem 1 is a combined result of Theorem $4, 5, 6, 7$, so after we have proved theorem $4, 5, 6, 7$, then go back to theorem 1.

13 Proof of Theorem 2: Special case for theorem 1

Theorem 13.1. *Let* $D \geq 2$ *and assume there is a single spherical scatterer with radius* $r < \frac{1}{2}$. *For* $n = 1$ *and for the set* $A \subset \mathbb{R}$ *with* $Leb(\partial A) = 0$,

$$\lim_{T\to\infty}\frac{\nu_0(\tau_1 > T, \tau_1 \in AT, i = 1)}{\nu_0(\tau_1 > T)} = \mathbb{P}(\mathcal{X}_1 \in A) \sim \frac{1}{A}.$$

Proof. To gain some insight into our main result, we prove it for the special case $n = 1$

first. In particular, we verify that $\mathcal{X}_1$ is distributed as the reciprocal of a uniform random varible on $[0, 1]$.

Fix any number $A \geq 1$. Then we have

$$\nu_0(\tau_1 > AT | \tau_1 > T) = \frac{\nu_0(\tau_1 > AT)}{\nu_0(\tau_1 > T)}$$

By [28, Theorem 1], we have

$$\nu_0(\tau_1 > T) \sim \Sigma_{H \in \mathbb{H}} F_H(t)$$

By [28], we have that in the case of a principal horizon H, I_H is an interval and the formula to compute $F_H(t)$ is:

$$F_H(t) \sim \frac{2\ volS_{d-2} |I_H|^2}{(1-\mathcal{P})\ volS_{d-1}\ vol(V^{\perp}/\mathcal{L}_V^{\perp})} \frac{1}{t}$$

where $\mathcal{P}$ is the volume fraction covered by scatters, which is a constant.

Then it is easy to verify

$$\nu_0(\tau_1 > T) \sim \frac{C}{T}$$

with $T \to \infty$ with a constant C depending only on the billiard table $\mathcal{D}$. Thus

$$\nu_0(\tau_1 > AT | \tau_1 > T) = \frac{\nu_0(\tau_1 > AT : \tau_1 > T)}{\nu_0(\tau_1 > T)} = \frac{\nu_0(\tau_1 > AT)}{\nu_0(\tau_1 > T)}$$

Therefore

$$\nu_0(\tau_1 > AT | \tau_1 > T) \sim \frac{1}{A}$$

This proves Theorem 11.1 for the case $n = 1$ and also proves that $\mathcal{X}_1$ is distributed as the reciprocal of a uniform random varible on $[0, 1]$. $\square$

14 Proof of Theorem 3: Special case for theorem 1

14.1 $D = 2$, single spherical scatterer

Theorem 14.1. *Let $D = 2$, $\mathcal{D} = \mathbb{R}^2 \setminus \cup_{z \in \mathbb{Z}^2} B(z, r)$ with $\sqrt{2}/4 < r < 1/2$. This condition ensures that principal corridors exist and they are all parallel to coordinate hyperplanes. There exists a stochastic process $\mathcal{X}_1, \mathcal{X}_2, \ldots$ so that for any finite n and for any sets $A_i \subset \mathbb{R}$ with $Leb(\partial A_i) = 0$,*

$$\lim_{T \to \infty} \frac{\nu_0(\tau_1 > T, \tau_i \in A_i T^{1/2^{i-1}}, i = 1, \ldots, n)}{\nu_0(\tau_1 > T)} = \mathbb{P}(\mathcal{X}_i \in A_i, i = 1, \ldots, n).$$

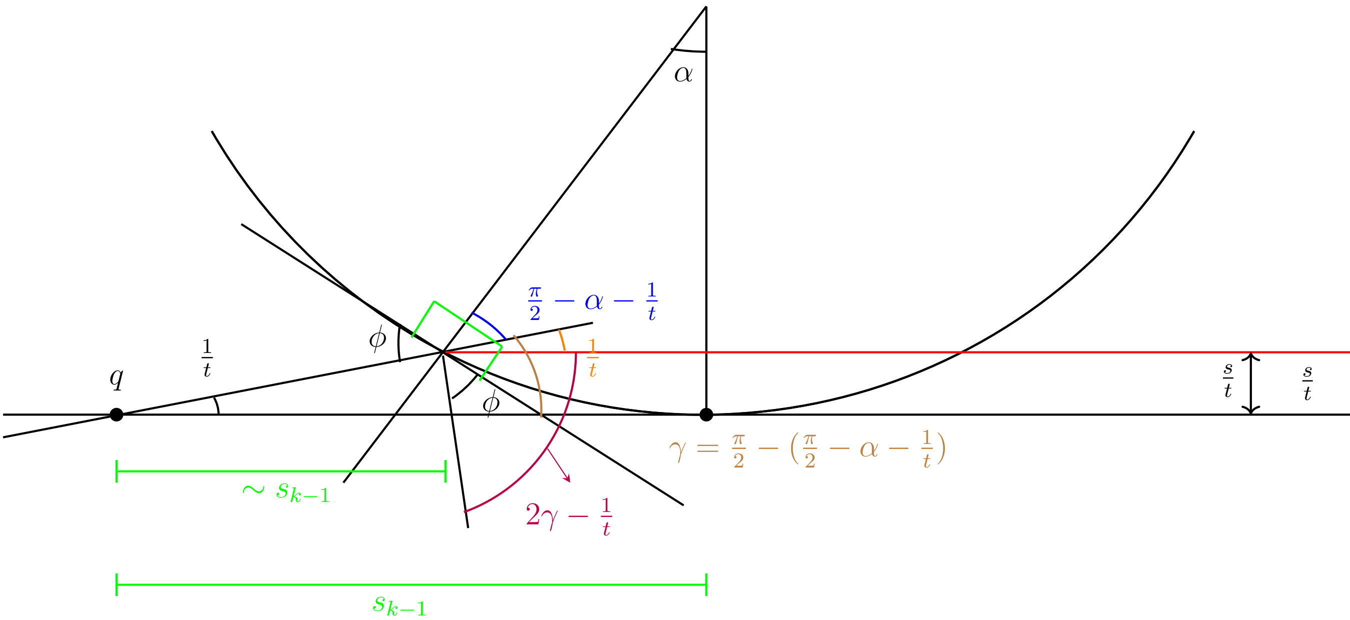


Figure 17: Collision after a long flight in dimension 2

Proof. Let us assume that the initial position q_1 is on the boundary of a horizontal corridor and the initial vector $v = (v_1, v_2)$ is so that $v_2 = 1/t.$ for some large t. Thus the initial angle with the horizontal direction is

$$\beta_1 \sim \frac{1}{t}.$$

Furthermore, q is at a distance $s_1 \neq 0$ from the intersection point of the boundary of the corridor and the scatterer, measured in the direction of the flight. See Figure 17.

From easy calculation, we have

$$\gamma = \frac{\pi}{2} - (\frac{\pi}{2} - \alpha - \frac{1}{t}) = \alpha + \frac{1}{t}$$

By the definition of circle, we have:

$$x^2 + (y - r)^2 = r^2 \Rightarrow (y - r)^2 = r^2 - x^2$$

Simplify it:

$$y^2 - 2yr + x^2 = 0$$

Solve this quadratic equation:

$$y(x) = r - \sqrt{r^2 - x^2}$$

Differentiate this quadratic equation to second order:

$$y'(x) = -\frac{1}{2}\frac{1}{\sqrt{r^2 - x^2}}(-2x) = \frac{x}{\sqrt{r^2 - x^2}}$$

$$y''(x) = \frac{\sqrt{r^2 - x^2} + \frac{x^2}{\sqrt{r^2-x^2}}}{r^2 - x^2}$$

Find the value of second order differentiation at $x = 0$:

$$y''(0) = \frac{r}{r^2} = \frac{1}{r}$$

Using second order Taylor series expansion, we have

$$\frac{s}{t} = \frac{1}{2r}\varepsilon^2$$

where ε is the width joining the hitting position with the vertical line starting from origin to south pole. Then with easy calculation, we have:

$$\varepsilon = \sqrt{\frac{2rs}{t}}$$

$$\alpha \approx sin\alpha = \frac{\varepsilon}{r}$$

$$\alpha \approx \sqrt{\frac{2}{r}}\sqrt{\frac{s}{t}}$$

So

$$\gamma \approx \sqrt{\frac{2}{r}}\sqrt{\frac{s}{t}}$$

Next angle would be

$$2\gamma - \frac{1}{t} \approx 2\sqrt{\frac{2}{r}}\sqrt{s}\frac{1}{\sqrt{t}}$$

Choose

$$s_i \sim UNI[0, 1]$$

Then we have

$$\beta_1 = \frac{1}{t}$$

$$t\beta_1 = 1 = y_1$$

$$\beta_2 = 2\sqrt{\frac{2}{r}}\sqrt{s_2}\frac{1}{\sqrt{t}}$$

An elementary computation, hinted on Figure 17, shows that the postcollisional angle with the horizontal direction (denoted by $2\gamma - 1/t$ on the figure) is asymptotic to

$$\beta_2 \sim \sqrt{\frac{8}{r}}\sqrt{s_1}\frac{1}{\sqrt{t}} \sim \sqrt{\frac{8}{r}}\sqrt{s_1}\sqrt{\beta}_1$$

as $t \to \infty$. Continue doing the iterations of computation:

$$\sqrt{t}\beta_2 = 2\sqrt{\frac{2}{r}}\sqrt{s_2} = y_2$$

$$\beta_3 = 2\sqrt{\frac{2}{r}}\sqrt{s_3}\frac{1}{\sqrt{t_2}}$$

$$t^{\frac{1}{4}}\beta_3 = 2\sqrt{\frac{2}{r}}\sqrt{s_3} = y_3$$

$$\cdots$$

We conclude

$$\beta_i = 2\sqrt{\frac{2}{r}}\sqrt{s_i}\sqrt{\beta_{i-1}}$$

$$y_i = 2\sqrt{\frac{2}{r}}\sqrt{s_i}\sqrt{y_{i-1}}$$

Note that these asymptotics hold for any fixed $s_1 \in (0, 1)$ and the convergence is uniform for s_1 chosen from any compact subset of $(0, 1)$. The convergence is not uniform for all $s_1 \in (0,1)$. Indeed, if $s_1 = 0$, then $\beta_2 = \beta_1$ and if $s_1 \ll 1/t$, then $\beta_1 \sim \beta_2$. Likewise, the convergence is not uniform for s_1 close to 1.

Let us write

$$y_1 = t\beta_1 \sim 1$$

and

$$y_2 = \sqrt{t}\beta_2.$$

Then we have

$$y_2 \sim \sqrt{\frac{8}{r}}\sqrt{s_1}\sqrt{y_1}$$

Now let q_2 be the point where the trajectory crosses the other boundary of the corridor (this would be the bottom boundary of the corridor whose top is depicted on Figure 17). Now let $s_2 \in [0,1)$ be the distance between q_2 and the next intersection of the boundary of the corridor with the scatterers (measured in the direction of the flight), etc. Arguing as before, we find that after collision $i-1$, the outgoing angle with the horizontal direction is

$$\beta_i \sim \sqrt{\frac{8}{r}}\sqrt{s}_{i-1}\sqrt{\beta}_{i-1}$$

and so with the notation

$$y_i = \beta_i t^{1/2^{i-1}},$$

we also have

$$y_i \sim \sqrt{\frac{8}{r}}\sqrt{s}_{i-1}\sqrt{y}_{i-1}$$

We claim that for any fixed n, in the limit $t \to \infty$, $s_1, ..., s_n$ converge to n iid random variables each uniform on the interval $[0,1]$. Assume now that the claim holds. Then noting that for $k \geq 2$

$$\tau_k \sim \frac{1-2r}{\beta_k}$$

we find

$$\mathcal{X}_k = \frac{1-2r}{y_k} = \frac{1-2r}{\sqrt{\frac{8}{r}\sqrt{s_{k-1}}\sqrt{y_{k-1}}}} = \frac{\sqrt{1-2r}}{\sqrt{\frac{8}{r}s_{k-1}}}\frac{\sqrt{1-2r}}{\sqrt{y_{k-1}}}$$

That means

$$\mathcal{X}_k = \sqrt{\frac{1-2r}{\frac{8}{r}s_{k-1}}}\sqrt{\mathcal{X}_{k-1}} = \sqrt{\frac{r-2r^2}{8s_{k-1}}}\sqrt{\mathcal{X}_{k-1}}$$

This identifies the stochastic process $\mathcal{X}_k$ with the recursive definition

$$\ln \mathcal{X}_k = \frac{1}{2}\ln\frac{r-2r^2}{8} - \frac{1}{2}\ln s_{k-1} + \frac{1}{2}\ln(\mathcal{X}_{k-1}).$$

for $k \geq 2$ and $\mathcal{X}_1 = 1/s_0$ where s_0 is uniform on $[0,1]$ and independent from $s_1, ..., s_n$. Sometimes such processes are called autoregressive of order 1.

It remains to prove the above claim. It is sufficient to prove it for $n = 2$ the general case being similar by induction. For $n = 2$, it is sufficient to prove that for any $u \in (0,1)$ and for any small $\epsilon > 0$, for sufficiently large t, the conditional distribution of s_2 given that $u < s_1 < u + \epsilon$ is $\epsilon^{1/3}$ close to the uniform distribution on $[0,1]$. To this end, note that under this assumption, the outgoing angle β_2 satisfies $C\sqrt{u/t} \leq \beta_2 \leq C\sqrt{(u+\epsilon)/t}$ (with $C = \sqrt{8/r}$) and hence the particle will reach the bottom of the corridor in a horizontal distance v satisfying

$$C^{-1}\sqrt{t/(u+\epsilon)} \leq v \leq C^{-1}\sqrt{t/u}. \tag{14.1}$$

Furthermore, if we consider the pushfoward of the initial uniform measure on the interval $[u, u+\epsilon]$ on the top of the corridor to the bottom of the corridor by the particle's next hitting dynamics, then the uniform measure will be distorted by a factor $C'\sqrt{\epsilon} \ll \epsilon^{1/4}$. Noting that

(14.1) contains a big number of copies of the unit interval, the result follows. □

It is also very important to stress in $D = 2$, we could generalize the result to any n and any $r \in (0, \frac{1}{2})$:

14.2 $D = 2$, any n and any r

In Section 20.1, we considered the case when $D = 2$, n is arbitrary and $r < 1/\sqrt{8}$. Now we generalize that proof for any $r \in (0, 1/2)$.

As discussed in [11, Section 6], given $r \in (0, 1/2)$, we have a bijection between the corridors and the set

$$\mathcal{H}_r = \{[0,1]^T, [1,0]^T\} \cup \{[p,q]^T, p > 0, GCD(p,q) = 1, \sqrt{p^2+q^2} < \frac{1}{2r}\}$$

where $v \in \mathcal{H}_r$ is identified with the vector connecting two consecutive points on the intersection of one side of the corridor and $\partial\mathcal{D}$. In [11, Section 6], $\|v\|$ is denoted by L and $1/L$ is denoted by $\nu_H^\perp$.

By the results of [11, 28], for any $v \in \mathcal{H}_r$,

$$\nu_0(\text{flight is in the corridor } H \text{ with } V_H = span(v) | \tau_1 > T) \sim$$

$$\frac{\|v\|(1/\|v\| - 2r)^2}{\sum_{w \in \mathcal{H}_r} \|w\|(1/\|w\| - 2r)^2} =: \mathcal{P}_v$$

as $t \to \infty$. Now we can describe the stochastic process $\mathcal{X}$. First, we choose a $v \in \mathcal{H}_r$ according to the probability distribution $\mathcal{P}_v$. Then once v is fixed, $\mathcal{X}_1$ is the reciprocal of a uniform random variable on $[0, 1]$ (independent of the choice of v) and

$$\ln \mathcal{X}_k = \frac{1}{2} \ln \frac{r/\|v\| - 2r^2}{8} - \frac{1}{2} \ln s_{k-1} + \frac{1}{2} \ln(\mathcal{X}_{k-1}).$$

for $k \geq 2$, $s_1, ..., s_{n-1}$ are iid random variables, uniform on $[0, \|v\|]$. The proof is identical to that in Section 20.1 with the exception that the corridor is changed: now the distance between consecutive boundary points is $\|v\|$ and the width of the corridor is $1/\|v\| - 2r$.

15 Proof of Theorem 4: Marklof-Strömbergsson theory

Theorem 15.1. *There exists a continuous function* $\Psi : \mathbb{R}_+ \to \mathbb{R}$ *so that for all* ξ*,*

$$\lim_{T\to\infty} (\hat{\nu} \times \lambda_T)(\tau > \xi T^{\frac{D-2}{D}}) = \int_\xi^\infty \Psi(\xi') d\xi'.$$

To explain the scaling $T^{\frac{D-2}{D}}$ in Theorem 13.1, assume that the free flight is of order t with $1 \ll t \ll T$. Then up to time t, the position q_t will be close to H. Namely, $q_t \in V + (b + h)V^\perp$ where $h = h(t) = t/T$. The scatterers intersect the hyperplane $V + (b + h)V^\perp$, at some small sets, which by assumption (H2) are approximate $D - 1$ dimensional spheres centered at $\mathcal{L}_b + hV^\perp$ and of radius $r = r(t) := \sqrt{2h/\kappa}$. In order to have a collision, we need q_t to enter one of these small spheres.It is reasonable to expect this to happen once the $r(t)$ neighborhood of the line segment $\Pi_V(q_0 - q_t)$ has volume ≈ 1, where Π_V is the orthogonal projection to the hyperplane V. This means

$$t \left(\sqrt{\frac{t}{T}} \right)^{D-2} \approx 1$$

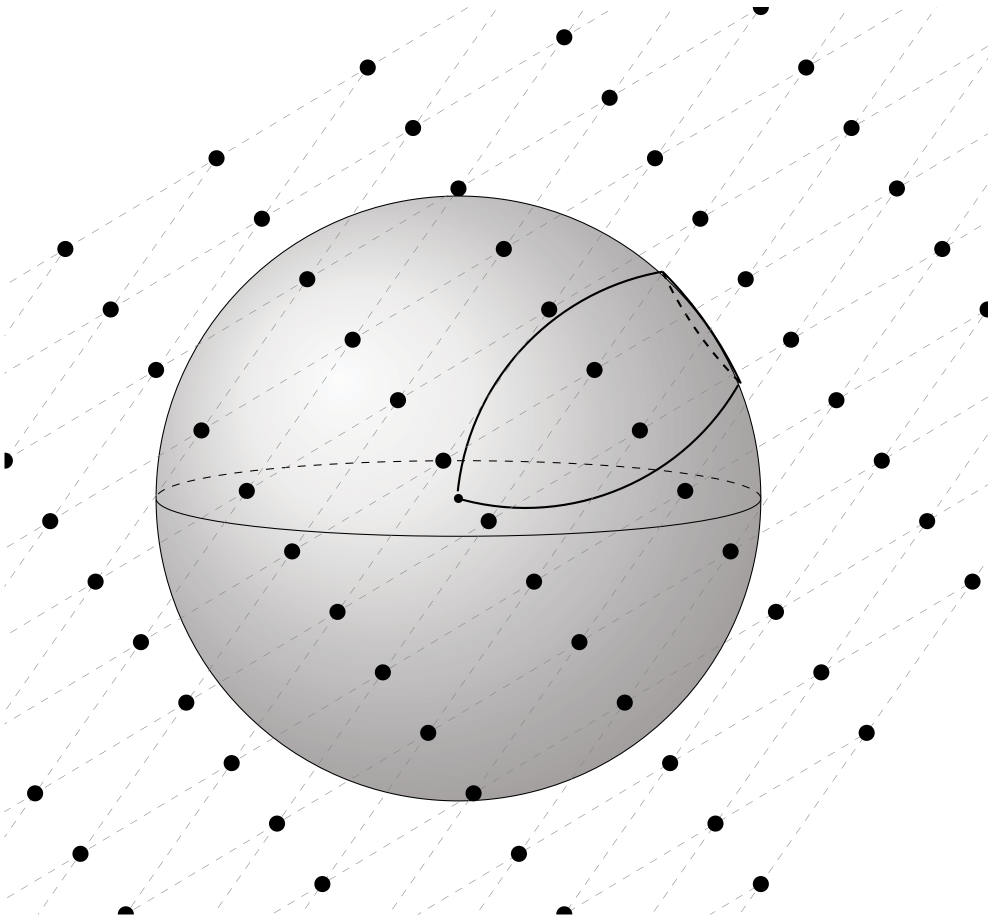


Figure 18: Paraboloid

whence

$$t \approx T^{\frac{D-2}{D}}.$$

Proof. It will be convenient to write $d = D - 1$. Let us fix some matrix $M_0 \in \mathbb{Z}^{d\times(d+1)}$ so that $\mathbb{Z}^d M_0 = \mathcal{L}_b$.

Now let us fix $q \in V + bV^{\perp}/\mathcal{L}_b$. Then since $\mathcal{L}_b$ is an affine lattice, by basic matrix computation, there is some $\alpha(q) \in \mathbb{R}^d$ so that $(\mathbb{Z}^d + \alpha)M_0 = \mathcal{L}_b$ Then we claim that

$$\lim_{T\to\infty} \lambda_T((q,v) : \tau > \xi T^{\frac{D-2}{D}}) = A(\xi, \alpha(q)) \tag{15.1}$$

for a function $A(\xi, \alpha)$ to be defined next.

In order to define $A(\xi, \alpha)$ we need some definitions. Let $S(d, \mathbb{R})$ be the special linear group of degree d over $\mathbb{R}$ (i.e. the set of $d \times d$ matrices with determinant 1). Likewise, let $S(d,\mathbb{Z})$ be the special linear group of degree d over $\mathbb{Z}$ (i.e. the set of $d \times d$ integer matrices with determinant 1).

Next, let

$$\Gamma(l) := \{\gamma \in SL(d,\mathbb{Z}) : \gamma \equiv 1_d \bmod l\}$$

also,

$$ASL(d,\mathbb{R}) = SL(d,\mathbb{R}) \ltimes \mathbb{R}^d$$

$$ASL(d,\mathbb{Z}) = SL(d,\mathbb{Z}) \ltimes \mathbb{Z}^d,$$

where $\ltimes$ is the semidirect product group with multiplication law

$$(M,\xi)(M',\xi') = (MM', \xi M' + \xi').$$

Next we define the homogeneous spaces

$$X_1 = SL(d,\mathbb{Z}) \setminus SL(d,\mathbb{R}); X_l = \Gamma(l) \setminus SL(d,\mathbb{R}); X = ASL(d,\mathbb{Z}) \setminus ASL(d,\mathbb{R}).$$

Now we are ready to define the function $A(\xi,\alpha)$ as

$$A(\xi,\alpha) = \begin{cases} \mu_1(\big\{M \in X_1 : \#(\mathbb{Z}^d_* M \cap \Upsilon(\xi)) = 0\big\}) & \text{if } \alpha \in \mathbb{Z}^d \\ \mu_l(\big\{M \in X_l : \#((\mathbb{Z}^d + \frac{k}{l})M \cap \Upsilon(\xi)) = 0\big\}) & \text{if } \alpha = \frac{k}{l} \in \mathbb{Q}^d, \alpha \notin \mathbb{Z}^d \\ \mu(\big\{(M,\eta) \in X : \#((\mathbb{Z}^d M + \eta) \cap \Upsilon(\xi)) = 0\big\}) & \text{if } \alpha \notin \mathbb{Q}^d \end{cases}$$

where μ_1, μ_l and μ are normalized Haar measure on the spaces X_1, X_l and X, respectively and

$$\Upsilon(\xi) = \left\{(x_1, ..., x_d) \in \mathbb{R}^d : 0 < x_1 < \frac{\xi}{[\mathrm{covol}(\mathcal{L})]^{1/d}}, \|(x_2, ..., x_d)\| \leq \frac{\sqrt{2x_1}}{\sqrt{\kappa}[\mathrm{covol}(\mathcal{L})]^{1/2d}}\right\} \tag{15.2}$$

The proof of (15.1) follows closely the proof of Theorems 2.1 and 3.1 in [22].

The only difference between our setup and the setup of [22] is that we consider the paraboloid Υ while [22] uses cylinder (Theorem 3.1) and cone (Theorem 2.1).

For fixed $T \gg 1$, we have $\tau(q, v) > \xi T^{\frac{D-2}{D}}$ if and only if for all $t < \xi T^{\frac{D-2}{D}}$, we have $q_t \in \mathcal{D}$. That is,

$$q_t \notin \mathcal{B}_{t,T},$$

where

$$\mathcal{B}_{t,T} = (\mathbb{R}^D \setminus \mathcal{D}) \cap (V + (b + t/T)V^{\perp}).$$

That is, we need

$$\Pi_{V+bV^{\perp}} q_t \notin \Pi_{V+bV^{\perp}} \mathcal{B}_{t,T} =: \cup_{\ell \in \mathcal{L}_b} (\mathfrak{b}_{t,T} + \ell) \tag{15.3}$$

for all $t \in [0, \xi T^{(D-2)/D}]$. Note that $\mathfrak{b}_{t,T}$ is approximately a sphere centered at zero and of radius $\sqrt{t/T}/\kappa$. Clearly, (15.3) is equivalent to the lattice $\mathcal{L}_b - q_0$ being disjoint to the set

$$\mathcal{C}_T := \cup_{t \in [0, \xi T^{(D-2)/D}]} (\Pi_{V+bV^{\perp}}(q_t - q_0) - \mathfrak{b}_{t,T}).$$

Let us identify $V + bV^{\perp}$ with $\mathbb{R}^d$ with the origin being in $\mathcal{L}_b$. Then in particular, $\mathcal{C}_T \subset \mathbb{R}^d$.

Let $M_1 = \mathrm{covol}(\mathcal{L}_b)^{-1/d} Id_d$ that is colvol$(\mathcal{L}_b)^{-1/d}$ times the identity matrix of size $d \times d$. Now $\mathcal{L}' = M_1 \mathcal{L}_b$ is a lattice of covolume 1. We need this transformation since the theory of Marklof and Strömbergsson is applicable to lattices of covolume 1.

Let us pick some pair (q, v) according to the measure λ_T. Given v, we define an orthogonal matrix $K \in \mathbb{R}^{d\times d}$ so that $K\Pi_V v = [\|\Pi_V v\|, 0, ..., 0]^T$. Note that $\|\Pi_V v\| = 1 + O(1/T)$.

Next, define the diagonal matrix

$$D_T = \begin{pmatrix} T^{(2-D)/D} & 0 & \cdots & 0 \\ 0 & T^{1/D} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T^{1/D} \end{pmatrix} \in S(d, \mathbb{R}) \tag{15.4}$$

Applying the linear transformation $D_T K M_1$ to the set $\mathcal{C}_T$ we obtain

$$\Upsilon_T := D_T K M_1(\mathcal{C}_T)$$

Now we claim that the open set Υ_T satisfies

$$\lim_{T\to\infty} Leb_{\mathbb{R}^d}(\Upsilon_T \Delta \Upsilon) = 0 \tag{15.5}$$

where Δ denotes symmetric difference.

To prove (15.5), note that by condition (H2), for all $\xi > 0$ and $\epsilon > 0$ there is some T_0 so that for all $T > T_0$ and for all $t \in [0, \xi T^{(D-2)/D}]$, we have

$$B_d\left(0, \sqrt{\frac{2t}{T\kappa}}(1-\epsilon)\right) \subset \mathfrak{b}_{t,T} \subset B_d\left(0, \sqrt{\frac{2t}{T\kappa}}(1+\epsilon)\right)$$

where $B_d(0, \rho)$ is the ball of radius ρ centered at $0 \in \mathbb{R}^d$. Let $\Upsilon_{T,\pm}$ be defined as Υ_T except that $\mathfrak{b}_{t,T}$ is replaced by $B_d\left(0, \sqrt{\frac{2t}{T\kappa}}(1 \pm \epsilon)\right)$.

Recall

$$0 \leq t \leq \xi T^{(D-2)/D}$$

Writing

$$s = \frac{t}{T^{(D-2)/D}}$$

we have

$$0 \leq s \leq \xi, \quad t = sT^{(D-2)/D}$$

Following the definition of $\mathcal{C}_T$, we define $\mathcal{C}_{T,\pm}$ as

$$\mathcal{C}_{T,\pm} = \cup_{t\cdots} \left(\Pi_{V+bV^\perp}(q_t - q_0) - B_d \left(0, \sqrt{\frac{2t}{T\kappa}}(1 \pm \epsilon) \right) \right)$$

and so by the definition of K,

$$K\mathcal{C}_{T,\pm} = \cup_{t\cdots} \left((t\|\Pi_V v\|, 0, 0, \cdots, 0) - B_d \left(0, \sqrt{\frac{2t}{T\kappa}}(1 \pm \epsilon) \right) \right)$$

where $\|\Pi_V v\| = 1 + O(\frac{1}{T})$. Next, we apply the diagonal matrix D_T and find

$$D_T K\mathcal{C}_{T,\pm} = \cup_{t\ldots} \left((tT^{(2-D)/D}\|\Pi_V v\|, 0, 0, \cdots, 0) - \mathcal{E}_d(t, T, \pm) \right)$$

where $\mathcal{E}_d(t, T\pm)$ is a d dimensional ellipsoid defined by

$$\mathcal{E}_d(t, T, \pm) = \{(x_1, ..., x_d) : x_1^2 T^{-2(2-D)/D} + \sum_{i=1}^{d} x_i^2 T^{-2/D} \leq (1 \pm \epsilon)^2 \frac{2t}{T\kappa}\}.$$

Now let

$$B_{d-1}(0, \rho) = \{(x_1, ..., x_d) : x_1 = 0, \sum_{i=2}^{d} x_i^2 \leq \rho^2\}$$

be the $d-1$ dimensional sphere of radius ρ embedded into the d dimensional space. Then we conclude that for T sufficiently large,

$$D_T K\mathcal{C}_{T,+} \subset \cup_{t...} \left((tT^{(2-D)/D} \|\Pi_V v\|, 0, 0, \cdots, 0) + B_{d-1}\left(0, \sqrt{\frac{2t}{T\kappa}}(1+2\epsilon)\right)\right)$$

Now using the definition of s, we see that

$$D_T K\mathcal{C}_{T,+} \subset \cup_{s\in[0,\xi]} \left((s\|\Pi_V v\|, 0, 0, \cdots, 0) + B_{d-1}\left(0, \sqrt{\frac{2s}{\kappa}}(1+2\epsilon)\right)\right)$$

Finally, since M_1 is constant times the identity matrix, we conclude

$$\Upsilon_{T,+} = D_T K M_1(\mathcal{C}_{T,+}) \subset$$

$$\cup_{s\in[0,\xi]} \left(\frac{s\|\Pi_V v\|}{[\text{covol}(\mathcal{L})]^{1/d}}(1,0,...,0) + B_{d-1}\left(0, \frac{\sqrt{2s}}{\sqrt{\kappa}[\text{covol}(\mathcal{L})]^{1/d}}(1+2\epsilon)\right)\right)$$

and likewise

$$\Upsilon_{T,-} \supset$$

$$\cup_{s\in[0,\xi]} \left(\frac{s\|\Pi_V v\|}{[\text{covol}(\mathcal{L})]^{1/d}}(1,0,...,0) + B_{d-1}\left(0, \frac{\sqrt{2s}}{\sqrt{\kappa}[\text{covol}(\mathcal{L})]^{1/d}}(1-2\epsilon)\right)\right)$$

Since ϵ is arbitrary, we have verified (15.5). With (15.5) verified, we can repeat the argument in Section 9.2 of [22] to deduce the q-averaged version (15.1). This completes the proof. $\square$

Now we proceed to describe the joint distribution of the free flight time and the relative position of the collision point on the scatterer. To this end, we introduce a coordinate system $x_1, ..., x_d$ in the hyperplane of collision, relative to the flight and a variable $w \in [0,1]$. The definition is the following. Let P be the point of collision after the long flight as in Theorem

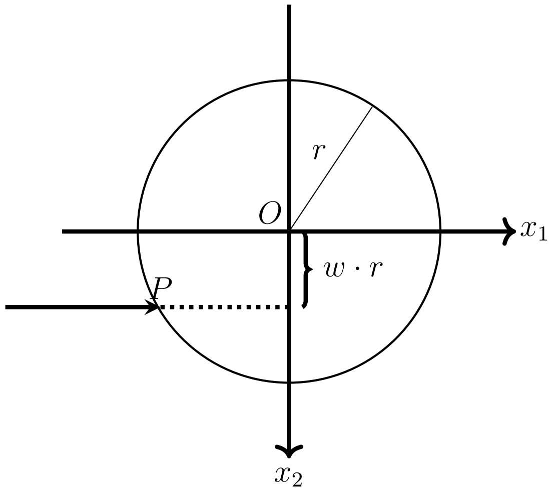


Figure 19: Coordinate system in the plane E^*

13.1. Consider the hyperplane V^* through P parallel with the corridor. The intersection of the scatterer of collision with this hyperplane is a sphere of dimension $d = D - 1$ and some radius $r \ll 1$. Let Q be the center of this sphere and let us choose coordinate system $x_1, ...x_d$ with origin Q in V^*. First we choose x_1 as the unit vector in the direction of the projection of the incoming flight to V^*. Now the vector $\overrightarrow{QP}$ and the direction x_1 define a two dimensional plane E^* inside V^* (except for the degenerate case when $\overrightarrow{QP}$ is parallel with x_1, in which case we choose $x_2, ..., x_d$ arbitrarily so that $x_1, ..., x_d$ is an orthonormal basis and write $w = 0$). Now chose x_2 unit vector perpendicular to x_1 in E^* so that the angle between $\overrightarrow{QP}$ and x_2 is acute. Then we define $w \in [0, 1]$ so that P in the coordinate system x_1, x_2 has coordinates $(-\sqrt{1 - w^2}r, wr)$, see Figure 19. Finally, we choose $x_3, ..., x_d$ arbitrarily so that $x_1, ..., x_d$ is an orthonormal basis in V^*.

Note that by definition $w \in [0, 1]$ where intuitively $w = 0$ means ”heads on” collision on the given latitude at height h and $w = 1$ means grazing collision.

16 Proof of Theorem 5

Theorem 16.1. *There exists a continuous density function Φ so that for all $\underline{\xi}, \bar{\xi}$, with $0 < \underline{\xi} < \bar{\xi} < \infty$ and for all $[a, b] \subset [0, 1]$*

$$\lim_{T\to\infty} (\nu \times \lambda_T)(\tau \in [\underline{\xi} T^{\frac{D-2}{D}}, \bar{\xi} T^{\frac{D-2}{D}}], w \in [a, b]) = \int_{\underline{\xi}}^{\bar{\xi}} \int_a^b \Phi(\xi, w) dw d\xi.$$

We can give a precise definition of the function Φ appearing in Theorem 16.1. To this end, we first need some definitions.

Recall that we have defined the homogeneous spaces

$$X = ASL(d, \mathbb{Z}) \setminus ASL(d, \mathbb{R}).$$

$X(y)$ is the submanifold of X such that

$$X(y) := \{g \in X : y \in \mathbb{Z}^d g\}$$

Write $\Gamma = ASL(x, \mathbb{Z})$, since $\mathbb{Z}^d = 0\Gamma$, we have

$$X(y) = \{\Gamma g : g \in ASL(d, \mathbb{R}), 0g = y\}$$

Thus

$$X(y) = \{\Gamma(M, y) : M \in SL(d, \mathbb{R})\}$$

Furthermore, we have

$$\Gamma(M_1, y) = \Gamma(M_2, y)$$

in X if and only if $SL(d,\mathbb{Z})M_1 = SL(d,\mathbb{Z})M_2$

Hence we get an identification of the sets $X(y)$ and $X_1 = SL(d,\mathbb{Z})\backslash SL(d,\mathbb{R})$, through

$$X(y) = \{(M,y) : M \in X_1\}$$

This gives $X(y)$ the structure of an embedded submanifold of X, of dimensional $d^2 - 1$.

We endow $X(y)$ with the Borel probability measure ν_y which comes from μ_1 on X_1 under the identification.

Then we define the density function as:

$$\Phi(\xi,w) = \nu_y(\{g \in X(y) : \mathbb{Z}^d g \cap \Upsilon(\xi) = \emptyset\})$$

where $y = \xi e_1 + \frac{\sqrt{2\xi}}{\sqrt{\kappa covol(\mathcal{L})^{1/d}}} we_2$, and Υ is defined by (15.2).

Now we give an intuitive explanation of the the above formulae. Assume that the have a collision exactly at time $\xi T^{(D-2)/D}$. Recall the coordinate directions $x_1, ..., x_d$ defined in V^*. Assume that the initial position is $q_0 \in U_{H,c}$. The flight is in the direction

$$v \approx (1 - 1/(2t^2), 0, ..., 0, 1/t)$$

when using the coordinate system $x_1, ..., x_d, x_D$. Projecting to the hyperplane V^* and applying the transformation D_T as in (15.4), we will have a collision at $\approx \xi e_1$ on a d dimensional sphere with center at $y = \xi e_1 + \frac{\sqrt{2\xi}}{\sqrt{\kappa covol(\mathcal{L})^{1/d}}} we_2$. To ensure that no other collision happened before, we need prohibit lattice points inside the paraboloid $\Upsilon(\xi)$. See Figure 20

We don't give a formal proof of Theorem 16.1 because it requires little changes to the very similar proof of [22, Theorem 4.4] and those changes are identical to the ones used in

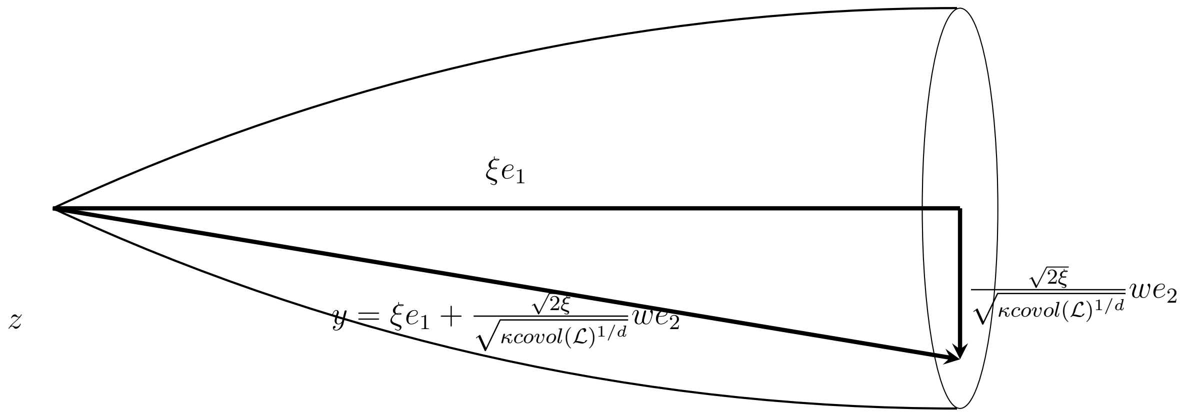
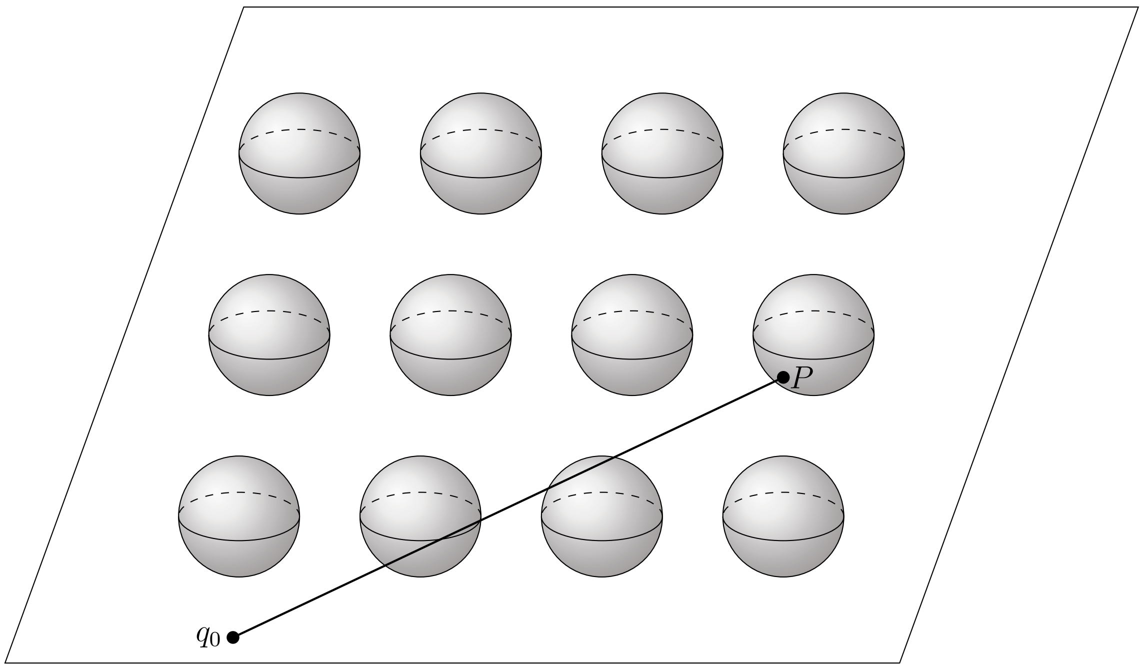


Figure 20: $\xi e_1 + we_2$

the proof of our Theorem 13.1. We do comment on the fact that we simplified the notation from [22, Theorem 4.4]. Indeed, with their notation, we have $w = \|\mathbf{w}\|$ and $\alpha \notin \mathbb{Q}^{\mathbf{d}}$. Noting that by [22, Remark 4.5], in case $\alpha \notin \mathbb{Q}^{\mathbf{d}}$, the function denoted by Φ_α in [22] is only a function of the two variables ξ, w. This is the simplification that we used.



17 Proof of Theorem 6

Theorem 17.1. *There exists a continuous density function Φ, ψ so that for all ξ, ξ',*

$$\lim_{T\to\infty}(\nu \times \lambda_T)(\tau_1 \in [\underline{\xi}T^{\frac{D-2}{D}}, \bar{\xi}T^{\frac{D-2}{D}}], w_1 \in [a,b]$$

$$\tau_2 \in [\underline{\xi}'T^{\frac{D-2}{2D}}, \bar{\xi}'T^{\frac{D-2}{2D}}], w_2 \in [a', b'])$$

$$= \int_{\underline{\xi}}^{\bar{\xi}} \int_a^b \Phi(\xi, w) \int_{\underline{\xi}'}^{\bar{\xi}'} \int_{a'}^{b'} \psi(\xi', w', \xi, w) dw d\xi.$$

where

$$\psi(\xi', w', \xi, w) = \Phi(\sqrt{\frac{\kappa}{8}}\frac{1}{\sqrt{\xi(1-w)}}\xi', \sqrt{\frac{8}{\kappa}}\sqrt{\xi(1-w)}w')$$

Proof. We use sphere in 3-dimensional to prove this theorem. We have a point particle following the inward vector and touches a specific position on the sphere, then bounces away tracing in the outward vector, until it touches another sphere. Suppose the touching point on another sphere is P. Project the inward vector on the first sphere and normalize it as z, also, project the outward vector on the second sphere and normalize it as w. South pole of the second sphere is $S = (0, 0, 0)$, the origin of the sphere is $O = (0, 0, \frac{1}{\kappa})$, and the inward vector is $V = (1, 0, \frac{1}{T})$.

In 2-dimensional, choose x_1 such that

$$V = (1, 0, 0, \cdots, \frac{1}{T})$$

Choose x_2 such that

$$\overrightarrow{w} = (0, w, 0, \cdots, 0)$$

From easy calculation, we know

$$P = (-r\sqrt{1-w^2}, rw, 0, ..., 0, h)$$

remind that

$$h = \frac{t}{T} = \xi T^{\frac{D-2}{D}-1} = \xi T^{-\frac{2}{D}}$$

So then we know

$$\overrightarrow{OP} = (-r\sqrt{1-w^2}, rw, \cdots, h - \frac{1}{\kappa})$$

Approximate $\|OP\| \approx \frac{1}{\kappa}$ since r,w and h is sufficiently small.

Decompose V as parallel vector and perpendicular vector,

$$V = V_{\|} + V_{\perp}$$

$$V_{\perp} = \frac{< V, \overrightarrow{OP} >}{\|OP\|} \cdot \overrightarrow{OP} =< -r\sqrt{1-w^2} + \frac{1}{T}(h - \frac{1}{\kappa}) > \cdot \kappa \cdot \overrightarrow{OP}$$

Assuming $\xi > \epsilon$ and $w < 1 - \epsilon$

We know

$$V^{'} = V - 2V_{\perp} \approx (1, 0, \frac{1}{T}) + 2\kappa r\sqrt{1-w^2}\overrightarrow{OP}$$

The last component of $V^{'}$ is

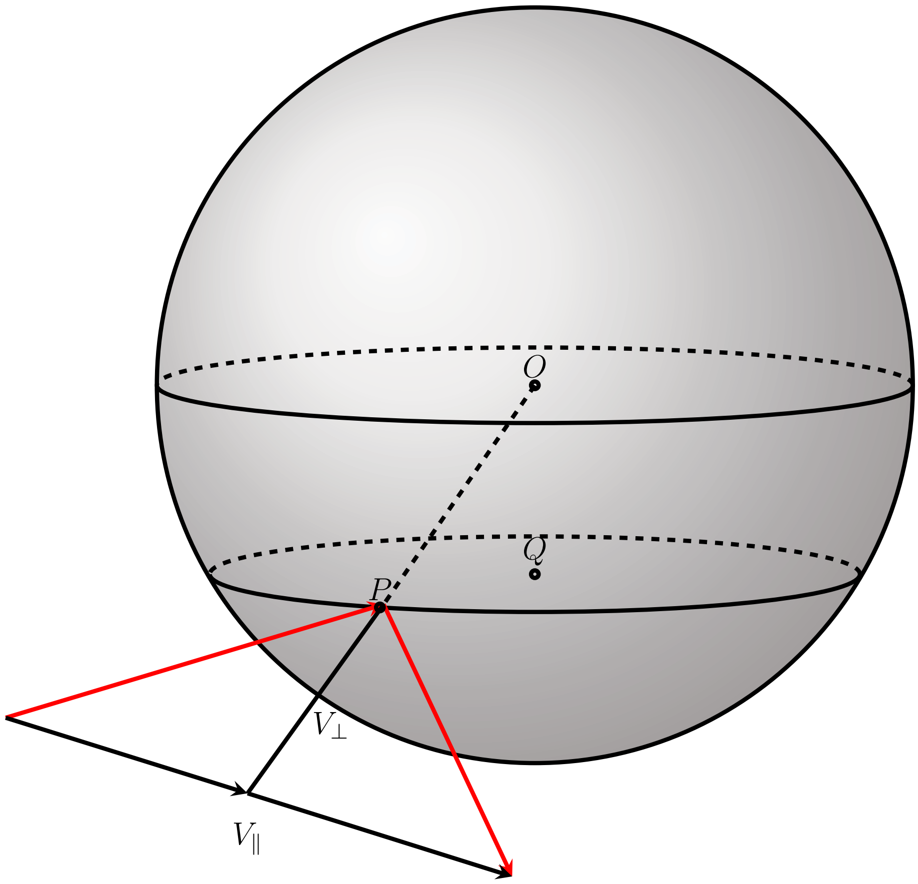


Figure 21: Decompose V

$$V_3^{'} = \frac{1}{T} + 2\kappa r\sqrt{1-w^2}(h - \frac{1}{\kappa})$$

Since

$$\frac{1}{T} - 2r\sqrt{1-w^2} \approx -2r\sqrt{1-w^2} \approx -2\sqrt{\frac{2\xi}{\kappa}}\sqrt{1-w^2}T^{-\frac{1}{D}} \approx -\sqrt{\frac{8}{\kappa}}\sqrt{\xi}\sqrt{1-w^2}T^{-\frac{1}{D}}$$

Suppose

$$w_1 = 1$$

Also from the calculation we know

$$w_2 = -\sqrt{\frac{8}{\kappa}}\sqrt{\xi_1}\sqrt{1-w_1^2}$$

Then

$$T^{'} = \frac{1}{-V_3^{'}} = \sqrt{\frac{\kappa}{8}}\frac{1}{\sqrt{\xi_1(1-w_1^2)}}T^{\frac{1}{D}}$$

$$V_3^{''} = \sqrt{\frac{8}{\kappa}}\sqrt{\xi_2}\sqrt{1-w^2}(T^{'})^{-\frac{1}{D}}$$

$\square$

18 Proof of Theorem 7

Theorem 18.1. *There exists continuous density functions* $\Phi, \psi, \cdots$ *so that for all* $\xi_1, \xi_2, \cdots, \xi_k$,

$$\lim_{T\to\infty}(\nu \times \lambda_T)(\tau_1 \in [\underline{\xi} T^{\frac{D-2}{D}}, \bar{\xi} T^{\frac{D-2}{D}}], w_1 \in [a,b], \cdots,$$

$$\tau_k \in [\underline{\xi_k} T^{\frac{D-2}{2D}}, \bar{\xi_k} T^{\frac{D-2}{2D}}], w_k \in [a_k, b_k])$$

$$= \int_{\underline{\xi_1}}^{\bar{\xi_1}} \int_{a_1}^{b_1} \Phi(\xi_1, w_1) \int_{\underline{\xi_2}}^{\bar{\xi_2}} \int_{a_2}^{b_2} \psi(\xi_1, w_1, \xi_2, w_2)$$

$$\int_{\underline{\xi_3}}^{\bar{\xi_3}} \int_{a_3}^{b_3} \psi(\xi_2, w_2, \xi_3, w_3) \cdots \int_{\underline{\xi_k}}^{\bar{\xi_k}} \int_{a_k}^{b_k} \psi(\xi_{k-1}, w_{k-1}, \xi_k, w_k)$$

$$dw_k d\xi_k dw_{k-1} d\xi_{k-1} \cdots dw_1 d\xi_1$$

Proof. Suppose

$$w_1 = 1$$

Also from the calculation we know

$$w_2 = -\sqrt{\frac{8}{\kappa}}\sqrt{\xi_1}\sqrt{1-w_1^2}$$

Then

$$T' = \frac{1}{-V_3'} = \sqrt{\frac{\kappa}{8}}\frac{1}{\sqrt{\xi_1(1-w_1^2)}}T^{\frac{1}{D}}$$

$$V_3'' = \sqrt{\frac{8}{\kappa}}\sqrt{\xi_2}\sqrt{1-w_2^2}(T')^{-\frac{1}{D}}$$

Following from above calculation, we derive

$$w_1(T) = V_3 \cdot T$$

$$w_2(T) = V_3^{'} \cdot T^{\frac{1}{D}}$$

$$w_3(T) = V_3^{''} \cdot T^{\frac{1}{D^2}}$$

The long flight vector satisfies the following law:

$$|w_k| = \alpha_{k-1} |w_{k-1}|^{\frac{1}{D}}$$

Take the logarithm of both side, we derive

$$\log |w_k| = \frac{1}{D} \log |w_{k-1}| + \log(\alpha_{k-1})$$

which mimics the autoregressive model. □

References

[1] Bleher, P. M., Statistical properties of two-dimensional periodic Lorentz gas with infinite horizon, *J. Statist. Phys.* **66** 315–373 (1992).

[2] Boca & Zaharescu: The distribution of the free path lengths in the periodic two-dimensional Lorentz gas in the small-scatterer limit, Comm. Math. Phys. 269 (2007), 425–471. MR 2008f:37013 Zbl 1143.37002

[3] Bowen, R., Markov partitions for Axiom A diffeomorphisms *Amer. J. Math.* **92** 725–747 (1970).

[4] Bunimovich, L. A., Sinai, Ya. G, Markov Partitions for Dispersed Billiards *Commun. Math.Phys* **78** 247–280 (1980).

[5] Bunimovich, L. A., Sinai, Ya. G, Statistical properties of Lorentz gas with periodic configuration of scatterers *Commun. Math.Phys* **78** 479–497 (1981).

[6] Bunimovich, L. A., Chernov, N. I., Sinai, Ya. G., Markov partitions for two-dimensional hyperbolic billiards *Russian Math. Surveys* **45** 105–152 (1990).

[7] Bunimovich, L. A., Chernov, N. I., Sinai, Ya. G., Statistical properties of two-dimensional hyperbolic billiards *Russian Math. Surveys* **46** 47–106 (1991).

[8] Chernov, N.,Young, LS.: Decay of correlations for Loretnz gases and hard balls. in Springer, EMS Vol. 101, Ed. Szasz, D. 89 – 120 (2000)

[9] D. Dolgopyat, D.Szäsz, and T.Varjú, Recurrence properties of planar Lorentz process, Duke Math. J. 142 (2008), 241–281. MR 2009h:37074 Zbl 1136.37022

[10] D.Domokos Szäsz, Markov approximations and statistical properties of billiards, Duke Math. J. 142 (2018), 241–281. MR 2009h:37074 Zbl 1136.37022

[11] Dettmann, C. P., New horizons in multidimensional diffusion: The Lorentz gas and the Riemann Hypothesis, *Journal of Statistical Physics* **146** 181–204, (2012).

[12] D. Szäsz and T.Varjú, Limit laws and recurrence for the planar Lorentz process with infinite horizon, J. Stat. Phys. 129 (2007), 59–80. MR 2008k:37087 Zbl 1128.82011

[13] E. Caglioti and F. Golse, On the distribution of free path lengths for the periodic Lorentz gas. III, Comm. Math. Phys. 236 (2003), 199–221. MR 2004h:82060 Zbl 1041.82016

[14] F. Golse and B. Wennberg, On the distribution of free path lengths for the periodic Lorentz gas. II, M2AN Math. Model. Numer. Anal. 34 (2000), 1151–1163. MR 2002a:82092 Zbl 1006.82025

[15] F. P. Boca, R. N. Gologan, and A. Zaharescu, The statistics of the trajectory of a certain billiard in a flat two-torus, Comm. Math. Phys. 240 (2003), 53–73. MR 2004f:37007 Zbl 1078.37006

[16] G.Pölya.,Zahlentheoretisches und Wahrscheinlichkeitstheoretisches über die Sichtweite im Walde, Arch. Math. Phys. **27** (1918), 135–142.

[17] H.Lorentz, Le mouvement des électrons dans les métaux, Arch. Néerl **10** (1905), 336–371.

[18] I.Melbourne and M.Nicol, and T.Varjú, Almost sure invariance principle for nonuniformly hyperbolic systems, Comm. Math. Phys. 260 (2005), 131–146. MR 2006h:37047 Zbl 1084.37024

[19] I.Melbourne and M.Nicol, and T.Varjú, A vector-valued almost sure invariance principle for hyperbolic dynamical systems, Ann. Probab. 37 (2009), 478–505. MR 2010i:37016 Zbl 1176.37006

[20] J. Bourgain, F. Golse, and B. Wennberg, On the distribution of free path lengths for the periodic Lorentz gas, Comm. Math. Phys. 190 (1998), 491–508. MR 98k:82146 Zbl 0910.60082

[21] L.A. Bunimovich and Y. G. Sinai, Statistical properties of Lorentz gas with periodic configuration of scatterers, Comm. Math. Phys. 78 (1980/81), 479–497. MR 82m:82007 Zbl 0459.60099

[22] Marklof, J., Strömbergsson, A., The distribution of free path lengths in the periodic Lorentz gas and related lattice point problems, *Ann. of Math.* **172** 1949–2033 (2010).

[23] Marklof, J., Strömbergsson, A., The Boltzmann-Grad limit of the periodic Lorentz gas, Ann. of Math.

[24] Marklof, J., Strömbergsson, A., The periodic Lorentz gas in the Boltzmann-Grad limit: Asymptotic estimates. 1008.3811

[25] Marklof, J., Strömbergsson, A., Kinetic transport in the two-dimensional periodic Lorentz gas, Nonlinearity 21 (2008), 1413–1422. MR 2009j:82029 Zbl 1153.82020 1008.3811

[26] M. Ratner.,On Raghunathan's measure conjecture, Ann. of Math. (2)134.3 (1991), 545–607. issn: 0003-486X

[27] M. Ratner.,Raghunathan's topological conjecture and distributions of unipotent flows, Duke Math. J. 63.1 (1991), 235–280. issn: 0012-7094

[28] Nándori, P., Szász, D., Varjú, T., Tail asymptotics of free path lengths for the periodic Lorentz process: On Dettmann's geometric conjectures.*Communications in Mathematical Physics* **331** 1: 111-137, (2014).

[29] Nelson, E., Dynamical Theories of Brownian Motion. *Princeton University Press*(1967).

[30] N.I.Chernov, Statistical properties of the periodic Lorentz gas. Multidimensional case, J.Statist. Phys. 74 (1994), 11–53. MR 95d:58083 Zbl 0946.37500

[31] P. Bálint and I. P.Tóth, Exponential decay of correlations in multi-dimensional dispersing billiards, Ann. Henri Poincaré 9 (2008), 1309–1369. MR 2010a:37069 Zbl 1162.37018

[32] Pène, F., Terhesiu, D., Sharp Error Term in Local Limit Theorems and Mixing for Lorentz Gases with Infinite Horizon Communications in Mathematical Physics volume 382, pages 1625–1689 (2021)

[33] P.M. Bleher, Statistical properties of two-dimensional periodic Lorentz gas with infinite horizon, J. Statist. Phys. 66 (1992), 315–373. MR 93b:82052 Zbl 0925.82147

[34] Sinai, Ya. G., Construction of Markov partitions. *Funct. Anal. its Appl* **2** 245–253 (1968).

[35] Sinai, Ya. G., Dynamical systems with elastic reflections. *Russian Mat.Surveys* **25** 137–189 (1970).

[36] Sinai, Ya. G., Markov partitions and C-diffeomorphisms. *Funct. Anal. its Appl.* **2** 61–82 (1968).

[37] Sinai, Ya. G., Gibbs measures in ergodic theory. *Uspehi Mat. Nauk* **27** 21–64 (1972).

[38] Szász, D., Varjú, T., Limit laws and recurrence for the planar Lorentz process with infinite horizon,*J. Stat. Phys.* **129** 59–80 (2007).

[39] Young, L-S.: Statistical properties of dynamical systems with some hyperbolicity. Ann. of Math. 147, 585–650 (1998)

19 Appendix

Proposition 19.1. *In dimensional-D, for $\cup_{\mathbb{Z}\subset\mathbb{Z}^D} B_r(z)$, if there is some $r > \frac{\sqrt{2}}{4}$, then $\mathcal{L}_+ = \mathcal{L}_- = \mathbb{Z}^d = \mathbb{Z}^{D-1}$.*

Proof. For spheres of radius r centered at all points of $\mathbb{Z}^D$. For $\cup_{z\in\mathbb{Z}^D} B_r(z)$, in 2-dimensional, we have two circles centered at $(1,0),(0,1)$. Define v as the line joining $(0,0)$ and $(1,1)$, we project $(1,0),(0,1)$ orthogonally onto v respectively:

$$Proj_v(1,0) = \frac{(1,0)\cdot v}{|v|}v = \frac{1}{2}$$

$$Proj_v(0,1) = \frac{(0,1)\cdot v}{|v|}v = \frac{1}{2}$$

when the diagonal line is greater than $4r$, that is $4r < \sqrt{2}$, $r < \frac{\sqrt{2}}{4}$, the diagonal corridor exists, we denote the diagonal corridor as $V^{\perp}$.

When the diagonal line is less than $4r$, that is $4r > \sqrt{2}$, $r > \frac{\sqrt{2}}{4}$, then all coincides H, that is x line $V^{\perp} = (0,1)$ or y line $V^{\perp} = (1,0)$, $\mathcal{L}_+ = \mathcal{L}_- = \mathbb{Z}^d = \mathbb{Z}^{D-1}$.

So $V^{\perp}$ opens when $r = \frac{\sqrt{2}}{4} \approx 0.35$

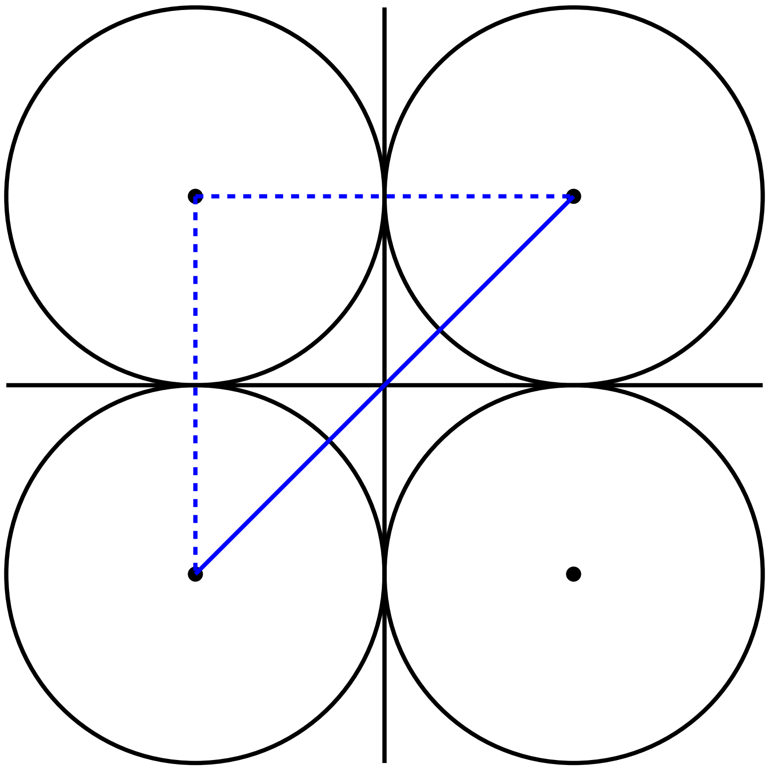


Figure 22: $4r > \sqrt{2}$

In 3-dimensional, we have six balls centered at

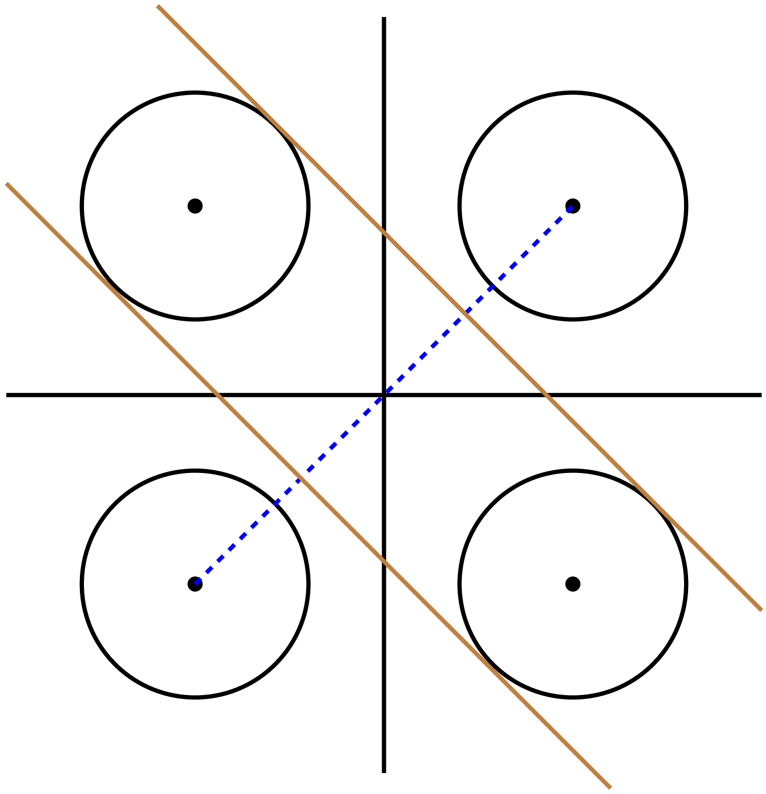


Figure 23: $4r < \sqrt{2}$

$(1,0,0),(0,1,0),(0,0,1),(1,0,0),(0,1,0),(0,0,1)$.

Define v as the line joining $(0,0,0)$ and $(1,1,1)$, we project $(1,0,0),(0,1,0),(0,0,1)$ orthogonally onto v respectively:

$$Proj_v(1,0,0) = \frac{(1,0,0)\cdot v}{|v|}v = \frac{1}{3}$$

$$Proj_v(0,1,0) = \frac{(0,1,0)\cdot v}{|v|}v = \frac{1}{3}$$

$$Proj_v(0,0,1) = \frac{(0,0,1)\cdot v}{|v|}v = \frac{1}{3}$$

Also, we project $(1,1,0),(0,1,1),(1,0,1)$ orthogonally onto v respectively:

$$Proj_v(1,1,0) = \frac{(1,1,0)\cdot v}{|v|}v = \frac{2}{3}$$

$$Proj_v(0,1,1) = \frac{(0,1,1)\cdot v}{|v|}v = \frac{2}{3}$$

$$Proj_v(1,0,1) = \frac{(1,0,1)\cdot v}{|v|}v = \frac{2}{3}$$

When the diagonal line is greater than $6r$, that is $6r < \sqrt{3}$, $r < \frac{\sqrt{3}}{6}$, $V^{\perp}$ exists. See Figure 24.

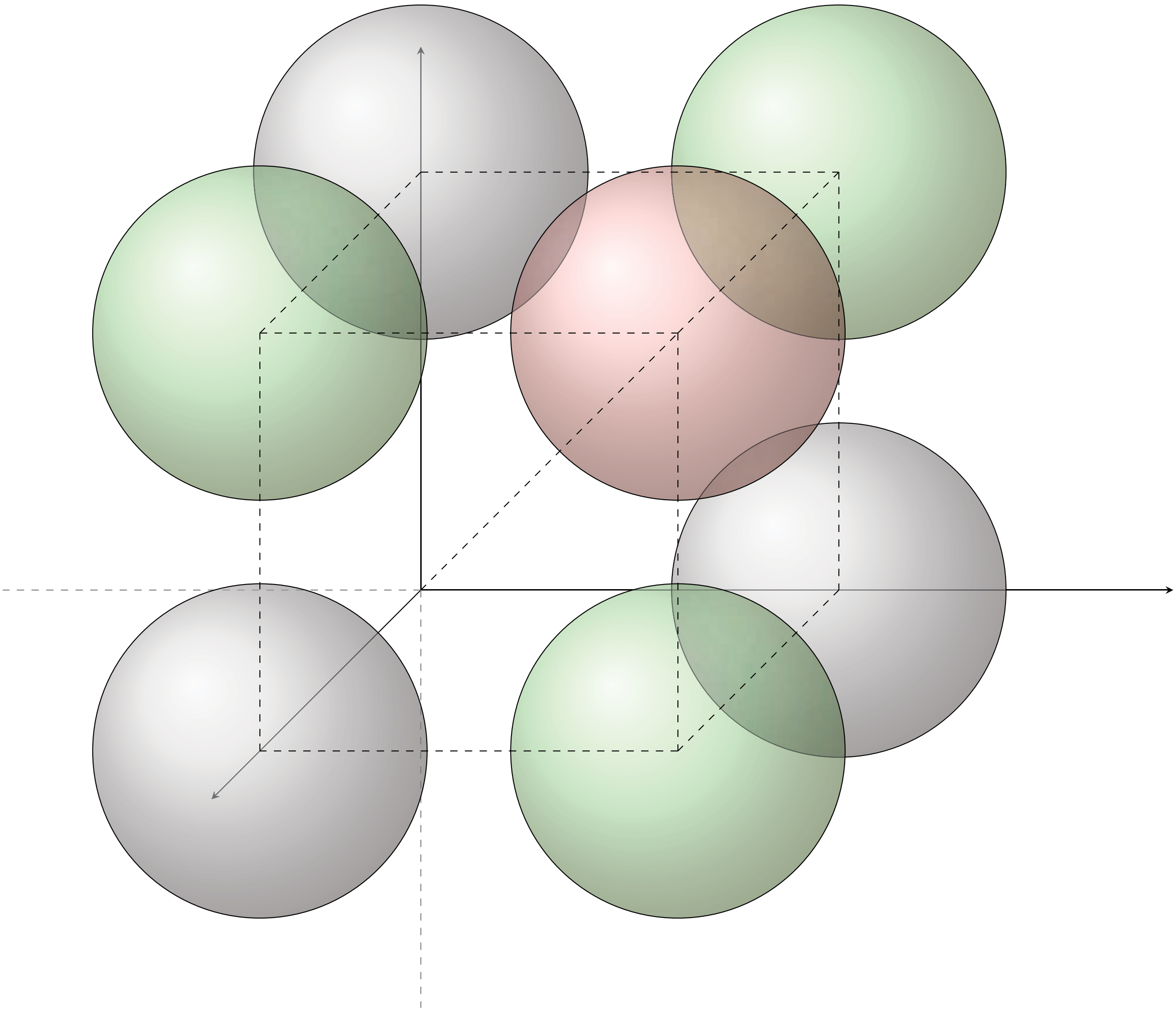


Figure 24: $Corridor\ r = 0.4$

When the diagonal line is less than $6r$, that is $6r > \sqrt{3}$, $r > \frac{\sqrt{3}}{6}$, then all coincides H, that is yz plane $V^{\perp} = (1,0,0)$ or xz plane $V^{\perp} = (0,1,0)$ or xz plane $V^{\perp} = (0,0,1)$, $\mathcal{L}_+ = \mathcal{L}_- = \mathbb{Z}^d = \mathbb{Z}^{D-1}$.

So $V^\perp$ opens when $r = \frac{\sqrt{3}}{6} \approx 0.29$

in d-dimensional, when the diagonal line is greater than $2dr$, that is $2dr < \sqrt{d}$, $r < \frac{\sqrt{d}}{2d}$, $V^\perp$ exists.

And we see $\frac{\sqrt{d}}{2d} < \frac{\sqrt{d-1}}{2(d-1)}$, so if $r > \frac{\sqrt{2}}{4}$, $V^\perp$ would be blocked in all dimensions.

That means, if there is some $r > \frac{\sqrt{2}}{4}$, then we have $V^\perp = e_i$ for some i, where:

$$e_i = \begin{bmatrix} 0 \\ 0 \\ \\ \cdots \\ 1 \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix} \text{ the } i-th \text{ element is } 1.$$

If there is some $\frac{\sqrt{2}}{4} > r > \frac{\sqrt{3}}{6}$, then there is some i such that $V_i^\perp = 0$.

If there is some $r < \frac{\sqrt{3}}{6}$, then $V^\perp = (1, 1, 1)$ is possible. $\square$