NON EQUILIBRIUM DENSITY PROFILES IN LORENTZ TUBES WITH THERMOSTATED BOUNDARIES

DMITRY DOLGOPYAT AND PÉTER NÁNDORI

ABSTRACT. We consider a long Lorentz tube with absorbing boundaries. Particles are injected to the tube from the left end. We compute the equilibrium density profiles in two cases: the semi-infinite tube (in which case the density is constant) and a long finite tube (in which case the density is linear). In the latter case, we also show that convergence to equilibrium is well described by the heat equation. In order to prove these results, we obtain new results for the Lorentz particle which are of independent interest. First, we show that a particle conditioned not to hit the boundary for a long time converges to the Brownian meander. Second, we prove several local limit theorems for particles having a prescribed behavior in the past.

1. INTRODUCTION

An important problem in mathematical physics is to understand the emergence of macroscopic equations from deterministic microscopic laws (see e.g. reviews [**BLRB00**,**Bu00**,**ChD06**,**LSp83**,**Sp80**,**Sp91**,**Sz00**]). In particular, one would like to derive the Fourier law for transport of conserved quantities. So far, this task has only been achieved for one deterministic system: Lorentz gas [**BBS83**,**BSC91**,**Ga69**,**Sp80**]. Even in that case our understanding is not complete. First, the Fourier law is derived for the ideal gas of non-interacting particles which is assumed to be at equilibrium. However, the ideal gas can not reach the equilibrium since in the absence of interactions the energy of each particle is conserved. Therefore, it is desirable to understand how the Lorentz gas achieves the equilibrium if the particles interact weakly with each other. Second, there are several ways to define the transport coefficients. In particular, one can consider

(i) particles in the whole space

(ii) particles confined to vessel with impenetrable boundaries

(iii) particles in a certain region whose boundary is kept at a given temperature by means of a thermostat.

For physicists, those definitions are clearly equivalent but mathematically they are different. In particular, boundary layers need to be studied in the second and third case. Case (i) has been analyzed in [**BSC91**] for periodic Lorentz gas and in [**Ga69**,**Sp78**,**BBS83**] for random Lorentz gas in Boltzmann-Grad limit. Case (ii) has been studied in [**DSzV09**] for periodic Lorentz gas and in [**LSp78**] for random Lorentz gas in Boltzmann-Grad limit. The present paper deals with case (iii).

We consider a strip on a plane with a periodic configuration of convex scatters removed. We assume that the domain has finite horizon (that is, the particle can not move indefinitely without hitting a scatterer) since an anomalous transport takes place in the infinite horizon case [**Bl92**,**SzV07**,**ChD09A**, **MS10**]. Moving particles are injected from the left end of the tube according to a Poisson process with constant intensity. We assume that the particles move with the unit speed and that their initial position and direction are random. When the particle hits an end of the tube it disappears from the system. First, we consider a semi-infinite tube and show (Theorem 1) that at equilibrium (that is, if we start injecting the particles at time $-\infty$) the density of particles approaches a finite limit as the distance from the boundary tends to infinity. The physical meaning of this result is that the particle density at the boundary is well defined. Next, we show (Theorem 2) that if we have have a large finite tube, then the equilibrium density profile is linear interpolating between the limiting densities at the end points (by the superposition principle it suffices to consider the case where particles are injected only from the left). Finally, we show (Theorem 3) that that if we start from a non-equilibrium profile then the approach to equilibrium is described by the heat equation.

To derive Theorems 1, 2 and 3, we obtained several new results for one Lorentz particle. First, we show (Theorem 5) that a particle conditioned not to hit the boundary for a long time converges to the Brownian meander. Second, we prove several local limit theorems for particles having a prescribed behavior in the past (see Section 3.3 for precise formulations). There are two novel features of our local limit theorems. First, since our system has no translational symmetry (due to the presence of the boundaries) we can not use Fourier analysis. Second, we are able to obtain local limit theorems conditioning on events of small probability in both past and future. These results seem to be of independent interest. First, the fact that we can gain a very precise information about the distribution of the particle at a given time t can be useful for studying weakly interacting particles. Secondly, local limit theorems have been used in [**DSzV08**] to compute the limiting distribution of ergodic averages for certain infinite measure preserving transformations related to the Lorentz system and we can hope to get similar results for the semi-infinite tubes. Third, our result should be helpful for analyzing Lorentz process with small deterministic holes (see [**NSz12**] for the case of random holes).

The layout of our paper is the following. In Section 2 we provide the necessary definitions and review the results from the theory of Sinai billiards which will be used in the sequel. Section 3 contains precise formulations of our results. In Section 4 we prove the equilibrium profile in a semi-infinite tube. Section 5 treats the convergence to Brownian meander. Section 6 contains the proofs of the new local limit results we need. In Section 7 we study the equilibrium profile in a long finite tube. In Section 8 we discuss the convergence to equilibrium.

The paper has two appendices. In Appendix A we extend the usual Local Limit Theorem for Lorentz particle to ensure the uniformity with respect to a large class of initial measures and also to provide the bound for cells which are further from the origin than predicted by diffusive scaling. Appendix B contains some computations involving the density of the Brownian meander.

2. PRELIMINARIES

2.1. **Notation.** In this paper we denote every universal constant by C, thus each occurrence of C may stand for a different number. We also write $\mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\mathbb{P}(B)$.

2.2. **Sinai billiard.** Here, we summarize briefly the most important notions from the theory of Sinai billiards needed in the present work. For a much ampler description, consult [**ChM06**]. Define $\mathcal{D} = \mathbb{R} \times \mathcal{S}^1 \setminus \cup_{i=1}^{\infty} B_i$, where $B_1, \ldots, B_k$ are disjoint strictly convex domains inside the unit torus, whose boundaries are C^3-smooth and whose curvatures are bounded from below. $B_{k+1}, B_{k+2}, \ldots$ are the translational copies of $B_1, \ldots, B_k$ with translations in $\mathbb{Z}$. The billiard flow is the dynamics of a point particle in $\mathcal{D}$, which consists of free flight inside $\mathcal{D}$ and specular reflection on $\partial\mathcal{D}$. Since the speed is constant, is it assumed to be 1. Thus the billiard flow Φ^t acts on the space $\mathcal{D} \times \mathcal{S}^1$. For $(x_1, x_2) \in \mathcal{D}$, $v \in \mathcal{S}^1$, and $\Phi^t((x_1, x_2), v) = ((x_1', x_2'), v')$, we will write $\hat{X}(t) = \hat{X}((x_1, x_2), v, t) = x_1'(t)$, for the horizontal component of the position at (continuous) time t.

It is common to take the Poincaré section on the boundaries of the scatterer, and switch to a discrete time dynamics, which is called the billiard map. The phase space of the billiard map is

$$\mathcal{M} = \{x = (q, v) \in \partial\mathcal{D} \times S^1, \langle v, n \rangle \geq 0\},$$

where n is the normal vector of ∂D at the point q pointing inside $\mathcal{D}$, and the map itself is denoted by $\mathcal{F} : \mathcal{M} \to \mathcal{M}$. The natural invariant measure on $\mathcal{M}$, denoted by μ, is the projection of the Lebesgue measure on the phase space of the billiard flow. In fact, $d\mu = \text{const} \cos\phi \, dr \, d\phi$, where r is the arc length parameter on $\partial\mathcal{D}$ and $\phi \in [-\pi/2, \pi/2]$ is the angle between v and n. We will write $q(x)$ for the the projection of the point x to its first coordinate (that is $q(x) \in \partial\mathcal{D}$). The free flight vector $\kappa(x)$ is the lifted version of $q(\mathcal{F}(x)) - q(x)$ form $\mathbb{R} \times \mathcal{S}^1$ to $\mathbb{R}^2$ (that would be the same as $q(\mathcal{F}(x)) - q(x)$ if the Lorentz process was defined in the plane, i.e. $B_{k+1}, B_{k+2}, \ldots$ where translational copies of $B_1, \ldots, B_k$ with translations in $\mathbb{Z}^2$). We assume that κ is bounded, i.e. $\kappa_{\min} \leq |\kappa| \leq \kappa_{\max}$ (the so-called finite horizon condition), and write

$$(1) \qquad\qquad X_k = X_k(q, v) = \Pi \sum_{i=0}^{k-1} \kappa(\mathcal{F}^i(q, v)),$$

where Π is the projection to the horizontal direction (that is, X_k is the discrete counterpart of $\hat{X}(t)$). We also denote by

$$(2) \qquad F_k = F_k(q, v) = \sum_{i=0}^{k-1} |\kappa(\mathcal{F}^i(q, v))|,$$

the time of the k-th collision.

Analogously, one can define the Sinai billiard on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Then one needs to introduce $\mathcal{D}_0 = \mathbb{T}^2 \setminus \cup_{i=1}^k B_i$, and define $\mathcal{M}_0$, $\mathcal{F}_0$ and μ_0 as before. μ is the periodic extension of μ_0. Since μ is infinite and μ_0 is finite, we choose the constant in the definition of μ so that μ_0 is a probability measure. Finally, we write $\bar{\kappa} = \int |\kappa| d\mu_0$ for the mean free path length.

Since we are going to consider tubes with absorbing walls, hitting times are very important. Let $\hat{\tau}_L$ denotes the first time instant, when the particle reaches the horizontal distance L, i.e. $\hat{\tau}_L = \inf\{s > 0 : \hat{X}(s) = L\}$, and τ_L is its discrete counterpart, i.e. $\tau_L = \min\{k : \lfloor X_k \rfloor = L\}$. We also write $\hat{\tau}^* = \hat{\tau}_0$ and $\tau^* = \tau_{-1}$ (this is the time of absorption in the case of semi infinite tube). Hyperbolicity and ergodicity of $\mathcal{F}_0$ (nice properties) were proven by Sinai [S70]. An unpleasant property of the billiard map is the presence of singularities (corresponding to grazing collisions). To overcome the technical difficulties caused by the singularities, we use the so-called standard pair method developed in [ChD09B]. Below we present an informal description of this method, see [ChM06] for more details.

For almost every $x \in \mathcal{M}_0$, stable and unstable manifolds through x exist. There is a factor of stretching in the unstable direction, which is bounded from below by some $\Lambda > 1$. Nevertheless, these factors are not bounded from above (if x is very close to a grazing collision where $\{\cos \phi = 0\}$, the expansion is very big), which makes it difficult to control the distortion of unstable manifolds. That is why it is common to introduce the following additional (secondary) singularities

$$S_{\pm k} = \{(r, \phi) : \phi = \pm \pi/2 \mp k^{-2}\}$$

for k larger than some k_0, yielding bounded distortion of an unstable manifold disjoint to all singularities. An unstable curve is some curve $W \subset \mathcal{M}_0$ such that at every point $x \in W$, the tangent space $T_x W$ is in the unstable cone (slightly weaker property than the unstable manifold). Further, W is homogeneous, if does not intersect any singularity. A pair $\ell = (W, \rho)$ is called a standard pair, if W is a homogeneous unstable curve and ρ is a regular probability measure supported on W. Precisely, the regularity required for the measures is the following:

$$\left| \log \frac{d\rho}{dLeb}(x) - \log \frac{d\rho}{dLeb}(y) \right| \leq C_0 \frac{|W(x, y)|}{|W|^{2/3}},$$

where C_0 is a fixed constant and $|W(x, y)|$ is the arc length of the segment of W lying between x and y (see [ChD09B] for more details). In particular, the

logarithm of the density of ρ is uniformly Hölder continuous. For a standard pair $\ell = (W, \rho)$, we write $\mathbb{E}_\ell$ for the integral with respect to ρ, $\mathbb{P}_\ell(A) = \mathbb{E}_\ell(\mathbf{1}_A)$ and $\mathrm{length}(\ell) = \mathrm{length}(W)$. Once we have a standard pair, its image under the map $\mathcal{F}_0$ is a bunch of unstable curves and some measures living on them.

A nice property of standard pairs is that this image is in fact a weighted sum of standard pairs. That is why we call weighted sums of standard pairs standard families. Formally, a standard family is a set $\mathcal{G} = \{(W_a, \nu_a)\}, a \in \mathfrak{A}$ of standard pairs and a probability measure $\lambda_\mathcal{G}$ on the index set $\mathfrak{A}$. This family defines a probability measure on $\mathcal{M}_0$ by

$$\mu_\mathcal{G}(B) = \int_\mathfrak{A} \nu_a(B \cap W_a) d\lambda_\mathcal{G}(a).$$

We will also write $\mathbb{E}_\mathcal{G}$ for the integral with respect to $\mu_\mathcal{G}$ and $\mathbb{P}_\mathcal{G}(A) = \mu_\mathcal{G}(A)$. Every $x \in W_a$ (for some $a \in \mathfrak{A}$), chops W_a into two pieces. The length of the shorter one is denoted by $r_\mathcal{G}(x)$. The $\mathcal{Z}$-function of $\mathcal{G}$ is defined by

$$\mathcal{Z}_\mathcal{G} = \sup_{\varepsilon > 0} \frac{\mu_\mathcal{G}(r_\mathcal{G} < \varepsilon)}{\varepsilon}.$$

Note that if $\mathcal{G}$ consists of one standard pair, then $\mathcal{Z}_\mathcal{G} = 2/|W|$. In any case, we assume $\mathcal{Z}_\mathcal{G} < \infty$.

While the unstable curves are expanded due to hyperbolicity, they are also cut by the singularities of $\mathcal{F}_0$. An important nice property of the billiard map is that the expansion prevails over the fragmentation. Namely, the following Growth lemma holds true:

Lemma 1. *(see [**ChD09B**, Prop 4.9 and 4.10]) Let $\ell = (W, \rho)$ be some standard pair. Then*

$$(3) \qquad \mathbb{E}_\ell(A \circ \mathcal{F}_0^n) = \sum_a c_{a,n} \mathbb{E}_{\ell_{an}}(A),$$

where $c_{a,n} > 0$, $\sum_a c_{a,n} = 1$; $\ell_{an} = (W_{an}, \rho_{an})$ are standard pairs such that $\cup_a W_{an} = \mathcal{F}_0^n W$ and ρ_{an} is the push-forward of ρ by $\mathcal{F}_0^n$ up to a multiplicative factor. Finally, there are universal constants $\varkappa, C_1$ (depending only on $\mathcal{D}$), such that if $n > \varkappa |\log \mathrm{length}(W)|$, then

$$(4) \qquad \sum_{\mathrm{length}(\ell_{an}) < \varepsilon} c_{a,n} < C_1 \varepsilon.$$

We call the decomposition (3) Markov decomposition. The proof of Lemma 1 depends on the fact that there are universal constants $\theta < 1, C_2, C_3$ (depending only on $\mathcal{D}$) such that for a standard family $\mathcal{G} = \{(W_a, \nu_a)\}, a \in \mathfrak{A}$, and $\mathcal{G}_n = \mathcal{F}_0^n(\mathcal{G})$, one has

$$\mathcal{Z}_{\mathcal{G}_n} < C_2 \theta^n \mathcal{Z}_\mathcal{G} + C_3.$$

If we fix some large constant C_p and call a standard family proper if its $\mathcal{Z}$ function is smaller than C_p, then briefly one can say that the image of $\mathcal{G}$ becomes proper in $\log \mathcal{Z}_\mathcal{G}$ steps.

The essence of the standard pair technique is that the measures carried on two proper standard families can be coupled together exponentially fast. When one of the two standard families is chosen to be μ_0 itself (it can be proven that there exists $\mathcal{G}$ such that $\mu_{\mathcal{G}} = \mu_0$) one obtains the following Equidistribution statement. Recall that a function f on $\mathcal{M}_0$ is called dynamically Holder continuous if there are constants $K > 0$ and $\theta < 1$ such that $|f(x) - f(y)| \leq K\theta^{s(x,y)}$ where $s(x,y)$ is the first number n such that either $\mathcal{F}_0^n x$ and $\mathcal{F}_0^n y$ belong to a different scatterer or $\mathcal{F}_0^{-n} x$ and $\mathcal{F}_0^{-n} y$ belong to a different scatterer.

Lemma 2 ([**Ch06**] Theorem 4). *Let $\mathcal{G}$ be a proper standard family. For any dynamically Hölder continuous f there exists some $\theta_f < 1$ such that for any $n \geq 0$,*

$$\left| \int_{\mathcal{M}_0} f \circ \mathcal{F}_0^n d\mu_{\mathcal{G}} - \int_{\mathcal{M}_0} f d\mu_0 \right| \leq B_f \theta_f^n.$$

We will also use standard pairs and standard families on $\mathcal{M}$ instead of $\mathcal{M}_0$. If ℓ is a standard pair supported on the mth translational copy of the unit torus, then we write $[\ell] = m$.

2.3. **Statistical properties of the Lorentz process.** In [**Ch06**], Lemma 2 is used to prove the invariance principle for Lorentz processes of finite horizon. In particular, Lemma 5.4 in [**Ch06**] implies the following strengthening of [**BSC91**]

Lemma 3. *There is a positive constant $\sigma = \sigma(\mathcal{D})$ such that if $\mathcal{G}$ be a proper standard family and x is distributed according to $\mathcal{G}$, then, as $n \to \infty$, $\left(\frac{X_{\lfloor nt \rfloor}(x)}{\sqrt{n}} \right)_{t \in [0,1]}$ converges weakly to a Brownian motion with variance σ^2.*

It is simple to derive the following continuous time version of Lemma 3 (see for example Theorem 5 in [**DSzV09**]).

Lemma 4. *Let $\mathcal{G}$ be a proper standard family, x be distributed according to $\mathcal{G}$, and write $\hat{\sigma} = \hat{\sigma}(\mathcal{D}) = \sigma/\sqrt{\bar{\kappa}}$. Then, as $n \to \infty$, $\left(\frac{\hat{X}(tT)(x)}{\sqrt{T}} \right)_{t \in [0,1]}$ converges weakly to a Brownian motion with variance $\hat{\sigma}^2$.*

We will use the following result on moderate deviations (called Proposition 3.7 (d) in [**DSzV08**]).

Lemma 5. *Fix some $\delta > 0$. There are constants c_1, c_2 such that for any dynamically Hölder continuous function A, for any positive integer n, for any R with $1 < R < n^{1/6-\delta}$ and for any standard pair ℓ with $|\log length(\ell)| < n^{1/2-\delta}$,*

$$\mathbb{P}_\ell \left(\left| \sum_{j=0}^{n-1} A \circ \mathcal{F}_0^j(x) - n \int A d\mu_0 \right| > R\sqrt{n} \right) < c_1 e^{c_2 R^2}.$$

Finally, we need a technical estimate (Lemma 11.1 (c) in [**DSzV08**]).

Lemma 6. *There exists a constant C such that for any standard pair ℓ and for any positive integers n and K,*

$$\mathbb{P}_\ell\left(\tau_n < \tau^* \text{ and } \tau_n > Kn^2\right) < \frac{C|\log length(\ell)|}{K^{100}n}.$$

2.4. **Local limit theorem for Lorentz processes.** Here we present a variant of the local version of Lemma 3 (called local limit theorem for Lorentz processes).

For brevity, let us write $\varphi_\rho(x) = \frac{1}{\sqrt{2\pi}\rho}\exp(-\frac{x^2}{2\rho^2})$ for $\rho > 0$, and $\varphi = \varphi_1$. Further, if Σ is a positive definite matrix (of size 2×2 in our case), then $\varphi_\Sigma(x) = \frac{1}{2\pi\sqrt{\det(\Sigma)}}\exp(-\frac{x^T\Sigma^{-1}x}{2})$ for $x \in \mathbb{R}^2$.

Fix some x, y real numbers and some standard pair ℓ supported on the zeroth cell. With the notation introduced in (1) and (2), let us write ϑ_n for the distribution of

$$\left(\lfloor X_n(q, v) - x\sqrt{n}\rfloor, F_n(q, v) - n\bar{\kappa} - y\sqrt{n}, \mathcal{F}_0^n(q, v)\right),$$

where (q, v) is chosen with respect to ℓ. That is, ϑ_n is a measure on $\mathbb{Z}\times\mathbb{R}\times\mathcal{M}_0$.

We also fix the set $\mathcal{A} \subset \mathbb{Z} \times \mathbb{R} \times \mathcal{M}_0$ such that $(n, -t, (q, v)) \in \mathcal{A}$ if and only if $t \geq 0$, the configuration component of $\Phi^t(q + n, v)$ is in the zeroth cell, and $|\kappa(q, v)| > t$. That is, $\mathcal{A}$ contains the possible positions of the particle at the last collision time before time 0, when it arrives at the zeroth cell (and also the time spent after the last collision). Due to the finite horizon assumption $\mathcal{A}$ is bounded.

Lemma 7. *There exist some positive definite 2×2 matrix Σ with $\Sigma_{11} = \sigma^2$, and some finite constants C, C_1, C_2 such that for any standard pair ℓ with $|\log length(\ell)| < n^{1/4}$ the following hold uniformly.*

(a) *for any real numbers x, y,*

$$n\vartheta_n(\mathcal{A}) \to \bar{\kappa}\varphi_\Sigma(x, y),$$

as $n \to \infty$ uniformly for x, y chosen from a compact set.

(b) *for any real numbers x, y and any positive integer n,*

$$n\vartheta_n(\mathcal{A}) < C_1\varphi_{\Sigma'}(x, y) + C_2n^{-1/2},$$

where $\Sigma' = C\Sigma$.

Note that in Lemma 7 (a), we fix x and y and then let $n \to \infty$, while the estimate in Lemma 7 (b) is valid for every x, y, n. In particular, we will use Lemma 7 (b) with x or y being roughly of order $n^{0.1}$. In this case clearly $C_2n^{-1/2} \gg C_1\varphi_{\Sigma'}(x, y)$.

Lemma 7 (a) is related to the result of [**SzV04**] and to Proposition 3.7 (e) in [**DSzV08**]. The main difference is that here, we use an observable that involves the free flight time and we also take standard pairs as initial measures. The latter means that we compute probabilities involving the future conditioned on some event of small probability in the past. Lemma 7 (b) is

related to the last formula on page 834 in [**P09**]. The main difference is again the fact that we use standard pairs as initial measures. In Appendix A we review the results of [**SzV04**] for the reader's convenience and give a proof of Lemma 7.

2.5. **Local time.** Here we present limit theorems involving the local time at the origin.

Lemma 8. *Let $\mathcal{G}$ be a proper standard family supported on the zeroth cell and write L_k for the discrete time spent in the zeroth cell up to time k. If x is distributed according to $\mathcal{G}$, then*

$$\left(\frac{X_{\lfloor tn \rfloor}(x)}{\sqrt{n}}, \frac{L_{\lfloor tn \rfloor}(x)}{\sqrt{n}} \right)_{0 < t < 1}$$

jointly converges to the Brownian motion with variance σ^2 and its local time process at the origin.

Lemma 8 is proven for the invariant measure in Proposition 3 of [**NSz12**]. Its proof uses only the local limit theorem, which can be extended to proper standard families by Proposition 3.7 (e) in [**DSzV08**] (or by our Lemma 7). Hence the lemma holds in the generality stated above.

The above result obtains local time as the asymptotic number of collisions which occur in the zeroth cell. We can also count the continuous time. Namely, let $\hat{L}_k$ be the continuous time spent at the zeroth cell between the kth and the $(k+1)$st collisions.

Lemma 9. *Let $\mathcal{G}$ be a proper standard family supported on the zeroth cell. If x is distributed according to $\mathcal{G}$, then*

$$\left(\frac{X_{\lfloor tn \rfloor}(x)}{\sqrt{n}}, \frac{\sum_{k=0}^{\lfloor tn \rfloor - 1} \hat{L}_k(x)}{\sqrt{n}} \right)_{0 < t < 1}$$

jointly converges to the Brownian motion with variance σ^2 and $\bar{\kappa}$ times its local time process at the origin.

This lemma can be proven by the same argument used in [**NSz12**] to prove Lemma 8. Namely the proof proceeds by computing the moments of the local time using the representation $L_n = \sum_{i=0}^{n-1} \mathbf{1}_{X_i \in [0,1]}$ and the local limit theorem (which is finite-dimensional distribution version of Theorem 10 from Appendix A.1). The local limit theorem also says that conditioning on $X_{ns_1}, ..., X_{ns_k}$ being in zeroth cell the asymptotic distribution of $(\mathcal{F}_0^{ns_1}(x), \ldots, \mathcal{F}_0^{ns_k}(x))$ is μ_0^k (here, $n \to \infty$ while $s_1, ..s_k \in [0, 1]$ are fixed numbers). Thus

$$\mathbb{E}_{\mathcal{G}} \left(\prod_{i=1}^{k} \hat{L}_{ns_i} \right) = \mathbb{P}_{\mathcal{G}}(X_{ns_1}, ..., X_{ns_k} \in [0, 1]) \bar{\kappa}^k.$$

Here, we have used the fact that $(Counting \times Leb \times \mu_0)(\mathcal{A}) = \bar{\kappa}$, (see (54) in Appendix (A.2)). With the above observations, Lemma 9 can be proven in the same way as Lemma 8.

2.6. **Brownian meander.** Informally, the Brownian meander is a Brownian motion on $[0, 1]$ conditioned to stay strictly positive on $(0, 1]$.

A formal definition is the following. Consider the Wiener measure on $C[0, 1]$ conditioned on functions whose minimum is bigger than $-\varepsilon$. The weak limit of these measures as $\varepsilon \to 0$ exists and defines the process called Brownian meander (see [**DIM77**] for more details).

Let $\mathfrak{X}_\rho$ be a Brownian meander with variance ρ^2, and $\mathfrak{M}_\rho$ is its maximum (i.e. $\mathfrak{M}_\rho(t) = \max_{0<s<t} \mathfrak{X}_\rho(s)$) with respect to some abstract probability measure P. For simplicity, we omit the subscript when $\rho = 1$. The joint distribution function of a Brownian meander and its maximum is the following:
(5)

$$P(\mathfrak{X}(1) < x, \mathfrak{M}(1) < y) = \sum_{k=-\infty}^{\infty} \left[\exp\left(-(2ky)^2/2\right) - \exp\left(-(2ky + x)^2/2\right) \right],$$

for any $y \geq x \geq 0$ (see [**Ch76**]). In order to prove Theorem 2, we will need the density in the first coordinate, i.e. the following quantity:

$$\phi_\rho(x, y) = \lim_{dt \to 0} \frac{1}{dt} P(\mathfrak{X}_\rho(1) \in [x, x + dt], \mathfrak{M}_\rho(1) < y).$$

An elementary computation yields that for any $y \geq x \geq 0$,

(6) $$\phi_\rho(x, y) = \sum_{k=-\infty}^{\infty} \frac{2ky + x}{\rho^2} \exp\left(-\frac{(2ky + x)^2}{2\rho^2}\right).$$

3. Results

3.1. **Density profile.** In this section, we formulate our results precisely. First, we clarify how we emit the particles. Let us fix some proper standard family $\mathcal{G}$ on the phase space $\mathcal{M}_0$ to be the distribution of the particle at its first collision. Then at each time instant T_j of a Poisson point process on the time interval $[-T, 0]$ with intensity 1, we put a Lorentz particle with a position distributed as $\mu_{\mathcal{G}}$. These initial positions are independent (and the particles do not interact with each other). Obviously, not all the standard families are interesting, since for some, $X_1 < 0$ almost surely. Thus for the rest of this paper, we assume that

(7) $$\bar{c}(\mathcal{G}) = \lim_{n \to \infty} L\mathbb{P}_{\mathcal{G}}(\tau_L < \tau^*)$$

exists and is positive (all the admissible standard families satisfy this condition, see the remark after the proof of Lemma 11.2 in [**DSzV08**]). We also write

(8) $$c(\mathcal{G}) = \frac{2\bar{c}(\mathcal{G})\bar{\kappa}}{\sigma^2}.$$

In the case of the semi-infinite tube, the expected number of particles in the interval $[L, L + 1]$ at time 0 is

$$g_{L,T} = \int_0^T \mathbb{P}_{\mathcal{G}}(\hat{X}(t) \in [L, L + 1], \hat{\tau}^* > t)dt$$

Theorem 1. $\lim_{L\to\infty} \lim_{T\to\infty} g_{L,T} = c(\mathcal{G})$, where $c(\mathcal{G})$ is given by (8).

In the case of finite tube, we ask a similar question, namely the density of the particle profile. More precisely, we are interested in the following quantity

$$h_{x,L,T} = \int_0^T \mathbb{P}_{\mathcal{G}}(\hat{X}(t) \in I^{xL}, \min\{\hat{\tau}^*, \hat{\tau}_L\} > t)dt,$$

where $I^{x,L} = [\lfloor xL \rfloor, \lfloor xL \rfloor + 1]$, $L >> 1$ is the length of the tube and $0 < x < 1$.

We have the following

Theorem 2. For every $0 < x < 1$,

$$\lim_{L\to\infty} \lim_{T\to\infty} h_{x,L,T} = c(\mathcal{G})(1 - x),$$

where $c(\mathcal{G})$ is given by (8).

Finally, we describe the evolution of a density profile when starting from a smooth initial configuration. Namely, we take a Lorentz tube of length L with absorbing boundaries and inject particles with rate λ_0 and with initial measure $\mu_{\mathcal{G}_0}$ from the left end and with rate λ_1 and with initial measure $\mu_{\mathcal{G}_1}$ from the right end. We assume that $\mathcal{G}_i$ are proper standard families supported on $\mathcal{M}_0$ and $\mathcal{M}_L$ respectively. Write $f_i = \lambda_i c(\mathcal{G}_i)$ (where $c(\mathcal{G}_i)$ is given by (8)). Take a non-negative function $f \in C^2[0, 1]$ with $f(i) = f_i$ for $i = 0, 1$. At time zero, place an independent $Poi(f(k/L))$ amount of particles into $\mathcal{D}|_{[k,k+1]} \times \mathcal{S}^1$ to some positions chosen by Lebesgue measure for every positive integer k with $k < L$ and also start to emit particles from both ends as prescribed above. Let

$$u_L(t, x) = \mathbb{E}(\text{number of particles at time } tL^2 \text{ in cell } \lfloor xL \rfloor),$$

where $\mathbb{E}$ is the measure generated by the initial particles and the sources.

Theorem 3. The function $u(t, x) = \lim_{L\to\infty} u_L(t, x)$ is the solution of the heat equation with Dirichlet boundary condition, i.e.

$$u'_t(t, x) = \frac{\hat{\sigma}^2}{2} u''_{xx}(t, x), \quad u(0, x) = f(x), \quad u(t, 0) = f_0, \quad u(t, 1) = f_1.$$

We remark that in the case of Theorems 1, 2, and 3 the limiting distribution of the particles in the cell L (and xL) is Poissonian with the above computed parameter. Let us consider for example the setting of Theorem 1. Note that for any finite T and L, the distribution of particles which have not been absorbed by time 0 is Poissonian. Indeed the emitted particles (T_i, x_i) form a Poisson process of $\mathbb{M} = [-T, 0] \times \mathcal{D} \times \mathcal{S}^1$. Consider a function $\mathbb{G} : \mathbb{M} \to (\mathcal{D} \times \mathcal{S}^1) \cup \{\infty\}$ where $\mathbb{G}(t, x) = X(0)$ if the particle has not been absorbed by time 0 and $\mathbb{G}(t, x) = \infty$ otherwise. Combining the Mapping

and Restriction Theorems for Poisson processes (see [**Ki93**], Sections 2.2 and 2.3) we see that $\{\mathbb{G}(T_i, x_i)\}_{G(T_i,x_i)\neq\infty}$ form a Poisson process on $\mathcal{D} \times \mathcal{S}^1$. Since the expected number of particles in $[L, L + 1]$ converges as $L \to \infty, T \to \infty$ the limit process is also Poisson. Thus Theorems 1, 2, and 3 provide the complete description of limiting distribution. For example the weak Law of Large Numbers for Poisson processes with large intensity gives the following.

Corollary 4. *Fix $0 < \beta < 1$. In the setting of Theorem 3 let $N(t, x, L)$ denote the number of particles with $|X(t) - tL| < L^\beta$ at time tL^2. Then $\frac{N(x,t,L)}{L^\beta} \to u(t, x)$ in probability as $L \to \infty$.*

3.2. **Convergence to Brownian meander.** In order to prove the above results, we need convergence to the Brownian meander, which precisely means the following.

Theorem 5. *The process $\left(\frac{\hat{X}(tT)}{\sqrt{T}}\right)_{0<t<1}$ with respect to the measure $\mathbb{P}_{\mathcal{G}}(.|\hat{\tau}^* > T)$ converges weakly to the Brownian meander with variance $\hat{\sigma}^2$.*

Note that the proof of Theorem 8 in [**DSzV08**] implies that there exists some constant $c_1(\mathcal{G}) > 0$ with

$$(9) \qquad \mathbb{P}_{\mathcal{G}}(\tau^* > N) \sim c_1(\mathcal{G})/\sqrt{N}.$$

Let

$$(10) \qquad \hat{c}_1(\mathcal{G}) = c_1(\mathcal{G})\sqrt{\kappa}.$$

The following corollaries will be derived from Theorem 5 in Sections 5.2 and 5.3 respectively.

Corollary 6. *Recalling (7) we have*

$$c_1(\mathcal{G}) = \bar{c}(\mathcal{G})\frac{\sqrt{2}}{\sqrt{\pi}\sigma}.$$

Corollary 7.

$$\mathbb{P}_{\mathcal{G}}(\hat{\tau}^* > t) \sim \hat{c}_1(\mathcal{G})/\sqrt{t}.$$

3.3. **Local Limit Theorems.** In order to prove Theorems 2, and 3 we will need several new local limit theorems for the Lorentz particle in the infinite tube. For this, recall the notation $\phi_s(x, y)$, $\mathfrak{X}$ and $\mathfrak{M}$ from Section 2.6.

Proposition 1. *Fix some $x < y$ positive real numbers. Then*

$$T\mathbb{P}_{\mathcal{G}}\left(\lfloor\hat{X}(T)\rfloor = \lfloor x\sqrt{T}\rfloor, \forall t, 0 < t < T, \hat{X}(t) \in [0, y\sqrt{T}]\right) \to \hat{c}_1(\mathcal{G})\phi_{\hat{\sigma}}(x, y),$$

as $T \to \infty$. Furthermore, for any δ, the convergence is uniform for x, y such that $\delta < x < x + \delta < y < 1/\delta$.

Proposition 2. *Fix real numbers x, y in $(0, 1)$ and $t \in \mathbb{R}_+$. Let $\mathcal{G}$ be a proper standard family such that on $\mathcal{G}$ $\lfloor \hat{X}(0) \rfloor = \lfloor xL \rfloor$. Then*

$$L^2 \mathbb{P}_{\mathcal{G}} \left(\lfloor \hat{X}(tL^2) \rfloor = \lfloor yL \rfloor, \forall s, 0 < s < t, \hat{X}(sL^2) \in [0, L] \right) \to \psi(t, x, y)$$

as $L \to \infty$ where $\psi(t, x, y)$ is the density at y of a Brownian motion at time t which is started from x and killed at 0 and 1. Furthermore, for any δ, the convergence is uniform for $x, y \in [\delta, 1 - \delta]$ and $\delta \leq t \leq 1/\delta$.

Proposition 3. *Fix some real number x. Then*

$$\sqrt{T} \mathbb{P}_{\mathcal{G}} \left(\lfloor \hat{X}(T) \rfloor = \lfloor x\sqrt{T} \rfloor \right) \to \varphi_{\hat{\sigma}}(x),$$

as $T \to \infty$. Furthermore, the convergence is uniform for x chosen from some compact set.

4. Proof of Theorem 1

4.1. **Proof for discrete time.** Here, we prove Theorem 1 without using Brownian meanders (but using Lemma 8). In Remark 8 we will sketch another argument using Brownian meanders but not using Lemma 8. For brevity, we will write $I = [L, L + 1]$. By Fubini's theorem,

$$g_{L,T} = \mathbb{E}_{\mathcal{G}} \int_0^{(-T) \wedge \hat{\tau}^*} \mathbf{1}(X(t) \in I) dt.$$

Thus by monotone convergence,

$$\lim_{T \to -\infty} g_{L,T} = \mathbb{E}_{\mathcal{G}} \int_0^{\hat{\tau}^*} \mathbf{1}(X(t) \in I) dt.$$

In order to prove that this is convergent as $L \to \infty$, let us switch to discrete time first, and prove that the following limit exists

$$(11) \qquad \lim_{L \to \infty} \mathbb{P}_{\mathcal{G}}(\tau_L < \tau^*) \mathbb{E}_{\mathcal{G}}(\#\{k < \tau^* : X_k \in I\} | \tau_L < \tau^*) = c'(\mathcal{G}).$$

Observe that due to our basic assumption (7), in order to prove (11), it suffices to verify

$$(12) \qquad \mathbb{E}_{\mathcal{G}}(\#\{k < \tau^* : X_k \in I\} | \tau_L < \tau^*) = c_B L(1 + o(1)).$$

To establish (12), write

$$\mathbb{E}_{\mathcal{G}}(\#\{k < \tau^* : X_k \in I\} | \tau_L < \tau^*)$$
$$= \sum_{m=1}^{\infty} \mathbb{E}_{\mathcal{G}}(\#\{k < \tau^* : X_k \in I\} \mathbf{1}(m = \tau_L) | \tau_L < \tau^*)$$
$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{\alpha \in J_{m,n}} c_\alpha \mathbb{E}_{\ell_\alpha}(\#\{k < \tau^* : X_k \in I\}),$$

where $\ell_\alpha = (\gamma_\alpha, \rho_\alpha)$ is a standard pair in the Lth copy of the unit torus ($[\ell_\alpha] = L$) and $\text{length}(\ell_\alpha) \in [2^{-n}, 2^{-n-1})$, if $\alpha \in J_{m,n}$. We have by definition

$$\text{(13)} \qquad \sum_{m=1}^\infty \sum_{n=1}^\infty \sum_{\alpha \in J_{m,n}} c_\alpha = 1.$$

The growth lemma implies that

$$\text{(14)} \qquad \sum_{n>N} \sum_{\alpha \in J_{m,n}} c_\alpha < C 2^{-N} L$$

holds uniformly in m. Indeed, the term 2^{-N} comes from by the growth lemma and since we condition on $\{\tau_L < \tau^*\}$ (which has probability of order $1/L$), we have a factor of L on the right hand side. Similarly, Lemma 6 implies

$$\text{(15)} \qquad \sum_{m>KL^2} \sum_n \sum_{\alpha \in J_{m,n}} c_\alpha < C K^{-100} L.$$

We will need the following lemma.

Lemma 10. *There is a constant c_B and a sequence η_L with $\eta_L/L \to 0$ such that for any standard pair ℓ with $[\ell] = L$ and $|\log \text{length}(\ell)| < \sqrt{L}$,*

$$|\mathbb{E}_\ell(\#\{k < \tau^* : X_k \in I\}) - c_B L| < \eta_L + C |\log \text{length}(\ell)|.$$

For any standard pair ℓ with $[l] = L$ and $|\log \text{length}(\ell)| > \sqrt{L}$,

$$\mathbb{E}_\ell(\#\{k < \tau^* : X_k \in I\}) < C(L + |\log \text{length}(\ell)|).$$

First, we prove that (11) follows from Lemma 10.
Observe that Lemma 10 implies

$$|\mathbb{E}_{\ell_\alpha}(\#\{k < \tau^* : X_k \in I\}) - c_B L| < Cn + o(L) = o(L)$$

uniformly for $\alpha \in J_{m,n}$ with $n < \sqrt{L}$. Similarly, for $\alpha \in J_{m,n}$ with arbitrary m and n,

$$|\mathbb{E}_{\ell_\alpha}(\#\{k < \tau^* : X_k \in I\}) - c_B L| < C(L + n).$$

Using (13) we conclude that in order to prove (12), it suffices to show

$$\text{(16)} \qquad \sum_m \sum_{n>\sqrt{L}} (n + L) \sum_{\alpha \in J_{m,n}} c_\alpha = o(L).$$

(16) follows by an elementary computation. Namely, (14) implies

$$\sum_{m<1.99\sqrt{L}} \sum_{n>\sqrt{L}} (n+L) \sum_{\alpha \in J_{m,n}} c_\alpha = o(1)$$

and

$$\sum_{m>1.99\sqrt{L}} \sum_{n>\log m/\log 1.99} (n+L) \sum_{\alpha \in J_{m,n}} c_\alpha = o(1).$$

On the other hand (15) implies

$$\sum_{n>\sqrt{L}} (n+L) \sum_{m>1.99^n} \sum_{\alpha \in J_{m,n}} c_\alpha = o(1).$$

Thus, assuming Lemma 10 we have proved (16) and finished the proof of (11).

Proof of Lemma 10. Write $\ell = (\gamma, \rho)$ and assume first that length$(\ell) > \delta$ with some fixed δ. Note that Lemma 8 implies that

$$\frac{\#\{k < \tau^* : X_k \in I\}}{L}$$

converges weakly to a limit distribution ξ, when the initial measure is ℓ. Here, ξ is the local time of a Brownian motion of variance σ^2 at 1 up to its first hitting of the origin assuming that it starts from 1. However, we need to prove that the expectations also converge. To this end, choose $K >> 1$ and observe that

$$\mathbb{E}_\ell \left(\frac{\#\{k < \tau^* \wedge KL^2 : X_k \in I\}}{L} \right) \to E(\xi_K),$$

as $L \to \infty$, where ξ_K is defined in a similar way as ξ except for 'the first hitting of the origin' being replaced by 'the minimum of the first hitting of the origin and K'. We also have

(17) $$\lim_K E(\xi_K) = E(\xi) = c_B.$$

It remains to prove that

(18) $$\limsup_L \frac{1}{L} \mathbb{E}_\ell \left(1_{\tau^* > KL^2} \#\{k : KL^2 < k < \tau^*, X_k \in I\} \right)$$

is small if K is big.

In order to do that, we need one more lemma.

Fix a standard pair ℓ' in the zeroth cell with $\lim_{L\to\infty} L\mathbb{P}_{\ell'}(\tau_{-L} < \tau^*) > 0$. Then there is a rectangle $\mathfrak{R}$ fully crossed by ℓ' and a constant c such that for any standard pair ℓ'' fully crossing $\mathfrak{R}$ and any L, we have $L\mathbb{P}_{\ell''}(\tau_{-L} < \tau^*) > c$ (see the Appendix of [**Ch06**]).

Now for any $\ell'' = (\gamma'', \rho'')$ and any x in γ'', write ν_k for the kth return to I and $\bar{n}$ for the first such time when the curve in $\mathcal{F}^{\nu_{\bar{n}}}\gamma''$ containing $\mathcal{F}^{\nu_{\bar{n}}}x$ fully crosses $\mathfrak{R} + L$ (i.e. the translated copy of $\mathfrak{R}$ to the Lth cell). Finally, let us write $\bar{n}_0 = 0$, $\bar{n}_1(x) = \bar{n}$ and $\bar{n}_k(x) = \bar{n}_1(\mathcal{F}^{\nu_{\bar{n}_{k-1}}}x)$.

Lemma 11. *There are constants C, C' and $\theta < 1$ such that for any standard pair ℓ'',*

$$\mathbb{P}_{\ell''}(\bar{n} - C|\log \text{length}(\ell'')| > n) < C'\theta^n.$$

Lemma 11 is almost the same as Lemma 11 in [**DSzV09**]. The only difference is that in [**DSzV09**] the curve containing $\mathcal{F}^{\nu_{\bar{n}}}x$ can be anywhere in I as long as it has length at least δ_0. The iterated version of that Lemma (via the coupling algorithm of [**Ch06**], as it was also pointed out in [**DSzV09**]) proves

our Lemma 11.

Now we apply Lemma 11 to those standard pairs in the standard family $\mathcal{F}^{KL^2}\ell$, which have not visited the zeroth cell yet. Let ℓ'' be such standard pair. Then we have

$$\mathbb{E}_{\ell''}(\#\{k < \tau^* : X_k \in I\}) - c|\log \operatorname{length}(\ell'')|$$

$$(19) \quad \leq \mathbb{E}_{\ell''}\left(\sum_{j=1}^{\infty}(\bar{n}_j - \bar{n}_{j-1})\mathbf{1}_{\{\tau^* > \nu_{\bar{n}_{j-1}}\}}\right) = \sum_{j=1}^{\infty}\mathbb{E}_{\ell''}\left((\bar{n}_j - \bar{n}_{j-1})\mathbf{1}_{\{\tau^* > \nu_{\bar{n}_{j-1}}\}}\right)$$

Now for any j we can consider Markov decomposition at time $\nu_{\bar{n}_{j-1}}$. Every standard pair in this decomposition is longer than a uniform δ by the definition of $\bar{n}$. Thus we can apply Lemma 11 and can also neglect the term $C|\log \operatorname{length}(\ell'')|$. It is not hard to show that if the function $\bar{n}$ satisfies $\mathbb{P}(\bar{n} > n) < C\theta^n$, then there is a universal constant C such that $\int_A \bar{n}d\mathbb{P} < C[\mathbb{P}(A)]^{0.9}$ for every set A. Thus (19) is bounded by

$$C\sum_{j=1}^{\infty}\left(\mathbb{P}_{\ell''}(\tau^* > \nu_{\bar{n}_{j-1}})\right)^{0.9} \leq C\sum_{j=1}^{\infty}\left(1 - \frac{c}{L}\right)^{0.9j} < CL.$$

Next,

$$\mathbb{E}_{\ell}\left(1_{\tau^* > KL^2}\#\{k : KL^2 < k < \tau^*, X_k \in I\}\right) \leq L\mathbb{P}(\tau^* > KL^2) + \mathbb{E}_{\ell}\left(\ln r_{F^{KL^2}\ell}(x)\right).$$

The first term is $o_{K\to\infty}(L)$ since $\mathbb{P}(\tau^* > KL^2) \to 0$ as $K \to \infty$. On the other hand the second term is $O(1)$ due to the Growth Lemma. This proves Lemma 10 if $\operatorname{length}(\ell) > \delta$.

In the general case let $\hat{n}(x)$ be the first time when $F_0^{\hat{n}}(x)$ belongs to a component which is longer than δ. We then split all visits to the zeroth cell into visits before and after $\hat{n}$. The later are estimated the same way as above. The former contribute at most $\mathbb{E}_{\ell}(\hat{n}) \leq C|\log(\operatorname{length}(\ell))|$ proving Lemma 10 in the general case. $\qquad \square$

Finally, we identify the constant in the limit. Let us denote a standard two dimensional Brownian motion by $W(t)$. Also write $\mathcal{L}_{\varrho}^a(T)$ for the local time at position a up to the first hitting of the origin of a one dimensional Brownian motion with variance ϱ^2 starting from a. Thus with the notation in (17), we have

$$c_B = \mathbb{E}(\xi) = \mathbb{E}\left(\mathcal{L}_{\sigma}^1(T)\right) = \frac{1}{\sigma}\mathbb{E}\left(\mathcal{L}_1^{1/\sigma}(T)\right).$$

Observe that due to the Ray-Knight theorem (see [**R63**] and [**Kn63**]), we have

$$c_B = \frac{1}{\sigma}\mathbb{E}\|W(1/\sigma)\|^2 = \frac{2}{\sigma^2}.$$

Thus for the constant defined in (11), we have $c'(\mathcal{G}) = 2\bar{c}(\mathcal{G})/\sigma^2$.

4.2. **Proof for continuous time.** Our proof for the case of continuous time is similar to the proof in Subsection 4.1. Thus we only highlight the differences.

Recall the notation $\hat{L}_k$ introduced in Section 2.3. Note that in order to finish the proof of Theorem 1, it suffices to verify the following analogue of (12)

$$(20) \qquad \mathbb{E}_{\mathcal{G}}\left(\sum_{k<\tau^*} \hat{L}_k | \tau_L < \tau^*\right) = \bar{\kappa}\frac{2}{\sigma^2}L(1+o(1)).$$

Indeed, (20) and the computations in Subsection 4.1 yield

$$\lim_{L\to\infty}\lim_{T\to\infty} g_{L,T} = \lim_{L\to\infty}\mathbb{E}_{\mathcal{G}}\int_0^{\hat{\tau}^*} \mathbf{1}(X(t)\in I)dt =$$

$$\lim_{L\to\infty}\mathbb{P}_{\mathcal{G}}(\tau_L < \tau^*)\mathbb{E}_{\mathcal{G}}\left(\sum_{k<\tau^*} \hat{L}_k|\tau_L<\tau^*\right) = \lim_{L\to\infty}\frac{\bar{c}(\mathcal{G})}{L}\bar{\kappa}\frac{2}{\sigma^2}L(1+o(1)) = c(\mathcal{G}).$$

The proof of (20) is similar to that of (12) except that Lemma 10 should be replaced by the following

Lemma 12. *There is a sequence η_L with $\eta_L/L \searrow 0$ such that for any standard pair ℓ with $[\ell] = L$ and $|\log\text{length}(\ell)| < \sqrt{L}$,*

$$\left|\mathbb{E}_\ell\left(\sum_{k<\tau^*} \hat{L}_k|\tau_L<\tau^*\right) - \bar{\kappa}\frac{2}{\sigma^2}L\right| < \eta_L + C|\log\text{length}(\ell)|.$$

For any standard pair ℓ with $[\ell] = L$ and $|\log\text{length}(\ell)| > \sqrt{L}$,

$$\mathbb{E}_\ell\left(\sum_{k<\tau^*} \hat{L}_k|\tau_L<\tau^*\right) < C(L+|\log\text{length}(\ell)|).$$

The proof of Lemma 12 follows the same lines as the proof of Lemma 10 except that instead of referring to Lemma 8 we use Lemma 9. This completes the proof of Theorem 1.

5. BROWNIAN MEANDER AS A LIMIT

5.1. **Proof of Theorem 5.** First, we prove the theorem for discrete time, i.e. the statement that $\left(\frac{X_{\lfloor tN\rfloor}}{\sqrt{N}}\right)_{0<t<1}$ with respect to the measure $\mathbb{P}_{\mathcal{G}}(.|\tau^* > N)$ converges weakly to a Brownian meander.

Let us begin with a lemma. Let τ_{-L} denote the first time the particle reaches $-L$ for the system in the doubly infinite tube without the absorption at the origin.

Lemma 13. *There exist some constants $\theta < 1$ and $C < \infty$, such that for $K \leq n^{10}$ and for a proper standard family $\mathcal{G}$, with n large enough,*

$$\mathbb{P}_{\mathcal{G}}\left(\min\{\tau_n, \tau_{-n}\} > Kn^2\right) \leq \theta^K + \frac{CK}{n^{1000}}.$$

For $K \geq n^{10}$ and K large enough,

$$\mathbb{P}_{\mathcal{G}}\left(\min\{\tau_n, \tau_{-n}\} > Kn^2\right) \leq \theta^{K^{0.8}} + \frac{C}{K^{99}}.$$

Proof. To prove the first statement it suffices to show that if ℓ is a standard pair with length$(\ell) > n^{-1000}$ then

$$\mathbb{P}_{\ell}\left(\min\{\tau_n, \tau_{-n}\} > Kn^2\right) \leq \theta^K + \frac{CK}{n^{1000}}.$$

We prove this by induction on K. For $K = 1$, the statement is true due to the invariance principle for Lorentz process (Lemma 3). Here θ is the probability that the maximum of a Brownian motion up to time 1 is smaller than 1. To apply Lemma 3 we use the fact that by Lemma 1 the image of $\mathbb{P}_{\ell}$ becomes proper after $\bar{K} \log N$ iterations while due to finite horizon property the particle travels distance $O(\log N)$ during the time $\bar{K} \log N$.

Assume that the statement is true for some K. Then with the notation

$$\mathbb{P}_{\mathcal{G}}\left(\min\{\tau_n, \tau_{-n}\} > (K+1)n^2\right) =$$
$$\mathbb{P}_{\mathcal{G}}\left(\min\{\tau_n, \tau_{-n}\} > Kn^2\right) \mathbb{P}_{\mathcal{G}}\left(\min\{\tau_n, \tau_{-n}\} > (K+1)n^2 \mid \min\{\tau_n, \tau_{-n}\} > Kn^2\right)$$
$$= I * II,$$

I is estimated by the inductive hypothesis. In order to bound II we use the Markov decomposition at time Kn^2. For standard pairs which are longer than n^{-1000}, we simply use the statement for $K = 1$ while the contribution of the short pairs is estimated by Lemma 1. We obtain

$$I * II < \left(\theta^K + \frac{CK}{n^{1000}}\right)\theta + \frac{C'}{n^{1000}} < \theta^{K+1} + \frac{C(K+1)}{n^{1000}},$$

assuming that C is large enough.

To prove the second statement we use the first one with $n_{new} = K^{0.1}$ and $K_{new} = K^{0.8}$. Thus

$$\mathbb{P}_{\mathcal{G}}\left(\min\{\tau_n, \tau_{-n}\} > K\right) < \mathbb{P}_{\mathcal{G}}\left(\min\{\tau_{K^{0.1}}, \tau_{K^{0.1}}\} > K\right)$$
$$= \mathbb{P}_{\mathcal{G}}\left(\min\{\tau_{n_{new}}, \tau_{-n_{new}}\} > K_{new}n_{new}^2\right) < \theta^{K^{0.8}} + \frac{C}{K^{99}}. \quad \square$$

Lemma 14. *For any $\varepsilon > 0$, with N large enough, we have*

$$\mathbb{P}_{\mathcal{G}}\left(\tau_{\varepsilon\sqrt{N}} > \varepsilon N \mid \tau^* > N\right) < C\theta^{1/\varepsilon}$$

Proof. We have

$$\mathbb{P}_{\mathcal{G}}\left(\tau_{\varepsilon\sqrt{N}} > \varepsilon N, \tau^* > N\right) <$$
$$\mathbb{P}_{\mathcal{G}}\left(\tau^* > \varepsilon N/2\right) \mathbb{P}_{\mathcal{G}}\left(\min\{\tau_{\varepsilon\sqrt{N}}, \tau^*\} > \varepsilon N \mid \tau^* > \varepsilon N/2\right) = I * II.$$

I is bounded by $c/\sqrt{\varepsilon N}$ by (9). In order to estimate II, we use Markov decomposition at time $\varepsilon N/2$ and the first part of Lemma 13 to conclude

$$II < \frac{1}{N^{100}} + \sum_\alpha c_\alpha \mathbb{P}_{\ell_\alpha} \left(\min\{\tau_{\varepsilon\sqrt{N}}, \tau_{-\varepsilon\sqrt{N}}\} > \frac{\varepsilon N}{2} \right)$$

$$< \quad \frac{1}{N^{100}} + \theta'^{\frac{1}{2\varepsilon}} + \frac{c}{\varepsilon^{1001} N^{500}} < \theta^{1/\varepsilon},$$

where the ℓ_α's are those standard pairs in the $\varepsilon N/2$-fold iterate of $\mathcal{G}$, which are longer than N^{-100} (or more precisely, their shifted version to the zeroth cell). The statement follows. $\square$

We are now ready to prove the discrete time version of Theorem 5. Namely, let us fix some distance in the space of probability measures on $C([0,1])$. Take a small δ. Choose ε so that $C\theta^{1/\varepsilon} < \delta$ and such that the Brownian Motion started from ε and conditioned on not hitting 0 before time 1 is δ-close in distribution to the Brownian meander. Then by Lemma 14 there is a set $\mathbb{P}(\cdot|\tau^* > N)$ measure at least $1 - \delta$ where $\tau_{\varepsilon\sqrt{N}} < \varepsilon N$. If x is in this set and $t > \varepsilon N$ then we can write

$$\frac{X_{\lfloor tN \rfloor}}{\sqrt{N}} = \frac{X_{\lfloor tN \rfloor - \tau_{\varepsilon\sqrt{N}}}(x_{\tau_{\varepsilon\sqrt{N}}})}{\sqrt{N}}$$

and observe that by the invariance principle for the Lorentz process the distribution of the RH'S is close to the distribution of the Brownian Motion started from ε. Applying the conditioning we obtain that the distribution of $\frac{X_{\lfloor tN \rfloor}}{\sqrt{N}}$ under $\mathbb{P}(\cdot|\tau^* > N)$ is close to the distribution of the Brownian meander.

The extension of the convergence to continuous time is straightforward. The finite horizon condition implies that the time needed for the first εN collisions is bounded by $\kappa_{\max}\varepsilon N$. In the discrete time interval $[\varepsilon N, N]$ we used the invariance principle for Lorentz process (Lemma 3); now we can apply its continuous time counterpart (Lemma 4). Thus we have finished the proof of Theorem 5.

5.2. **Proof of Corollary 6.** Let us write

$$A_N = \{\tau^* > N\} \text{ and } B_{N,\varepsilon} = \{\tau_{\varepsilon\sqrt{N}} < \tau^*\}.$$

Using Lemma 14 we conclude that

$$\lim_{N \to \infty} \mathbb{P}_\mathcal{G}(A_N | B_{N,\varepsilon})$$

is asymptotic (as $\varepsilon \to 0$) to the probability that the minimum of a Brownian motion of variance σ^2 up to time 1 is bigger than $-\varepsilon$. Thus an elementary computation shows

$$\lim_{\varepsilon \to 0} \lim_{N \to \infty} \sqrt{N}\mathbb{P}_\mathcal{G}(A_N \cap B_{N,\varepsilon}) = \lim_{\varepsilon \to 0} \lim_{N \to \infty} \sqrt{N}\mathbb{P}_\mathcal{G}(B_{N,\varepsilon})\mathbb{P}_\mathcal{G}(A_N | B_{N,\varepsilon})$$

$$(21) \quad = \quad \lim_{\varepsilon \to 0} \frac{\bar{c}(\mathcal{G})}{\varepsilon} \frac{\sqrt{2}}{\sqrt{\pi}\sigma} \varepsilon = \bar{c}(\mathcal{G}) \frac{\sqrt{2}}{\sqrt{\pi}\sigma}.$$

On the other hand, the definition of $c_1(\mathcal{G})$ and Theorem 5 imply that

$$
\begin{aligned}
\lim_{\varepsilon \to 0} \lim_{N \to \infty} \frac{\mathbb{P}_{\mathcal{G}}(A_N \cap B_{N,\varepsilon})}{c_1(\mathcal{G})/\sqrt{N}} &= \lim_{\varepsilon \to 0} \lim_{N \to \infty} \frac{\mathbb{P}_{\mathcal{G}}(A_N \cap B_{N,\varepsilon})}{\mathbb{P}_{\mathcal{G}}(A_N)} \\
&= \lim_{\varepsilon \to 0} \lim_{N \to \infty} \mathbb{P}_{\mathcal{G}}(B_{N,\varepsilon}|A_N) = 1.
\end{aligned}
$$

The statement follows.

5.3. **Proof of Corollary 7.** Analogously to the proof of Corollary 6, let

$$
A = \{\tau^* > T\}, \quad B = \{\hat{\tau}_{\frac{\varepsilon\sqrt{T}}{\kappa_{\max}}} < \hat{\tau}^*\} \text{ and } C = \{\hat{\tau}_{\frac{\varepsilon\sqrt{T}}{\kappa_{\max}}} < \varepsilon T\}.
$$

By the definition of $\kappa_{\min}$ and $\kappa_{\max}$ and by Lemma 14, we have

$$
\mathbb{P}_{\mathcal{G}}(\bar{C}|A) = \frac{\mathbb{P}_{\mathcal{G}}\left(\hat{\tau}_{\frac{\varepsilon\sqrt{T}}{\kappa_{\max}}} > \varepsilon T \text{ and } \hat{\tau}^* > T\right)}{\mathbb{P}_{\mathcal{G}}(\hat{\tau}^* > T)} \leq \frac{\mathbb{P}_{\mathcal{G}}\left(\tau_{\frac{\varepsilon\sqrt{T}}{\kappa_{\max}}} > \frac{\varepsilon T}{\kappa_{\max}} \text{ and } \tau^* > \frac{T}{\kappa_{\max}}\right)}{\mathbb{P}_{\mathcal{G}}\left(\tau^* > \frac{T}{\kappa_{\min}}\right)}
$$

$$
= \mathbb{P}_{\mathcal{G}}\left(\tau_{\frac{\varepsilon\sqrt{T}}{\kappa_{\max}}} > \frac{\varepsilon T}{\kappa_{\max}} \middle| \tau^* > \frac{T}{\kappa_{\max}}\right) \sqrt{\frac{\kappa_{\max}}{\kappa_{\min}}}(1 + o_T(1)) \leq C\theta^{1/\varepsilon}.
$$

Since $\mathbb{P}_{\mathcal{G}}(ABC) = \mathbb{P}_{\mathcal{G}}(AC)$, we conclude

$$
\lim_{\varepsilon \to 0} \lim_{T \to \infty} \frac{\mathbb{P}_{\mathcal{G}}(ABC)}{\mathbb{P}_{\mathcal{G}}(A)} = 1. \tag{22}
$$

Now we can use Markov decomposition at $\tau_{\frac{\varepsilon\sqrt{T}}{\kappa_{\max}}}$ and Lemma 4 to deduce the following analogue of (21):

$$
\lim_{\varepsilon \to 0} \frac{\kappa_{\max}}{\varepsilon} \lim_{T \to \infty} \mathbb{P}_{\mathcal{G}}(A|BC) = \frac{\sqrt{2}}{\sqrt{\pi}\frac{\sigma}{\sqrt{\bar{\kappa}}}}. \tag{23}
$$

Notice that by Lemma 6 we have

$$
\lim_{\varepsilon \to 0} \lim_{T \to \infty} \mathbb{P}_{\mathcal{G}}(C|B) = 1. \tag{24}
$$

Since by definition

$$
\lim_{\varepsilon \to 0} \lim_{T \to \infty} \frac{\varepsilon\sqrt{T}}{\kappa_{\max}} \mathbb{P}_{\mathcal{G}}(B) = \bar{c}(\mathcal{G}),
$$

we can finish the proof by combining (24), (23), (22) and Corollary 6.

6. PROOFS OF THE LOCAL LIMIT THEOREMS.

Here we prove Proposition 1. The proofs of Proposition 2 and Proposition 3 are similar but easier so we leave them to the reader.

6.1. **Upper bound.** First, we prove the upper bound. The strategy of our proof is the following. First, we write

$$(25) \qquad N = \frac{T}{\bar{\kappa}}, \quad N_1 = (1 - \delta_t)\frac{T}{\bar{\kappa}}.$$

We choose δ_t, δ_s small positive numbers, and chop the interval $[0, y\sqrt{N\bar{\kappa}}]$ into pieces of length $\delta_s y\sqrt{N\bar{\kappa}}$. Using Theorem 5, we can estimate the probability of arriving into one of these intervals at time $N_1 = (1 - \delta_t)N$. For the upper bound, we simply omit the condition that the particle should stay in the interval $[0, y\sqrt{N}]$ between time $(1 - \delta_t)N$ and T. Fix a large constant A. We expect that typically there are n collisions with

$$(26) \qquad n \in \mathcal{I} = [\delta_t T/\bar{\kappa} - A\sqrt{T}, \delta_t T/\bar{\kappa} + A\sqrt{T}] \cap \mathbb{N}$$

between discrete time N_1 and continuous time T. The contribution of n's chosen from $\mathcal{I}$ can be computed with Lemma 7 (a). The contribution of n's from $\mathbb{N} \setminus \mathcal{I}$ is small, which can be verified by using Lemma 7 (b).

We use the following simple property of Brownian meanders proven in Appendix B.

Lemma 15. *The Brownian meander satisfies the following.*

$$(27) \qquad \phi_{\hat{\sigma}}(x, y) = \lim_{\delta_t \to 0} \lim_{\delta_s \to 0} \sum_{h=1}^{\lfloor 1/\delta_s \rfloor} \wp_{1,h} \wp_{2,h},$$

where

$$\wp_{1,h} = P\left(\mathfrak{X}_{\hat{\sigma}}(1 - \delta_t) \in [hy\delta_s, (h+1)y\delta_s], \mathfrak{M}_{\hat{\sigma}}(1 - \delta_t) \le y\right),$$

$$\wp_{2,h} = \varphi_{\hat{\sigma}\sqrt{\delta_t}}(x - y_h)$$

and $y_h \in [hy\delta_s, (h+1)y\delta_s]$ is arbitrary.

Let us fix some small $\varepsilon > 0$, choose small positive numbers δ_t, δ_s (to be specified later) and write $\{\ell_{h,\alpha}\}_{\alpha \in \mathfrak{A}_h}$ for the set of standard pairs in $\mathcal{F}_*^{N_1}\mathcal{G}$ satisfying

$$\tau^*(x) > N_1 \text{ and } [\ell_{h,\alpha}] \in [h\delta_s y\sqrt{N\bar{\kappa}}, (h+1)\delta_s y\sqrt{N\bar{\kappa}}].$$

By Theorem 5, we have the Markov decomposition

$$\mathbb{P}_{\mathcal{G}}(x \in \mathcal{F}^{-N_1}\mathcal{B} \text{ and } \tau^*(x) > N_1) = \sum_{h=1}^{1/\delta_s} \sum_{\alpha \in \mathfrak{A}_h} c_{h,\alpha} \ell_{h,\alpha}(\mathcal{B}),$$

where $\mathcal{B} \subset \mathcal{M}$ is measurable and $\sum_{\alpha \in \mathfrak{A}_h} c_{h,\alpha}$ is asymptotic to

$$(28) \qquad \frac{\sqrt{\bar{\kappa}}c_1(\mathcal{G})}{\sqrt{T}} P\left(\mathfrak{X}_{\sigma}(1 - \delta_t) \in \left[h\delta_s y\sqrt{\bar{\kappa}}, (h+1)\delta_s y\sqrt{\bar{\kappa}}\right], \mathfrak{M}_{\sigma}(1 - \delta_t) < y\sqrt{\bar{\kappa}}\right)$$

for every h. Since $\mathfrak{X}_\sigma / \sqrt{\bar\kappa}$ has the same distribution as $\mathfrak{X}_{\hat\sigma}$, we conclude that with the notation of Lemma 15,

$$\sum_{\alpha \in \mathfrak{A}_h} c_{h,\alpha} \sim \frac{1}{\sqrt{T}} \sqrt{\bar\kappa} c_1(\mathcal{G}) \wp_{1,h}. \tag{29}$$

Now let us fix some standard pair $\ell_{h,\alpha} = (\gamma_{h,\alpha}, \rho_{h,\alpha})$. We want to compute the probability of arriving in $[\lfloor x\sqrt{T} \rfloor, \lfloor x\sqrt{T} \rfloor + 1]$ at continuous time T assuming that at discrete time N_1 the point is distributed according to $\ell_{h,\alpha}$. Clearly, we will need to control the continuous time spent during discrete time N_1. Thus let us write

$$f_{h,\alpha} = \sum_{i=0}^{N_1-1} |\kappa\left(\mathcal{F}^{-i}(q,v)\right)|$$

with some fixed $(q,v) \in \gamma_{h,\alpha}$. Even though $f_{h,\alpha}$ depends on the choice of (q,v), in order to keep notation simple, we pretend it does not and explain at the end of the proof how the argument should be modified to treat non-constant $f_{h,\alpha}$. Observe that by Lemma 5, the complement of the event

$$|f_{h,\alpha} - \bar\kappa N_1| = |f_{h,\alpha} - (1 - \delta_t)T| < N^{0.6} \tag{30}$$

has superpolynomially small $\mathbb{P}_\mathcal{G}$-probability. Thus we can assume that (30) is true.

By the growth lemma, we can also neglect the contribution of standard pairs $\ell_{h,\alpha}$ with

$$|\log \text{length}(\ell_{h,\alpha})| > N^{1/4}. \tag{31}$$

Thus we can assume that Lemma 7 is applicable to $\ell_{h,\alpha}$. Since $f_{h,\alpha}$ is not exactly equal to $(1 - \delta_t)T$, we need to adjust the definition of $\mathcal{I}$. Namely, let us write

$$\mathcal{I}_{T,h,\alpha} = [(T - f_{h,\alpha})/\bar\kappa - A\sqrt{T}, (T - f_{h,\alpha})/\bar\kappa + A\sqrt{T}] \cap \mathbb{N} \tag{32}$$

Now by Lemma 7 (a), for every $n \in \mathcal{I}_{T,h,\alpha}$ with the notation $n = \lfloor (T - f_{h,\alpha})/\bar\kappa \rfloor + m$, we have

$$q_{T,h,\alpha,n} := \mathbb{P}_{\ell_{h,\alpha}}\left((X_n - x\sqrt{T} + [\ell_{h,\alpha}], F_n - T + f_{h,\alpha}, \mathcal{F}_0^n(q,v)) \in \mathcal{A} \right)$$

$$\sim \frac{\bar\kappa}{n} \varphi_\Sigma \left(\frac{x\sqrt{T} - [\ell_{h,\alpha}]}{\sqrt{n}}, \frac{m\bar\kappa}{\sqrt{n}} \right). \tag{33}$$

Note that by (30),

$$\min_h \min_{\alpha \in \mathfrak{A}_h} \mathcal{I}_{T,h,\alpha}$$

tends to infinity at a linear speed with T. Thus Lemma 7 a also implies that the convergence in (33) is **uniform** in h, α satisfying (30) and (31) and

$n \in \mathcal{I}_{T,h,\alpha}$. Also, we have

$$(34) \qquad \sqrt{n} \sim \sqrt{\frac{\delta_t T}{\bar{\kappa}}}$$

uniformly for h, α and $n \in \mathcal{I}_{T,h,\alpha}$. Hence with the notation

$$y_{h,\alpha} = \frac{[\ell_{h,\alpha}]}{\sqrt{T}} \in [h\delta_s y, (h+1)\delta_s y],$$

we also have

$$(35) \qquad \frac{x\sqrt{T} - [\ell_{h,\alpha}]}{\sqrt{n}} \sim \sqrt{\bar{\kappa}} \frac{x - y_{h,\alpha}}{\sqrt{\delta_t}}$$

uniformly for h, α. Thus summing up the estimation in (33) for $n \in \mathcal{I}_{T,h,\alpha}$, substituting a Riemann sum with the integral and using (35), we obtain that

$$\sum_{n \in \mathcal{I}_{T,h,\alpha}} q_{T,h,\alpha,n} \sim \frac{\bar{\kappa}^2}{\delta_t \sqrt{T}} \int_{-A}^{A} \varphi_\Sigma \left(\sqrt{\bar{\kappa}} \frac{x - y_{h,\alpha}}{\sqrt{\delta_t}}, \frac{\bar{\kappa}^{3/2}}{\sqrt{\delta_t}} y \right) dy$$

uniformly for h, α. With the notation of Lemma 15, by choosing $y_h = y_{h,\alpha}$, we have

$$\wp_{2,h} = \frac{\sqrt{\bar{\kappa}}}{\sigma\sqrt{2\pi\delta_t}} \exp \left(-\frac{\bar{\kappa}(x - y_{h,\alpha})^2}{2\sigma^2 \delta_t} \right).$$

Thus for any fixed positive numbers $\varepsilon, \delta_t, \delta_s$, by choosing a large $A = A(\varepsilon, \delta_t, \delta_s)$, we conclude

$$(36) \qquad \left| \sum_{n \in \mathcal{I}_{T,h,\alpha}} q_{T,h,\alpha,n} - \frac{1}{\sqrt{T}} \wp_{2,h} \right| < \frac{\delta_s \varepsilon}{\sqrt{T}}$$

for T large enough (uniformly in h, α).

Now, we want to bound

$$(37) \qquad T\mathbb{P}_{\mathcal{G}} \left(\lfloor \hat{X}(T) \rfloor = \lfloor x\sqrt{T} \rfloor, \forall t, 0 < t < T, \hat{X}(t) \in [0, y\sqrt{T}] \right)$$

from above by

$$(38) \qquad T \sum_{h=1}^{1/\delta_s} \sum_{\alpha \in \mathfrak{A}_h} c_{h,\alpha} \sum_{n \in \mathcal{I}_{T,h,\alpha}} q_{T,h,\alpha,n}.$$

Performing the summation over h, using (29), (36) and Lemma 15, we conclude that (38) is close to

$$c_1(\mathcal{G})\sqrt{\bar{\kappa}}\phi_{\hat{\sigma}}(x, y) = \hat{c}_1(\mathcal{G})\phi_{\hat{\sigma}}(x, y).$$

(Here $\hat{c}_1$ is defined by (10). See also Corollary 7.) More precisely, the closeness means ε-closeness when $\delta_t = \delta_t(\varepsilon)$, $\delta_s = \delta_s(\delta_t, \varepsilon)$, $A = A(\delta_s, \delta_t, \varepsilon)$, $T_0 = T_0(A, \delta_s, \delta_t, \varepsilon)$ are chosen appropriately and $T > T_0$.

In order to conclude the upper bound, we need to check two technical details which we treat in two separate Lemmas.

Lemma 16. *Given ε there exist constants A and T_0 such that if $T \geq T_0$ then the contribution of $n \notin \mathcal{I}_{T,h,\alpha}$ is bounded by $\varepsilon/\sqrt{T}$.*

Proof. Clearly, for $n < n_1 = \delta_t T/(2\kappa_{\max})$ and for $n > n_2 = 2\delta_t T/\kappa_{\min}$ we have $q_{T,h,\alpha,n} = 0$. Applying Lemma 5 to the function $|\kappa|$, we conclude that the contribution of indices $n \in [n_1, n_2]$ with

$$|n - (T - f_{h,\alpha})/\bar\kappa| > T^{0.6}$$

is bounded from above by a superpolynomial term:

$$\sum_{n:n_1<n<n_2,|n-(T-f_{h,\alpha})/\bar\kappa|>T^{0.6}} q_{T,h,\alpha,n} < CTe^{-cT^{0.2}}.$$

For the remaining n's, we will use Lemma 7 (b). Because of symmetry reasons, we only need to compute the contribution of

$$n \in \mathcal{I}'_{T,h,\alpha} = [(T - f_{h,\alpha})/\bar\kappa + A\sqrt{T}, (T - f_{h,\alpha})/\bar\kappa + T^{0.6}] \cap \mathbb{N}.$$

Thus, with the notation $n = \lfloor (T - f_{h,\alpha})/\bar\kappa \rfloor + m$, we have

$$\sum_{n\in\mathcal{I}'_{T,h,\alpha}} q_{T,h,\alpha,n}$$

(39)
$$< \sum_{n\in\mathcal{I}'_{T,h,\alpha}} \frac{C_1}{n}\varphi_{\Sigma'}\left(\frac{x\sqrt{T} - [\ell_{h,\alpha}]}{\sqrt{n}}, \frac{m\bar\kappa}{\sqrt{n}}\right) + \frac{C_2}{n^{3/2}}$$

Since (34) and (35) hold uniformly for $n \in \mathcal{I}'_{T,h,\alpha}$, we conclude that there are some positive finite constants $c = c(\delta_t), C_i = C_i(\delta_t)$ for $i = 3, 4, 5$ such that (39) is bounded by

$$\frac{C_3}{T}\left(\sum_{m=A\sqrt{T}}^{T^{0.6}} \exp(-cm^2/T)\right) + C_4 T^{-0.9}$$

$$< \frac{C_3}{T}\left(\sum_{m=A\sqrt{T}}^{\infty} \left(\exp\left(-\frac{c}{\sqrt{T}}\right)\right)^m\right) + C_4 T^{-0.9}$$

$$< \frac{C_3}{T}\exp(-cA)\frac{1}{1 - \exp\left(-\frac{c}{\sqrt{T}}\right)} + C_4 T^{-0.9} < \frac{C_5 e^{-cA}}{\sqrt{T}}$$

for T large enough. Thus by choosing $A = A(\varepsilon, \delta_t)$ large enough we can guarantee $C_5 e^{-cA} < \varepsilon$. $\qquad\square$

Lemma 17. *The above argument remains valid for (q, v)-dependent $f_{h,\alpha}$*

Proof. Note that by the Hölder continuity of $|\kappa|$, for every $\bar\epsilon > 0$ there exists some $\delta > 0$ such that

$$\text{dist}((q, v), (q', v')) < \delta \text{ implies } |f_{h,\alpha}(q, v) - f_{h,\alpha}(q', v')| < \bar\epsilon.$$

For any given $\delta > 0$ we can chop the standard pairs to smaller pieces by introducing artificial singularities so that any standard pair is shorter than δ.

Thus taking the real $f_{h,\alpha}(q, v)$ instead of the constant $\bar{f}_{h,\alpha}$ in (33), we have

$$
\mathbb{P}_{\ell_{h,\alpha}}\left((X_n - x\sqrt{T} + [\ell_{h,\alpha}], F_n - T + \bar{f}_{h,\alpha}, \mathcal{F}_0^n(q, v)) \in \mathcal{A}_{\bar{\epsilon}}\right)
$$

$$
(40) \quad \leq \quad \mathbb{P}_{\ell_{h,\alpha}}\left((X_n - x\sqrt{T} + [\ell_{h,\alpha}], F_n - T + f_{h,\alpha}(q, v), \mathcal{F}_0^n(q, v)) \in \mathcal{A}\right)
$$

$$
\leq \quad \mathbb{P}_{\ell_{h,\alpha}}\left((X_n - x\sqrt{T} + [\ell_{h,\alpha}], F_n - T + \bar{f}_{h,\alpha}, \mathcal{F}_0^n(q, v)) \in \mathcal{A}^{\bar{\epsilon}}\right),
$$

where

$$
\begin{aligned}
\mathcal{A}_{\bar{\epsilon}} &= \{(x, y, \omega) : \forall y', |y - y'| < \bar{\epsilon}, (x, y', \omega) \in \mathcal{A}\} \\
\mathcal{A}^{\bar{\epsilon}} &= \{(x, y, \omega) : \exists y', |y - y'| < \bar{\epsilon}, (x, y', \omega) \in \mathcal{A}\}.
\end{aligned}
$$

Thus applying the Local limit theorem for $\mathcal{A}_{\bar{\epsilon}}$ and $\mathcal{A}^{\bar{\epsilon}}$, and using the fact that $\partial \mathcal{A}$ has zero measure we see that the for T large enough, the ratio of (40) and (33) is in $[1 - \varepsilon, 1 + \varepsilon]$ (by choosing $\bar{\varepsilon} = \bar{\varepsilon}(\varepsilon)$ and $\delta = \delta(\bar{\varepsilon}, \varepsilon)$ small enough). With this adjustment, one can apply the above argument for (q, v)-dependent $f_{h,\alpha}$. $\qquad\square$

6.2. **Lower bound.** We use the notation of Subsection 6.1. Note that our previous argument for the upper bound was in fact an asymptotic equality except for one point: when we substituted (37) by (38). Thus the lower bound (and hence Proposition 1) will be established whenever we prove the following statement.

For every $\varepsilon > 0$ there exist $\delta_t = \delta_t(\varepsilon)$, and $T_0 = T_0(\delta_t)$ such that for every $T > T_0$ and for every h and α,

$$
(41) \quad \mathbb{P}_{\ell_{h,\alpha}}\left(\lfloor \hat{X}(T - f_{h,\alpha}) \rfloor = \lfloor x\sqrt{T} \rfloor, \exists s < T - f_{h,\alpha} : \hat{X}(s) \notin [0, y\sqrt{T}]\right)
$$

$$
< \varepsilon/\sqrt{T}
$$

In the remaining part of the subsection we prove (41).

Let us fix some $\ell_{h,\alpha} = (\gamma_{h,\alpha}, \rho_{h,\alpha})$. In order to keep the notation simple, we will discard the subscript and simply write $\ell = (\gamma, \rho) = \ell_{h,\alpha} = (\gamma_{h,\alpha}, \rho_{h,\alpha})$, $f = f_{h,\alpha}$.

Let us denote by $\tilde{n}_1$ the smallest integer (a random variable w.r.t. ℓ) such that at time $N_1 + \tilde{n}_1$ the particle is outside of the tube segment $[0, y\sqrt{T}]$. Let us write

$$
\mathbb{Q}_\ell(.) = \mathbb{P}_\ell\left(. \,\Big|\, \sum_{i=0}^{\tilde{n}_1 - 1} |\kappa \circ \mathcal{F}^i| < T - f\right),
$$

i.e. $\mathbb{Q}_\ell$ is the conditional probability under the condition that the particle leaves the tube segment $[0, y\sqrt{T}]$ before continuous time T. We have the

Markov decomposition at time $\tilde{n}_1$

$$\mathbb{Q}_\ell(\mathcal{F}^{\tilde{n}_1}(q,v) \in \mathcal{B}) = \sum_{\beta \in \mathfrak{B}} c_\beta \ell_\beta(\mathcal{B}).$$

Let us write $\mathcal{T}_\beta$ for the remaining continuous time until time T. More precisely, observe that for fixed β, for every $(q,v) \in l$ with $\mathcal{F}^{\tilde{n}_1}(q,v)$ being on the standard pair ℓ_β, $\tilde{n}_1$ is the same. Thus using that common $\tilde{n}_1$, we can write

$$\mathcal{T}_\beta = T - \sum_{i=0}^{N_1+\tilde{n}_1-1} |\kappa \circ \mathcal{F}^{-i}(q,v)|$$

with some $(q,v) \in \ell_\beta$. This definition depends slightly on the choice of (q,v), but for simplicity, we will ignore this issue (similarly to $T - f_{h,\alpha}$ in Subsection 6.1 - but this case is simpler since we only need to prove that (41) is small thus we can enlarge $\mathcal{A}$ instead of proving the analogue of Lemma 17). Clearly the event $\tilde{n}_1 < (T-f)/\kappa_{\min}$ has full $\mathbb{Q}_\ell$ probability, thus the growth lemma implies

$$\sum_{\beta: |\log \text{length}(\ell_\beta)| > T^{1/8}} c_\beta < \frac{CT \exp(-cT^{1/8})}{\mathbb{P}_\ell(\text{the particle leaves } [0, y\sqrt{T}])}.$$

Since we want to prove that (41) is less than $\varepsilon/\sqrt{T}$, we can clearly neglect the contribution of standard pairs ℓ_β with $|\log \text{length}(\ell_\beta)| > T^{1/8}$. In particular, we can assume that all of our standard pairs are long enough in the sense that $|\log \text{length}(\ell_\beta)| < (\sqrt{T})^{1/4}$ thus Lemma 7 and Lemma 5 are applicable with $n \geq \sqrt{T}$. Finally note that by definition $[\ell_\beta]$ is $\kappa_{\max}$-close to either $\lfloor y\sqrt{T} \rfloor$ or -1.

When estimating the probability

$$(42) \qquad \mathbb{P}_{\ell_\beta}\left(\lfloor \hat{X}(\mathcal{T}_\beta) \rfloor = \lfloor x\sqrt{T} \rfloor \right)$$

we distinguish two cases.

Case 1 $\mathcal{T}_\beta < T^{0.99}$

In this case we estimate a probability of a very unlikely event. Whence it is enough to estimate the 'global probability' instead of its local version. Namely, we can use Lemma 5. Note that if $\mathcal{T}_\beta < \frac{\min\{x, y-x\}}{\kappa_{\min}}\sqrt{T}$, then the probability we are computing is zero. Thus we can assume that the number of collisions before time $\mathcal{T}_\beta$ is bigger than $c\sqrt{T}$.

Let us write $n_0 = T^{0.995}$. Note that it is impossible to have $n > n_0$ collisions during continuous time $\mathcal{T}_\beta$ due to the finite horizon condition. If there are n collisions with $c\sqrt{T} < n < n_0$ before time $\mathcal{T}_\beta$, then it is very unlikely that the particle travels distance $\min\{x, y-x\}\sqrt{T}$ in discrete time n. Thus we can

bound the probability in (42) by

$$\sum_{n=c\sqrt{T}}^{n_0} \mathbb{P}_{\ell_\beta}(X_n > C\sqrt{T}) = \sum_{n=c\sqrt{T}}^{n_0} \mathbb{P}_{\ell_\beta}\left(X_n > C\sqrt{n}\sqrt{\frac{T}{n}}\right),$$

which is bounded by

$$\sum_{n=c\sqrt{T}}^{n_0} C\exp(-c\frac{T}{n}) < CT^{0.995}\exp(-cT^{0.005})$$

due to Lemma 5.

Case 2 $T^{0.99} < \mathcal{T}_\beta < \delta_t T$ where δ_t is from (25).
Similarly to the estimations in Subsection 6.1, we write

$$\mathcal{I}_{T,\beta} = [\mathcal{T}_\beta/\bar{\kappa} - \sqrt{T}, \mathcal{T}_\beta/\bar{\kappa} + \sqrt{T}] \cap \mathbb{N}$$

and use Lemma 7 (b) to derive that for every $n \in \mathcal{I}_{T,h,\alpha}$ with the notation $n = \lfloor (T - f_{h,\alpha})/\bar{\kappa} \rfloor + m$, we have

$$q_{T,\beta,n} := \mathbb{P}_{\ell_\beta}\left((X_n - x\sqrt{T} + [\ell_\beta], F_n - \mathcal{T}_\beta, \mathcal{F}_0^n(q,v)) \in \mathcal{A}\right) <$$

$$< \frac{C_1}{n}\varphi_{\Sigma'}\left(\frac{x\sqrt{T} - [\ell_\beta]}{\sqrt{n}}, \frac{m\bar{\kappa}}{\sqrt{n}}\right) + \frac{C_2}{n^{3/2}}.$$

Note that we also have

$$\left|\frac{x\sqrt{T} - [\ell_\beta]}{\sqrt{n}}\right| > \frac{\sqrt{\bar{\kappa}}\min\{x, y-x\}}{2}\sqrt{\frac{T}{\mathcal{T}_\beta}}.$$

Thus by simply using $\varphi_{\Sigma'}(x,y) < C\exp(-cx^2)$, we obtain

$$\sum_{n\in\mathcal{I}_{T,\beta}} q_{T,\beta,n} < \frac{C\sqrt{T}}{\mathcal{T}_\beta}\exp\left(-c\frac{T}{\mathcal{T}_\beta}\right) + C_2\sqrt{T}\mathcal{T}_\beta^{-3/2}.$$

Since the function $x\exp(-cx)$ tends to zero as $x \to \infty$ and $\mathcal{T}_\beta \in [T^{0.99}, \delta_t T]$, we have

$$\sqrt{T}\sum_{n\in\mathcal{I}_{T,\beta}} q_{T,\beta,n} < C\frac{T}{\mathcal{T}_\beta}\exp\left(-c\frac{T}{\mathcal{T}_\beta}\right) + C_2 T^{1-0.99*3/2} < \varepsilon$$

assuming that $\delta_t = \delta_t(\varepsilon)$ is small enough and T is large enough.

For the estimation of the remaining possible collision numbers $n \notin I_{T,\beta}$ we essentially need to repeat the proof of Lemma 16. Namely, observe that $\mathcal{T}_\beta^{0.6} > \sqrt{T}$ and by using that $\varphi_{\Sigma'}(x,y) < C\exp(-cy^2)$ we can bound the contribution of the n's in

$$\mathcal{I}'_{T,\beta} = [\mathcal{T}_\beta/\bar{\kappa} + \sqrt{T}, \mathcal{T}_\beta/\bar{\kappa} + \mathcal{T}_\beta^{0.6}] \cap \mathbb{N}$$

by

$$\frac{C}{\mathcal{T}_\beta}\left(\sum_{m=\sqrt{T}}^{\mathcal{T}_\beta^{0.6}} \exp(-cm^2/\mathcal{T}_\beta)\right) + C\mathcal{T}_\beta^{0.6-3/2} < C\frac{1}{\sqrt{\mathcal{T}}_\beta}\exp\left(-c\frac{T}{\mathcal{T}_\beta}\right) + CT^{-0.99*9/10}$$

for T large enough. As before, this expression is less than $\varepsilon/\sqrt{T}$ for $\delta_t = \delta_t(\varepsilon)$ small and $T = T(\delta_t)$ large enough. Finally, the case

$$|n - \mathcal{T}_\beta/\bar{\kappa}| > \mathcal{T}_\beta^{0.6}$$

is treated exactly the same way as in Lemma 16. We have finished the proof of (41) and hence that of Proposition 1.

7. Proof of Theorem 2

Since under the condition that a particle does not return to the origin it still diffuses, we expect that the main contribution to $h_{x,L} = \lim_{T\to\infty} h_{x,L,T}$ comes from the time interval $[\delta t^2, t^2/\delta]$. Thus with the notation $I^{xL} = [\lfloor xL\rfloor, \lfloor xL\rfloor + 1]$, define

$$I_{x,L,\delta} = \int_{\delta L^2}^{L^2/\delta} \mathbb{P}_\mathcal{G}(\hat{X}(t) \in I^{xL}, \min\{\hat{\tau}^*, \hat{\tau}_L\} > t)dt.$$

Using Proposition 1 (with T, x and y being replaced by $t, xL/\sqrt{t}$ and $L/\sqrt{t}$, respectively), we obtain

$$(43) \qquad I_{x,\delta} = \lim_{L\to\infty} I_{x,L,\delta} = \lim_{L\to\infty} \hat{c}_1(\mathcal{G}) \int_{\delta L^2}^{L^2/\delta} \frac{1}{t}\phi_{\hat{\sigma}}\left(\frac{xL}{\sqrt{t}}, \frac{L}{\sqrt{t}}\right)dt$$

Writing $s = \frac{L}{\sqrt{t}}$ we have

$$(44) \qquad I_{x,\delta} = 2\hat{c}_1(\mathcal{G}) \int_{\sqrt{\delta}}^{1/\sqrt{\delta}} \frac{1}{s}\phi_{\hat{\sigma}}(xs, s)ds.$$

Substituting formula (6), we conclude

$$\begin{aligned}
I_{x,\delta} &= \frac{2\hat{c}_1(\mathcal{G})}{\hat{\sigma}^2} \int_{\sqrt{\delta}}^{1/\sqrt{\delta}} \sum_{k=-\infty}^{\infty} (2k+x) \exp\left(-\frac{(2k+x)^2 s^2}{2\hat{\sigma}^2}\right) ds \\
&= \frac{2\hat{c}_1(\mathcal{G})}{\hat{\sigma}^2} \sum_{k=-\infty}^{\infty} \int_{\sqrt{\delta}}^{1/\sqrt{\delta}} (2k+x) \exp\left(-\frac{(2k+x)^2 s^2}{2\hat{\sigma}^2}\right) ds.
\end{aligned}$$

In order to establish that the equilibrium profile is linear, it remains to prove two lemmas.

Lemma 18.

$$I_x := \lim_{\delta\to 0} I_{x,\delta} = \frac{\hat{c}_1(\mathcal{G})\sqrt{2\pi}}{\hat{\sigma}}(1-x).$$

This Lemma is proved in Appendix B.

Lemma 19.

(45) $$\lim_{\delta \to 0} \lim_{L \to \infty} \int_{[0,\delta L^2] \cup [L^2/\delta, \infty)} \mathbb{P}_\mathcal{G}(\hat{X}(t) \in I^{xL}, \min\{\hat{\tau}^*, \hat{\tau}_L\} > t) dt = 0$$

Proof. For the case $t \in [0, \delta L^2]$, let us write

$$\int_0^{\delta L^2} \mathbb{P}_\mathcal{G}(\hat{X}(t) \in I^{xL}, \min\{\hat{\tau}^*, \hat{\tau}_L\} > t) dt$$

(46) $$\leq \int_0^{\delta L^2} \mathbb{P}_\mathcal{G}(\tau_{\frac{xL}{2}} < \tau^*) \mathbb{P}_\mathcal{G}(\hat{X}(t) \in I^{xL} | \tau_{\frac{xL}{2}} < \tau^*) dt$$

We have $\mathbb{P}_\mathcal{G}(\tau_{\frac{xL}{2}} > \tau^*) = O(1/L)$ for fixed x by (7). On the other hand the argument used in Section 6.2 to prove (41) shows that for every given $x \in (0,1)$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that for large enough L and for any $t < \delta L^2$,

$$\mathbb{P}_\mathcal{G}(\hat{X}(t) \in I^{xL} | \tau_{\frac{xL}{2}} < \tau^*) < \frac{\varepsilon}{L}.$$

Substituting these estimations to (46), we obtain

$$\lim_{\delta \to 0} \lim_{L \to \infty} \int_0^{\delta L^2} \mathbb{P}_\mathcal{G}(\hat{X}(t) \in I^{xL}, \min\{\hat{\tau}^*, \hat{\tau}_L\} > t) dt = 0.$$

Next, consider the case of $t > L^2/\delta$. Similarly to the proof of Lemma 13, we get that for any t with $L^2/\delta < t < L^{12}$,

$$\mathbb{P}_\mathcal{G}\left(\min\{\hat{\tau}^*, \hat{\tau}_L\} > t/2\right) \leq \frac{C}{L}\left(\theta^{t/L^2} + \frac{Ct}{L^{1002}}\right).$$

Indeed, the term $1/L$ comes from the fact that $\hat{\tau}^* > L^2$, while the other term on the right hand side comes from the same argument as the proof of the first case of Lemma 13 with $n = L$ and $K = t/(2L^2) - 1$ (possibly with some different θ and C). Let us denote by $\tilde{n}$ the smallest k when $F_k > t/2$. Applying Markov decomposition at time $\tilde{n}$ and using Proposition 3 we conclude that there is some $\theta < 1$ and $C < \infty$ such that

$$\mathbb{P}_\mathcal{G}\left(\min\{\hat{\tau}^*, \hat{\tau}_L\} > t/2, \hat{X}(t) \in I^{xL}\right) \leq \frac{C}{L}\left(\theta^{t/L^2} + \frac{Ct}{L^{1002}}\right)\frac{1}{\sqrt{t}} \leq \frac{C}{L^2}\left(\theta^{t/L^2} + \frac{Ct}{L^{1002}}\right)$$

For $t > L^{12}$ we simply use the second part of Lemma 13 to conclude

$$\lim_{L \to \infty} \int_{L^2/\delta}^{\infty} \mathbb{P}_\mathcal{G}(\hat{X}(t) \in I^{xL}, \min\{\hat{\tau}^*, \hat{\tau}_L\} > t) dt$$

$$< \lim_{L \to \infty} \left(\frac{C}{L^2}\int_{L^2/\delta}^{L^{12}}\left(\theta^{t/L^2} + \frac{Ct}{L^{1002}}\right) dt + \int_{L^{12}}^{\infty}\left(\theta^{t^{0.8}/L^{1.6}} + \frac{CL^{198}}{t^{99}}\right) dt\right) < \theta^{1/\delta}.$$

The proof of Lemma 19 is complete. $\square$

The last step in the proof is the identification of the constant. Corollary 6 implies

$$\frac{\hat{c}_1(\mathcal{G})\sqrt{2\pi}}{\hat{\sigma}} = \frac{c_1(\mathcal{G})\sqrt{\bar{\kappa}}\sqrt{2\pi}}{\sigma/\sqrt{\bar{\kappa}}} = \frac{2\bar{c}(\mathcal{G})\bar{\kappa}}{\sigma^2} = c(\mathcal{G}).$$

Thus we have finished the proof of Theorem 2.

Remark 8. *The argument used in this section can be adapted to prove Theorem 1. Observe that the main contribution to $h_{x,L,T}$ and $g_{xL,T}$ comes from particles whose age is of order x^2L^2. If $x \ll 1$ then such particles do not have enough time to reach the L-th cell so that $h_{x,L,T} \approx g_{xL,T}$. One can make this argument rigorous by combining (18) with the argument of the present section thus obtaining another proof of Theorem 1 using Brownian meanders but not using Lemma 8. This also explains the fact that the constants appearing in Theorems 1 and 2 are the same.*

Remark 9. *Note that $c_1(\mathcal{E})$ is computed on page 277 of [**DSzV08**], where $\mathcal{E}$ is the special standard family for which $\mu_{\mathcal{G}} = \mu_0$. Using their formula and Corollary 6 we conclude that*

$$c(\mathcal{E}) = 2.$$

Note also that in the case of general $\mathcal{G}$, there is no explicit formula for $c(\mathcal{G})$.

8. Proof of Theorem 3

In this section, we prove Theorem 3. Let us write

$$
\begin{aligned}
u_L^\delta(t,x) &= \sum_{k=\delta L}^{(1-\delta)L} f(k/L)\mu_k(\lfloor \hat{X}(tL^2) \rfloor = \lfloor xL \rfloor, \hat{X}(s) \in [0,L] \text{ for } s \leq tL^2) \\
&+ \int_{\delta L^2}^{tL^2} \lambda_0 \mathbb{P}_{\mathcal{G}_0}(\lfloor \hat{X}(s) \rfloor = \lfloor xL \rfloor, \hat{X}(u) \in [0,L] \text{ for } u \leq s)ds \\
&+ \int_{\delta L^2}^{tL^2} \lambda_1 \mathbb{P}_{\mathcal{G}_1}(\lfloor \hat{X}(s) \rfloor = \lfloor -(1-x)L \rfloor, \hat{X}(u) \in [-L,0] \text{ for } u \leq s)ds
\end{aligned}
$$

where μ_k denotes the measure μ_0 shifted to the cell k.

Note that by definition,

$$u_L(t,x) = u_L^0(t,x)$$

For every fixed small positive δ, we can apply Propositions 1 and 2 (as in the derivation of (44)) to conclude

$$u^\delta(t,x) := \lim_{L\to\infty} u_L^\delta(t,x) = \int_\delta^{1-\delta} f(z)\psi(t,z,x)dz +$$

$$(47) \quad 2\lambda_0\hat{c}_1(\mathcal{G}_0)\int_{1/\sqrt{t}}^{1/\sqrt{\delta}} \frac{1}{s}\phi_{\hat{\sigma}}\left(xs,s\right)ds + 2\lambda_1\hat{c}_1(\mathcal{G}_1)\int_{1/\sqrt{t}}^{1/\sqrt{\delta}} \frac{1}{s}\phi_{\hat{\sigma}}\left((1-x)s,s\right)ds.$$

Applying Proposition 3 for the first term and Lemma 19 for the second and third terms, we conclude that (47) also holds for $\delta = 0$ (with the identification $1/0 = \infty$). Namely,

$$u(t, x) = \lim_{L \to \infty} u_L(t, x) = \int_0^1 f(z)\psi(t, z, x)dz +$$

$$2\lambda_0 \hat{c}_1(\mathcal{G}_0) \int_{1/\sqrt{t}}^\infty \frac{1}{s} \phi_{\hat{\sigma}}(xs, s)\, ds + 2\lambda_1 \hat{c}_1(\mathcal{G}_1) \int_{1/\sqrt{t}}^\infty \frac{1}{s} \phi_{\hat{\sigma}}((1-x)s, s)\, ds =: I_1 + I_2 + I_3.$$

We need to check that all the integrals I_1, I_2, I_3 satisfy the heat equation. It is a well known fact about Gaussian densities that

$$\frac{\partial}{\partial t}\psi(t, z, x) = \frac{\hat{\sigma}^2}{2}\frac{\partial^2}{\partial x^2}\psi(t, z, x)$$

Since $\psi(t, z, 0) = \psi(t, z, 1) = 0$, I_1 satisfies the heat equation of Theorem 3 with constant 0 boundary conditions. Due to symmetry reasons, it remains to apply the following result proven in Appendix B.

Lemma 20.

$$u(t, x) = 2\lambda_0 \hat{c}_1(\mathcal{G}_0) \int_{1/\sqrt{t}}^\infty \frac{1}{s} \phi_{\hat{\sigma}}(xs, s)\, ds$$

solves the following Cauchy problem for the heat equation

$$u_t'(t, x) = \frac{\hat{\sigma}^2}{2}u_{xx}''(t, x), \quad u(0, x) = 0, \quad u(t, 0) = f_0, \quad u(t, 1) = 0.$$

Appendix A. Proof of Lemma 7

A.1. **Local Limit Theorem of Szász and Varjú.** Before proving Lemma 7 we briefly summarize the main statement of [**SzV04**].

Take a bounded Hölder function $f : \mathcal{M}_0 \to \mathbb{R}^d$ (in our case, $d = 2$) and consider the smallest closed subgroup of $\mathbb{R}^d$ which supports the values of the function $f - r$ for some constant r. Denote this subgroup by $S(f)$. Let us also write $f \sim g$ if there exists some measurable h with $f - g = h - h \circ \mathcal{F}_0$ (that is, f and g are cohomologous). With the notation

$$M(f) = \cap_{g:g \sim f} S(g)$$

we say that the function f is minimal if $M(f) = S(f)$. We say that f is non-degenerate if $span(M(f)) = \mathbb{R}^d$. In this case, there exists some lattice $\mathcal{L}$ of dimension $d' \leq d$ such that $M(f)$ is isomorphic to $\mathcal{L} \times \mathbb{R}^{d-d'}$.

Fix some vector $k \in \mathbb{R}^d$ and a sequence $k_n \in S(f) + nr$ such that

$$(48) \qquad\qquad \left\| \frac{k_n - n\mu_0(f)}{\sqrt{n}} - k \right\| \to 0.$$

Choose the initial point $x \in \mathcal{M}_0$ according to the measure μ_0 and denote by υ_n the distribution of the triple

$$\left(x, \sum_{i=0}^{n-1} f \circ \mathcal{F}^i(x) - k_n, \mathcal{F}_0^n(x) \right).$$

Thus the measure υ_n is supported on $\mathcal{M}_0 \times S(f) \times \mathcal{M}_0$. Finally, we denote by U the uniform measure (i.e. product of counting and Lebesgue measures) on $S(f)$. Here U is normalized so that constant in this uniform measure is chosen in such a way that $U(B(R)) \sim \mathrm{Leb}(B(R))$ for large R (in order words, the product of the usual counting and Lebesgue measures is multiplied by $\mathrm{vol}(\mathbb{R}^{d'}/\mathcal{L})$).

Theorem 10. (*[**SzV04**]*) *Assume that the function f is minimal and non-degenerate. Then there exists some positive definite $d \times d$ matrix Σ_f such that $n^{d/2} \upsilon_n$ converges vaguely to*

$$\varphi_{\Sigma_f}(k) \mu_0 \times U \times \mu_0.$$

Furthermore, for any fixed sequence $\delta_n \searrow 0$ and compact set $\mathcal{K} \subset \mathbb{R}^d$ the above convergence is uniform in the choice of k_n and $k \in \mathcal{K}$ if the sequence in (48) is bounded by δ_n.

In the proof of Lemma 7, we will use certain constructions from the papers [**Ch06**], [**Ch07**], [**P09**] [**SzV04**] and [**Y98**] without giving the original details. Our proof consists of three major steps.

A.2. **Proof of Lemma 7 (a) for the invariant measure.** First, let us replace the standard pair ℓ by the measure μ_0 and prove the convergence

(49) $$n \vartheta_n(\mathcal{A}) \to \varphi_{\Sigma}(x, y) \mathbf{c}_{\mathcal{A}}.$$

with some constant $\mathbf{c}_{\mathcal{A}}$. We are going to apply Theorem 10. First, take the function

$$f(q, v) = (\psi(q, v), |\kappa(q, v)| - \bar{\kappa}),$$

where ψ is the discretized version of $\Pi \kappa$ and Π is the projection to the horizontal direction (exactly as in Section 5 of [**SzV04**]). Clearly, the smallest closed subgroup of $\mathbb{R}^2$ that supports the values of f is $\mathbb{Z} \times \mathbb{R}$. In order to apply Theorem 10, we need to check that the function f is minimal. Note that by Theorem 3.1 of [**SzV04**], there exists a minimal function in each cohomology class. In particular, there is some $\tilde{f} = (\tilde{f}_1, \tilde{f}_2) \sim f$ with $S(\tilde{f}) = M(f)$.
Also note that the billiard flow can be represented as a suspension flow over $(\mathcal{M}_0, \mathcal{F}_0)$ with roof function $|\kappa|$. With this identification, the usual notation for the phase space of the billiard flow is

$$\Omega = \{(x, t) : x \in \mathcal{M}_0, 0 \leq t < |\kappa(x)|\}.$$

It also makes sense to take $(x, t) \in \Omega$, where $t > |\kappa(x)|$ with the identification $(x, t) = (\mathcal{F}_0 x, t - |\kappa(x)|)$. With this notation the billiard flow $\Phi_{|\kappa|}^t$ acts on Ω

by $\Phi^t_{|\kappa|}(x, s) = (x, s + t)$ and preserves the measure $\mu_0 \times Leb$. We need the following result.

Lemma 21. *For arbitrary positive constant b the suspension flow $\Phi^t_{|\kappa|+b}$ over $(\mathcal{M}_0, \mathcal{F}_0)$ with roof function $|\kappa| + b$ is weak mixing.*

Proof. In fact, for billiard flows one knows much stronger result. Namely the flow enjoys stretched exponential decay of correlations [**Ch07**]. The proof of this fact given in [**Ch07**] relies only on the properties of so called temporal distance function. Namely given x and y such that both $v_1 = W^u_{loc}(x) \cap W^s_{loc}(y)$ and $v_2 = W^s_{loc}(x) \cap W^u_{loc}(y)$ exist one can define

$$\Delta_{|\kappa|}(x, y) = \sum_{n=-\infty}^{\infty} [|\kappa(\mathcal{F}^n_0 x)| + |\kappa(\mathcal{F}^n_0 y)| - |\kappa(\mathcal{F}^n_0 v_1)| - |\kappa(\mathcal{F}^n_0 v_2)|]$$

(to see that this series converges note that for $n \to +\infty$ $\mathcal{F}^n_0 x$ and $\mathcal{F}^n v_2$ as well as $\mathcal{F}^n_0 y$ and $\mathcal{F}^n v_1$ become exponentially close while for $n \to -\infty$ $\mathcal{F}^n_0 x$ and $\mathcal{F}^n v_1$ as well as $\mathcal{F}^n_0 y$ and $\mathcal{F}^n v_2$ become exponentially close). The proof of mixing of the special flow with roof function $|\kappa|$ depends on the estimates on oscillations of the $\Delta_{|\kappa|}(x, y)$ when x and y are close. Since $\Delta_{|\kappa|} = \Delta_{|\kappa|+b}$ the argument of [**Ch07**] works for roof function $|\kappa| + b$ as well. $\square$

Now we can prove the following

Lemma 22. *The function f is minimal, i.e. $M(f) = \mathbb{Z} \times \mathbb{R}$.*

Proof. We claim that if $M(f)$ is a proper subgroup of $\mathbb{Z} \times \mathbb{R}$ then there exist numbers α, r and measurable functions $h : \mathcal{M}_0 \to \mathbb{R}$ and $g : \mathcal{M}_0 \to \mathbb{Z}$ such that

$$(50) \qquad\qquad |\kappa|(x) = h(x) - h(\mathcal{F}_0 x) + r + \alpha g(x).$$

Consider first the case when $M(f)$ is one-dimensional. By Theorem 5.1 in [**SzV04**], ψ is minimal, hence the projection of $M(f)$ to the first coordinate is $\mathbb{Z}$. Therefore if $M(f)$ is one dimensional, then the projection of $M(f)$ to the second coordinate is a discrete subgroup. Let us denote it by $\mathcal{L} = \alpha\mathbb{Z}$. Clearly, $S(\tilde{f}_2) = \mathcal{L}$ and $\tilde{f}_2 \sim |\kappa|$ proving (50) in this case.

Next, consider the case when $M(f)$ is a two dimensional discrete subgroup of $\mathbb{Z} \times \mathbb{R}$. We claim that the generators of $M(f)$ can be chosen of the form

$$(51) \qquad\qquad (0, \alpha) \text{ and } (1, \beta).$$

Indeed let $e_1 = (m_1, \alpha_1)$ and $e_2 = (m_2, \alpha_2)$ be arbitrary generators. If either m_1 or m_2 is 0 we are done. Otherwise m_1 and m_2 need to be coprime since otherwise the projection of $M(f)$ to the first coordinate would be a proper subgroup of $\mathbb{Z}$. Thus we can take $e = m_2 e_1 - m_1 e_2$ as one of the generators and it is of the form $(0, \alpha)$. So if $\tilde{e} = (\tilde{m}, \tilde{b})$ is a second generator, then because ψ is minimal we must have $\tilde{m} = \pm 1$ and we can ensure $+$ sign replacing $\tilde{e}$

by $-\tilde{e}$ if necessary. (51) tells us that for some measurable functions h_1, h_2 we have

(52) $\quad (\psi, |\kappa|)(x) - (r_1, r_2) = m'(0, \alpha) + n'(1, \beta) + (h_1, h_2)(x) - (h_1, h_2)(\mathcal{F}_0 x).$

Taking the first component of (52) we obtain

$$n' = \psi(x) - r_1 - h_1(x) + h_1(\mathcal{F}_0 x).$$

Now the second component of (52) takes form

(53) $\qquad\qquad |\kappa(x)| - r - \beta\psi(x) = m'\alpha + \tilde{h}(x) - \tilde{h}(\mathcal{F}_0 x)$

where $\tilde{h} = h_2 - \beta h_1$. Let $x = (q, v)$ and $\mathcal{F}_0(x) = (q_1, v_1)$. Then (53) for the original and the time reversed orbits read

$$|\kappa(q, v)| - r - \beta\psi(q, v) = m'\alpha + \tilde{h}(q, v) - \tilde{h}(q_1, v_1) \text{ and}$$

$$|\kappa(q, v)| - r + \beta\psi(q, v) = m''\alpha + \tilde{h}(q_1, -v_1) - \tilde{h}(q, -v).$$

Adding them together we get (50) with $h(q, v) = \frac{1}{2}[\tilde{h}(q, v) + \tilde{h}(q_1, -v_1)]$.

We now show that (50) contradicts to Lemma 21. Let us define the subset

$$\mathcal{C}_\delta = \{(x, t), x \in \mathcal{M}_0, t \in [h(x) - \delta, h(x) + \delta]\} \subset \Omega.$$

$\mathcal{C}_\delta$ is measurable since h and κ are measurable. Observe that h is only defined up to an additive constant in (50). Clearly one can choose this constant in such a way that for any $\delta > 0$, $\mathcal{C}_\delta$ has a positive $\mu_0 \times Leb$-measure. Now choose some $(x, h(x)) \in \mathcal{C}_0$ and write

$$\varsigma(x) = \min_{s>0}\{\Phi^s_{|\kappa|}(x, t) \in \mathcal{C}_0\}$$

Using (50) we conclude

$$\varsigma(x) = -nr + \alpha \sum_{i=1}^n g(\mathcal{F}_0^{i-1}x),$$

where $n = n(x)$ is the number of hits of the roof before time $\varsigma(x)$. Let us choose a positive ε such that $r - \varepsilon$ is a rational multiple of α. Let us denote by $\mathcal{L}'$ the lattice generated by the numbers $r - \varepsilon$ and α and write b for the smallest positive element of $\mathcal{L}'$. Using the canonical embedding of Ω to the phase space of $\Phi_{|\kappa|+\varepsilon}$, we conclude that for any $(x, h(x)) \in \mathcal{C}_0$, the first return time to $\mathcal{C}_0$ with the dynamics $\Phi_{|\kappa|+\varepsilon}$ is in $\mathcal{L}'$. Thus, taking $\delta > 0$ smaller than $b/2$, we conclude that for every $t > 0$ with $dist(t, \mathcal{L}') > 2\delta$,

$$(\mu_0 \times Leb)(\mathcal{C}_\delta \cap \Phi^{-t}_{|\kappa|+\varepsilon}\mathcal{C}_\delta) = 0.$$

This contradicts Lemma 21. Thus f is minimal. $\qquad\qquad\square$

Now we apply Theorem 10 to conclude that (49) holds uniformly for x, y chosen from a compact set and

(54) $\qquad \mathbf{c}_{\mathcal{A}} = (Counting \times Leb \times \mu_0)(\mathcal{A}) = \int \int_0^{\kappa(x)} 1 dt d\mu_0(x) = \bar{\kappa}$

where the second identity follows by time reversal.

A.3. **Proof of Lemma 7 (a) for standard pairs.** In this subsection, we prove that

(55) $$n\vartheta_n(\mathcal{A}) \to \bar{\kappa}\varphi_\Sigma(x, y)$$

holds when the initial measure is some standard pair $\ell = (\gamma, \rho)$. For brevity, let us write ϑ_n^ν for the distribution of

$$(\lfloor X_n(q, v) - x\sqrt{n} \rfloor, F_n(q, v) - n\bar{\kappa} - y\sqrt{n}, \mathcal{F}_0^n(q, v))$$

when the initial measure is some ν.

Fix a small $\varepsilon > 0$. As it was proven in [**Y98**], there exists some set $\mathfrak{R} \subset \mathcal{M}_0$ such that

(T1) $\mathfrak{R}$ is in the domain $\mathfrak{Q}$ bounded by two stable and two unstable manifolds $(W_1^s, W_2^s, W_1^u, W_2^u)$,

(T2) $\mu_0(\mathfrak{R}) > (1 - \varepsilon)\mu_0(\mathfrak{Q})$

(T3) for every $x \in \mathfrak{R}$, the local stable and unstable manifolds through x exist and both of them fully cross $\mathfrak{Q}$ (i.e. $W^u(x) \cap W_i^s \neq \emptyset$ and $W^s(x) \cap W_i^u \neq \emptyset$ hold for $i = 1, 2$).

(T4) for every $x, y \in \mathfrak{R}$ there is a unique $z \in \mathfrak{R}$ with $z = W^u(x) \cap W^s(y)$.

(T5) The diameter of $\mathfrak{Q}$ is small enough so that both the ratio of the density of μ_0 at different points of $\mathfrak{Q}$ and the Jacobian of the holonomy map is in the interval $[1 - \varepsilon, 1 + \varepsilon]$.

(T6) $\mathfrak{R}$ satisfies all Young's axioms ((P1) - (P5) in [**Y98**], their precise formulation is not needed for our argument).

Namely, it is shown in [**Y98**] that one can construct $\mathfrak{R}$ and $\mathfrak{Q}$ so that (T1), (T3), (T4) and (T6) are satisfied and, moreover, the diameter of $\mathfrak{Q}$ can be taken arbitrary small. It remains to take $\mathfrak{Q}$ so small that (T2) and (T5) hold.

Let us fix this set $\mathfrak{R}$. Following the notation of [**Ch06**], we write

$$\mathfrak{S} = \cup_{x \in \mathfrak{R}} W^s(x).$$

Further, let us fix an unstable manifold γ^* that fully crosses $\mathfrak{R}$ and write $\pi : \mathfrak{S} \to \gamma^*$ with $\pi(x) = y$ if $x \in \mathfrak{S}$ and $y \in \gamma^*$ lie on the same stable manifold.

We claim that with the notation $\nu^B(.) = \nu(.|B)$, we have

(56) $$n\vartheta_n^{\pi_*\mu_0^{\mathfrak{R}}}(\mathcal{A}) \to \bar{\kappa}\varphi_\Sigma(x, y).$$

Indeed, Theorem 4.1 in [**SzV04**] (which is our Theorem 10) is intrinsically proven for the so-called expanding Young tower, which is constructed over $\mathfrak{R}$ by factorizing along the stable direction. Hence the measure for which Theorem 4.1 in [**SzV04**] is first obtained is $\pi_*\mu_0^{\mathfrak{R}}$, which combined with Section A.2 gives (56).

Now we have the following

Lemma 23. *For every $\bar{\varepsilon} > 0$ there is some $\varepsilon > 0$ such that if $\mathfrak{R}$ satisfies (T1)–(T6) then the following statement is true. For every standard pair $\ell' = (\gamma', \rho')$ with γ' fully crossing $\mathfrak{S}$, the density $\frac{d\rho'}{dLeb_{\gamma'}}$ is in the interval $[1 - \bar{\varepsilon}, 1 + \bar{\varepsilon}]$ and we have*

$$|n\vartheta_n^{(\rho')^{\mathfrak{S}}}(\mathcal{A}) - \bar{\kappa}\varphi_\Sigma(x, y)| < \bar{\varepsilon} \tag{57}$$

for n large enough.

Proof. In fact, the conditions (T2) and (T5) are imposed exactly in order to enable the argument below.

By definition the densities of standard pairs are uniformly Hölder continuous, thus for $\varepsilon > 0$ small enough, $\frac{d\rho'}{dLeb_{\gamma'}}$ is in the interval $[1 - \bar{\varepsilon}, 1 + \bar{\varepsilon}]$. Whence by choosing $\varepsilon > 0$ small, and using the definition of $\mathfrak{R}$ one easily concludes that the measures $\pi_* \rho'|_{\mathfrak{S}}$ and $\pi_* \mu_0^{\mathfrak{R}}$ are close to each other in the sense that their Radon-Nikodym derivative w.r.t each other are in the interval $[1 - \bar{\varepsilon}, 1 + \bar{\varepsilon}]$. The Hölder continuity of f and the fact that stable manifolds are exponentially contracted by $\mathcal{F}_0^n$ implies that we can choose a small $\varepsilon > 0$ such that for any integer N, for any $x \in \mathfrak{S}$, and $y = \pi x$, we have

$$|\sum_{i=1}^{N} f \circ \mathcal{F}^i(x) - \sum_{i=1}^{N} f \circ \mathcal{F}^i(y)| < \bar{\varepsilon}.$$

Thus enlarging $\mathcal{A}$ a little bit to $\mathcal{A}^{\bar{\varepsilon}}$, where

$$\mathcal{A}^{\bar{\varepsilon}} = \{(n, r, \omega) : \exists r', \omega', |r - r'| < \bar{\varepsilon}, \operatorname{dist}(\omega, \omega') < \bar{\varepsilon}, (n, r', \omega') \in \mathcal{A}\},$$

we have both

$$n\vartheta_n^{(\rho')^{\mathfrak{S}}}(\mathcal{A}) < (1 + \bar{\varepsilon})n\vartheta_n^{\pi_* \mu_0^{\mathfrak{R}}}(\mathcal{A}^{\bar{\varepsilon}}) \tag{58}$$

and

$$n\vartheta_n^{\pi_* \mu_0^{\mathfrak{R}}}(\mathcal{A}) < (1 + \bar{\varepsilon})n\vartheta_n^{(\rho')^{\mathfrak{S}}}(\mathcal{A}^{\bar{\varepsilon}}).$$

Finally, observe that by construction the $(Counting \times Leb \times \mu_0)$ -measure of $\partial\mathcal{A}$ is 0. The statement follows. $\qquad\square$

Now we can prove the convergence of $n\vartheta_n(\mathcal{A}) = n\vartheta_n^\ell(\mathcal{A})$.
The Appendix of [**Ch06**] implies the existence of a function

$$\Upsilon = (\Upsilon_1, \Upsilon_2) : \gamma \to (\mathbb{N} \cup \infty) \times \mathbb{N}$$

such that

- There exist universal constants $\varkappa$, C and $\theta < 1$ depending only on the geometry of the billiard such that

$$\rho(\omega : \Upsilon_1(\omega) > \varkappa |\log \operatorname{length}(l)| + N) < C\theta^N.$$

  In particular, $\rho(\omega : \Upsilon_1(\omega) = \infty) = 0$
- For any $\omega \in \gamma$ with $\Upsilon_1(\omega) < \infty$, $\mathcal{F}^{\Upsilon_1(\omega)}(\omega)$ lies on the translational copy of $\mathfrak{S}$ in the cell $\Upsilon_2(\omega)$.

- For any $\omega \in \gamma$ with $\Upsilon_1(\omega) < \infty$, let us write $\gamma'_\omega \subset \gamma$ for the smallest subcurve of γ which contains ω and $\mathcal{F}_0^{\Upsilon_1(\omega)}\gamma'_\omega$ fully crosses $\mathfrak{S}$. Then for $\omega' \in \gamma'_\omega$, the equation $\Upsilon_1(\omega') = \Upsilon_1(\omega)$ holds if and only if $\mathcal{F}^{\Upsilon_1(\omega)}\omega'$ lies on the translational copy of $\mathfrak{S}$ in the cell $\Upsilon_2(\omega)$. In this case, $\Upsilon_2(\omega') = \Upsilon_2(\omega)$.

The meaning of the function Υ is that for a point ω, the first n such that $\mathcal{F}_0^n\gamma$ fully crosses $\mathfrak{S}$ and $\mathcal{F}_0^n(\omega)$ lies on $\mathfrak{S}$ is Υ_1. But when we apply $\mathcal{F}^{\Upsilon_1(\omega)}$ instead of $\mathcal{F}_0^{\Upsilon_1(\omega)}$, the point ω arrives at some cell $\Upsilon_2(\omega)$. Also note that by construction $|\Upsilon_2(\omega)| < \kappa_{\max}\Upsilon_1(\omega)$ for every ω.

Now pick a large n and some standard pair $\ell = (\gamma, \rho)$ with $|\log \operatorname{length}(\ell)| < n^{1/4}$. For any $\omega \in \gamma$ with

$$(59) \qquad \Upsilon_1(\omega) < \varkappa n^{1/4} + n^{1/5}$$

we want to apply Lemma 23 to the measure

$$\left(\mathcal{F}_0^{\Upsilon_1(\omega)}\right)_* \rho|_{\{\omega' \in \gamma'_\omega \text{ such that } \Upsilon_1(\omega')=\Upsilon_1(\omega)\}}.$$

More precisely, we need to adjust the parameters of Lemma 23 a little bit. Namely, we replace n, x and y by

$$(60) \qquad n' = n - \Upsilon_1(\omega)$$

$$(61) \qquad x' = \frac{x\sqrt{n} - \Upsilon_2(\omega)}{\sqrt{n - \Upsilon_1(\omega)}}$$

$$(62) \qquad y' = \frac{y\sqrt{n} - \sum_{i=0}^{\Upsilon_1(\omega)-1} |\kappa \circ \mathcal{F}_0^i(\omega)| - \bar{\kappa}\Upsilon_1(\omega)}{\sqrt{n - \Upsilon_1(\omega)}},$$

respectively. Note that by construction, $n \sim n'$ and the pairs x, x' and y, y' are close to each other when n is big, uniformly in ℓ and in the choice of ω as long as (59) is true. Also note that by the first property of Υ, the set of ω's not satisfying (59) has measure less than $C\theta^{n^{1/5}}$, which is negligible. Thus we conclude that

$$|n\vartheta_n^\ell(\mathcal{A}) - \bar{\kappa}\varphi_\Sigma(x, y)| < \bar{\varepsilon}$$

if n is large enough. Since $\bar{\varepsilon}$ was arbitrary, (55) follows. Finally, observe that all the estimations in this subsection are uniform for x, y chosen from a compact set. Thus the convergence in (55) is uniform for x, y chosen from a compact set.

A.4. **Proof of Lemma 7 (b).** Our argument is similar to the one in Section A.3, with the main difference of fixing a small enough $\varepsilon > 0$ (and not letting $\varepsilon \to 0$ in the end). We use the notation of Section A.3.

We apply a simplified version of Lemma 23 (since we only need (58)). Namely, by choosing some fixed $\bar{\varepsilon}$, say $\bar{\varepsilon} = 1$, we have by (58)

$$\vartheta_n^{\rho'|_\mathfrak{S}}(\mathcal{A}) < 2\vartheta_n^{\pi_*\mu_0^\mathfrak{R}}(\mathcal{A}^1).$$

Next, the computation of Appendix A.1-A.4 in [**P09**] implies that there exist some $C_1, C_2 < \infty$ depending only on the geometry of the billiard such that for every x, y and n,

$$(63) \qquad n\vartheta_n^{\pi_* \mu_0^{\mathfrak{R}}}(\mathcal{A}^1) < C_1 \varphi_\Sigma(x, y) + C_2 n^{-1/2}.$$

Indeed, even though the computation of Pène is formulated for the function κ instead of our f, her arguments are more general, since the computations are done on the expanding Young tower. Thus replacing her Lemma 11 by Lemma 4.1 of [**SzV04**] and enlarging $\mathcal{A}^1$ to $\mathcal{B}$ where $\mathcal{B} = \cup_{p \in \mathcal{A}^1} W^s(p)$ (so that $\mathcal{B}$ contains entire stable manifolds as required by Pène), we obtain (63) with some $C_1 > (counting \times Leb \times \mu_0)(\mathcal{B})$. So we have some constants $C_1', C_2' < \infty$ depending only on the geometry of the billiard such that for every x, y and n,

$$n\vartheta_n^{\rho'|_{\mathfrak{S}}}(\mathcal{A}) < C_1' \varphi_\Sigma(x, y) + C_2' n^{-1/2}.$$

Now using the same argument as in the end of Section A.3, we conclude that

$$(64) \quad (n - \varkappa n^{1/4} - n^{1/5})\vartheta_n^l(\mathcal{A}) < C_1' \varphi_\Sigma(\tilde{x}, \tilde{y}) + C_2'(n - \varkappa n^{1/4} - n^{1/5})^{-1/2} + C\theta^{n^{1/5}},$$

where $\tilde{x} = \tilde{x}(x, n), \tilde{y} = \tilde{y}(y, n)$ are such that $\varphi_\Sigma(\tilde{x}, \tilde{y})$ maximizes $\varphi_\Sigma(x', y')$ over all possible x', y' we can get in (61) and (62) when using some ω satisfying (59). Clearly, there is a finite constant C_2'' such that

$$(65) \qquad C_2'(n - \varkappa n^{1/4} - n^{1/5})^{-1/2} + C\theta^{n^{1/5}} < C_2'' n^{-1/2}.$$

It is also not hard to deduce from the formulas (60), (61) and (62) that there is a C depending only on the geometry of the billiard such that

$$|x - \tilde{x}| < C(|x|n^{-3/4} + n^{-1/4}), \quad |y - \tilde{y}| < C(|y|n^{-3/4} + n^{-1/4})$$

and hence there exists a constant N depending only on the geometry of the billiard such that for every $n > N$,

$$(66) \qquad \|(x, y) - (\tilde{x}, \tilde{y})\| < \frac{\|(x, y)\|}{3} + 1.$$

Note that the isocontours of the function $(x, y) \mapsto \varphi_\Sigma(x, y)$ are ellipsoids centered at the origin with ratio of axes $\sqrt{\lambda_1} : \sqrt{\lambda_2}$, where $\lambda_1 > \lambda_2 > 0$ are the eigenvalues of Σ. Let $R = \|(x, y)\|$. If $R > 6$ then $R/3 + 1 < R/2$ and considering two ellipsoids such that the major axis of the smaller one is $R/2$, the minor axis of the bigger one is R, and both are isocontours, we conclude that there are constants C, C_1'' depending only on the geometry of the billiard (e.g. C can be $4\lambda_1/\lambda_2$) such that

$$(67) \qquad \varphi_\Sigma(\tilde{x}, \tilde{y}) < C_1'' \varphi_{C\Sigma}(x, y),$$

provided that the vectors $(x, y), (\tilde{x}, \tilde{y})$ satisfy (66) and $\|(x, y)\| > 6$. Clearly the restriction $\|(x, y)\| > 6$ can be discarded by taking a bigger C_1''. Now substituting (67) into (64) and using (65) we conclude that there are constants

C, C_1''', C_2'' and N depending only on the geometry of the billiard such that for every x, y and $n > N$, we have

$$(n - \varkappa n^{1/4} - n^{1/5})\vartheta_n^l(\mathcal{A}) < C_1'''\varphi_{C\Sigma}(x, y) + C_2''n^{-1/2}.$$

Thus there exist constants $C_{1,final}, C_{2,final}$ depending only on the geometry of the billiard such that for every x, y and for every n,

$$n\vartheta_n^l(\mathcal{A}) < C_{1,final}\varphi_{C\Sigma}(x, y) + C_{2,final}n^{-1/2}.$$

APPENDIX B. PROPERTIES OF THE BROWNIAN MEANDER.

Proof of Lemma 15. Let us write $B_{\hat{\sigma}}^a(t)$ for a Brownian motion of variance $\hat{\sigma}^2$ with $B_{\hat{\sigma}}^a(0) = a$ and denote

$$M_{\hat{\sigma}}^a(t) = \max_{s \in [0,t]} B_{\hat{\sigma}}(s) \text{ and } m_{\hat{\sigma}}^a(t) = \min_{s \in [0,t]} B_{\hat{\sigma}}(s).$$

Let us compute $\phi_{\hat{\sigma}}(x, y)$ by conditioning on the value of $\mathfrak{X}_{\hat{\sigma}}(1 - \delta_t)$ (call it z). By the self-similar property of the Brownian motion,

$$
\begin{aligned}
\phi_{\hat{\sigma}}(x, y) &= \int_0^y \sqrt{1 - \delta_t} \phi_{\hat{\sigma}}\left(\frac{z}{\sqrt{1 - \delta_t}}, \frac{y}{\sqrt{1 - \delta_t}}\right) \\
&\quad \lim_{\Delta \to 0} \frac{1}{\Delta} P(B_{\hat{\sigma}}^z(\delta_t) \in [x, x + \Delta], M_{\hat{\sigma}}^z(\delta_t) < y | m_{\hat{\sigma}}^a(\delta_t) > 0) dz
\end{aligned}
$$

We can approximate this expression by

(1) Substituting the second line by $\frac{1}{\Delta} \lim_{\Delta \to 0} P(B_{\hat{\sigma}}^z(\delta_t) \in [x, x + \Delta])$, which is clearly good approximation if δ_t is small and $z \in [\delta, y - \delta]$ for some fixed δ (the contribution of other z's is small because of the first line)
(2) Replacing the integral by a sum over δ_s. $\qquad\square$

Proof of Lemma 18. We have

$$
(68) \qquad \int_{\sqrt{\delta}}^{1/\sqrt{\delta}} \frac{|2k + x|}{\sqrt{2\pi}\hat{\sigma}} \exp\left(-\frac{(2k + x)^2 s^2}{2\hat{\sigma}^2}\right) ds
$$

$$
= \Phi\left(\frac{|2k + x|}{\hat{\sigma}\sqrt{\delta}}\right) - \Phi\left(\frac{|2k + x|\sqrt{\delta}}{\hat{\sigma}}\right)
$$

where Φ is the cumulative distribution function of the standard Gaussian random variable. Computing the integral for $k = 0$, we obtain

$$
\begin{aligned}
I_x = \frac{\hat{c}_1(\mathcal{G})\sqrt{2\pi}}{\hat{\sigma}} &+ \lim_{\delta \to 0} \frac{2\hat{c}_1(\mathcal{G})\sqrt{2\pi}}{\hat{\sigma}} \sum_{k=1}^{\infty} \left[\Phi\left(\frac{(2k + x)}{\hat{\sigma}\sqrt{\delta}}\right) - \Phi\left(\frac{(2k - x)}{\hat{\sigma}\sqrt{\delta}}\right)\right. \\
(69) &\qquad\qquad + \left.\Phi\left(\frac{(2k - x)\sqrt{\delta}}{\hat{\sigma}}\right) - \Phi\left(\frac{(2k + x)\sqrt{\delta}}{\hat{\sigma}}\right)\right].
\end{aligned}
$$

It is clear that

$$\lim_{\delta \to 0} \sum_{k=1}^{\infty}\left[\Phi\left(\frac{(2k + x)}{\hat{\sigma}\sqrt{\delta}}\right) - \Phi\left(\frac{(2k - x)}{\hat{\sigma}\sqrt{\delta}}\right)\right] = 0.$$

Let us write

$$(70) \qquad d_{k,x,\delta} = \Phi\left(\frac{(2k-x)\sqrt{\delta}}{\hat{\sigma}}\right) - \Phi\left(\frac{(2k+x)\sqrt{\delta}}{\hat{\sigma}}\right).$$

Then

$$I_x = \frac{\hat{c}_1(\mathcal{G})\sqrt{2\pi}}{\hat{\sigma}} + \frac{2\hat{c}_1(\mathcal{G})\sqrt{2\pi}}{\hat{\sigma}} \lim_{M\to\infty} \lim_{\delta\to 0} \left\{ \sum_{k=1}^{\lfloor\frac{M}{\sqrt{\delta}}\rfloor} d_{k,x,\delta} + \sum_{k=\lfloor\frac{M}{\sqrt{\delta}}\rfloor}^{\infty} d_{k,x,\delta} \right\}.$$

Denote the above sums by $S_{1,M,\delta}$ and $S_{2,M,\delta}$. Since $x < 1$, it is easy to see that for $\delta < 1$

$$|S_{2,M,\delta}| < 1 - \Phi\left(\frac{(2M/\sqrt{\delta} - x)\sqrt{\delta}}{\hat{\sigma}}\right).$$

Thus

$$\lim_{M\to\infty} \lim_{\delta\to 0} S_{2,M,\delta} = 0.$$

Next,

$$d_{k,x,\delta} = -\frac{2x\sqrt{\delta}}{\hat{\sigma}}\varphi\left(\frac{(2k-x)\sqrt{\delta}}{\hat{\sigma}}\right) + o(\sqrt{\delta})$$

as $\delta \to 0$ uniformly for $k < M/\sqrt{\delta}$. Thus using the convergence of Riemannian sums, we conclude

$$\lim_{\delta\to 0} S_{1,M,\delta} = -\frac{2x}{\hat{\sigma}} \int_0^M \varphi\left(\frac{2y}{\hat{\sigma}}\right) dy.$$

Consequently,

$$\lim_{M\to\infty} \lim_{\delta\to 0} S_{1,M,\delta} = -x/2.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Lemma 20. Similarly to (68) we get that $u(t,x) = \sum_k \operatorname{sgn}(x+2k)v(t,x,k)$ with

$$v(t,x,k) = 1 - \Phi\left(\frac{x+2k}{\hat{\sigma}\sqrt{t}}\right).$$

An elementary computation shows that

$$v_t' = \frac{1}{2}\frac{x+2k}{\hat{\sigma}t^{3/2}}\varphi\left(\frac{x+2k}{\hat{\sigma}\sqrt{t}}\right) \text{ and } v_{xx}'' = -\frac{1}{\hat{\sigma}\sqrt{t}}\varphi\left(\frac{x+2k}{\hat{\sigma}\sqrt{t}}\right)\left(-\frac{x+2k}{\hat{\sigma}^2 t}\right).$$

Thus $u(t,x)$ satisfies our heat equation. In order to check the boundary conditions, we use the same argument as in the proof of Lemma 18. Namely, recalling (70) it is clear that

$$\lim_{x\searrow 0} \sum_{k=1}^{\infty} d_{k,x,1/t} = 0.$$

so the main contribution to u comes from $k = 0$ giving

$$\lim_{x \searrow 0} u(t, x) = \frac{2\lambda_0 \hat{c}_1(\mathcal{G})}{\hat{\sigma}^2} \lim_{x \searrow 0} \int_{1/\sqrt{t}}^{\infty} x \exp\left(-\frac{x^2 s^2}{2\hat{\sigma}^2}\right) ds = \frac{\lambda_0 \hat{c}_1(\mathcal{G})\sqrt{2\pi}}{\hat{\sigma}} = f_0.$$

Likewise $\lim_{x \nearrow 1} u(t, x) = 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

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(Dmitry Dolgopyat) UNIVERSITY OF MARYLAND
E-mail address: dmitry@math.umd.edu

(Péter Nándori) COURANT INSTITUTE
E-mail address: nandori@cims.nyu.edu