A central limit theorem for time-dependent dynamical systems

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Abstract

The work [8] established memory loss in the time-dependent (non-random) case of uniformly expanding maps of the interval. Here we find conditions under which we have convergence to the normal distribution of the appropriately scaled Birkhoff-like partial sums of appropriate test functions. A substantial part of the problem is to ensure that the variances of the partial sums tend to infinity (cf. the zero-cohomology condition in the autonomous case). In fact, the present paper is the first one where non-random, i. e. specific examples are also found, which are not small perturbations of a given map. Our approach uses martingale approximation technique in the form of [9].

1 Introduction

Time-dependent dynamical systems appear in various applications. Recently, [8] could establish exponential loss of memory for expanding maps and, moreover, for one-dimensional piecewise expanding maps with slowly varying parameters. It also provided interesting motivations and examples for the problem. For us - beside their work - an additional incentive was the question of J. Lebowitz [6]: bound the correlation decay for a planar finite-horizon Lorentz process which is periodic apart form the 0-th cell; in it, the Lorentz particle encounters a particular scatterer of the 0-th cell moderately displaced at its each subsequent return to the 0-th cell. (Slightly similar is the situation in the Chernov-Dolgopyat model of Brownian Brownian motion, where - between subsequent collisions of the light particle with the heavy one - the heavy particle slightly moves away, cf. [3].)

The results of [8] say that - for sequences of uniformly uniformly expanding maps - distances of images of a pair of different initial measures converge to 0 exponentially fast. In the same setup it is also natural to expect that probability laws of the Birkhoff-type partial sums of some given function - scaled, of course, by the square roots of their variances - are approximately Gaussian. The main theorem of our paper provides a positive answer though our conditions are surprisingly more restrictive than those of [8]. Let us explain the difficulty and some related results.

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In functional central limit theorems for functions of autonomous chaotic deterministic systems the zero-cohomology condition is - in quite a generality - known to be necessary and sufficient for the vanishing of the limiting variance (see [7] for instance). For time-dependent systems, however, such a condition is only known for almost all versions of random dynamical systems (see [1]) and for other models the situation can be and definitely is completely different. In fact, for time-dependent systems, first [2] had proved a Gaussian approximation theorem in quite a generality; he, however, assumed that the variances of the Birkhoff-type partial sums tend to ∞ sufficiently fast; the paper, however, did not provide any example when this condition would hold. The more recent work [4] proves under some reasonable conditions a dichotomy: either the variances are bounded or the Gaussian approximation holds; the article also provides an example for the latter in the case when the time dependent maps are smaller and smaller perturbations of a given map. But still there is no general method for ascertaining whether the variance is bounded or not. Finally we note that [5] has interesting results for higher order cohomologies but its setup is different.

The present work is, in fact, the first one where non-random, i. e. specific examples are also found, that are not small perturbations of a given map. The proof of our main theorem uses martingale approximation technique in the form introduced in [9] for treating additive functions of inhomogeneous Markov chains. The organization of our paper is simple: its section 2 contains our main theorem and provides examples when it is applicable. Section 3 is devoted to the proof of the theorem.

2 Results

Let A be a set of numbers and $(X, \mathcal{F}, \mu)$ a probability space. For each $a \in A$ define $T_a : X \to X$. Suppose that μ is invariant for all T_a's. Now consider a sequence of numbers from A, i.e. $\underline{a} : \mathbb{N} \to A$. Our aim is to prove some kind of central limit theorem for the sequence

$$f \circ T_{a_1}, f \circ T_{a_2} \circ T_{a_1}, ...$$

with some nice function $f : X \to \mathbb{R}$.
As usual,

$$\hat{T}_a g(x) = g(T_a x)$$

and $\hat{T}^*$ is the $L^2(\mu)$-adjoint of $\hat{T}$ (the so called Perron-Frobenius operator). Further, introduce the notation

$$\hat{T}_{[i..j]} = \begin{cases} \hat{T}_{a_i} \dots \hat{T}_{a_j} & \text{if } i \le j \\ Id & \text{otherwise} \end{cases}$$

and for simplicity write $\hat{T}_{[j]} = \hat{T}_{[1..j]}$.
Similarly, define

$$\hat{T}^*_{[i..j]} = \begin{cases} \hat{T}^*_{a_j} \dots \hat{T}^*_{a_i} & \text{if } i \le j \\ Id & \text{otherwise} \end{cases}$$

and $\hat{T}^*_{[j]} = \hat{T}^*_{[1..j]}$.
Further, let $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{F}_i = (T_{a_1})^{-1} \dots (T_{a_i})^{-1} \mathcal{F}_0$ and assume that there is a Banach space $\mathcal{B}$ of functions

on X such that $\|g\| := \|g\|_{\mathcal{B}} \geq \|g\|_\infty$ for all $g \in \mathcal{B}$.
Finally, for the fixed function f, introduce the notation

$$u_k = \sum_{i=1}^{k} \hat{T}^*_{[i+1..k]} f.$$

With the above notation, our aim is to prove limit theorem for $S_n(x) = \sum_{k=1}^n \hat{T}_{[k]} f(x)$.

Theorem 1 *Assume that f, $\underline{a}$ and T_b, $b \in A$ satisfy the following assumptions.*

1. $\int f d\mu = 0$.
2. *T_b is onto but not invertible for all $b \in A$.*
3. *$f \in \mathcal{B}$ and there exist $K < \infty$ and $\tau < 1$ such that for all $\underline{b}$ sequences and for all k, $\|\hat{T}^*_{b_1} ... \hat{T}^*_{b_k} f\| < K\tau^k \|f\|$.*
4. (accumulated transversality) *Define χ_k as the L^2-angle between u_k and the subspace of $(T_{a_{k+1}})^{-1}\mathcal{F}_0$-measurable functions. Then*
$$\sum_{k=1}^{N} \min_{j\in\{k,k+1\}} (1 - \cos^2(\chi_j))$$
converges to ∞ as $N \to \infty$.

Then

$$Var(S_n) \to \infty$$

and

$$\frac{S_n(x)}{\sqrt{Var(S_n)}}$$

converges weakly to the standard normal distribution, where x is distributed according to μ.

Assumption 3 roughly tells that there is an eventual spectral gap of the operators $\hat{T}^*_{a_j}$ which is quite a natural assumption. Assumption 4 guarantees that there is no much cancellation in S_n, for instance f cannot be in the cohomology class of the zero function when $|A| = 1$.
Before proving the statement let us examine a special case.

Example 2 *Define $(X, \mathcal{F}, \mu) = (S^1$, Borel, Leb), $A = \{2, 3, \dots\}$, $T_a(x) = ax(mod1)$, $\mathcal{B} = C^1 = C^1(S^1)$,*

$$\|g\| := \sup_{x\in S^1} |g(x)| + \sup_{x\in S^1} |g'(x)|.$$

Fix a non constant function $f \in C^1$ satisfying $\int f dx = 0$. Then there exists some integer $L = L(f)$ such that with all sequences $\underline{a}$ for which

$$\#\{k : \min\{a_k, a_{k+1}, a_{k+2}\} > L\} = \infty$$

the assumptions of Theorem 1 are fulfilled.

Proof of Example 2. It is easy to see that for all $g \in C^1$ with zero mean, and for all $\underline{b} : \mathbb{N} \to A$,

$$\|\hat{T}^*_{b_1} g\| \leq 2b_1^{-1}\|g\|$$

and similarly,

$$\|\hat{T}^*_{b_1} \dots \hat{T}^*_{b_k} g\| \leq 2 \cdot 2^{-k}\|g\|. \tag{1}$$

Hence Assumption 3 is fulfilled.

In order to check Assumption 4, select $x, y \in S^1$, $\varepsilon > 0, \delta > 0$ such that

$$\min_{z\in[x,x+\varepsilon]} f(z) > \delta + \max_{z\in[y,y+\varepsilon]} f(z).$$

This can be done since f is not constant. Now choose

$$L > \max\{\frac{16\|f\|}{\delta}, \frac{2}{\varepsilon}\}.$$

Whence

$$\|\hat{T}^*_L f\| \leq \delta/8.$$

Thus if $a_k > L$, then

$$\|\sum_{i=1}^{k-1} \hat{T}^*_{[i+1..k]} f\| < 3\delta/8$$

is true independently of the choice of $a_1, \dots a_{k-1}$. This yields

$$\min_{z\in[x,x+\varepsilon]} u_k(z) > \delta/4 + \max_{z\in[y,y+\varepsilon]} u_k(z).$$

Since $L > \frac{2}{\varepsilon}$, for all g which is $(T^L)^{-1}\mathcal{F}_0$ measurable, one can find $h : [0, \varepsilon/2) \to \mathbb{R}$ and $\varepsilon_1 \leq \varepsilon/2$ such that $g(y + \varepsilon_1 + z) = g(x + z) = h(z)$ for all $z \in [0, \varepsilon/2)$. Hence,

$$\begin{aligned}
& \|u_k - g\|_2^2 \\
\geq\ & \int_x^{x+\varepsilon/2} (u_k(z) - g(z))^2\, dz + \int_{y+\varepsilon_1}^{x+\varepsilon_1+\varepsilon/2} (u_k(z) - g(z))^2\, dz \\
=\ & \int_0^{\varepsilon/2} (u_k(x+z) - h(z))^2\, dz + \int_0^{\varepsilon/2} (u_k(y+\varepsilon_1+z) - h(z))^2\, dz \\
\geq\ & \frac{1}{2}\int_0^{\varepsilon/2} (u_k(x+z) - u_k(y+\varepsilon_1+z))^2\, dz \geq \frac{\delta^2\varepsilon}{64}
\end{aligned} \tag{2}$$

Since

$$\|u_k\|_2 < \|u_k\|$$

is bounded, (2) implies that $(1 - \cos^2(\chi_k))$ is uniformly bounded away from zero if $\min\{a_k, a_{k+1}\} > L$. Hence, Assumption 5 is fulfilled if there exist infinitely many indices k such that

$$\min\{a_k, a_{k+1}, a_{k+2}\} > L.$$

■

In Example 2, expanding maps with large derivative were needed in order to obtain the Gaussian approximation. Naturally arises the question that what happens in the case when one uses only finitely many dynamics, for instance, only T_2 and T_3 of Example 2. That is why we discuss the following example.

Example 3 *Define $X, \mathcal{F}, \mu, A, T_b, \mathcal{B}$ as in Example 2. If $\underline{a}$ is a sequence for which there is a $b \in A$ such that for all integer K, one can find a k for which*

$$a_k = a_{k+1} = ... = a_{k+K-1} = b,$$

and $f \in \mathcal{B}$, $\int f = 0$ is any function for which the equation $f = \hat{T}_b u - u$ has no solution u, then the assumptions of Theorem 1 are fulfilled.

Proof of Example 3. It is enough to verify Assumption 4. To do so, for $K \in \mathbb{Z}_+$ pick k such that

$$a_{k-K} = a_{k-K+1} = ... = a_{k+2} = b. \tag{3}$$

Then (1) implies that

$$\|u_j - \sum_{i=0}^{\infty} \left(\hat{T}_b^*\right)^i f\| < C2^{-K} \tag{4}$$

holds for $j = k, k+1$ with some C uniformly in K. Now, if $g := \sum_{i=0}^{\infty} \left(\hat{T}_b^*\right)^i f$ is not $(T_b)^{-1}\mathcal{F}_0$-measurable, then necessarily its L^2-angle with those functions is positive. Since (3) and (4) hold for infinitely many k's, $\min\{\chi_k, \chi_{k+1}\}$ has a positive lower bound infinitely many times, inferring Assumption 5. On the other hand, if g is $(T_b)^{-1}\mathcal{F}_0$-measurable, then $g = \hat{T}_b \hat{T}_b^* g$ and $g - \hat{T}_b^* g = f$ imply that for $u = \hat{T}_b^* g$, $\hat{T}_b u - u = f$. ■

Note, that in Example 3, $Var(S_n)$ can be arbitrary small. Indeed, pick a C^1 function f, for which $f = \hat{T}_3 u - u$ has no solution u, but there is some v such that $f = \hat{T}_2 v - v$. Now, pick a sequence of integers $d_l, l \in \mathbb{N}$, $d_l \to \infty$ fast enough, and define

$$a_k = \begin{cases} 3 & \text{if } \exists l : d_l \le k < d_l + l \\ 2 & \text{otherwise.} \end{cases}$$

It is easy to see that (1) implies $\mathbb{E}(|\hat{T}_{[i]} f \cdot \hat{T}_{[j]} f|) \le 2^{|i-j|+1} \|f\|^2$ (formally it follows from (14)), which in turn yields that $Var(S_k)$ is bounded by some constant times k. Now, with the notation $l_n := \max\{l : d_l \le n\}$, write

$$\begin{aligned} Var(S_n) \quad \le \quad & 4Var(S_{d_{l_n-1}+l_n}) + 4Var(S_{d_{l_n}} - S_{d_{l_n-1}+l_n}) \\ & +4Var(S_{d_{l_n}+l_n} - S_{d_{l_n}}) + 4Var(S_n - S_{d_{l_n}+l_n}). \end{aligned}$$

On the other hand, $f = \hat{T}_2 v - v$ implies that $\hat{T}_2 f + ... + \hat{T}_2^m f$ is uniformly bounded in m. Thus the second and the last term in the above sum are bounded. Whence $Var(S_n)$ is smaller than some constant times d_{l_n-1}. Especially, if $d_l = 2^{2^{2^l}}$, then

$$\frac{Var(S_n)}{n^\alpha} \to 0$$

as $n \to 0$ for any α positive. Note that in this case the conditions of [2] for the Gaussian approximation are not met.

3 Proof of Theorem 1

This section is devoted to the proof of Theorem 1.

As in [7], [9] and [4], the proof is based on martingale approximation. First, observe that

$$\hat{T}^*_{[n]}\hat{T}_{[n]} = Id$$

and

$$\hat{T}_{[n]}\hat{T}^*_{[n]}$$

is the orthogonal projection onto the $\mathcal{F}_n$ measurable functions (for the proof of the latter, see [7]). Now we introduce our approximating martingale, which is analogous to the one of [9]:

$$Z_k = \sum_{i=1}^{k} \mathbb{E}\left[\hat{T}_{[i]}f|\mathcal{F}_k\right] = \sum_{i=1}^{k} \hat{T}_{[k]}\hat{T}^*_{[k]}\hat{T}_{[i]}f = \sum_{i=1}^{k} \hat{T}_{[k]}\hat{T}^*_{[i+1..k]}f = \hat{T}_{[k]}u_k \tag{5}$$

Since

$$\begin{aligned} \hat{T}_{[i]}f &= Z_i - \mathbb{E}\left[Z_{i-1}|\mathcal{F}_i\right] && (6) \\ &= \left(Z_i - \mathbb{E}\left[Z_i|\mathcal{F}_{i+1}\right]\right) + \left(\mathbb{E}\left[Z_i|\mathcal{F}_{i+1}\right] - \mathbb{E}\left[Z_{i-1}|\mathcal{F}_i\right]\right), && (7) \end{aligned}$$

one obtains

$$S_n = \sum_{k=1}^{n-1} \left(Z_k - \mathbb{E}\left[Z_k|\mathcal{F}_{k+1}\right]\right) + Z_n.$$

Now,

$$\xi_k^{(n)} = \frac{1}{\sqrt{Var(S_n)}}\left(Z_k - \mathbb{E}\left[Z_k|\mathcal{F}_{k+1}\right]\right),$$

is a reverse martingale difference for the σ-algebras $\mathcal{F}_1, \dots \mathcal{F}_n$. Thus, in particular

$$Var(S_n) = \sum_{k=1}^{n-1} Var\left(Z_k - \mathbb{E}\left[Z_k|\mathcal{F}_{k+1}\right]\right) + Var(Z_n). \tag{8}$$

Using our martingale approximation and the well known martingale CLT (see [9] for instance), it is enough to prove that the difference between the martingale approximant and S_n is negligible,

$$\max_{1\le i\le n} \|\xi_i^{(n)}\|_\infty \to 0 \tag{9}$$

and

$$\|\sum_{i=1}^{n} \mathbb{E}\left[\left(\xi_i^{(n)}\right)^2|\mathcal{F}_{i+1}\right] - 1\|_2 \to 0. \tag{10}$$

To prove (9) and (10), we adopt the ideas of [9]. To verify (9), observe that by Assumption 4,

$$\begin{aligned} \|Z_k\|_\infty &\le \sum_{j=1}^{k} \|\hat{T}_{[k]}\hat{T}^*_{[j+1..k]}f\|_\infty \le \sum_{j=1}^{k} \|\hat{T}^*_{[j+1..k]}f\|_\infty \\ &\le \sum_{j=1}^{k} \|\hat{T}^*_{[j+1..k]}f\| \le \sum_{j=1}^{k} K\tau^{k-j}\|f\| \le C_f. && (11) \end{aligned}$$

Thus

$$\|\mathbb{E}\left[Z_k|\mathcal{F}_{k+1}\right]\|_\infty \leq C_f. \tag{12}$$

Now, we prove that the variance of S_n converges to infinity:

$$Var(S_n) = \mu(S_n^2) \to \infty \tag{13}$$

as $n \to \infty$. Since (11) implies that $Var(Z_n)$ is bounded, (8) can be written as

$$\begin{aligned} Var(S_n) &= O(1) + \sum_{k=1}^{n-1} \mathbb{E}(Z_k^2) + \mathbb{E}\left(\mathbb{E}[Z_k|\mathcal{F}_{k+1}]^2\right) - 2\mathbb{E}\left(Z_k \mathbb{E}[Z_k|\mathcal{F}_{k+1}]\right) \\ &= O(1) + \sum_{k=1}^{n-1} \mathbb{E}(Z_k^2) - \mathbb{E}\left(\mathbb{E}[Z_k|\mathcal{F}_{k+1}]^2\right) \\ &= O(1) + \sum_{k=1}^{n-1} \|u_k\|_2^2 - \|u_k\|_2^2 \cos^2 \chi_k. \end{aligned}$$

Here, we used (5), and the fact that $\hat{T}_{[k]}$ is $L^2(\mu)$-isometry. Now, since

$$Var(f) = Var(\hat{T}_{[i]} f) \leq 2Var(Z_i) + 2Var(\mathbb{E}[Z_{i-1}|\mathcal{F}_i]) \leq 2\|u_i\|_2^2 + 2\|u_{i-1}\|_2^2,$$

one obtains

$$Var(S_n) \geq O(1) + \frac{1}{4} Var(f) \sum_{k=1}^{n-1} \min_{j \in \{k,k+1\}} \left(1 - \cos^2 \chi_j\right),$$

which converges to infinity as $n \to \infty$ by Assumption 4. Thus we have verified (13).

Now, (11), (12) and (13) together imply (9) and that the difference between the martingale and S_n is negligible.

To verify (10), first observe that for $i > j$

$$\begin{aligned} \|\mathbb{E}\left[\hat{T}_{[j]} f|\mathcal{F}_i\right]\|_\infty &= \|\hat{T}_{[i]} \hat{T}^*_{[i]} \hat{T}_{[j]} f\|_\infty = \|\hat{T}_{[i]} \hat{T}^*_{[j+1..i]} f\|_\infty = \|\hat{T}^*_{[j+1..i]} f\|_\infty \\ &\leq K\tau^{i-j}\|f\|. \end{aligned} \tag{14}$$

Then one can prove the assertion obtained from Lemma 4.4 in [9] by replacing $v_l^{(n)}$ with

$$\mathbb{E}\left[\left(\xi_{n-l}^{(n)}\right)^2 |\mathcal{F}_{n-l+1}\right]$$

the same way as it was done in [9], which yields (10).

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