

Equidistribution for standard pairs in planar dispersing billiard flows

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Abstract

We prove exponential correlation decay in dispersing billiard flows on the 2-torus assuming finite horizon and lack of corner points. With applications aimed at describing heat conduction, the highly singular initial measures are concentrated here on 1-dimensional submanifolds (given by standard pairs) and the observables are supposed to satisfy a generalized Hölder continuity property. The result is based on the exponential correlation decay bound of Baladi, Demers and Liverani [1] obtained for Hölder continuous observables in these billiards. The model dependence of the bounds is also discussed.

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1 Introduction

1.1 General introduction

Decay of correlations is a most useful property when establishing probabilistic laws for stochastic processes. What is more - and this is the main reason of our interest - it has a pivotal role in non-equilibrium statistical physics since it also controls convergence to equilibrium. Usually correlation decay bounds are originally established when the system starts from a nice measure absolutely continuous with respect to the smooth invariant measure of the system. In contrast, our goal here is to present a *correlation bound for planar Sinai billiard flows in the case where the initial measure is determined by a standard pair*. Briefly saying a *standard pair* is a smooth unstable curve together with a nice probability density on it hence determining a singular measure in the phase space (the precise definition will be given in Section 2.5). The tool of standard pairs was initiated in [15, 16] where it appeared as a much effective and flexible variant of Markov approximations of hyperbolic dynamical systems. Since then, it has been widely used when tackling a variety of problems (see [3, 4, 8, 10, 11, 13, 17, 18, 19, 20, 21, 22]). In particular, in diverse approaches studying the Fourier law of heat conduction for Hamiltonian models, it occurs that correlation bounds - and the resulting convergence to equilibrium - for *a hyperbolic flow starting from precisely a standard pair* seem inevitable (see e. g.

[17, 19, 2]).¹ Although on the one hand, our prime motivation was the aforementioned application (for instance [19] also uses our result), we, on the other hand, also mean our work as a contribution to and a reference on the methodology of standard pairs applied to flows.

Our result is based on correlation bounds when the initial measure is a nice smooth one. Indeed, for planar Sinai billiards *exponential correlation bounds for the billiard ball map* had been known since the late 90's (see [30] – finite horizon case, [5] – infinite horizon case). However, it took time until, in 2007, Chernov [7] could derive a *stretched exponential correlation bound for the flow* in case of so-called generalized Hölder continuous observables. (Earlier [26] had obtained superpolynomial correlation bounds for functions smooth in the flow direction). Quite recently, Baladi, Demers and Liverani [1] were finally able to *achieve an exponential bound* for Hölder observables.² It is essential to note here that the aforementioned results were drawn up in terms of the invariant Liouville measure as an initial measure and they can be naturally extended to cases when the initial law is absolutely continuous with respect to it with a nice density.

As said above it is, however, substantial for some actual applications (see [19, 2] that the initial measure be given by a standard pair (and, moreover, the observable be Hölder in a weaker, so-called generalized sense, only). Such a result is the goal of this paper and will be the claim of our Main Theorem 1.4. Under these circumstances a natural way to derive a correlation bound starting from a standard pair is 1) to smear the singular measure given by the standard pair in a small neighbourhood of its unstable curve component and 2) apply a known result to this smooth initial measure. This idea works well for smooth dynamical systems, for instance in case of geodesic flows [17]. However, for singular systems, like billiards, there arises a substantial difficulty, and actually the bulk of the paper fights exactly this obstacle. We discuss the problem in detail in Section 1.3. We also note that though the Banach spaces used by [1] also contain singular measures nevertheless the standard pairs we need and consider here do not belong to those Banach spaces.

Here, we prove our results for the simplest possible class of Sinai billiards: for planar dispersing billiards with finite horizon and no corner points. However, in order to discuss possible extensions, we present a setup which is slightly more general, and also allows corner points. It is worth mentioning that in the recent work [14] a similar approach, i. e. taking iterates of standard pairs, is successfully used for constructing SRB measures for multidimensional, non-uniformly hyperbolic maps.

The billiard ball map has been extensively studied in the literature, see e.g. [12] and [9]. In the preliminary Section 2, we extend several notions and definitions *from the billiard ball map to the billiard flow* (with the exception of Section 2.3, where various notions of Hölder continuity are recalled). Section 3 contains the proof of our Main Theorem 1.4. In particular, certain substantial foliations are introduced in Sections 3.1-3.4. The approximating density is defined in Section 3.5. Then its regularity properties are studied and adjusted to the foliations in Sections 3.6 and 3.7. The proof is completed in Section 3.8. Finally, a possible strengthening and two possible generalizations of our result are discussed in Section 4. In particular, as we learnt from [9], standard pairs are most appropriate for the perturbative analysis of billiard-like systems therefore we pay special attention to the dependence on the model of the constants appearing in the bounds.³ This question is discussed in Subsection 4 (see also Subsection 2.2.3). The aforementioned and desirable generalizations are discussed in Subsections 4.2 and 4.3.

About the Appendix: Subsection A.1 is devoted to demonstrate three properties, formulated in Subsection 2.4, of holonomy maps along central-stable manifolds. In subsection A.2 we extend expo-

¹[17] uses the method of standard pairs for the study of systems with *slow-fast degrees of freedom* whereas [19, 2] apply it to treat the *rare interaction limit*.

² [1] also contains more details on earlier results providing correlation decay for planar dispersing billiard flows.

³We note, however, that the proof of the limit transition to a Markov jump process in the model of [2] does not need these uniform - in the appearing models - bounds of the constants but they are necessary for more complicated models.

ponential correlation decay known for Hölder observables to those for generalized Hölder observables. In particular we use Corollary 3.28 of a Theorem 3.27 by IP Tóth, which is proved in his separate paper [28]. Finally Subsection A.3 formulates statements about the extension of Hölder continuous function to larger sets.

1.2 Setup and Main Theorem

1.2.1 Billiard table

Our main discussion is restricted to hyperbolic billiards with smooth scatterers. Nevertheless, keeping for later purpose we introduce the model a bit more generally, i. e. we permit the boundaries of the scatterers to consist of smooth pieces (called walls, cf. [12], Section 2.1). Specifically, let \mathbb{T}^2 denote the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let $Q_0 \subset \mathbb{T}^2$ be open and connected and let $Q = \overline{Q_0}$ be its closure. Assume that the boundary of Q is a finite union of curves in \mathbb{T}^2 :

$$\partial Q = \Gamma = \Gamma_1 \cup \dots \cup \Gamma_r.$$

The curves Γ_i are assumed to be C^3 -smooth of finite length, and intersect only at the endpoints:

$$\Gamma_i \cap \Gamma_j \subset \partial\Gamma_i \cup \partial\Gamma_j \quad \text{for every } i \neq j.$$

The closed set $Q \subset \mathbb{T}^2$ is called the *billiard table* or *billiard flow configuration space*. The Γ_i are called *walls* (while the boundaries of the the connected components of $\mathbb{T}^2 \setminus Q_0$ are called *scatterers*). The billiard flow describes the motion of a point particle (called the billiard particle) that moves in Q with unit velocity, so *the phase space of the billiard flow* is

$$M = Q \times S^1 \subset \mathbb{T}^3$$

where S^1 is the unit circle. Geometrically, we view it as the set of unit velocity vectors: $S^1 = \{v \in \mathbb{R}^2 \mid |v| = 1\}$. From a metric perspective, it is better to view it as the 1-torus $S^1 = \mathbb{T}^1 = \mathbb{R}/\mathbb{Z} \cong \{e^{i2\pi\xi} \mid \xi \in \mathbb{R}\}$. The billiard particle moves uniformly (with constant speed) in Q until it hits the boundary ∂Q . When it reaches the boundary, it bounces back under the rules of elastic collision – so the velocity remains unit, and its direction changes like the direction of a reflected light ray in geometric optics: the angle of reflection equals the angle of incidence.

For $r \in Q$ and $v \in S^1$ and $x = (r, v) \in M$ we call r the position, v the velocity, and x the phase point of the particle. For a particle with phase point $x = (r, v) \in M$, let $\Phi^t(x)$ give the phase point of the same particle after it moves for time t . For $x = (r, v) \in M$ denote $\pi_Q x = r$. $\{\Phi^t : M \rightarrow M \mid t \in \mathbb{R}\}$ is called the billiard flow and this definition is unambiguous if we assume that, in addition, the trajectories are continuous from the right.⁴ For $x \in M$, let $\tau(x)$ denote the time of free flight for x until the first collision:

$$\tau(x) = \inf\{t > 0 \mid \pi_Q(\Phi^t(x)) \in \Gamma\}.$$

1.2.2 Assumptions and statement of the theorem

We will use the following assumptions:

Assumption 1.1 (No corner points). *All the walls Γ_i are closed C^3 -smooth curves, so they have no boundary and they are disjoint.*

⁴Another way to make it unambiguous is to identify the left and right limit points when the orbit hits the boundary, cf. [25, 9]. Both ways are equivalent.

Consequently the walls and the scatterers are identical.

Assumption 1.2 (Dispersing planar billiard). *All walls are strictly convex when viewed from the outside of Q .*

Consequently all scatterers are strictly convex.

Assumption 1.3 (Finite horizon). *There are no phase points which can fly indefinitely without a collision (or equivalently the free flight time $\tau(x)$ is finite for every x and consequently it is also bounded).*

Our main result is proven under the assumptions 1.2, 1.3 and 1.1.⁵ Most importantly, our main reference [1] only covers this case. However, our long term aim is to drop Assumption 1.1 and cover billiards with corner points, under the much weaker Assumption 4.4, which will be discussed in Section 4.2.

The more often used – and this was, indeed, traditionally the most favoured and more convenient – description of billiard dynamics relies upon the discrete phase space

$$\mathcal{M} = \{(q, v) | q \in \partial Q, |v| \in \mathbb{R}^2, |v| = 1, \langle n(q), v \rangle \geq 0\} \subset M,$$

where $n = n(q)$ is the normal vector of ∂Q at q pointing inwards into Q (that is, out of the scatterers). Then the related billiard ball map $T : \mathcal{M} \rightarrow \mathcal{M}$ is defined by $Tx = \Phi^{\tau(x)}(x)$. (We note that – in the sense of our convention made in Section 1.2.1 – actually $M = (Q_0 \times S^1) \cup \mathcal{M}$.) We also note that a convenient coordinate for the velocity component $(q, v) \in \mathcal{M}$ is the angle $\varrho \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ between v and n implying $\cos \varrho = \langle n, v \rangle$. It is sometimes convenient to use (q, ϱ) instead of (q, v) , so – with some abuse of notation – we can write

$$\mathcal{M} = \{(q, \varrho) | q \in \partial Q, -\frac{\pi}{2} \leq \varrho \leq \frac{\pi}{2}\}. \quad (1.1)$$

The connection between the two phase spaces is provided by the projection $\Pi : M \rightarrow \mathcal{M}$ that assigns to each point $x \in M$ the point of the previous collision. That is, if $\tau_-(x) = \min\{t \geq 0 \mid \Phi^{-t}(x) \in \mathcal{M}\}$, then $\Pi x = \Phi^{-\tau_-(x)}x$. The invariant, Liouville measure μ for the billiard flow is given by

$$\mu = \frac{1}{\text{Leb}(M)} \text{Leb}_M \quad , \text{ so } \quad d\mu = \frac{1}{2\pi \text{Vol}(Q)} dr dv,$$

whereas the natural invariant measure ν for the billiard ball map is given by

$$d\nu = \frac{1}{2|\partial Q|} \langle n, v \rangle dq dv \quad (= \frac{1}{2|\partial Q|} \cos \varrho dq dv).$$

In general we are going to follow the terminology of (and several facts from) [12] often without particular reference.

Now we present the main theorem of the paper. Most of the precise terminology will be introduced later, cf. Definition 2.24. For now assume that (W^u, φ) is a sufficiently regular standard pair where φ is a nice probability density with respect to the length measure m_{W^u} on the u -curve W^u . The essence is that if F is a sufficiently regular observable on M , then $\int_{W^u} \varphi(x) F(\Phi^t x) dx \rightarrow \int_{W^u} \varphi(x) dx \int_M F(x) dx$ as $t \rightarrow \infty$, exponentially fast.

Assume we are given a standard pair (W^u, φ) . In addition, for technical reasons we assume that there is some $\varepsilon_0 > 0$ such that the $2\varepsilon_0$ -neighbourhood of W^u is disjoint from the boundary of M . This condition is formulated in Definition 2.6, and discussed more in Section 4.3.

⁵According to the terminology of [12], billiards satisfying assumptions 1.2, 1.3 and 1.1 belong to category A.

Theorem 1.4. *Suppose the billiard table Q satisfies assumptions 1.2, 1.3 and 1.1. Let $0 < \Theta_\varphi < 1$, $0 < \alpha_F \leq 1$ and $\varepsilon_0 > 0$. Then there exist $\mathcal{C} < \infty$ and $a > 0$ with the following properties:*

Let (W^u, φ) be a standard pair with a dynamically Θ_φ -Hölder φ (cf. Definition 2.17). Assume that the $2\varepsilon_0$ -neighbourhood of W^u is disjoint from the boundary of M . Let $F : M \rightarrow \mathbb{R}$ be generalized α_F -Hölder continuous (cf. Definition 2.12). Then, for every $t \geq 0$,

$$\left| \int_{W^u} (F \circ \Phi^t) \varphi d\mu_{W^u} - \int_M F d\mu \right| \leq \mathcal{C} \cdot \|\varphi\|_{\Theta_\varphi; dH} \cdot \text{var}_{\alpha_F} F \cdot e^{-at}.$$

Here

- $\|\varphi\|_{\Theta_\varphi; dH}$ is the dynamical Hölder norm of φ (see (2.8)).
- $\text{var}_{\alpha_F} F$ is the generalized Hölder seminorm of F (see (2.7)).
- $a = a(Q, \mathcal{R}_u, \Theta_\varphi, \alpha_F) < \infty$ and $\mathcal{C} = \mathcal{C}(Q, \mathcal{R}_u, \Theta_\varphi, \alpha_F) < \infty$ depend on the billiard table Q , the regularity of good u -curves quantified in \mathcal{R}_u , and the regularity classes of φ and F given by Θ_φ and α_F . In other words, they depend on W^u , φ and F through the aforementioned parameters, only.

Moreover, \mathcal{C} depends on Q only through \mathcal{R}_Q from (2.1) and, furthermore, through $\mathcal{C}_{BDL}(Q, \alpha)$ from Theorem 1.5 with some $\alpha = \alpha(\mathcal{R}_Q, \mathcal{R}_u, \Theta_\varphi) > 0$. Similarly, a depends on $(Q, \mathcal{R}_u, \Theta_\varphi)$ only through $a'(Q, \alpha)$ from Theorem 1.5 with the same $\alpha = \alpha(\mathcal{R}_Q, \mathcal{R}_u, \Theta_\varphi) > 0$. That is,

$$\mathcal{C} = \mathcal{C}(\mathcal{R}_Q, \mathcal{R}_u, \alpha_F, \mathcal{C}_{BDL}(Q, \alpha(\mathcal{R}_Q, \mathcal{R}_u, \Theta_\varphi)))$$

and

$$a = a(\alpha_F, a'(Q, \alpha(\mathcal{R}_Q, \mathcal{R}_u, \Theta_\varphi))).$$

1.3 Some words about the proof

As mentioned before, we will use the recent result of Baladi, Demers and Liverani on the exponential correlation bound for Hölder observables. In fact, Corollary 1.3 (of Theorem 1.2) in their work [1] says the following:

Theorem 1.5. *Consider a billiard like the one introduced above. Assume $0 < \alpha \leq 1$. Then there exist $a' = a'(Q, \alpha) > 0$ and $\mathcal{C}_{BDL} = \mathcal{C}_{BDL}(Q, \alpha) < \infty$ such that for any $F, G : M \rightarrow \mathbb{R}$ α -Hölder functions with $\int_M F d\mu = 0$ and any $t \geq 0$ one has*

$$\left| \int_M (F \circ \Phi^t) G d\mu \right| \leq \mathcal{C}_{BDL} \|F\|_{\alpha; H} \|G\|_{\alpha; H} e^{-a't}.$$

Here $\|\cdot\|_{\alpha; H}$ denotes the α -Hölder norm defined in (2.4). (BDL stands for Baladi-Demers-Liverani.)

Once a bound for correlations under the initial Liouville measure for Hölder observables is known, then, for obtaining a similar one for less regular objects (standard pairs vs. generalized Hölder observables) it is a natural idea – as this was also sketched in [7] – to smear both the initial measure and the observable to improve regularity, and then use Theorem 1.5. This is exactly what we will do.

We first generalize Theorem 1.5 to the case when F is only generalized Hölder continuous. This is done in Section A.2 with a standard approximation argument. The essence is that in $L^1(\mu)$ a generalized Hölder continuous function can easily be approximated by a truly Hölder one. This is sometimes done by the method of mollification (see, for instance, [23]).

The second step – namely, replacing $Gd\mu$ with a measure given by a standard pair, is by far less trivial. No L^1 approximation makes sense, so we have to carefully make use of the regularity of the integrand. However, the integrand $F \circ \Phi^t$ has very bad regularity properties when t is big – except in the central-stable directions. Consequently the mollification (in other words the smearing of the density component of the standard pair) should act along the central-stable direction. However, not every point has a long enough central-stable manifold. Indeed, for any unstable curve W^u and for any neighbourhood U of W^u , those points of the unstable curve, along which smooth pieces of central-stable manifolds fully cross U (i. e. have no boundary points inside U), form a Cantor-like subset only. Similar is the situation with the union of the aforementioned smooth pieces inside the given neighbourhood. The bulk of the paper fights exactly this difficulty. (In fact, in case of geodesic flows, this idea of smearing works much more simply, since this difficulty does not arise there and e.g. [17] completes the proof in half a page.)

We note that doing the two steps the other way round – i.e. first allowing standard pairs instead of G and then allowing F to be generalized Hölder instead of Hölder, would not work (or at least not naively). Indeed, in the second step, approximating a generalized Hölder F with a Hölder one in $L^1(\mu)$ does not help: their integrals w.r.t. the singular measure can be very different.

Finally we note that the title of our work follows the terminology of [9] where correlation bounds, in case when the initial measure is determined by a standard pair, are called *equidistribution properties*.

2 Preliminaries

Below we suppose that Assumptions 1.1, 1.2 and 1.3 hold.

2.1 Singularities, homogeneity layers, central-stable manifolds

As well-known, although dispersing billiards are hyperbolic dynamical systems, they possess singularities that necessarily cut the invariant manifolds thus making the mathematical treatment harder. In our case the so-called primary singularities correspond to grazing collisions. For the billiard ball map they are

$$\begin{aligned} S_0 &= \{x = (q, v) \in \mathcal{M} \mid \langle n(q), v \rangle = 0\} \\ S_n &= \{x = (q, v) \in \mathcal{M} \mid T^{-n}x \in S_0\} \quad n \in \mathbb{Z} \end{aligned}$$

These set are submanifolds of \mathcal{M} which can only terminate on each other or on the boundary of \mathcal{M} :

Lemma 2.1. $\partial S_n \subset \bigcup_{0 \leq k < n} S_k$.

A detailed analysis of singularity curves also providing the proof of the lemma was provided in Sinai's classical paper [29].

In order to control the unbounded expansion in the vicinity of the tangential collisions, it is useful to introduce secondary singularities (cf. [12], chapter 5, in particular Definitions 5.8 and 5.11). These partition neighbourhoods of primary singularities into so-called homogeneity layers. As a consequence one uses local manifolds (unstable, stable or central-stable ones) as pieces contained in a single layer and call them local homogeneous - unstable, stable or central-stable - manifolds. (For instance, local central-stable manifolds (and their holonomy maps, see Section 2.4) will play a central

role in our argument.) Specifically, the phase space \mathcal{M} is partitioned into homogeneity layers

$$\begin{aligned}\mathbb{H}_k &= \left\{ (q, \varrho) \in \mathcal{M} \mid \frac{\pi}{2} - \frac{1}{k^2} \leq \varrho < \frac{\pi}{2} - \frac{1}{(k+1)^2} \right\} \quad (k > k_0), \\ \mathbb{H}_{-k} &= \left\{ (q, \varrho) \in \mathcal{M} \mid \frac{\pi}{2} - \frac{1}{k^2} \leq -\varrho < \frac{\pi}{2} - \frac{1}{(k+1)^2} \right\} \quad (k > k_0), \\ \mathbb{H}_0 &= \left\{ (q, \varrho) \in \mathcal{M} \mid |\varrho| \leq \frac{\pi}{2} - \frac{1}{(k_0)^2} \right\}.\end{aligned}$$

where $k_0 > 0$ is an appropriately fixed integer. Just like in analogous constructions for the billiard map (see for instance Appendix A in [9]), we have to introduce the homogeneous manifolds (see Definition 2.2 below) to guarantee the required regularity properties.

As said, for obtaining appropriate distortion control, the boundaries of the homogeneity layers are regarded as artificial (or secondary) singularities. A *homogeneous local stable manifold* for the map $T : \mathcal{M} \rightarrow \mathcal{M}$ is a local stable manifold γ such that for any $n \geq 0$ $T^n \gamma$ belongs to a single homogeneity strip. Here we define a possible extension for the billiard flow.

Definition 2.2. *A homogeneous local central-stable manifold for the billiard flow $\Phi^t : M \rightarrow M$ is a local central-stable manifold $W \subset M$ such that $\gamma = \Pi W$ is a homogeneous local stable manifold for the map $T : \mathcal{M} \rightarrow \mathcal{M}$.*

2.2 Global constants and regularity parameters

In the billiards literature, numbers that depend on the billiard table Q only, are often called “global constants” and are simply denoted by C . Their precise value is usually not important and not studied. However, in some cases it is good to know if such a “constant” depends on Q only through some regularity parameters, like bounds on free flight time, scatterer curvature, etc. We will keep track of such dependence. Moreover, the notion of “unstable curve” plays a key role in our study. The definition of this notion includes a number of arbitrary choices of regularity constants. These could be chosen as global constants – i.e. depending on Q only, – but for the sake of applicability, we will keep track of these choices as well. As a result, what we will call “constants” actually depend on both Q and a number of further input parameters. The precise form of the dependence is unimportant, but we record what they depend on.

2.2.1 Regularity parameters of the billiard table

Let $\tau_{min} > 0$ and $\tau_{max} > \infty$ be lower and upper bounds for the free flight. Let $\kappa_{min} > 0$ and $\kappa_{max} < \infty$ be lower and upper bounds for the curvature of the scatterers. Let $\kappa'_{max} < \infty$ be an upper bound for the derivative of the curvature (as a function of the base point with respect to the arc length parametrization). Let $K_{max} < \infty$ be an upper bound for the number of scatterers. Let A_{min} , A_{max} be lower and upper bounds for the area of Q (NB: we think of situations where our result is applied to a family of billiards and we need uniform bounds). Let d_Q be an upper bound for the diameter of the configuration space. The data

$$\mathcal{R}_Q := \{\tau_{min}, \tau_{max}, \kappa_{min}, \kappa_{max}, \kappa'_{max}, K_{max}, A_{min}, A_{max}, d_Q\} \quad (2.1)$$

describe the regularity of the billiard table Q for our purposes. In most of our calculations, constants that depend only on the billiard table Q , will actually depend on Q only through these regularity parameters. Note that the bounds $\tau_{min}, \tau_{max}, \kappa_{min}, \kappa_{max}, \kappa'_{max}, K_{max}, d_Q$ need not be sharp, so the estimates we give are uniform for the class of billiard tables satisfying the same bounds.

Remark 2.3. *So far as we understand the proof of Theorem 1.5 in [1], in its present form, does not provide that $a' = a'(Q, \alpha)$ and $\mathcal{C}_{BDL} = \mathcal{C}_{BDL}(Q, \alpha)$ only depend on Q through \mathcal{R}_Q . However, we believe these are true and it would also be desirable to know them. This is discussed in more detail in Section 4.*

2.2.2 Hyperbolicity, cone fields and regularity parameters of u -curves

When studying hyperbolicity of planar dispersing billiards in discrete time, there is a natural choice of stable and unstable cone fields: $\mathcal{C}_u(x) := \{(dq, d\varphi) \in T_x\mathcal{M} \mid dqd\varphi > 0\} \cup 0$ for the unstable ones and $\mathcal{C}_s(x) := \{(dq, d\varphi) \in T_x\mathcal{M} \mid dqd\varphi < 0\} \cup 0$ for the stable ones. These cone fields are strictly invariant, meaning that $DT(\mathcal{C}_u(x)) \subset \text{int}(\mathcal{C}_u(Tx)) \cap \{0\}$ and $DT^{-1}(\mathcal{C}_s(x)) \subset \text{int}(\mathcal{C}_s(T^{-1}x)) \cap \{0\}$. However, for applications it is often convenient to use smaller cone fields, e.g. by applying the (derivative of the) dynamics to the above. We will stick to the simplest definition above, and use this unstable cone field to define u -curves below. Using a smaller cone field would result in a more restrictive definition of u -curves, so our result still applies.

A discrete time u -curve or *unstable curve*, is defined in [12]:

Definition 2.4. *A discrete time u -curve is a twice differentiable curve $w \subset \mathcal{M}$ such that its tangent vector is in the unstable cone $\mathcal{C}_u(x)$ at any point x belonging to w , and its curvature is at most some $B_{max} < \infty$ everywhere.*

$B_{max} = B_{max}(\mathcal{R}_Q)$ is chosen big enough to make sure that u -curves evolve into u -curves under the billiard map T , apart from being cut by singularities. However, one may want to choose B_{max} bigger than what is necessary for this invariance. We allow that, and record when our “constants” may depend on B_{max} .

To define a u -curve in continuous time, we take a discrete time u -curve $w \subset \mathcal{M}$ and let its points move with the flow for some place-dependent time:

Definition 2.5. *A u -curve is a curve $W \subset M$ obtained as*

$$W := \{\Phi^{t(x)}(x) \mid x \in w\},$$

where w is a discrete time u -curve and the flight time function $t : w \rightarrow \mathbb{R}^+$ is chosen so that the following regularity properties are satisfied:

- (1) $W \subset M$ is also a twice differentiable curve.
- (2) The angle of the tangent vector of W with the flow direction is at least some $\alpha_{min} > 0$.
- (3) The curvature of W is at most some $0 < \Gamma_{max} < \infty$ everywhere.
- (4) The length of W (to be denoted as $L = L(W^u)$) is at most some $L_{max} \leq \frac{1}{100\Gamma_{max}}$.

We note that the notion of u -curve for the flow in [19] slightly differs from ours. Our notion of u -curve depends on the data

$$\{B_{max}, \alpha_{min}, \Gamma_{max}, L_{max}\}.$$

For several reasons, we restrict to u -curves that are sufficiently far from the scatterers as expressed in the following definition. For example, the proof of certain regularity properties (discussed in Section 2.4) is easier this way.

Definition 2.6. *Fix some $\varepsilon_0 \geq 10L_{max}$. Given a u -curve W^u , let $\text{dist}(W^u, \partial M)$ be its distance from the boundary of M . We say that W^u is a good u -curve if $\text{dist}(W^u, \partial M) > \varepsilon_0$.*

So our notion of good u -curve depends on the data

$$\mathcal{R}_u := \{B_{max}, \alpha_{min}, \Gamma_{max}, L_{max}, \varepsilon_0\}, \tag{2.2}$$

and it is assumed that $10L_{max} \leq \varepsilon_0$.

2.2.3 Convention on the notation for constants

What we call “constants” or “global constants” in this paper, are numbers that depend on Q and \mathcal{R}_u only. We usually denote these by C , often with an index. This index will often refer to the role of the constant, like C_h to the regularity constant of the holonomy in Theorem 2.22, of $C_{G;u}$ to the regularity constant of our function G along u -curves in Proposition 3.30. In other cases, we just use C_1, C_2, \dots to denote different numbers. We also use the unindexed C , which may denote different constants at each appearance – even within a line. In all cases, anything denoted by C depends on (Q, \mathcal{R}_u) only. Also, some global constants may be denoted by other letters for reasons of tradition, like λ for the hyperbolic expansion factor in Theorem 2.7.

Actually, all the global constants that appear in this paper are known to depend on Q only through the regularity parameters \mathcal{R}_Q . (Unfortunately, this is not known for the coefficients of the correlation decay estimates denoted by \mathcal{C} and a , see later.) Also, in many cases, a global constant is known to depend only on Q and not on \mathcal{R}_u . We keep track of these dependences, with future applications in mind. As mentioned above, equidistribution theorems like ours are sometimes applied to a class of models simultaneously, see e.g. [9]. Then it is important to know if the same bounds hold for all models in the class.

We also note that in the literature of correlation decay, it is common to use the word “constant” and the notation C for something which is not a global constant in our sense. A typical example is an exponential correlation decay statement for Hölder observables, of the form

$$\text{Cov}(F, G, t) \leq C \|F\|_\alpha \|G\|_\alpha e^{-at}.$$

Here the “constant” C does not depend on F, G and t , but it does depend on α – i.e. the class of regularity of the observables – so it is not a global constant. In this paper, such quantities will not be denoted by C , but by \mathcal{C} .

2.2.4 Hyperbolic properties

Below we state three theorems on the hyperbolic properties of the billiard flow. Our main reference for these statements is [12]. Although not formulated exactly as in our Theorems, [12] contains some estimates from which these properties immediately follow. Below we point out these connections. Note also that similar properties are discussed in [1], too. Yet, the discussion of [1] does not literally apply, as our notion of u -curve is different from that of [1] since we allow variations in the flow direction. Nonetheless, our u -curves are uniformly transversal to the flow direction, see item (2) in Definition 2.5. Accordingly, all the constants that appear in the statements below depend on the class of u -curves \mathcal{R}_u , in particular on the choice of the constant α_{min} .

Theorem 2.7 (Uniform hyperbolicity). *There are constants $\lambda = \lambda(\mathcal{R}_Q, \mathcal{R}_u) > 1$ and $c_{hyp} = c_{hyp}(\mathcal{R}_Q, \mathcal{R}_u) > 0$ such that if W is a u -curve, $x, y \in W$ and Φ^t is smooth on W , then*

$$\text{dist}_{\Phi^t W}(\Phi^t x, \Phi^t y) \geq c_{hyp} \lambda^t \text{dist}_W(x, y).$$

Proof. In this whole section we assume that $F : M \rightarrow \mathbb{R}$ is generalized α_F -Hölder continuous (cf. Definition 2.12) and that $\int_M F d\mu = 0$. The analogous property for the billiard map is stated in Corollary 4.20 and Formula (4.19) in [12]. As formulated in Theorem 2.7, the statement follows from the definition of α_{min} and the expansion properties of dispersing wave fronts, in particular Formulas (3.35) and (4.9) in [12]. See also Lemma 3.3 in [1]. \square

Theorem 2.8 (Transversality). *There is a constant $c_{tr} = c_{tr}(\mathcal{R}_Q, \mathcal{R}_u) > 0$ such that any u -curve and any central-stable manifold intersecting it have an angle at least c_{tr} at their intersection point.*

Proof. The analogous property for the billiard map is stated in Formulas (4.14) and (4.21) in [12]. To discuss the property for the flow, note that central-stable manifolds are two dimensional. Here we describe two linearly independent directions tangential to central stable manifolds such that the plane they span is uniformly transversal to u -curves. Tangent vectors in the flow phase space are conveniently described in the Jacobi coordinates $(d\xi, d\eta, d\omega)$, see section 3.6 in [12], or Section A.1 in the present paper. On the one hand, the flow direction $(0, 1, 0)$ is tangential to central stable manifolds, and uniformly transversal to u -curves by the definition of α_{min} . On the other hand, the stable direction for the billiard flow is $(1, 0, -B_s)$ for some $B_s > 0$, that is, stable manifolds are associated with *convergent* wave fronts, see Formula (4.45) in [12]. u -curves, in turn, correspond to dispersing wave fronts that have tangent vector $(1, 0, B_u)$ for some $B_u > 0$. For the claimed uniform transversality, we need to see that the plane $\{(b, a, -bB_s) | (a, b) \in \mathbb{R}^2\}$ is uniformly transversal to the vector $(1, 0, B_u)$. This follows as both B_s and B_u are uniformly bounded away from 0, and even though neither of these quantities is bounded from above, they cannot approach infinity simultaneously. In particular, B_u can be arbitrarily large just *after* tangential collisions, while B_s can be arbitrarily large just *before* tangential collisions. See also [1], in particular Remark 2.1 and the discussion following it. \square

To state one more property we need some terminology: the set $\{t \geq 0 | \Phi^{-t}S_0\}$, the singularity set for *the flow* Φ^t , $t \geq 0$ is a countable collection of smooth, one-codimensional (i.e. two dimensional) submanifolds. We will refer to these submanifolds as singularity manifolds.

Theorem 2.9 (Alignment). *There is a constant $c_{al} = c_{al}(\mathcal{R}_Q, \mathcal{R}_u) > 0$ such that if W is a u -curve, S_i is any singularity manifold which is a pre-image of a tangential collision, and W and S_i intersect, then they have an angle at least c_{al} at their intersection point.*

Proof. The argument is essentially the same as in the proof of Theorem 2.8. Just like central-stable manifolds, the singularity manifolds are two dimensional, and the flow direction is tangential to them. Furthermore, singularity manifolds can be associated to convergent wave fronts, see Proposition 4.41 (more precisely, its time reversal counterpart) in [12]. That is, for an appropriate choice of B_{Sing} , the vector $(1, 0, -B_{Sing})$ is tangential to the singularity manifold. Here B_{Sing} is uniformly bounded away from 0 and can approach infinity just *before* tangential collisions. The required transversality follows as in the case of Theorem 2.8. \square

Remark 2.10. *Note that, for dispersing billiards with corner points, the above properties are more subtle as there is no lower bound on the free flight.*

However, uniform hyperbolicity and transversality (theorems 2.7 and 2.8) extend to dispersing billiards with corner points, under the weaker Assumption 4.4, without any problem. Alignment is more problematic, because it fails at specific points of the singularity set that corresponds to a collision at the corner. However, in Theorem 2.9 we only claim alignment for tangential singularities. This does remain true under Assumption 4.4, and this is exactly what we need in this paper. (Alignment is used only once, in the proof of Proposition 3.33.) For further details, see section 9 in [5].

2.3 Notions of Hölder continuity

2.3.1 Hölder continuity

Let $f : X \rightarrow \mathbb{R}$, where $(X, dist)$ is some metric space. Let $0 < \alpha \in \mathbb{R}$ and $0 \leq C < \infty$. The function f is said to be Hölder continuous with exponent α and Hölder constant C if for any $x, y \in X$

$$|f(x) - f(y)| \leq C dist(x, y)^\alpha. \quad (2.3)$$

We also say that f is α -Hölder with constant C , or that f is Hölder with constants (α, C) . The best constant

$$|f|_{\alpha;H} := \inf\{C \in \mathbb{R} \mid \forall x, y \in X \ |f(x) - f(y)| \leq C \operatorname{dist}(x, y)^\alpha\}$$

is a seminorm on the space of α -Hölder functions. Correspondingly, the α -Hölder norm of f is

$$\|f\|_{\alpha;H} := \sup_{x \in X} |f(x)| + |f|_{\alpha;H}. \quad (2.4)$$

We will use this notion with $X = M$ or $X = W$ where W is a u -curve or a central-stable manifold. We need the following easy quantitative properties:

Lemma 2.11. *Let $0 < \alpha \leq 1$ and let $f, g : X \rightarrow \mathbb{R}$ be α -Hölder. Then*

(i) *fg is also α -Hölder and*

$$|fg|_{\alpha;H} \leq \sup_X |f| |g|_{\alpha;H} + \sup_X |g| |f|_{\alpha;H}.$$

(ii) *If $0 < \alpha' \leq \alpha$, then f is also α' -Hölder and*

$$|f|_{\alpha';H} \leq \operatorname{diam}(X)^{\alpha-\alpha'} |f|_{\alpha;H}.$$

(iii) *If $\inf_X f > 0$, then $\frac{1}{f}$ is also α -Hölder and*

$$\left| \frac{1}{f} \right|_{\alpha;H} \leq \frac{|f|_{\alpha;H}}{\inf_X f^2}.$$

Proof. Trivial calculation using only the definition. □

2.3.2 Generalized Hölder continuity

Following Keller [24], Saussol [27] and Chernov [7], we generalize the above notion so that (2.3) need not hold for every pair (x, y) , only “on average” w.r.t the natural invariant measure μ .

For $x \in M$ and $0 \leq r \in \mathbb{R}$, $B_r(x)$ denotes the ball of radius r centred at x :

$$B_r(x) := \{y \in M : \operatorname{dist}(x, y) \leq r\}.$$

For a function $f : M \rightarrow \mathbb{R}$ we use $(\operatorname{osc}_r f) : M \rightarrow \mathbb{R}$ to denote its “ r oscillation”:

$$(\operatorname{osc}_r f)(x) := \sup_{y \in B_r(x)} f(y) - \inf_{y \in B_r(x)} f(y). \quad (2.5)$$

For $0 < \alpha \leq 1$ the generalized α -Hölder seminorm of f is

$$|f|_{\alpha;gH} := \sup_{r>0} \frac{1}{r^\alpha} \int_M (\operatorname{osc}_r f)(x) d\mu(x). \quad (2.6)$$

It is easy to see that $\operatorname{osc}_r f$ is μ -measurable.

Definition 2.12. *Let $f : M \rightarrow \mathbb{R}$ be Borel measurable. We say that f is generalized α -Hölder if $|f|_{\alpha;gH} < \infty$.*

Remark 2.13. *This definition coincides with the one given by Chernov in [7]. It is also similar to what Saussol calls the “quasi-Hölder property” in [27] (which is a special case of the notion defined by Keller in [24]). However, it is not exactly the same. The difference is that Keller [24] and Saussol [27] use essential supremum and infimum in the definition (2.5) of the oscillation, so their definition does not notice the difference between functions that are equal almost everywhere - w.r.t some distinguished (e.g. Lebesgue) measure. This is in accordance with using absolutely continuous measures only, when integrating f .*

From our point of view, two functions, which are equal μ -almost everywhere, may be very different. The measures we use for integration are given by standard pairs, so they are singular w.r.t. μ - actually, concentrated on submanifolds. So, for us, the notion of oscillation with the true sup and inf is the good one.

Since M is compact, f can only be generalized α -Hölder if it is bounded, so

$$\text{var}_\alpha(f) := |f|_{\alpha;gH} + \sup_M f - \inf_M f < \infty \quad (2.7)$$

as well. This is still not a norm, since $\text{var}_\alpha(\text{const}) = 0$, but it is the good quantity to measure the regularity of f for the purpose of our statements.

Lemma 2.14. *If the function f is α -Hölder then it is also generalized α -Hölder and*

$$\begin{aligned} |f|_{\alpha;gH} &\leq 2|f|_{\alpha;H}, \\ \text{var}_\alpha(f) &\leq 2|f|_{\alpha;H} + \sup_M f - \inf_M f \leq 2\|f\|_{\alpha;H}. \end{aligned}$$

Proof. This is immediate from the definition. We use that $\mu(M) = 1$. □

Lemma 2.15. *If $0 < \alpha' \leq \alpha \leq 1$ and $f : M \rightarrow \mathbb{R}$ is generalized α -Hölder, then it is also generalized α' -Hölder and*

$$\text{var}_{\alpha'}(f) \leq \mathcal{C}(\mathcal{R}_Q, \alpha, \alpha') \text{var}_\alpha(f)$$

where $\mathcal{C}(\mathcal{R}_Q, \alpha, \alpha') = \max\{(\text{diam}(M))^{\alpha-\alpha'}, 1\}$.

Proof. Setting $R := \text{diam}(M)$, clearly $\text{osc}_r f \leq \text{osc}_R f$, so

$$|f|_{\alpha;gH} = \sup_{r>0} \frac{1}{r^\alpha} \int_M (\text{osc}_r f)(x) d\mu(x) = \sup_{0<r<R} \frac{1}{r^\alpha} \int_M (\text{osc}_r f)(x) d\mu(x).$$

So

$$|f|_{\alpha';gH} = \sup_{0<r<R} r^{\alpha-\alpha'} \frac{1}{r^\alpha} \int_M (\text{osc}_r f)(x) d\mu(x) \leq R^{\alpha-\alpha'} |f|_{\alpha;gH}.$$

Now this implies

$$\text{var}_{\alpha'}(f) := |f|_{\alpha';gH} + \sup_M f - \inf_M f \leq R^{\alpha-\alpha'} |f|_{\alpha;gH} + \sup_M f - \inf_M f \leq \max\{R^{\alpha-\alpha'}, 1\} \text{var}_\alpha(f).$$

□

2.3.3 Dynamical Hölder continuity

The billiard ball map can be extended to M in a natural way: $T : M \rightarrow \mathcal{M}$, $T(x) := \Phi^{\tau(x)}(x)$.

Definition 2.16. *Let W be a u -curve. For any $x, y \in W$, their separation time $s^+(x, y)$ is the smallest $n > 0$ for which T^n is not continuous on the subcurve of W connecting x and y .*

This is a convenient definition to describe the partitioning of W during its time evolution due to the presence of singularities, but we note that $s^+(x, y)$ does not depend on the u -curve W connecting them, as long as they can be connected with some u -curve. Also note that this separation time is connected to the discrete time billiard map. In this paper, no continuous version of the separation time will be used.

Since W is expanded by T , $s^+(x, y) < \infty$ for every $x \neq y \in W$. If we fix some $\vartheta < 1$, then

$$d_\vartheta(x, y) := \vartheta^{s^+(x, y)}$$

is a metric on W , and is called the dynamical distance.

In this section we concentrate on functions f that are not defined on all of M , but instead, on a u -curve W only. Then the notion of dynamical distance makes sense for every $x, y \in W$, and the usual notion of Hölder continuity, w.r.t. this metric, is called *dynamical Hölder continuity*: $f : W \rightarrow \mathbb{R}$ is called dynamically Hölder continuous if there are constants $0 \leq C < \infty$ and $0 < \alpha \leq 1$, such that

$$|f(x) - f(y)| \leq C d_\vartheta(x, y)^\alpha$$

for every $x, y \in W$. Since $d_\vartheta(x, y)^\alpha = (\vartheta^{s^+(x, y)})^\alpha = \theta^{s^+(x, y)}$ with $\theta := \vartheta^\alpha$, the notion of dynamical Hölder continuity is independent of the choice of ϑ – or, in other words, only the power $\theta := \vartheta^\alpha$ has physical meaning. This justifies the following formal definition:

Definition 2.17. *Let W be a u -curve, $0 < \theta < 1$ and $C < \infty$. The function $f : W \rightarrow \mathbb{R}$ is dynamically Hölder with constants (C, θ) (or dynamically θ -Hölder) if for any $x, y \in W$*

$$|f(x) - f(y)| \leq C \theta^{s^+(x, y)}.$$

The dynamical θ -Hölder seminorm of f is defined as the best constant

$$|f|_{\theta; dH} := \inf\{C \in \mathbb{R} \mid \forall x, y \in W \mid f(x) - f(y) \mid \leq C \theta^{s^+(x, y)}\}. \quad (2.8)$$

The dynamical θ -Hölder norm of f is defined as

$$\|f\|_{\theta; dH} := |f|_{\theta; dH} + \sup_W f. \quad (2.9)$$

A few easy properties:

Lemma 2.18. *Let W be a u -curve, $0 < \theta < 1$ and $f, g : W \rightarrow \mathbb{R}$ dynamically θ -Hölder. Then*

(i) *fg is also dynamically θ -Hölder and*

$$|fg|_{\theta; dH} \leq \sup_W |f| \cdot |g|_{\theta; dH} + \sup_W |g| \cdot |f|_{\theta; dH}.$$

(ii) *If $\theta' < \theta < 1$, then f is also dynamically θ' -Hölder and*

$$|f|_{\theta'; dH} \leq |f|_{\theta; dH}.$$

(iii) If $\inf_W f > 0$, then $\frac{1}{f}$ is also dynamically θ -Hölder and

$$\left| \frac{1}{f} \right|_{\theta; dH} \leq \frac{|f|_{\theta; dH}}{\inf_W f^2}.$$

Proof. Trivial calculation using only the definition. \square

A relation between Hölder continuity and dynamical Hölder continuity is shown in the following lemma.

Lemma 2.19. *If W is a u -curve, $0 < \alpha \in \mathbb{R}$ and $f : W \rightarrow \mathbb{R}$ is Hölder continuous with exponent α , then f is also dynamically Hölder continuous with some constant $\theta = \theta(\alpha, \mathcal{R}_Q, \mathcal{R}_u)$ and*

$$|f|_{\theta; dH} \leq \mathcal{C}(\alpha, \mathcal{R}_Q, \mathcal{R}_u) |f|_{\alpha; H}.$$

Proof. We only need to estimate the Euclidean distance with the dynamical distance from above. The dynamics is uniformly expanding along u -curves by uniform hyperbolicity (Theorem 2.7), but the length of possible smooth components of $T^n W$ is bounded: indeed, they are increasing curves in the discrete time phase space viewed as (1.1), and must terminate on negative time singularities. So $|T^n W| \leq C_1(\mathcal{R}_Q, \mathcal{R}_u)$, and points x and y with a long separation time $s^+(x, y)$ have to be close: if $n < s^+(x, y)$, then

$$C_1 \geq |T^n W| \geq \text{dist}_{T^n W}(T^n x, T^n y) \geq c_{hyp} \lambda^n \text{dist}_W(x, y).$$

Using this with $n = s^+(x, y) - 1$, we get

$$\text{dist}_W(x, y) \leq \frac{C_1 \lambda}{c_{hyp}} \left(\frac{1}{\lambda} \right)^{s^+(x, y)}.$$

Choosing

$$\theta = \theta(\alpha, \mathcal{R}_Q, \mathcal{R}_u) := \left(\frac{1}{\lambda} \right)^\alpha < 1, \quad \mathcal{C} = \mathcal{C}(\alpha, \mathcal{R}_Q, \mathcal{R}_u) := \left(\frac{C_1 \lambda}{c_{hyp}} \right)^\alpha$$

this becomes

$$\text{dist}_W(x, y)^\alpha \leq \mathcal{C} \theta^{s^+(x, y)}.$$

So since f is α -Hölder,

$$|f(x) - f(y)| \leq |f|_{\alpha; H} \text{dist}(x, y)^\alpha \leq |f|_{\alpha; H} \text{dist}_W(x, y)^\alpha \leq |f|_{\alpha; H} \mathcal{C} \theta^{s^+(x, y)}.$$

This is exactly the statement to prove. \square

A comparison in the other direction is not so easy: dynamical Hölder continuity obviously doesn't imply Hölder continuity – in fact, it doesn't even imply continuity. However, a dynamically Hölder function can be made Hölder at the cost of modifying it on a small set. We will use this in our construction (in Proposition 3.33).

Lemma 2.20. *If W_1^u and W_2^u are u -curves, $h : H \subset W_1^u \rightarrow W_2^u$ is the holonomy map along central-stable manifolds, $x, y \in H$ and s^+ is the separation time from Definition 2.16, then*

$$s^+(h(x), h(y)) = s^+(x, y).$$

Proof. x and $h(x)$ are on the same central-stable manifold, so for every discrete time step of the collision map, their images are on the same continuity component of the map. Similarly for y and $h(y)$. So x and y are separated exactly when $h(x)$ and $h(y)$ are separated. \square

2.4 Holonomy along central-stable manifolds

We state some regularity properties of homogeneous local central stable-manifolds and their holonomy which will play a crucial role in our argument. In particular, we consider u -curves in the sense of Definition 2.5, and we want to ensure that there are many homogeneous local central-stable manifolds that connect these u -curves, and that the holonomy maps obtained by sliding along the central stable manifolds are sufficiently regular. For this purpose, we only consider *good* u -curves in the sense of Definition 2.6. Analogous properties in the map context are discussed in [12], Chapter 5; hence our task is to reduce the flow statements to the map statements. The proofs are given in Section A.1.

For any $x \in M$, let $r^{c-s}(x)$ denote the inner diameter of the homogeneous local central stable manifold of x – meaning the supremum of the radii of those disks in the homogeneous local central-stable manifold, centred at x , which fit completely into the manifold without reaching its boundary.

Theorem 2.21. *There is a constant $C_W = C_W(\mathcal{R}_Q, \mathcal{R}_u)$ such that for any good u -curve W^u and any $0 < \varepsilon$,*

$$m_{W^u}(\{x \in W^u \mid r^{c-s}(x) \leq \varepsilon\}) \leq C_W \varepsilon.$$

Consider two good u -curves W_1^u and W_2^u sufficiently close to each other. For $H_1 \subset W_1^u$ denote by $h : H_1 \rightarrow W_2^u$ the holonomy map defined with *homogeneous central-stable* manifolds and let $H_2 = h(H_1)$. Let finally m_{H_1} and m_{H_2} denote the corresponding arc length measures. Our next theorem claims that the holonomy maps between good u -curves along homogeneous central-stable manifolds are absolutely continuous with uniformly bounded and dynamically Hölder continuous densities.

Theorem 2.22. *There exist constants $C_h = C_h(\mathcal{R}_Q, \mathcal{R}_u) < \infty$, $\Theta_h = \Theta_h(\mathcal{R}_Q, \mathcal{R}_u) < 1$ and a function $Jh : H_1 \rightarrow \mathbb{R}^+$ such that for any Borel $B \subset H_1$,*

$$m_{H_2}(h(B)) = \int_B Jh dm_{H_1}. \quad (2.10)$$

Furthermore, this Jh satisfies that for any $x, y \in H_1$

$$|Jh(x)| \leq C_h$$

and

$$|Jh(x) - Jh(y)| \leq C_h \Theta_h^{s^+(x,y)}$$

where $s^+(x, y)$ is the separation time of x and y . The function Jh being, in fact, the Radon-Nikodym derivative of $m_{H_2} \circ h$ w.r.t. m_{H_1} is called the *Jacobian of the holonomy*.

We will need one more property to ensure that the Jacobian of the holonomy varies sufficiently regularly along the central stable direction, as formulated in the following theorem.

Theorem 2.23. *Let W_1 and W_2 be two good u -curves, and $p_1 \in W_1$ and $p_2 \in W_2$ two points on them that lie on the same homogeneous central stable manifold W_{cs} . We introduce the following quantities:*

- δ , the distance of p_1 and p_2 along W_{cs} (in the natural Riemannian metric on W_{cs} as a submanifolds of M);
- α , the angle of the tangent vectors $T_{p_1}W_1$ and $T_{p_2}W_2$,
- $J_h(p_1)$, the Jacobian of the holonomy along central stable manifolds from W_1 to W_2 , evaluated at p_1 .

With these notations, there is a constant $C_{h2} = C_{h2}(\mathcal{R}_Q, \mathcal{R}_u)$ such that

$$|J_h(p_1) - 1| \leq C_{h2}(\alpha + \delta^{1/3}).$$

2.5 Standard pairs

The method of standard pairs was introduced in [16]. Somewhat later in [6] it was already utilized for obtaining various stochastic properties of dispersing billiards whereas in [9] Chernov and Dolgopyat presented far-reaching novel applications of the method. A standard pair is a u -curve W and a probability density φ on W , with certain regularity properties. It can be pictured as a measure on the phase space which is concentrated on a single u -curve – i.e. it is highly singular w.r.t. Riemannian volume. The precise notion to use may depend on the application, and there are many slightly different versions in the literature.

In applications, time evolution of standard pairs plays a crucial role, so the regularity properties required are such that standard pairs evolve into standard pairs in some sense. In the present work we will not consider such a time evolution: instead, we will approximate a standard pair with an absolutely continuous measure. So, for the purpose of applicability, we choose the notion of standard pair as general as we can.

Definition 2.24. *A standard pair is a pair (W, φ) where*

- $W \subset M$ is a u -curve in the sense of Definition 2.5.
- $\varphi : W \rightarrow \mathbb{R}^+$ is a probability density function w.r.t. arc length on W .
- φ is dynamically Hölder continuous with some constants $C_\varphi < \infty$ and $\theta_\varphi < 1$

3 Proof of the main theorem

In the whole section we assume that $F : M \rightarrow \mathbb{R}$ is generalized α_F -Hölder continuous (cf. Definition 2.12) and that $\int_M F d\mu = 0$.

The proof is based on Theorem 1.5, i.e. on Corollary 1.3 of Theorem 1.2 from [1].

We will prove our theorem by approximating the singular measure $\tilde{\varphi}$ concentrated on W^u with an absolutely continuous one, which has some Hölder continuous density G w.r.t. μ . The choice of G will depend on t , which we sometimes emphasize by writing $G = G_t$.

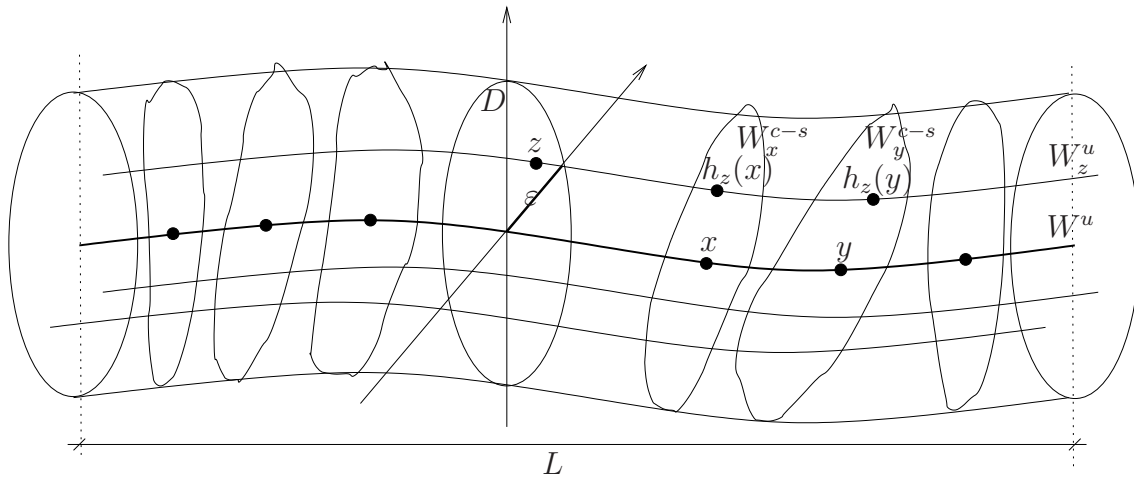
Our approximating density G_t will be supported on some ε -neighbourhood of W^u , where $\varepsilon = \varepsilon(t) \leq \varepsilon_0$ will be specified later. The delicate construction of G uses two foliations of this ε -neighbourhood, which correspond to some kind of “product structure”, at least for the vast majority of points. Both foliations will be measurable partitions w.r.t. μ , and the regularity properties of the factor and conditional measures will play a crucial role – although not always exploited formally.

This approximate product structure, with some of the notation, is shown in Figure 1.

3.1 Conditional measures

During the construction, we will use foliations of (subsets of) the phase space with lower dimensional submanifolds. Such a foliation can also be viewed as a partition into subsets indexed with some index set. We use the notion of *measurability* of such a partition in the usual sense:

Definition 3.1. *Let (X_1, \mathcal{F}_1) and (X_2, \mathcal{F}_2) be measurable spaces. The function $k : X_1 \times \mathcal{F}_2 \rightarrow [0, \infty]$ is called a kernel (from X_1 to X_2) if $k(x, \cdot) : \mathcal{F}_2 \rightarrow [0, \infty]$ is a measure for every $x \in X_1$ and $k(\cdot, A) : X_1 \rightarrow [0, \infty]$ is measurable for every $A \in \mathcal{F}_2$. It is called a probability kernel if also $k(x, X_2) = 1$ for every $x \in X_1$.*

Figure 1: Product structure of a neighbourhood of a u -curve

Definition 3.2. Let (X_1, \mathcal{F}_1) and (X_2, \mathcal{F}_2) be measurable spaces, ν a measure on (X_1, \mathcal{F}_1) and $k : X_1 \times \mathcal{F}_2 \rightarrow [0, \infty]$ a kernel. The composition of ν and k is the measure $\nu \otimes k$ on (X_2, \mathcal{F}_2) defined by

$$(\nu \otimes k)(B) := \int_{X_1} k(x, B) d\nu(x)$$

for every $B \in \mathcal{F}_2$.

Definition 3.3. Let X_1, X_2 be measurable spaces and $X_2 = \bigcup_{i \in X_1} E_i$ a partition of X_2 . We say that the partition is measurable w.r.t. the measure μ on X_2 if there is a measure ν on X_1 and a kernel k from X_1 to X_2 such that $\mu = \nu \otimes k$ and each measure $k(i, \cdot)$ is concentrated on E_i . Then ν is called factor measure, and the measures $k_i(\cdot) := k(i, \cdot)$ are called conditional measures.

If a partition is measurable, the factor measure and conditional measures are not unique. However, if μ is finite, then there is a canonical choice. Let $\pi : X_2 \rightarrow X_1$ be the natural projection defined by $\pi(x) = i$ for $x \in E_i$, so $\pi^{-1}A = \cup_{i \in A} E_i$ for $A \subset X_1$. Then ν can be chosen to be the push-down of μ to X_1 by π :

$$\nu(A) = \pi_* \mu(A) = \mu(\pi^{-1}A)$$

for every measurable $A \subset X_1$. With this choice

$$\nu(A) = \mu(\pi^{-1}A) = (\nu \otimes k)(\pi^{-1}A) = \int_{X_1} k(x, \cup_{i \in A} E_i) d\nu(i) = \int_A k(i, X_2) d\nu(i),$$

so k becomes a probability kernel.

In our discussion we always work with finite measures. This justifies the following convention:

Convention 3.4. When decomposing a measure into factor measure and conditional measures, we always choose the factor measure to be the push-down by the natural projection, and the conditional measures to be probabilities.

3.2 u -foliation

First, we construct a foliation of a tube around W^u , about ε thick ($\varepsilon < \varepsilon_0$), with 1-dimensional curves which are “nearly parallel” to W^u . By “nearly parallel” we mean that if two points are sufficiently close, then the tangent vectors of the curves in those points are guaranteed to be arbitrarily close, even if the two points are not on the same curve.

This foliation can be chosen to have very nice regularity properties. Set $D \subset \mathbb{R}^2$ to be the disk of radius ε centred at 0. We place this disk orthogonally to W^u at one of its points (say, the centre point of W^u), such that D is centred at this intersection point. Then, for any $z \in D$, W_z^u will be the copy of W^u shifted with z in Euclidean space: $W_z^u = W^u + z$. Now we set $U := \bigcup_{z \in D} W_z^u$ to be the tube formed by these shifted versions of W^u . The foliation $\{W_z^u\}_{z \in D}$ of U has nice regularity properties. In particular, the partition is measurable, and the conditional measures on the W_z^u are smooth. However, since W^u is not (necessarily) straight, these conditional measures do not coincide with arc length, and we have to be careful about their precise regularity.

The specific form of the construction above is not important – other smooth foliations would also do. However, with this special choice, some calculations can be simplified by introducing the following notation:

Notation 3.5. Let L be the length of the orthogonal projection of W^u to the normal vector of D . Let m_D be Lebesgue measure on D . For $r \in U$ let $\psi(r)$ be the angle of the tangent vector of W_z^u at r with the normal vector of D .

With this notation, $\text{Leb}(U) = Lm_D(D)$ and $\mu(U) = \frac{L}{\text{Leb}(M)}m_D(D)$. The properties we need are summarized in the following lemma.

Lemma 3.6. The foliation $\{W_z^u\}_{z \in D}$ of $U := \bigcup_{z \in D} W_z^u$ has the following properties:

1. $W^u = W_0^u$.
2. U is contained in the ε -neighbourhood of W^u .
3. The partition $\{W_z^u\}_{z \in D}$ is measurable w.r.t. the invariant measure μ restricted to U (i. e. to $\mu(\cdot) = \mu(\cdot \cap U)$), in the sense of Definition 3.3. Denote the factor measure on D by μ_D^{factor} and the conditional (probability) measures on the foliation leaves W_z^u by $\mu_{W_z^u}^{\text{cond}}$.
4. $\mu_D^{\text{factor}} = \frac{L}{\text{Leb}(M)}m_D$.
5. $\mu_{W_z^u}^{\text{cond}}$ is absolutely continuous w.r.t. arc length, with density

$$\frac{d\mu_{W_z^u}^{\text{cond}}}{dm_z}(r) = \frac{1}{L} \cos \psi(r). \quad (3.1)$$

Proof. All items are obvious from the construction. The factors L and $\frac{1}{L}$ are in accordance with Convention 3.4. \square

For a technical reason (mainly of convenience) we need the following

Lemma 3.7. For every $z \in D$, W_z^u is a good u -curve (as in Definition 2.6).

Proof. By definition, the u -curve W^u is obtained from a discrete time u -curve $w^u \subset \mathcal{M}$ by lifting to M with the flow. Since W^u is at least $2\varepsilon_0$ far from ∂M , the shifted version W_z^u is still at least ε_0 far, so it is good, provided that it is a u -curve.

Now, by shifting W^u into W_z^u , the curvature does not change, but the trace on \mathcal{M} does: W_z^u is obtained from some w_z^u in the same way as W_z^u is obtained from w^u . We need that this w_z^u is still a discrete time u -curve. But, strictly speaking, w_z^u is not a shifted version of w^u , so the tangent vectors and curvature can change slightly. At this point, being absolutely precise would result in overly complicated notation and no real ideas presented. One could introduce a more restricted class of u -curves into which w^u is requested to belong (smaller cones, smaller curvature bound), and a less restricted class into which the nearby w_z^u falls automatically if ε_0 – and thus $|z|$ – is small enough. We omit these details. \square

As mentioned in Section 3.1, the choice of the normalization for the factor measure and the conditional measures is somewhat arbitrary. In our choice we follow Convention 3.4. However, other choices of normalization would also be possible, and in some sense, only the “product of the two” has a physical meaning. Accordingly, what we will really use is the following immediate corollary of Lemma 3.6:

Lemma 3.8. *For any $z \in D$ and any $r \in W_z^u$*

$$\frac{d\mu_D^{factor}}{dm_D}(z) \frac{d\mu_{W_z^u}^{cond}}{dm_z}(r) = \frac{1}{Leb(M)} \cos \psi(r).$$

These functions of r on the different W_z^u are obviously uniformly bounded and Lipschitz continuous:

Lemma 3.9. *For any $z \in D$ and any $r_1, r_2 \in W_z^u$*

$$\frac{1}{2} \leq \cos \psi(r_1) \leq 1$$

and

$$|\cos \psi(r_1) - \cos \psi(r_2)| \leq \Gamma_{max} dist_{W_z^u}(r_1, r_2).$$

Proof. Each W_z^u is a shifted version of W^u , so it is obviously enough to prove the statement for $z = 0$ (meaning $r_1, r_2 \in W_0^u = W^u$). W^u is a u -curve, so by Definition 2.5 its curvature is at most Γ_{max} and its length is at most $L_{max} \leq \frac{1}{100\Gamma_{max}}$. So

$$|\cos \psi(r_1) - \cos \psi(r_2)| \leq |\psi(r_1) - \psi(r_2)| \leq \Gamma_{max} dist_{W_z^u}(r_1, r_2).$$

Choosing r_2 to be the point of W_z^u that lies on D , we have $\psi(r_2) = 0$, so

$$|\cos \psi(r_1) - 1| \leq \Gamma_{max} L_{max} \leq \frac{1}{100},$$

so $\cos \psi(r_1) \geq \frac{1}{2}$. □

This has easy consequences:

Lemma 3.10. *For any $z_1, z_2 \in D$, $dist(W_{z_1}^u, W_{z_2}^u) \geq \frac{1}{2}|z_2 - z_1|$.*

Proof. Trivial geometry using $\cos \psi \geq \frac{1}{2}$. □

Lemma 3.11. *Let $x_1, x_2 \in W^u$, let $z_1, z_2 \in D$ and let $r_1 := x_1 + z_1$, $r_2 := x_2 + z_2$ (so $r_1, r_2 \in U$). Then $dist(r_1, r_2) \geq \frac{1}{2} dist_{W^u}(x_1, x_2)$.*

Proof. Trivial geometry using $\cos \psi \geq \frac{1}{2}$. □

Lemma 3.12. *ψ is Lipschitz continuous on D with constant $2\Gamma_{max}$. That is, for any $r_1, r_2 \in U$,*

$$|\psi(r_2) - \psi(r_1)| \leq 2\Gamma_{max} dist(r_1, r_2).$$

Proof. Let $r_1 := x_1 + z_1$, $r_2 := x_2 + z_2$ where $x_1, x_2 \in W^u$ and $z_1, z_2 \in D$. Then, by construction, $\psi(r_1) = \psi(x_1)$ and $\psi(r_2) = \psi(x_2)$, so

$$|\psi(r_2) - \psi(r_1)| = |\psi(x_2) - \psi(x_1)| \leq \Gamma_{max} dist_{W^u}(x_1, x_2).$$

Using Lemma 3.11 we get the statement. □

3.3 Central-stable foliation

The other foliation we use to get the product structure of U consists of (homogeneous) central-stable manifolds. This is crucial, since we will use in our calculations that the points on such a manifold stay close to each other for any long time. For this, a high price has to be paid: this foliation has much worse regularity properties than the u -foliation above, and not every point can be covered with the product structure.

We will use H to denote the set of those points in W^u whose central-stable manifold crosses U properly. That is, the definition will ensure that for every $x \in H$

- The central-stable manifold of x is long enough in every direction (meaning $r^{c-s}(x)$ is big enough) so that it surely reaches the boundary of U .
- The central-stable manifold of x should not hit the circular faces of the tube U , but rather cross W_z^u for every $z \in D$.

Notation 3.13.

$$H := \left\{ x \in W^u \mid r^{c-s}(x) \geq \frac{10\varepsilon}{\sin c_{tr}} \text{ and } W^{c-s}(x) \cap W_z^u \neq \emptyset \text{ for every } z \in D \right\}. \quad (3.2)$$

It is easy to see that if ε is small, then H contains the vast majority of points in W^u :

Lemma 3.14. *There is a global constant $C_2 < \infty$ such that $m_{W^u}(W^u \setminus H) \leq C_2\varepsilon$.*

Proof. Due to transversality (Theorem 2.8), u -curves and central-stable manifolds have an angle at least c_{tr} , so if $r^{c-s}(x) \geq \frac{10\varepsilon}{\sin c_{tr}}$ then $W^{c-s}(x)$ either crosses every W_z^u , or it hits one of the circular faces at the end of the tube. This latter can only happen if x is less than $\frac{11\varepsilon}{\sin c_{tr}}$ away from one of the endpoints of W^u . On the other hand, not reaching far enough also only happens to a small set of x due to Theorem 2.21. \square

However, we need a little more than that.

Notation 3.15. *From now on, for every $x \in U$ we use W_x^{c-s} to denote the central-stable manifold of x , intersected with U .*

So

$$U_0 := \cup_{x \in H} W_x^{c-s} \subset U$$

is the set of points in U that can be covered with these properly crossing central-stable manifolds. For any $z \in D$, set $H_z := W_z^u \cap U_0$. Actually, this H_z is nothing else than H shifted from W^u to W_z^u by the holonomy map. For future use, we introduce two notations for this holonomy:

Notation 3.16. *Let $h : H \times D \rightarrow U$ be defined so that for any $x \in H$ and $z \in D$ $h(x, z)$ is the single element of $W_x^{c-s} \cap W_z^u$. We will also use the notation $h_z(x) := h(x, z)$, so $h_z : H \rightarrow W_z^u$ is the usual holonomy map.*

With this notation, $U_0 = h(H \times D)$ and $H_z = h_z(H)$.

Lemma 3.17. *There is a global constant $C_3 < \infty$ such that for any $z \in D$ $m_{W_z^u}(W_z^u \setminus H_z) \leq C_3\varepsilon$.*

Proof. Just like before. Due to transversality (Theorem 2.8), u -curves and central-stable manifolds have an angle at least c_{tr} , so if $y \in W_z^u$ and $r^{c-s}(y) \geq \frac{20\varepsilon}{\sin c_{tr}}$, then the central-stable manifold of y either crosses U properly, or it hits one of the circular faces at the end of the tube. This latter can only happen if y is less than $\frac{22\varepsilon}{\sin c_{tr}}$ away from one of the endpoints of W_z^u . Not reaching far enough also only happens to a small set of y due to Lemma 3.7 and Theorem 2.21. Now if the central-stable manifold of $y \in W_z^u$ crosses U properly, then it also intersects W^u at some $x \in W^u$, so it has to coincide with the central-stable manifold of this x and surely $r^{c-s}(x) \geq \frac{10\varepsilon}{\sin c_{tr}}$. So $x \in H$ and $y \in U_0$. \square

We only use this through the following immediate corollary:

Lemma 3.18. *There is a global constant $C_4 < \infty$ such that $\mu(U \setminus U_0) \leq C_4 \varepsilon^3$.*

Proof. By Lemma 3.8, since $\cos \psi(r) \leq 1$,

$$\begin{aligned} \mu(U \setminus U_0) &= \int_D \mu_{W_z^u}^{cond}(W_z^u \setminus H_z) d\mu_D^{factor}(z) = \\ &= \int_D \int_{W_z^u \setminus H_z} \frac{d\mu_{W_z^u}^{cond}(r)}{dm_{W_z^u}(r)} dm_{W_z^u}(r) \frac{d\mu_D^{factor}(z)}{dm_D(z)} dm_D(z) = \\ &= \int_D \int_{W_z^u \setminus H_z} \frac{1}{Leb(M)} \cos \psi(r) dm_{W_z^u}(r) dm_D(z) \leq \\ &\leq \frac{1}{Leb(M)} \int_D m_{W_z^u}(W_z^u \setminus H_z) dm_D(z). \end{aligned}$$

So by Lemma 3.17

$$\mu(U \setminus U_0) \leq \frac{1}{Leb(M)} \int_D C_3 \varepsilon dm_D(z) = \frac{C_3}{Leb(M)} \varepsilon m_D(D) = \frac{C_3 \pi}{Leb(M)} \varepsilon^3.$$

□

Definition 3.19. *Let π denote the natural projection of U_0 to H by the holonomy, meaning $\pi(r) := x$ for $r \in W_x^{c-s}$. Let the measure γ on H be the push-down of $\mu|_{U_0}$ from U_0 to H by π , and let $m := Leb_{W^u}|_H$ denote the restriction of the Lebesgue measure of W^u to H .*

Lemma 3.20. $\gamma \ll m$ with density

$$\beta := \frac{d\gamma}{dm} : H \rightarrow \mathbb{R}^+. \quad (3.3)$$

satisfying

$$\frac{1}{C_\beta} \varepsilon^2 \leq \beta \leq C_\beta \varepsilon^2,$$

where $C_\beta = C_\beta(\mathcal{R}_Q, \mathcal{R}_u)$ is a global constant.

The proof of the lemma also gives a formula for β , but we do not need that.

Proof. Let h_z be the holonomy from W^u to W_z^u . So for $B \subset H$, $(\pi^{-1}B) \cap W_z^u = h_z(B)$. So, disintegrating the μ -measure of $\pi^{-1}B$ w.r.t. the u -foliation, we get

$$\begin{aligned} \gamma(B) &= \mu_U(\pi^{-1}B) = \int_D \mu_{W_z^u}^{cond}(h_z(B)) d\mu_D^{factor}(z) = \\ &= \int_D \int_{W_z^u} \mathbf{1}_{h_z(B)}(\tilde{x}) \frac{d\mu_{W_z^u}^{cond}(\tilde{x})}{dm_{W_z^u}(\tilde{x})} dm_{W_z^u}(\tilde{x}) d\mu_D^{factor}(z). \end{aligned}$$

In the inner integral we substitute $\tilde{x} = h_z(x)$, which is exactly pulling back the integral to W^u from W_z^u . We get

$$\gamma(B) = \int_D \int_{W^u} \mathbf{1}_B(x) \frac{d\mu_{W_z^u}^{cond}(h_z(x))}{dm_{W_z^u}(h_z(x))} Jh_z(x) dm_{W^u}(x) d\mu_D^{factor}(z).$$

We can now exchange the integrals to get

$$\gamma(B) = \int_{W^u} \mathbf{1}_B(x) \int_D \frac{d\mu_{W_z^u}^{cond}(h_z(x))}{dm_{W_z^u}(h_z(x))} Jh_z(x) d\mu_D^{factor}(z) dm_{W^u}(x),$$

which means exactly that

$$\beta(x) = \frac{d\gamma}{dm}(x) = \int_D \frac{d\mu_{W_z^u}^{cond}}{dm_{W_z^u}}(h_z(x)) Jh_z(x) d\mu_D^{factor}(z). \quad (3.4)$$

is indeed the density of γ w.r.t. m . Using Lemma 3.8 this can be written as

$$\beta(x) = \frac{d\gamma}{dm}(x) = \int_D \frac{1}{Leb(M)} \cos \psi(h_z(x)) Jh_z(x) dm_D(z). \quad (3.5)$$

To see the bounds on β , we use that $\frac{1}{2} \leq \cos \psi(h_z(x)) \leq 1$ by Lemma 3.9, $\frac{1}{C_h} \leq Jh_z(x) \leq C_h$ by Theorem 2.22 and $m_D(D) = \varepsilon^2 \pi$ by definition. \square

3.4 Product structure

We used U_0 to denote the set of points where leaves of the u -foliation and long central-stable manifolds intersect. Indeed, points of U_0 can be identified with a pair of “coordinates” (x, z) , where x identifies the central-stable manifold and z identifies the u -curve. This can be formulated in many ways – see Notation 3.16:

$$\begin{aligned} U_0 &= \{h_z(x) | x \in H, z \in D\} = \\ &= \bigcup_{z \in D} h_z(H) = \\ &= \bigcup_{x \in H} W_x^{c-s} = \\ &= \bigcup_{x \in H} \bigcup_{z \in D} (W_x^{c-s} \cap W_z^u) = \\ &= h(H \times D). \end{aligned}$$

For measurable subsets $A \subset H$ and $B \subset D$ we will use the notation $A * B$ to denote this approximate product in U_0 , while reserving the notation $A \times B$ for the usual Cartesian product:

$$A * B := h(A \times B) = \bigcup_{x \in A} \bigcup_{z \in B} (W_x^{c-s} \cap W_z^u).$$

With this notation, we have

$$U_0 = H * D.$$

Lemma 3.20 immediately implies that

Lemma 3.21. *For any measurable $A \subset H$ we have $\mu(A * D) \leq C_\beta \varepsilon^2 m_{W^u}(A)$, where $C_\beta = C_\beta(\mathcal{R}_Q, \mathcal{R}_u) < \infty$ is the global constant from Lemma 3.20.*

Proof. By Definition 3.19 and Lemma 3.20 we can write

$$\mu(A * D) = \mu(\pi^{-1}(A)) = \gamma(A) = \int_A 1 d\gamma = \int_A \beta(x) dm(x).$$

Lemma 3.20 also gives the bound for the integrand, so

$$\mu(A * D) \leq \int_A C_\beta \varepsilon^2 dm = C_\beta \varepsilon^2 m_{W^u}(A).$$

\square

3.5 Construction of the approximating density

We choose $q = q_\varepsilon : D \rightarrow \mathbb{R}^+$ be a smooth enough probability density. The specific form is not important, but for easier calculations we choose the function

$$q(z) = q_\varepsilon(z) := \frac{3}{\varepsilon^2\pi} \left(1 - \frac{|z|}{\varepsilon}\right) \quad \text{for } z \in D.$$

we will use the following regularity properties:

Lemma 3.22. 1. q is a probability density on D with respect to Lebesgue measure m_D ,

2. $q(z) \leq \frac{3}{\pi\varepsilon^2}$ for any $z \in D$,

3. $|q(z_1) - q(z_2)| \leq \frac{3}{\pi\varepsilon^3}|z_1 - z_2|$ for any $z_1, z_2 \in D$,

4. q vanishes on the boundary of D .

Proof. Easy calculation. Remember that $D = \{z \in \mathbb{R}^2 : |z| \leq \varepsilon\}$. □

Let \tilde{q} denote the measure on D with density q (with respect to Lebesgue measure m_D). We construct our approximating density as a “product” of φ in the x direction and q in the z direction using the product structure. Specifically, let \tilde{G}_0 be the measure on U_0 which is defined on approximate product sets $A * B$ as the push-forward of $\tilde{q} \otimes \tilde{\varphi}$ from $H \times D$ to U_0 by h :

$$\tilde{G}_0(A * B) = \tilde{G}_0(h(A \times B)) := (\tilde{q} \otimes \tilde{\varphi})(A \times B) = \int_A \varphi dm \cdot \int_B q dm_D. \quad (3.6)$$

Let $G_0 : U_0 \rightarrow \mathbb{R}^+$ be defined as

$$G_0 := \frac{d\tilde{G}_0}{d\mu}. \quad (3.7)$$

This definition makes sense, since \tilde{G}_0 is indeed absolutely continuous with respect to μ , as we will see in Lemma 3.23.

We will use this definition only through two consequences. The first consequence is an explicit formula that allows us to check the regularity of G_0 , and we obtain it by disintegrating (3.6) w.r.t. the u -foliation.

Lemma 3.23. For $x \in H$, $z \in D$ and $r = h_z(x)$

$$G_0(r) = \text{Leb}(M)\varphi(x)q(z) \frac{1}{Jh_z(x)} \frac{1}{\cos \psi(r)}.$$

(for almost every r).

Proof. For any measurable $A \subset H$ and $B \subset D$ we calculate $\tilde{G}_0(A * B)$ from (3.7) by disintegrating w.r.t. the u -foliation:

$$\begin{aligned} \tilde{G}_0(A * B) &= \int_B \int_{H_z(A)} G_0(r) d\mu_{W_z^u}^{\text{cond}}(r) d\mu_D^{\text{factor}}(z) = \\ &= \int_B \int_{H_z(A)} G_0(r) \frac{d\mu_{W_z^u}^{\text{cond}}}{dm_z}(r) dm_z(r) d\mu_D^{\text{factor}}(z). \end{aligned}$$

In the inner integral we use the substitution $r = h_z(x)$ and notice that $dm_z(r) = Jh_z(x)dm(x)$ by definition (2.10):

$$\begin{aligned}\tilde{G}_0(A * B) &= \int_B \int_A G_0(h_z(x)) \frac{d\mu_{W_z^u}^{cond}}{dm_z}(h_z(x)) Jh_z(x) dm(x) d\mu_D^{factor}(z) = \\ &= \int_B \int_A G_0(h_z(x)) \frac{d\mu_{W_z^u}^{cond}}{dm_z}(h_z(x)) Jh_z(x) dm(x) \frac{d\mu_D^{factor}}{dm_D}(z) dm_D(z)\end{aligned}$$

Comparing this with (3.6) we find that

$$\begin{aligned}\int_{A \times B} \varphi(x) q(z) d(m \otimes m_D)(x, z) &= \\ &= \int_{A \times B} G_0(h_z(x)) \frac{d\mu_{W_z^u}^{cond}}{dm_z}(h_z(x)) Jh_z(x) \frac{d\mu_D^{factor}}{dm_D}(z) d(m \otimes m_D)(x, z)\end{aligned}$$

for every measurable $A \subset H$ and $B \subset D$, so the integrands have to be equal:

$$\varphi(x) q(z) = G_0(h_z(x)) \frac{d\mu_{W_z^u}^{cond}}{dm_z}(h_z(x)) Jh_z(x) \frac{d\mu_D^{factor}}{dm_D}(z)$$

for $m \otimes m_D$ -a.e. $(x, z) \in H \times D$, which also means μ -a.e. $r \in U_0$. Using Lemma 3.8 we get exactly the statement to be proven. \square

The second consequence says that for functions that are constant along central-stable manifolds, integrating against G_0 on U_0 is exactly the same as integrating against φ on H . We get it by projecting down to H along central-stable manifolds. Remember that π denotes the natural projection from U_0 to H so when $r = h_z(x) \in U_0$ for some $x \in H$ and $z \in D$, we have $\pi(r) = x$. So a function that is constant along central-stable manifolds is a function which depends on r through $\pi(r)$ only.

Lemma 3.24. *For $f : U_0 \rightarrow \mathbb{R}$ (or $f : H \rightarrow \mathbb{R}$) and any $A \subset H$,*

$$\int_{A * D} (f \circ \pi) G_0 d\mu = \int_A f \varphi dm.$$

Proof. We first use the definition of G_0 to replace integration w.r.t. μ by integration w.r.t. \tilde{G}_0 . Then we use the definition of \tilde{G}_0 to perform the integral substitution $r := h(x, z)$:

$$\int_{A * D} (f \circ \pi) G_0 d\mu = \int_{A * D} f(\pi(r)) d\tilde{G}_0(r) = \int_{A \times D} f(\pi(h(x, z))) d(\tilde{\varphi} \otimes \tilde{q})(x, z).$$

Since $\pi(h(x, z)) = x$, the integrand depends on x only (that's how it was designed), so the integral factorizes and the integral w.r.t. z becomes 1:

$$\int_{A * D} (f \circ \pi) G_0 d\mu = \int_A f(x) \left[\int_D 1 d\tilde{q}(z) \right] d\tilde{\varphi}(x) = \int_A f d\tilde{\varphi}.$$

Using the definition of $\tilde{\varphi}$ gives the statement. \square

Remark 3.25. *It is not hard to see that the foliation of U_0 with central-stable manifolds is also integrable (w.r.t. $\mu|_{U_0}$). We will not use this fact, so we do not prove it. However, for better understanding, we give the following statement (which we will not use), demonstrating how the function G_0 is “well related” to the central-stable foliation. It says that “the integral of G_0 on each central-stable manifold is exactly what it should be”, and we obtain it by disintegrating (3.6) w.r.t. the central-stable foliation. Let ν_x denote the conditional measures of $\mu|_{U_0}$ on the W_x^{c-s} (the existence of which we do not show). The factor measure is γ . Then for m -a.e. $x \in H$*

$$\int_{W_x^{c-s}} G_0(r) d\nu_x(r) = \frac{\varphi(x)}{\beta(x)}.$$

Indeed, applying (3.6) and (3.7) with $B = D$, and then disintegrating w.r.t. the central-stable foliation, we get

$$\int_A \varphi(x) dm(x) = \tilde{G}_0(A * D) = \int_{A * D} G_0 d\mu = \int_A \int_{W_x^{c-s}} G_0(r) d\nu_x(r) d\gamma(x).$$

By (3.3) $\beta(x) = \frac{d\gamma(x)}{dm(x)}$, so this gives

$$\int_A \varphi(x) dm(x) = \int_A \int_{W_x^{c-s}} G_0(r) d\nu_x(r) \beta(x) dm(x)$$

for every measurable $A \subset H$, which means that

$$\varphi(x) = \int_{W_x^{c-s}} G_0(r) d\nu_x(r) \beta(x)$$

for m -a.e. $x \in H$.

The following proposition is the key to the approximation of the singular measure with the density. It says that if we substitute the density φ on W^u with the density G_0 on M , we make little mistake when integrating $F \circ \Phi^t$, if only F is sufficiently regular. This is a strong statement, since $F \circ \Phi^t$ is far from inheriting the regularity of F , at least in unstable directions. So the essence of the proposition is that this particular approximating density is insensitive to irregularity in the unstable direction.

Proposition 3.26. *There is a constant $1 \leq C_\pi = C_\pi(\mathcal{R}_Q, \mathcal{R}_u) < \infty$ such that for any measurable $A \subset H$,*

$$\left| \int_{A * D} (F \circ \Phi^t) G_0 d\mu - \int_A (F \circ \Phi^t) \varphi dm \right| \leq \int_{A * D} [(osc_{C_\pi \varepsilon} F) \circ \Phi^t] G_0 d\mu.$$

Proof. Let $C_\pi = \frac{10}{\sin c_{tr}}$, where c_{tr} is the transversality bound from Theorem 2.8. Let us consider the error we make if we substitute the function $F(\Phi^t(r))$ with the constant $F(\Phi^t(x))$ on each W_x^{c-s} . In other words, we are comparing the function $F \circ \Phi^t$ to the function $F \circ \Phi^t \circ \pi$. By the construction of H in (3.2), for any $r \in U_0$, r and $\pi(r)$ are $C_\pi \varepsilon$ close, and they are also on the same central-stable manifold, so their distance does not increase in time. This means that $dist(\Phi^t(r), \Phi^t(\pi(r))) \leq C_\pi \varepsilon$, or in other words $\Phi^t(\pi(r)) \in B_{C_\pi \varepsilon}(\Phi^t(r))$. This implies that

$$|F \circ \Phi^t - F \circ \Phi^t \circ \pi| \leq (osc_{C_\pi \varepsilon} F) \circ \Phi^t. \quad (3.8)$$

Using Lemma 3.24 from right to left with $f = F \circ \Phi^t$, we get

$$\begin{aligned} & \left| \int_{A^*D} (F \circ \Phi^t) G_0 d\mu - \int_A (F \circ \Phi^t) \varphi dm \right| = \\ & \left| \int_{A^*D} (F \circ \Phi^t) G_0 d\mu - \int_{A^*D} (F \circ \Phi^t \circ \pi) G_0 d\mu \right| \leq \\ & \leq \int_{A^*D} |F \circ \Phi^t - F \circ \Phi^t \circ \pi| G_0 d\mu. \end{aligned}$$

Substituting (3.8) gives the result. \square

The above proposition points to a technical difficulty we have to fight: we will need to show that the right hand side is small when t is large and ε is chosen appropriately. This does not follow immediately from norm estimates. Indeed, if F were Hölder continuous, then $\text{osc}_{C_\pi\varepsilon} F$ would be uniformly small, and a good upper bound would be immediate. However, we only assume that F is generalized Hölder, so $\text{osc}_{C_\pi\varepsilon} F$ is only small on average. The upper bound on the right hand side – which is a time correlation function – will follow from the correlation decay in Theorem A.5. For this, the generalized Hölder regularity of $\text{osc}_{C_\pi\varepsilon} F$ needs to be shown. This is done in the separate paper [28]. The main theorem there is the following:

Theorem 3.27. *For any Lebesgue measurable $D \subset \mathbb{R}^d$, any bounded $f : D \rightarrow \mathbb{R}$, any $r > 0$ and any $0 < \alpha \leq 1$*

$$|\text{osc}_r f|_{\alpha;gH} \leq 2(\sup_D f - \inf_D f) \mu(\text{Conv}(D)) \left(\frac{2d+1}{r} \right)^\alpha,$$

where $\text{Conv}(D)$ denotes the convex hull of D .

We use this via the following corollary:

Corollary 3.28. *There is a global constant $C_5 = C_5(\mathcal{R}_Q) < \infty$ such that for any $0 < \alpha \leq 1$ and $0 < \varepsilon \leq \text{diam}(M)$*

$$\text{var}_\alpha(\text{osc}_{C_\pi\varepsilon} F) \leq \frac{C_5}{\varepsilon^\alpha} (\sup_M F - \inf_M F).$$

Proof. By the definition of var_α in (2.7),

$$\text{var}_\alpha(\text{osc}_{C_\pi\varepsilon} F) = |\text{osc}_{C_\pi\varepsilon} F|_{\alpha;gH} + \sup_M (\text{osc}_{C_\pi\varepsilon} F) - \inf_M (\text{osc}_{C_\pi\varepsilon} F).$$

The second term is $\sup_M (\text{osc}_{C_\pi\varepsilon} F) \leq \sup_M F - \inf_M F$, while the third is $\inf_M (\text{osc}_{C_\pi\varepsilon} F) \geq 0$, so

$$\text{var}_\alpha(\text{osc}_{C_\pi\varepsilon} F) \leq |\text{osc}_{C_\pi\varepsilon} F|_{\alpha;gH} + \sup_M F - \inf_M F. \quad (3.9)$$

To bound the first term, we would like to apply Theorem 3.27 with $d = 3$, $D = M \subset \mathbb{R}^3$, $f = F$, $r = C_\pi\varepsilon$ and $\mu = \frac{1}{\text{Leb}(M)} \text{Leb}$. The only minor problem is that this theorem is about functions on subsets of \mathbb{R}^d , while our F has domain $M \subset \mathbb{T}^3$. This can easily be treated at the cost of some non-optimal constant factor $C = 125$, see Remark 3.29. We get

$$|\text{osc}_{C_\pi\varepsilon} F|_{\alpha;gH} \leq 2(\sup_M F - \inf_M F) \frac{1}{\text{Leb}(M)} 125 \text{Leb}(\mathbb{T}^3) \left(\frac{2 \cdot 3 + 1}{C_\pi\varepsilon} \right)^\alpha.$$

Since $0 < \alpha \leq 1$ and $C_\pi \geq 1$, an upper bound independent of α can be given (although this is not important for us): $\left(\frac{2 \cdot 3 + 1}{C_\pi}\right)^\alpha \leq 7$. We assumed $\varepsilon \leq \text{diam}(M)$, so $1 \leq \left(\frac{\text{diam}(M)}{\varepsilon}\right)^\alpha \leq \frac{\max\{\text{diam}(M), 1\}}{\varepsilon^\alpha}$. Writing these back to (3.9), we get the statement of the corollary with $C_5 = 14 \frac{125 \text{Leb}(\mathbb{T}^3)}{\text{Leb}(M)} + \max\{\text{diam}(M), 1\}$. \square

Remark 3.29. *Theorem 3.27 is about $f : D \rightarrow \mathbb{R}$ where $D \subset \mathbb{R}^d$, while we have $F : M \rightarrow \mathbb{R}$ where $M \subset \mathbb{T}^3$. A non-optimal, but easy way out is the following:*

If $r := C_\pi \varepsilon \geq \text{diam}(M)$ (which is unrealistic anyway), then $\text{osc}_r F \equiv \sup_M F - \inf_M F$ is constant, so $|\text{osc}_r F|_{\alpha; gH} = 0$.

So assume $r = C_\pi \varepsilon < \text{diam}(M)$. We view $M \subset \mathbb{T}^3$ as $M \subset [0, 1]^3 \subset \mathbb{R}^3$, and extend F from M to $M + \mathbb{Z}^3$ periodically. Then set $D := (M + \mathbb{Z}^3) \cap [-2, 3]^3 \subset \mathbb{R}^3$, and restrict F to D . In words: we extend F from a single copy of M to the neighbouring $5 \cdot 5 \cdot 5$ fundamental cells, $5^3 = 125$ copies of M all together, to get some $\tilde{F} : D \rightarrow \mathbb{R}$ where $D \subset \mathbb{R}^3$ is still bounded. On the central $3 \cdot 3 \cdot 3 = 27$ copies of M in $[-1, 2]^3$, $\text{osc}_r \tilde{F}$ is the same as $\text{osc}_r F$. So, as long as $\delta < \text{diam}(M)$, $\text{osc}_\delta(\text{osc}_r \tilde{F})$ and $\text{osc}_\delta(\text{osc}_r F)$ coincide on the central copy $M \subset [0, 1]^3$. So, for $\delta < \text{diam}(M)$,

$$\int_M \text{osc}_\delta(\text{osc}_r F) = \int_{D \cap [0, 1]^3} \text{osc}_\delta(\text{osc}_r \tilde{F}) \leq \int_D \text{osc}_\delta(\text{osc}_r \tilde{F}).$$

(For $\delta \geq \text{diam}(M)$ the oscillation doesn't grow further, meaning $\text{osc}_\delta(\text{osc}_r F) = \text{osc}_{\text{diam}(M)}(\text{osc}_r F)$, so the same bound trivially holds.)

Now we can apply Theorem 3.27 to the extended function $\tilde{F} : D \rightarrow \mathbb{R}$ to get the bound on $|\text{osc}_{C_\pi \varepsilon} \tilde{F}|_{\alpha; gH}$. Clearly $\text{Leb}(\text{Conv}(D)) \leq \text{Leb}([-2, 3]^3) = 125 \text{Leb}(\mathbb{T}^3)$.

3.6 Regularity of the approximating density

Proposition 3.30. *If φ is Θ_φ -dynamically Hölder continuous, then G_0 is uniformly dynamically Hölder continuous when restricted to any element of the u -foliation: There exist constants $C_{G;u} = C_{G;u}(\mathcal{R}_Q, \mathcal{R}_u) < \infty$ and $\Theta_G = \Theta_G(\Theta_\varphi, \mathcal{R}_Q, \mathcal{R}_u) < 1$ such that for any $z \in D$ and $r_1, r_2 \in H_z$*

$$|G_0(r_1) - G_0(r_2)| \leq C_{G;u} \frac{1}{\varepsilon^2} \|\varphi\|_{\Theta_\varphi; dH} \Theta_G^{s^+(r_1, r_2)}.$$

In particular, $C_{G;u}$ and Θ_G do not depend on φ and ε (but Θ_G depends on Θ_φ).

Proof. Lemma 3.23 gives an explicit formula for G_0 :

$$G_0(r) = \text{Leb}(M) \varphi(x) q(z) \frac{1}{Jh_z(x)} \frac{1}{\cos \psi(r)},$$

where $x = \pi(r)$ is the projection of r to H by the holonomy. All factors are dynamically Hölder, thus so is the product. Quantitatively:

a.) Since we fix z , $q(z)$ is just a constant, and Lemma 3.22 ensures that $0 \leq q(z) \leq \frac{3}{\pi} \frac{1}{\varepsilon^2}$.

b.) $\varphi(x) = \varphi(\pi(r))$ has the same dynamical Hölder regularity as φ , due to Lemma 2.20, so for $\varphi \circ \pi : H_z \rightarrow \mathbb{R}$

$$\sup_{H_z} (\varphi \circ \pi) = \sup_H \varphi \leq \sup_{W^u} \varphi \quad , \quad |\varphi \circ \pi|_{\Theta_\varphi; dH} = |\varphi|_{H; \Theta_\varphi; dH} \leq |\varphi|_{\Theta_\varphi; dH}.$$

c.) $\frac{1}{Jh_z(x)} = Jh_{H_z \rightarrow H}(r)$ is exactly the Jacobian of the holonomy from H_z to H , so Theorem 2.22 ensures that

$$\sup_{H_z} \frac{1}{Jh_z} \leq C_h \quad , \quad \left| \frac{1}{Jh_z} \right|_{\Theta_h; dH} \leq C_h.$$

d.) By Lemma 3.9 $\cos \psi(r)$ is α -Hölder with $\alpha = 1$ (which is Lipschitz continuity), so it is also dynamically Hölder by Lemma 2.19 (ii) with some $\Theta_{\cos \psi} = \Theta_{\cos \psi}(\alpha = 1, \mathcal{R}_Q, \mathcal{R}_u) = \Theta_{\cos \psi}(\mathcal{R}_Q, \mathcal{R}_u) < 1$ and $C_{\cos \psi} = C_{\cos \psi}(\alpha = 1, \mathcal{R}_Q, \mathcal{R}_u) = C_{\cos \psi}(\mathcal{R}_Q, \mathcal{R}_u) \in \mathbb{R}$. In turn, Lemma 2.18(iii) ensures that $\frac{1}{\cos \psi(r)}$ is also dynamically Hölder:

$$\left| \frac{1}{\cos \psi(r)} \right| \leq 2 \quad , \quad \left| \frac{1}{\cos \psi} \right|_{\Theta_{\cos \psi}; dH} \leq 4C_{\cos \psi}.$$

So let us choose

$$\Theta_G = \Theta_G(\Theta_\varphi, \mathcal{R}_Q, \mathcal{R}_u) := \max\{\Theta_\varphi, \Theta_h, \Theta_{\cos \psi}\} < 1.$$

With this choice, Lemma 2.18(ii) ensures that the above three dynamical Hölder regularity statements remain valid with Θ_φ , Θ_h and $\Theta_{\cos \psi}$ replaced by Θ_G . Now Lemma 2.18(i) and the definition (2.9) of the dynamical Hölder norm ensure that

$$\begin{aligned} |G_0|_{H_z} |_{\Theta_G; dH} &\leq \text{Leb}(M) \frac{3}{\pi} \frac{1}{\varepsilon^2} (\sup \varphi C_h 4C_{\cos \psi} + \sup \varphi 2C_h + C_h 2|\varphi|_{\Theta_\varphi; dH}) \\ &\leq \text{Leb}(M) \frac{3}{\pi} \frac{1}{\varepsilon^2} C_h (4C_{\cos \psi} + 2 + 2) \|\varphi\|_{\Theta_\varphi; dH}. \end{aligned}$$

Choosing $C_{G;u} := \frac{12}{\pi} \text{Leb}(M) C_h (C_{\cos \psi} + 1)$ completes the proof. \square

Proposition 3.31. *With $\alpha_{G_0} := \frac{1}{3}$, G_0 is uniformly α_{G_0} -Hölder continuous when restricted to any element of the central-stable-foliation: There is a constant $C_{G;cs} = C_{G;cs}(\mathcal{R}_Q, \mathcal{R}_u) < \infty$ such that for any $x \in H$ and any $r_1, r_2 \in W_x^{c-s}$*

$$|G_0(r_1) - G_0(r_2)| \leq C_{G;cs} \sup_{W^u} \varphi \frac{1}{\varepsilon^3} \text{dist}_{W_x^{c-s}}(r_1, r_2)^{\alpha_{G_0}}.$$

Proof. Lemma 3.23 gives an explicit formula for G_0 :

$$G_0(r) = \text{Leb}(M) \varphi(x) q(z) \frac{1}{Jh_z(x)} \frac{1}{\cos \psi(r)}, \quad (3.10)$$

when $r = h_z(x) \in W_x^{c-s}$. For x fixed, the first two terms are constant, the third and last are Lipschitz. Let us estimate $\frac{1}{Jh_z(x)}$.

Let $z_1, z_2 \in D$ and let $r_1 = h_{z_1}(x)$, $r_2 = h_{z_2}(x)$. Then

$$\frac{Jh_{z_2}(x)}{Jh_{z_1}(x)} = Jh_{H_{z_1} \rightarrow H_{z_2}}(r_1)$$

is exactly the Jacobian of the holonomy from $H_{z_1} \subset W_{z_1}^u$ to $H_{z_2} \subset W_{z_2}^u$. We estimate this using Theorem 2.23:

$$\left| \frac{Jh_{z_2}(x)}{Jh_{z_1}(x)} - 1 \right| \leq C_{h2} \left(|\psi(r_2) - \psi(r_1)| + \text{dist}_{W_x^{c-s}}(r_1, r_2)^{\frac{1}{3}} \right). \quad (3.11)$$

Now by Lemma 3.12

$$|\psi(r_2) - \psi(r_1)| \leq 2\Gamma_{\max} \text{dist}(r_1, r_2) \leq 2\Gamma_{\max} \text{dist}(r_1, r_2)^{\frac{2}{3}} \text{dist}(r_1, r_2)^{\frac{1}{3}} \leq \quad (3.12)$$

$$\leq 2\Gamma_{\max} \text{diam}(M)^{\frac{2}{3}} \text{dist}_{W_x^{c-s}}(r_1, r_2)^{\frac{1}{3}}. \quad (3.13)$$

We substitute to (3.11), and use $\frac{1}{C_h} \leq Jh_{z_2}(x)$ from Theorem 2.22 to get

$$\left| \frac{1}{Jh_{z_1}(x)} - \frac{1}{Jh_{z_2}(x)} \right| \leq C_{h2}C_h \left(2\Gamma_{max} \text{diam}(M)^{\frac{2}{3}} + 1 \right) \text{dist}_{W_x^{c-s}}(r_1, r_2)^{\frac{1}{3}}.$$

So $\frac{1}{Jh_z(x)}$ is Hölder continuous in its variable $r = h_z(x)$ along W_x^{c-s} , with exponent $\frac{1}{3}$ and constants

$$\left| \frac{1}{Jh_z(x)} \right| \leq C_h \quad , \quad \left| \frac{1}{Jh_z(x)} \right|_{\frac{1}{3}; H} \leq C_{h2}C_h \left(2\Gamma_{max} \text{diam}(M)^{\frac{2}{3}} + 1 \right).$$

As mentioned before, the other factors in (3.10) are easy:

- a.) Since we fix x , $\varphi(x)$ is just a constant, and of course $0 \leq \varphi(x) \leq \sup_{W^u} \varphi$.
- b.) By Lemma 3.22 q is α -Hölder with $\alpha = 1$ (which is Lipschitz continuity) on D , so by Lemma 2.11 it is also $\frac{1}{3}$ -Hölder and

$$|q|_{\frac{1}{3}; H} \leq \text{diam}(D)^{\frac{2}{3}} |q|_{1; H} = (2\varepsilon)^{\frac{2}{3}} \frac{3}{\pi} \frac{1}{\varepsilon^3} = \frac{3 \cdot 2^{2/3}}{\pi} \frac{1}{\varepsilon^{2\frac{1}{3}}}.$$

Now Lemma 3.10 says that $|z_2 - z_1| \leq 2\text{dist}(r_1, r_2) \leq 2\text{dist}_{W_x^{c-s}}(r_1, r_2)$, so $q(z)$ is also $\frac{1}{3}$ -Hölder as a function of $r = h_z(x)$:

$$|q(z_2) - q(z_1)| \leq \frac{3 \cdot 2^{2/3}}{\pi} \frac{1}{\varepsilon^{2\frac{1}{3}}} |z_2 - z_1|^{\frac{1}{3}} \leq \frac{6}{\pi} \frac{1}{\varepsilon^{2\frac{1}{3}}} \text{dist}_{W_x^{c-s}}(r_1, r_2)^{\frac{1}{3}}.$$

We also have $q(z) \leq \frac{3}{\pi} \frac{1}{\varepsilon^2}$ by Lemma 3.22.

- c.) By Lemma 3.12 $\psi(r)$ is Lipschitz on U , thus so is $\cos \psi(r)$. So it is also $\frac{1}{3}$ -Hölder by Lemma 2.11 (ii). Also $\cos \psi(r) \geq \frac{1}{2}$ by Lemma 3.9, so Lemma 2.11 (iii) says that $\frac{1}{\cos \psi(r)}$ is also $\frac{1}{3}$ -Hölder. Quantitatively,

$$\begin{aligned} \left| \frac{1}{\cos \psi} \right| &\leq 2, \\ \left| \frac{1}{\cos \psi} \right|_{\frac{1}{3}; H} &\leq 4 |\cos \psi|_{\frac{1}{3}; H} \leq 4 \text{diam}(M)^{\frac{2}{3}} |\cos \psi|_{1; H} \leq \\ &\leq 4 \text{diam}(M)^{\frac{2}{3}} |\psi|_{1; H} \leq 4 \text{diam}(M)^{\frac{2}{3}} 2\Gamma_{max}. \end{aligned}$$

Since $\text{dist}(r_1, r_2) \leq \text{dist}_{W_x^{c-s}}(r_1, r_2)$, $\frac{1}{\cos \psi}$ is also $\frac{1}{3}$ -Hölder on W_x^{c-s} with the same constants.

Putting the estimates for the factors in (3.10) together, Lemma 2.11 (i) says that G_0 is $\frac{1}{3}$ -Hölder and

$$|G_0|_{\frac{1}{3}; H} \leq \text{Leb}(M) \sup_{W^u} \varphi \left(\frac{3}{\pi} \frac{1}{\varepsilon^2} C_h 8 \text{diam}(M)^{\frac{2}{3}} \Gamma_{max} + \frac{3}{\pi} \frac{1}{\varepsilon^2} C_{h2} C_h \left(2\Gamma_{max} \text{diam}(M)^{\frac{2}{3}} + 1 \right) 2 + \frac{6}{\pi} \frac{1}{\varepsilon^{2\frac{1}{3}}} C_h 2 \right)$$

Since $\varepsilon \leq \text{diam}(M)$, choosing

$$C_{G;cs} := \text{Leb}(M) \frac{6}{\pi} C_h \left(2\Gamma_{max} (2 + C_{h2}) \text{diam}(M)^{\frac{5}{3}} + C_{h2} \text{diam}(M) + 2 \text{diam}(M)^{\frac{2}{3}} \right)$$

completes the proof. \square

Proposition 3.32. G_0 is uniformly bounded: there is a constant $C_{G;b} = C_{G;b}(\mathcal{R}_Q, \mathcal{R}_u) < \infty$ such that

$$0 \leq G_0(r) \leq \frac{C_{G;b}}{\varepsilon^2} \sup_{W^u} \varphi$$

for every $r \in U_0$.

Proof. This is actually included in the proofs of both previous lemmas. Lemma 3.23 gives an explicit formula for G_0 :

$$G_0(r) = \text{Leb}(M)\varphi(x)q(z)\frac{1}{Jh_z(x)}\frac{1}{\cos\psi(r)},$$

where $x = \pi(r)$ is the projection of r to H by the holonomy. All the factors multiplying φ have known bounds. Quantitatively:

a.) Lemma 3.22 ensures that $0 \leq q(z) \leq \frac{3}{\pi} \frac{1}{\varepsilon^2}$.

b.) $\frac{1}{Jh_z(x)} = Jh_{H_z \rightarrow H}(r)$ is exactly the Jacobian of the holonomy from H_z to H , so Theorem 2.22 ensures that $0 < \frac{1}{Jh_z} \leq C_h$.

c.) By Lemma 3.9, $0 < \frac{1}{\cos\psi(r)} \leq 2$.

Choosing $C_{G;b} := \frac{6}{\pi} \text{Leb}(M)C_h$ completes the proof. \square

3.7 Smoothing the approximating density

Our approximating density G will be a slight modification of G_0 to ensure that it has the required regularity. First, we restrict G_0 from U_0 to a smaller set U_1 to improve its regularity from dynamically Hölder continuous to truly Hölder continuous:

Proposition 3.33. *There exist $C_6 = C_6(\mathcal{R}_Q, \mathcal{R}_u) < \infty$, $\alpha_G = \alpha_G(\mathcal{R}_Q, \mathcal{R}_u, \Theta_\varphi) \leq 1$ and a set $H_1 \subset H$ with the following properties: Let $U_1 = H_1 * D$. Then*

1. $m_{W^u}(W^u \setminus H_1) \leq C_6\varepsilon$,
2. $\mu(U \setminus U_1) \leq C_6\varepsilon^3$,
3. G_0 restricted to U_1 is α_G -Hölder continuous: for any $r_1, r_2 \in U_1$

$$|G_0(r_1) - G_0(r_2)| \leq C_6 \frac{\|\varphi\|_{\Theta_\varphi; dH}}{\varepsilon^3} |r_1 - r_2|^{\alpha_G}.$$

Proof. The main input is the dynamical Hölder continuity of G_0 , as stated in Proposition 3.30. We will construct H_1 by cutting out some neighbourhood of every singularity from H . If we do this appropriately, the total set we cut out will be small, and G_0 restricted to the remaining set will be Hölder (and not only dynamically Hölder). To obtain this, take $c > 0$ and $\theta < 1$ to be specified later. For every singularity of order $n \geq 0$, we cut out a neighbourhood (in the metric on W_z^u) of radius at least $\frac{c}{2}\theta^n$ (meaning an interval of length $c\theta^n$) from every W_z^u .

To make the argument precise, we have to take into account secondary singularities, meaning that in principle, for every n , there are infinitely many intervals we need to cut out around singularities of order n . As usual, this only costs some power of $\frac{1}{\varepsilon}$ in the measure of the set cut out, since the infinitely many secondary singularities accumulate at finitely many primary ones, so the intervals overlap heavily. An easy (not optimal) way to do this is the following: Let $y_k \in W_z^u$ be on the k -th secondary singularity near a primary singularity. Then, by alignment (Theorem 2.9) y_k it is at most

some $\frac{C_7}{k^2}$ far from $y \in W_z^u$, in the metric of W_z^u , where y is on the primary singularity – or, possibly, an endpoint of W_z^u . (Here $C_7 = C_7(\mathcal{R}_Q, \mathcal{R}_u)$.) So if

$$k > k_{crit}(n) := \frac{\sqrt{C_7}}{\sqrt{\frac{c}{2}\theta^n}}, \quad (3.14)$$

then this distance is less than $\frac{c}{2}\theta^n$, meaning that the entire $\frac{c}{2}\theta^n$ -neighbourhood of y_k is contained in the $c\theta^n$ -neighbourhood of y . So, with some generosity, we cut out neighbourhoods of radius $c\theta^n$ around the primary singularity and the first $k_{crit}(n)$ secondary singularities (and possibly the endpoints of W_z^u), and these finitely many intervals cover the $\frac{c}{2}\theta^n$ -neighbourhood of every (primary and secondary) singularity.

Now let $y \in W_{z_y}^u$ be a singular point of order $n(y)$ (either primary or secondary). Due to the continuation property of singularity curves, Lemma 2.1, the singularity containing y – or its continuation – intersects W^u in a single point x , which is singular of order $n(x) \leq n(y)$.⁶

Let $I(y) \subset W_{z_y}^u$ be the neighbourhood of radius $c\theta^{n(y)}$ around y in $W_{z_y}^u$, in the metric of $W_{z_y}^u$. Now the set we cut out from H near x is

$$I_x := \bigcup_y h_{z_y}^{-1}(I(y) \cap H_{z_y}),$$

where the union is over all y that give the same x as above. This is an uncountable union, but the members of the union are all intervals around x in W^u (intersected by H), so the union is just the longest interval. (More precisely, the longest half-interval has to be taken in both directions.) So

$$m_{W^u}(I_x) \leq 2 \sup_y m_{W^u}(h_{z_y}^{-1}(I(y) \cap H_{z_y})) \leq 2 \sup_y C_h m_{W_{z_y}^u}(I(y)) \leq C_h \sup_y c\theta^{n(y)} \leq C_h c\theta^{n(x)}$$

by the absolute continuity, Theorem 2.22. Again, $C_h = C_h(\mathcal{R}_Q, \mathcal{R}_u)$. Now we set

$$H_1 := H \setminus \bigcup_x I_x,$$

where the union is over all singular points $x \in W^u$. Since the number of primary singularities of order n is at most K_{max}^n (see Section 2.2.1), the total length we cut out is at most

$$m_{W^u}(H \setminus H_1) \leq \sum_x m_{W^u}(I_x) \leq \sum_{n=0}^{\infty} k_{crit}(n) K_{max}^n C_h c\theta^n.$$

Using (3.14), this gives

$$m_{W^u}(H \setminus H_1) \leq \sum_{n=0}^{\infty} C_8 K_{max}^n \sqrt{c\theta^n} = C_8 \frac{\sqrt{c}}{1 - K_{max}\sqrt{\theta}}$$

with $C_8 = \sqrt{2C_7C_h}$ if $K_{max}\sqrt{\theta} < 1$. So let us choose $\theta \leq \theta(\mathcal{R}_Q) := \frac{1}{4K_{max}^2}$, which means that

$$m_{W^u}(H \setminus H_1) \leq C_8 \frac{\sqrt{c}}{1 - \frac{1}{2}} = 2C_8\sqrt{c}.$$

Now take $c := \varepsilon^2$. So item 1 is shown, because $W^u \setminus H$ is also small by Lemma 3.14.

⁶Actually, if y is close to the end of the u -curve W_z^u , it may happen that W_z^u terminates before intersecting the singularity. However, this can only happen if all points of $H_z \subset W_z^u$ are on the same side of y , since central-stable manifolds cannot cross singularities. As it will be clear below, such singular points y are of no interest for us.

Item 2 follows from Lemma 3.18 and Lemma 3.21:

$$\mu(U \setminus U_1) = \mu(U \setminus U_0) + \mu((H \setminus H_1) * D) \leq C_4 \varepsilon^3 + C_\beta \varepsilon^2 2C_8 \varepsilon.$$

We are left to prove the last item, which is Hölder continuity of the restriction. We start with Hölder continuity along each W_z^u . By construction, if $r_1, r_2 \in W_z^u \cap U_1 = h_z(H_1)$ and their separation time is $s^+(r_1, r_2) = n$, then there is a singularity of order n separating them, around which we already cut out an interval of length $c\theta^n$, meaning that $\text{dist}_{W_z^u}(r_1, r_2) \geq c\theta^n$. In short,

$$\text{dist}_{W_z^u}(r_1, r_2) \geq c\theta^{s^+(r_1, r_2)}. \quad (3.15)$$

Now Hölder continuity of the restriction of G_0 to any $h_z(H_1)$ follows from Proposition 3.30: for any $r_1, r_2 \in h_z(H_1)$

$$\begin{aligned} |G_0(r_1) - G_0(r_2)| &\leq C_{G;u} \frac{1}{\varepsilon^2} \|\varphi\|_{\Theta_\varphi; dH} \Theta_G^{s^+(r_1, r_2)} = C_{G;u} \frac{1}{\varepsilon^2} \|\varphi\|_{\Theta_\varphi; dH} \left(\theta^{s^+(r_1, r_2)}\right)^{\frac{\ln \Theta_G}{\ln \theta}} \leq \\ &\leq C_{G;u} \frac{1}{\varepsilon^2} \|\varphi\|_{\Theta_\varphi; dH} \left(\frac{1}{c} \text{dist}_{W_z^u}(r_1, r_2)\right)^{\frac{\ln \Theta_G}{\ln \theta}} = C_{G;u} \frac{1}{\varepsilon^2} \|\varphi\|_{\Theta_\varphi; dH} \left(\frac{1}{c}\right)^{\alpha'} \text{dist}_{W_z^u}(r_1, r_2)^{\alpha'} \end{aligned}$$

with $\alpha' := \frac{\ln \Theta_G}{\ln \theta} > 0$. We can choose $\theta = \theta(\mathcal{R}_Q, \Theta_G)$ sufficiently small to make sure that that $\alpha' \leq \frac{1}{2}$, so by our earlier choice $c = \varepsilon^2$ we get (using $\varepsilon \leq \text{diam}(M)$) that

$$|G_0(r_1) - G_0(r_2)| \leq C_9 \frac{\|\varphi\|_{\Theta_\varphi; dH}}{\varepsilon^3} \text{dist}_{W_z^u}(r_1, r_2)^{\alpha'} \quad (3.16)$$

(with $C_9 = C_{G;u} \max\{\text{diam}(M), 1\}$.)

So we are able to compare function values if r_1 and r_2 are on the same u -curve. On the other hand, for two points on the same central-stable manifold, Proposition 3.31 can be applied directly. For arbitrary $r_1, r_3 \in U_1$ we combine the two by setting r_2 to be the only intersection point of the u -curve of r_1 and the central-stable manifold of r_3 : if $r_1 = h_{z_1}(x)$ and $r_3 = h_{z_2}(y)$ with $x, y \in H_1$ and $z_1, z_2 \in D$, then $r_2 := h_{z_1}(y)$. Transversality (by Theorem 2.8) of $W_{z_1}^u$ and W_y^{c-s} guarantees that, with some $C_{10} = C_{10}(\mathcal{R}_Q, \mathcal{R}_u)$

$$\text{dist}_{W_{z_1}^u}(r_1, r_2) \leq C_{10} |r_1 - r_3|$$

and

$$\text{dist}_{W_x^{c-s}}(r_2, r_3) \leq C_{10} |r_1 - r_3|.$$

So (3.16) and Proposition 3.31 give

$$\begin{aligned} |G_0(r_1) - G_0(r_3)| &\leq |G_0(r_1) - G_0(r_2)| + |G_0(r_2) - G_0(r_3)| \leq \\ &\leq C_9 \frac{\|\varphi\|_{\Theta_\varphi; dH}}{\varepsilon^3} \text{dist}_{W_z^u}(r_1, r_2)^{\alpha'} + \frac{C_{G-cs} \sup \varphi}{\varepsilon^3} \text{dist}_{W_x^{c-s}}(r_2, r_3)^{\alpha_{G_0}} \leq \\ &\leq C_9 C_{10}^{\alpha'} \frac{\|\varphi\|_{\Theta_\varphi; dH}}{\varepsilon^3} |r_1 - r_3|^{\alpha'} + \frac{C_{G-cs} \sup \varphi}{\varepsilon^3} C_{10}^{\alpha_{G_0}} |r_1 - r_3|^{\alpha_{G_0}}. \end{aligned}$$

Choosing $\alpha_G := \min\{\alpha', \alpha_{G_0}\}$, we use $|r_1 - r_3| \leq \text{diam}(M)$ and $\sup \varphi \leq \|\varphi\|_{\Theta_\varphi; dH}$ from (2.9) to get the result. \square

Remark 3.34. So far we only defined and considered G_0 at points of the product set $U_0 = H * D$. However, Lemma 3.23 shows that $G_0(r) = 0$ whenever $r \in U_0 \cap W_z^u$ with $|z| = \varepsilon$, since then $q(z) = 0$. So it is reasonable to set $G_0 = 0$ on all of $\bigcup\{W_z^u \mid |z| = \varepsilon\}$, the curved surface of U . With this extension, G_0 is still clearly Hölder along the u -curves W_z^u , and Proposition 3.33 remains true, with the proof unchanged.

Now we extend $G_0|_{U_1}$ from U_1 to all of M .

Proposition 3.35. *There is a global constant $C_{11} = C_{11}(\mathcal{R}_Q, \mathcal{R}_u) < \infty$ and a function $G : M \rightarrow \mathbb{R}^+$ such that*

1. $G = G_0$ on U_1 ,
2. $\mu(\text{supp}(G) \setminus U_1) \leq C_{11}\varepsilon^3$,
3. G is Hölder continuous with constants as G_0 : for any $r_1, r_2 \in M$

$$|G(r_1) - G(r_2)| \leq C_6 \frac{\|\varphi\|_{\Theta_\varphi; dH}}{\varepsilon^3} |r_1 - r_2|^{\alpha_G},$$

where $C_6(\mathcal{R}_Q, \mathcal{R}_u) < \infty$ and $\alpha_G(\mathcal{R}_Q, \mathcal{R}_u, \Theta_\varphi) \leq 1$ are given by Proposition 3.33,

4. $0 \leq G \leq \frac{C_{11}}{\varepsilon^2} \sup_{W^u} \varphi$,
5. $|\int_M G d\mu - 1| \leq C_{11}\varepsilon \sup_{W^u} \varphi$,
6. $\|G\|_{\alpha_G; H} \leq \frac{C_{11}}{\varepsilon^3} \|\varphi\|_{\Theta_\varphi; dH}$.

Proof. Let us attach two semi-spheres of radius ε to the two flat faces of U to get the set U^+ , which is now a neighbourhood of W^u . We set $G := 0$ outside this U^+ , so $\text{supp}(G) \subset U^+$. So

$$\mu(\text{supp}(G) \setminus U_1) \leq \mu(U^+ \setminus U_1) = \mu(U^+ \setminus U) + \mu(U \setminus U_1) \leq \frac{4\varepsilon^3\pi}{3} + C_6\varepsilon^3$$

by Proposition 3.33, so item 2 holds. Setting $G = 0$ outside U^+ does not spoil Hölder continuity of G_0 : the function $G^* : (M \setminus U^+) \cup U_1$ defined as

$$G^*(r) := \begin{cases} G_0(r), & \text{if } r \in U_1 \\ 0, & \text{if } r \in M \setminus U^+ \end{cases}$$

satisfies $0 \leq G^* \leq \sup G_0 \leq \frac{C_{G,b}}{\varepsilon^2} \sup_{W^u} \varphi$ by Proposition 3.32, and it is Hölder continuous with the same constants as G_0 on U_1 as in Proposition 3.33, item 3. This is so because for any point r where G^* is defined and possibly non-zero, the nearest point outside U^+ is surely on $\bigcup\{W_z^u \mid |z| = \varepsilon\}$, so the regularity of G^* follows from Remark 3.34.

Now the abstract extension lemma Lemma A.8 ensures that G^* can be extended to some G defined on all of M with all the required Hölder continuity and upper bound, so items 1, 3 and 4 are shown.

To see item 5 we use that φ is a probability density so $\int_{W^u} \varphi dm_{W^u} = 1$ and that $\int_{U_1} G d\mu = \int_{H_1} \varphi dm_{W^u}$ by (3.6), (3.7) and item 1. We get that

$$\left| \int_M G d\mu - 1 \right| = \left| \int_M G d\mu - \int_{W^u} \varphi dm_{W^u} \right| \leq \int_{M \setminus U_1} G d\mu + \int_{W^u \setminus H_1} \varphi dm_{W^u}.$$

The first term is bounded due to items 2 and 4 of the present proposition, while the second term is bounded due to item 1 of Proposition 3.33, giving the statement of item 5.

Finally, item 6 follows from items 3, 4 and the definitions (2.4) and (2.9). \square

3.8 Using the approximating density

3.8.1 Total error of the approximation

Lemma 3.36. *There is a global constant $C_{12} = C_{12}(\mathcal{R}_Q, \mathcal{R}_u) < \infty$ such that*

$$\left| \int_M (F \circ \Phi^t) G d\mu - \int_M F \circ \Phi^t d\tilde{\varphi} \right| \leq \int_M [(osc_{C_{\pi\varepsilon}} F) \circ \Phi^t] G d\mu + C_{12}(\sup |F|)(\sup_{W^u} \varphi)\varepsilon,$$

where $C_{\pi}(\mathcal{R}_Q, \mathcal{R}_u) < \infty$ if from Proposition 3.26.

Proof. We understand the integrals on U_1 and H_1 well, and the measure of the rest is small. So we write

$$\begin{aligned} & \left| \int_M (F \circ \Phi^t) G d\mu - \int_M F \circ \Phi^t d\tilde{\varphi} \right| = \\ & = \left| \int_{supp(G)} (F \circ \Phi^t) G d\mu - \int_{W^u} F \circ \Phi^t d\tilde{\varphi} \right| \leq \\ & \leq \left| \int_{U_1} (F \circ \Phi^t) G_0 d\mu - \int_{H_1} F \circ \Phi^t d\tilde{\varphi} \right| + \\ & \quad \left| \int_{supp(G) \setminus U_1} (F \circ \Phi^t) G d\mu \right| + \left| \int_{W^u \setminus H_1} F \circ \Phi^t d\tilde{\varphi} \right|. \end{aligned}$$

The first term is estimated using Proposition 3.26. For the second we use Proposition 3.35, items 2 and 4. For the third we use Proposition 3.33, item 1. We get

$$\begin{aligned} & \left| \int_M (F \circ \Phi^t) G d\mu - \int_M F \circ \Phi^t d\tilde{\varphi} \right| \leq \\ & \leq \int_{U_1} [(osc_{C_{\pi\varepsilon}} F) \circ \Phi^t] G_0 d\mu + \\ & \quad + (\sup |F|)(\sup_M G)\mu(supp(G) \setminus U_1) + (\sup |F|)(\sup_{W^u} \varphi)m_{W^u}(W^u \setminus H_1) \leq \\ & \leq \int_M [(osc_{C_{\pi\varepsilon}} F) \circ \Phi^t] G d\mu + \\ & \quad + (\sup |F|) \frac{C_{11}}{\varepsilon^2} (\sup_{W^u} \varphi) C_{11} \varepsilon^3 + (\sup |F|)(\sup_{W^u} \varphi) C_6 \varepsilon = \\ & = \int_M [(osc_{C_{\pi\varepsilon}} F) \circ \Phi^t] G d\mu + (C_{11}^2 + C_6)(\sup |F|)(\sup_{W^u} \varphi)\varepsilon. \end{aligned}$$

□

3.8.2 Completing the proof

We now have all the regularity estimates to complete the proof of our main Theorem 1.4 using the approximating density G .

Remark 3.37. *The “moreover” part of Theorem 1.4 discusses the dependence of the correlation decay on the billiard domain Q . To prove it, we needed and need to keep track of the Q -dependence of our “constants”. Up to this point, every constant and every estimate depended on Q through \mathcal{R}_Q (from (2.1)) only. It is only the remaining final step where a (possibly) more complicated dependence appears, through the application of Theorem 1.5 (more precisely, its corollary Theorem A.5).*

Proof of Theorem 1.4. We need to estimate $\left| \int_M F \circ \Phi^t d\tilde{\varphi} \right|$ from above. We do this using Lemma 3.36, which implies

$$\left| \int_M F \circ \Phi^t d\tilde{\varphi} \right| \leq \left| \int_M (F \circ \Phi^t) G d\mu \right| + \int_M [(osc_{C_{\pi\varepsilon}} F) \circ \Phi^t] G d\mu + C_{12}(\sup |F|)(\sup_{W^u} \varphi)\varepsilon. \quad (3.17)$$

The first term is estimated using Theorem A.5 and Proposition 3.35, item 6. Since $\int_M F d\mu = 0$, the result is

$$\left| \int_M (F \circ \Phi^t) G d\mu \right| \leq \mathcal{C}_{BDL;g} var_{\alpha_F}(F) \|G\|_{\alpha_G;H} e^{-a''t} \leq \mathcal{C}_{BDL;g} var_{\alpha_F}(F) \frac{C_{11}}{\varepsilon^3} \|\varphi\|_{\Theta_\varphi;dH} e^{-a''t}. \quad (3.18)$$

Here $\mathcal{C}_{BDL;g} = \mathcal{C}_{BDL;g}(Q, \alpha_F, \alpha_G)$ and $a'' = a''(Q, \alpha_F, \alpha_G)$ both depend on Q , \mathcal{R}_u , α_F and Θ_φ , since $\alpha_G = \alpha_G(\mathcal{R}_Q, \mathcal{R}_u, \Theta_\varphi)$.

To estimate the second term, we use Theorem A.5 again, with the same α_F and α_G (this choice is for convenience only). We get

$$\int_M [(osc_{C_{\pi\varepsilon}} F) \circ \Phi^t] G d\mu \leq \int_M osc_{C_{\pi\varepsilon}} F d\mu \int_M G d\mu + \mathcal{C}_{BDL;g} var_{\alpha_F}(osc_{C_{\pi\varepsilon}} F) \|G\|_{\alpha_G;H} e^{-a''t}. \quad (3.19)$$

The definitions (2.6) and (2.7) give

$$\int_M osc_{C_{\pi\varepsilon}} F d\mu \leq |f|_{\alpha_F;gH}(C_{\pi\varepsilon})^{\alpha_F} \leq C_{\pi}^{\alpha_F} var_{\alpha_F} \varepsilon^{\alpha_F}.$$

Item 5 of Proposition 3.35 and (2.9) give

$$\int_M G d\mu \leq 1 + C_{11}\varepsilon \sup_{W^u} \varphi \leq (L_{max} + C_{11} diam(M)) \|\varphi\|_{\Theta_\varphi;dH}.$$

Corollary 3.28 with $\alpha = \alpha_F$ and (2.7) give

$$var_{\alpha_F}(osc_{C_{\pi\varepsilon}} F) \leq \frac{C_5}{\varepsilon^{\alpha_F}} (\sup_M F - \inf_M F) \leq \frac{C_5}{\varepsilon^{\alpha_F}} var_{\alpha_F}(F).$$

Finally, item 6 of Proposition 3.35 gives $\|G\|_{\alpha_G;H} \leq \frac{C_{11}}{\varepsilon^3} \|\varphi\|_{\Theta_\varphi;dH}$. Substituting to (3.19), we get

$$\int_M [(osc_{C_{\pi\varepsilon}} F) \circ \Phi^t] G d\mu \leq C_{13} \cdot (1 + \mathcal{C}_{BDL;g}) var_{\alpha_F}(F) \|\varphi\|_{\Theta_\varphi;dH} \left(\varepsilon^{\alpha_F} + \frac{e^{-a''t}}{\varepsilon^{3+\alpha_F}} \right). \quad (3.20)$$

The third term of (3.17) is trivially bounded using (2.7) and (2.9) as

$$C_{12}(\sup |F|)(\sup_{W^u} \varphi)\varepsilon \leq C_{12} var_{\alpha_F}(F) \|\varphi\|_{\Theta_\varphi;dH} \varepsilon.$$

Substituting this, (3.18) and (3.20) back to (3.17), we obtain

$$\left| \int_M F \circ \Phi^t d\tilde{\varphi} \right| \leq C_{14} \cdot (1 + \mathcal{C}_{BDL;g}) var_{\alpha_F}(F) \|\varphi\|_{\Theta_\varphi;dH} \left(\varepsilon^{\alpha_F} + \frac{e^{-a''t}}{\varepsilon^{3+\alpha_F}} \right). \quad (3.21)$$

This holds for every $t \geq 0$ and every $0 < \varepsilon \leq \varepsilon_0$. Now to minimize the sum $\varepsilon^{\alpha_F} + \frac{e^{-a''t}}{\varepsilon^{3+\alpha_F}}$, we choose $\varepsilon := \varepsilon(t)$ so that the two terms are equal, whenever this is allowed by the restriction $\varepsilon \leq \varepsilon_0$. Specifically, let

$$a := \frac{\alpha_F}{3 + 2\alpha_F} a'' \quad (3.22)$$

and

$$\varepsilon := \varepsilon(t) := \min \left\{ e^{-\frac{a''}{3+2\alpha_F}t}, \varepsilon_0 \right\}.$$

With this choice, if $e^{-\frac{a''}{3+2\alpha_F}t} \leq \varepsilon_0$, then

$$\varepsilon^{\alpha_F} + \frac{e^{-a''t}}{\varepsilon^{3+\alpha_F}} = 2\varepsilon^{\alpha_F} = 2e^{-at}.$$

If $e^{-\frac{a''}{3+2\alpha_F}t} \geq \varepsilon_0$, then a short calculation gives

$$\varepsilon^{\alpha_F} + \frac{e^{-a''t}}{\varepsilon^{3+\alpha_F}} = \left[\left(\frac{\varepsilon_0}{e^{-\frac{a''}{3+2\alpha_F}t}} \right)^{\alpha_F} + \left(\frac{e^{-\frac{a''}{3+2\alpha_F}t}}{\varepsilon_0} \right)^{3+\alpha_F} \right] e^{-at} \leq \left[1 + \left(\frac{1}{\varepsilon_0} \right)^{3+\alpha_F} \right] e^{-at}.$$

Substituting to (3.21), we get

$$\left| \int_M F \circ \Phi^t d\tilde{\varphi} \right| \leq C_{14} \cdot (1 + \mathcal{C}_{BDL;g}) \left[2 + \left(\frac{1}{\varepsilon_0} \right)^{3+\alpha_F} \right] \text{var}_{\alpha_F}(F) \|\varphi\|_{\Theta_\varphi; dH} e^{-at}. \quad (3.23)$$

So the main statement of the theorem is proven with

$$\mathcal{C} = C_{14} \cdot (1 + \mathcal{C}_{BDL;g}) \left[2 + \left(\frac{1}{\varepsilon_0} \right)^{3+\alpha_F} \right]. \quad (3.24)$$

To see the “moreover” part of the theorem, note that the Q -dependence of $\mathcal{C}_{BDL;g}$ and a'' is described by the “moreover” part of Theorem A.5. With this, (3.22) gives

$$a = \frac{\alpha_F}{3 + 2\alpha_F} \frac{\alpha_F}{\alpha_F + 1} a'(Q, \alpha_G(\mathcal{R}_Q, \mathcal{R}_u, \Theta_\varphi))$$

and (3.24) gives

$$\mathcal{C} = \mathcal{C}(\mathcal{R}_Q, \mathcal{R}_u, \alpha_F, \mathcal{C}_{BDL}(Q, \alpha_G(\mathcal{R}_Q, \mathcal{R}_u, \Theta_\varphi))).$$

Now the “moreover” part is shown with $\alpha = \alpha_G$. □

4 Possible extension, open problems

4.1 Dependence of constants on the billiard table

As mentioned in the introduction, equidistribution theorems, like the one in this paper, are sometimes applied to a class of models simultaneously. In such situations, it is useful to know that the same estimate holds for all the models in the class. Unfortunately, our main Theorem 1.4 says little about the dependence of the “constants” \mathcal{C} and a on the billiard table Q . The only reason for this is that our main reference, Theorem 1.5 from [1] does not say anything about the Q -dependence of the constants \mathcal{C}_{BDL} and a' . If this dependence was better understood – say, we would know that \mathcal{C}_{BDL} and a' depend on Q through \mathcal{R}_Q only, – then a better understanding of the Q -dependence of \mathcal{C} and a in Theorem 1.4 would be automatic – see the “moreover” part of Theorem 1.4.

However, there is good reason that [1] does not discuss the Q -dependence of constants. At the heart of their functional analytic proof is a compactness argument, which allows only finitely many eigenvalues of the transfer operator in a neighbourhood of the imaginary axis. This ensures a spectral gap, but gives no control on the size of that gap. Getting explicit bounds seems difficult at least.

Another possible way to control the Q -dependence of equidistribution would be to give up exponential decay, and prove stretched exponential equidistribution only. This is clearly not optimal, but also very useful technically, and can be proven with different methods. Indeed, if we want stretched exponential decay only, we can, instead of [1], refer to Theorem 1.1 for [7]:

Theorem 4.1. *Consider a billiard like the one in Theorem 1.4. Assume $0 < \alpha \leq 1$. Then there exist $a' = a'(Q, \alpha) > 0$ and $\mathcal{C}_{Ch} = \mathcal{C}_{Ch}(Q, \alpha) < \infty$ such that for any $F, G : M \rightarrow \mathbb{R}$ generalized α -Hölder functions with $\int_M F d\mu = 0$ and any $t \geq 0$ one has*

$$\left| \int_M (F \circ \Phi^t) G d\mu \right| \leq \mathcal{C}_{Ch} \text{var}_\alpha(F) \text{var}_\alpha(G) e^{-a' \sqrt{t}}.$$

Here $\text{var}_\alpha(\cdot)$ denotes the generalized α -Hölder seminorm defined in (2.7). (Ch stands for Chernov.)

This theorem, combined with lemmas 2.14 and 2.15 has the following obvious corollary:

Corollary 4.2. *A planar billiard flow with finite horizon and no corner points enjoys stretched exponential correlation decay for a pair of Hölder and a generalized Hölder observables. Quantitatively, let $0 < \alpha_F, \alpha_G \leq 1$. Then there exists an $a'' = a''(Q, \alpha_F, \alpha_G) > 0$ and a $\mathcal{C}_{Ch;g} = \mathcal{C}_{Ch;g}(Q, \alpha_F, \alpha_G) < \infty$ such that if $F : M \rightarrow \mathbb{R}$ is α_F -generalized Hölder and $G : M \rightarrow \mathbb{R}$ is α_G -Hölder, then for any $t \geq 0$*

$$\left| \int_M (F \circ \Phi^t) G d\mu - \int_M F d\mu \int_M G d\mu \right| \leq \mathcal{C}_{Ch;g} \text{var}_{\alpha_F}(F) \|G\|_{\alpha_G; H} e^{-a'' \sqrt{t}}.$$

(Here again Ch stands for Chernov.)

Moreover, $a''(Q, \alpha_F, \alpha_G) = a'(Q, \min\{\alpha_F, \alpha_G\})$ and

$$\mathcal{C}_{Ch;g}(Q, \alpha_F, \alpha_G) = \max\{(\text{diam}(M))^{|\alpha_G - \alpha_F|}, 1\} \mathcal{C}_{Ch}(Q, \min\{\alpha_F, \alpha_G\})$$

where a' and \mathcal{C}_{Ch} are from Theorem 4.1.

Now the stretched exponential version of our main Theorem 1.4 follows:

Theorem 4.3. *Suppose the billiard table Q satisfies assumptions 1.2, 1.3 and 1.1. Let $0 < \Theta_\varphi < 1$, $0 < \alpha_F \leq 1$ and $\varepsilon_0 > 0$. Then there exist $\mathcal{C} < \infty$ and $a > 0$ with the following properties:*

Let (W^u, φ) be a standard pair with a dynamically Θ_φ -Hölder φ . Assume that W^u is a good u -curve. Let $F : M \rightarrow \mathbb{R}$ be generalized α_F -Hölder continuous. Then, for every $t \geq 0$,

$$\left| \int_{W^u} (F \circ \Phi^t) \varphi dm_{W^u} - \int_M F d\mu \right| \leq \mathcal{C} \cdot \|\varphi\|_{\Theta_\varphi; dH} \cdot \text{var}_{\alpha_F} F \cdot e^{-a \sqrt{t}}.$$

Here

- $a = a(Q, \mathcal{R}_u, \Theta_\varphi, \alpha_F) < \infty$ and $\mathcal{C} = \mathcal{C}(Q, \mathcal{R}_u, \Theta_\varphi, \alpha_F) < \infty$ depend on the billiard table Q , the regularity of good u -curves quantified in \mathcal{R}_u , and the regularity classes of φ and F given by Θ_φ and α_F . They do not depend on W^u , φ and F .

Moreover, \mathcal{C} depends on Q only through \mathcal{R}_Q from (2.1) and $\mathcal{C}_{Ch}(Q, \alpha)$ from Theorem 4.1 with some $\alpha = \alpha(\mathcal{R}_Q, \mathcal{R}_u, \Theta_\varphi, \alpha_F) > 0$. Similarly, a depends on $(Q, \mathcal{R}_u, \Theta_\varphi)$ only through $a'(Q, \alpha)$ from Theorem 4.1 with the same $\alpha = \alpha(\mathcal{R}_Q, \mathcal{R}_u, \Theta_\varphi, \alpha_F) > 0$. That is,

$$\mathcal{C} = \mathcal{C}(\mathcal{R}_Q, \mathcal{R}_u, \alpha_F, \mathcal{C}_{Ch}(Q, \alpha(\mathcal{R}_Q, \mathcal{R}_u, \Theta_\varphi, \alpha_F)))$$

and

$$a = a(\alpha_F, a'(Q, \alpha(\mathcal{R}_Q, \mathcal{R}_u, \Theta_\varphi, \alpha_F))).$$

Note that there are only two differences between this theorem and the main Theorem 1.4. First, there is \sqrt{t} instead of t in the exponent. Second, the description of the Q -dependence of constants in the “moreover” part now refers to Theorem 4.1 instead of Theorem 1.5.

Proof. The proof of Theorem 1.4 from Section 3.8.2 applies up to the proof of the main statement in (3.24), with the following minor modifications:

- every t should be replaced by \sqrt{t} ,
- every $\mathcal{C}_{BDL;g}$ should be replaced by $\mathcal{C}_{Ch;g}$,
- every reference to Theorem A.5 should be replaced by a reference to Corollary 4.2.

To see the “moreover” part, we use the “moreover” part of Corollary 4.2. With this, (3.22) gives

$$a = \frac{\alpha_F}{3 + 2\alpha_F} a'(Q, \min\{\alpha_F, \alpha_G(\mathcal{R}_Q, \mathcal{R}_u, \Theta_\varphi)\})$$

and (3.24) gives

$$\mathcal{C} = \mathcal{C}(\mathcal{R}_Q, \mathcal{R}_u, \alpha_F, \mathcal{C}_{Ch}(Q, \min\{\alpha_F, \alpha_G(\mathcal{R}_Q, \mathcal{R}_u, \Theta_\varphi)\})).$$

Now the “moreover” part is shown with $\alpha = \min\{\alpha_F, \alpha_G\}$. \square

Just like in the case of our main Theorem 1.4, if we were able to better understand the functions a' and \mathcal{C}_{Ch} in Theorem 4.1, then we would also know the Q -dependence of a and \mathcal{C} better. This problem is currently open: as a first step, a detailed analysis of the proof in [7] is needed to see what Q -dependence can be read out of it.

4.2 Corner points

A weakness of the main Theorem 1.4 is that it requires Assumption 1.1, so it does not cover the case of corner points. Again, the main reason is that our main reference Theorem 1.5 does not cover corner points either. However, several regularity properties that we use, are also known only in the case of no corner points. We do not attempt to generalize these here. Instead, we discuss the points where Assumption 1.1 is used, and state two conditional theorems, describing what needs to be checked to cover the case of corner points.

If there are corner points, there is no lower bound τ_{min} for the free flight. Instead of Assumption 1.1, we require Assumption 4.4 below to rule out cusps.⁷

Assumption 4.4 (No cusps). *For every corner point $r \in \Gamma_i \cap \Gamma_j$, the angle of Γ_i and Γ_j at x is non-zero.*

Then there is some lower bound ξ_{min} for the angle of the two smooth components of ∂Q that meet at a corner. Also, let d_{min} be the minimal free flight between two scatterers that do *not* form a corner. (If d_{min} is very small, there is a “bottleneck” situation between two scatterers that almost touch. Such a situation is close to a cusp.)

So the (hopefully sufficient) set of regularity parameters for the table Q is now

$$\mathcal{R}_Q^{corner} := \{\tau_{max}, d_{min}, \xi_{min}, \kappa_{min}, \kappa_{max}, \kappa'_{max}, K_{max}, A_{min}, A_{max}, d_Q\} \quad (4.1)$$

(compare (2.1)).

The inputs of our argument – apart from the main reference [1] – are the regularity properties of u -curves stated in sections 2.2.4 and 2.4. Of these, the elementary ones in Section 2.2.4 are known

⁷According to the terminology of [12], billiards satisfying assumptions 1.2, 1.3 and 4.4 belong to category C.

under Assumption 4.4 (no cusps) instead of Assumption 1.1, so there is nothing to do, see also Remark 2.10. On the other hand, the advanced regularity properties of Section 2.4 are proven in Section A.1 with reference to Theorem 5.67, Proposition 5.48 and Theorem 5.42 from [12]. These are definitely known only for billiards with no corner points.

To get the three statements of Section 2.4 from these statements of [12], our arguments (in Section A.1) do not use Assumption 1.1. That is, to prove them for billiards with corner points, it would be enough to prove the corner point generalizations of the three statements of [12]. This justifies the following conditional extension of our main theorem:

Theorem 4.5. *Suppose the billiard table Q satisfies assumptions 1.2, 1.3 and 4.4. Assume that*

1. *the generalization of Theorem 2.21, or alternatively, Theorem 5.67 from [12] is proven for this class of billiards, with \mathcal{R}_Q replaced by $\mathcal{R}_Q^{\text{corner}}$,*
2. *the generalization of Theorem 2.22, or alternatively, Proposition 5.48 from [12] is proven for this class of billiards, with \mathcal{R}_Q replaced by $\mathcal{R}_Q^{\text{corner}}$,*
3. *the generalization of Theorem 2.23, or alternatively, Theorem 5.42 from [12] is proven for this class of billiards, with \mathcal{R}_Q replaced by $\mathcal{R}_Q^{\text{corner}}$,*
4. *the generalization of Theorem 1.5 (which is Corollary 1.3 in [1]) is proven for this class of billiards.*

Then there is exponential equidistribution for the evolution of the standard pair under the flow:

Let $0 < \Theta_\varphi < 1$, $0 < \alpha_F \leq 1$ and $\varepsilon_0 > 0$. Then there exist $\mathcal{C} < \infty$ and $a > 0$ with the following properties:

Let (W^u, φ) be a standard pair with a dynamically Θ_φ -Hölder φ . Assume that the $2\varepsilon_0$ -neighbourhood of W is disjoint from the boundary of M . Let $F : M \rightarrow \mathbb{R}$ be generalized α_F -Hölder continuous. Then, for every $t \geq 0$,

$$\left| \int_{W^u} (F \circ \Phi^t) \varphi dm_{W^u} - \int_M F d\mu \right| \leq \mathcal{C} \cdot \|\varphi\|_{\Theta_\varphi; dH} \cdot \text{var}_{\alpha_F} F \cdot e^{-at}.$$

Here

- *$a = a(Q, \mathcal{R}_u, \Theta_\varphi, \alpha_F) < \infty$ and $\mathcal{C} = \mathcal{C}(Q, \mathcal{R}_u, \Theta_\varphi, \alpha_F) < \infty$ depend on the billiard table Q , the regularity of good u -curves quantified in \mathcal{R}_u , and the regularity classes of φ and F given by Θ_φ and α_F . They do not depend on W^u , φ and F .*

Moreover, \mathcal{C} depends on Q only through $\mathcal{R}_Q^{\text{corner}}$ from (4.1) and $\mathcal{C}_{BDL}(Q, \alpha)$ from the generalization of Theorem 1.5 with some $\alpha = \alpha(\mathcal{R}_Q^{\text{corner}}, \mathcal{R}_u, \Theta_\varphi) > 0$. Similarly, a depends on $(Q, \mathcal{R}_u, \Theta_\varphi)$ only through $a'(Q, \alpha)$ from the generalization of Theorem 1.5 with the same $\alpha = \alpha(\mathcal{R}_Q^{\text{corner}}, \mathcal{R}_u, \Theta_\varphi) > 0$. That is,

$$\mathcal{C} = \mathcal{C}(\mathcal{R}_Q^{\text{corner}}, \mathcal{R}_u, \alpha_F, \mathcal{C}_{BDL}(Q, \alpha(\mathcal{R}_Q^{\text{corner}}, \mathcal{R}_u, \Theta_\varphi)))$$

and

$$a = a(\alpha_F, a'(Q, \alpha(\mathcal{R}_Q^{\text{corner}}, \mathcal{R}_u, \Theta_\varphi))).$$

Similarly to the situation in Section 4.1, we don't know if the generalization of Theorem 1.5 to the case of corner points can reasonably be done. An alternative can be to treat the corner points with the method of [7], and get stretched exponential equidistribution only. This would be good enough for most applications. The analogue of Theorem 4.5 for this case is straightforward.

4.3 Standard pairs close to scatterers

Theorem 1.4, as we stated and proved it, requires that the u -curve W^u is at least ε_0 far away from collision points. This is clearly a weakness from the application point of view. However, it is not a serious restriction, because if ε_0 is small enough, then any short u -curve W^u , which is possibly too close to a scatterer (or even in the process of being reflected from it), will soon evolve (under Φ^{t_0} with some small t_0) either into a curve which is more than ε_0 away, or possibly a bounded number of such curves. (At most two if there are no corner points.) At the same time, φ evolves into some density φ' on $\Phi^{t_0}W^u$. Now we can apply Theorem 1.4 to the pair $(\Phi^{t_0}W^u, \varphi')$, provided that it is a standard pair, and the regularity can be quantitatively checked.

This requires a study of the time evolution of standard pairs (for finite times), especially the evolution of their regularity parameters. This was not needed for the proof of our theorem, and is thus not done here. The notion of standard pairs (including the notion of u -curves) was designed with this need in mind, and we expect the notion to be time-invariant in some good sense. We leave this to a future paper that applies this result.

5 Acknowledgement

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A Appendix

A.1 Holonomy

Here we provide the proofs for the three theorems stated in Section 2.4, namely theorems 2.21, 2.22 and 2.23.

Notation A.1. *Since the arguments in this section are independent from the bulk of the paper, we will use slightly different notation for better readability. In particular, we will use φ for the angle of the velocity with the normal vector of the scatterer at collision points, as usual in the billiard literature.*

These theorems formulate regularity properties of u -curves (of the flow phase space). For simplicity we reduce their proofs to regularity properties of discrete time u -curves. The direct reduction to those of u -curves of the billiard ball map raises a technical problem since the transition between the discrete time and the flow u -curves has bad regularity near tangent collisions: the flight time that connects them has unbounded derivative. There is an additional fact we want to emphasize: our reference [12] states the theorems in a form which is a little weaker than what they actually prove, and what we need. In particular, they use the term “constant” for numbers which depend on the billiard table Q only, but they do not discuss the form of this dependence. To see that their constants actually depend on Q through \mathcal{R}_Q from (2.1), we need to look into the proofs.

Taking the aforementioned circumstances into consideration our approach will still be to reduce our statements to the discrete time analogues by introducing so-called *transparent walls* and relying on regularity properties of u -curves living on them. In this way we can still use the results of [12] by also avoiding unbounded derivatives. This means that we extend the discrete time phase space \mathcal{M} with an extra component $\mathcal{M}_{tr} = \partial Q_{tr} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. The motion of the particle remains unchanged, but when $\mathcal{M}_{tr} \subset M$ is reached, that is treated as a collision (note that \mathcal{M}_{tr} is in the phase space, i.e. we only keep track of passing through the transparent wall in one direction). Since the statements

we want to prove are local, it is enough to set up the transparent wall near the u -curves we study. See Figure 2.

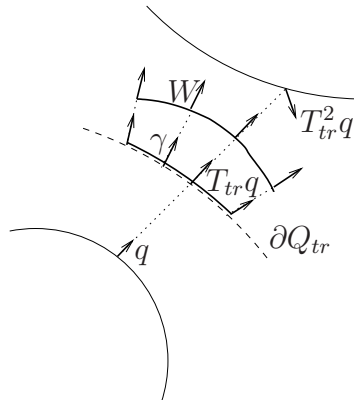


Figure 2: A transparent wall with the trace of a u -curve

The new map $T_{tr} : \mathcal{M} \cup \mathcal{M}_{tr} \rightarrow \mathcal{M} \cup \mathcal{M}_{tr}$ is very similar to the billiard dynamics, and the theorems we use from [12] remain valid. The new domain $\partial Q \cup \partial Q_{tr}$ will depend on the u -curves under consideration, but we take care that the constants in the estimates remain uniform.

In return, we can always choose the new wall so that it has a big enough angle with the trajectories considered. This makes it easy to show the necessary regularity properties for the transition map between the flow curves and their traces on the transparent wall.

Before the specific calculations, we ask the reader to recall the notations \mathcal{R}_Q and \mathcal{R}_u (formulas (2.1) and (2.2)).

Convention A.2. *As discussed in Section 2.2.3, C denotes constants that depend only on Q and \mathcal{R}_u , also called geometric constants. In this section, dependence on Q is always through \mathcal{R}_Q only, so $C = C(\mathcal{R}_Q, \mathcal{R}_u)$, even if this is not indicated. On the other hand, the exact value of C may change from line to line (even within a line).*

Throughout, we consider *good u -curves* W for the flow (cf. Definitions 2.5 and 2.6). In particular

$$W = \{p(s) = (x(s), y(s), v_x(s), v_y(s)) | s \in I\}.$$

This generates $W_0 = \{(x(s), y(s))\}$, a curve in the configuration domain. In fact, we have

$$(v_x(s), v_y(s)) = (\cos \omega(s), \sin \omega(s)),$$

as the the speed is one. Note that the curvature of W_0 (as a curve in the configuration space) is bounded from below by some constant $B_{\min} = B_{\min}(\mathcal{R}_Q, \mathcal{R}_u)$. To describe the tangent vectors in $\mathcal{T}_p W$ it is convenient to use the Jacobi coordinates $(d\xi, d\eta, d\omega)$, where $d\xi = -\sin \omega dx + \cos \omega dy$ and $d\eta = \cos \omega dx + \sin \omega dy$. Note that the arc length along W can be computed by integrating $\sqrt{d\xi^2 + d\eta^2 + d\omega^2}$.

Now we construct the transparent wall associated to the good u -curve W . Take one particular point $p(s_0) = (x(s_0), y(s_0))$ in the interior of W_0 , and consider the unit (velocity) vector $v(s_0) = (v_x(s_0), v_y(s_0))$. Consider, furthermore the point $p' = p(s_0) - Lv(s_0)$ in the configuration space. Let ∂Q_{tr} denote the circular arc of length $10L$, and curvature $B_{\min}/2$ centred at p' with normal vector $v(s_0)$ at p' . Then ∂Q_{tr} is our transparent wall, which will be regarded as a scatterer. Accordingly, the billiard map phase space \mathcal{M} can be extended with the associated component \mathcal{M}_{tr} , which is parametrized by the usual (r, φ) (configuration and velocity) coordinates. ∂Q_{tr} has the following properties:

- The trajectories emerging from the points of W necessarily intersect ∂Q_{tr} for some negative time.
- Let $\gamma = \{q(s) = (r(s), \varphi(s))\}$ denote the resulting trace of W on \mathcal{M}_{tr} . Then γ is a u -curve for this Poincaré section.
- From now on, with a slight abuse of notation, let s denote arc length along γ . We have $W = \{p(s) = (q(s), \tau(s))\}$ where, by bounded curvature, $\tau(s)$ is C^1 and $\tau'(s)$ is Lipschitz with some constant C independent of W . The canonical projection π provides a one-to-one correspondence between γ and W .
- The above properties hold not only for W but for any other good u -curve \bar{W} sufficiently close to W : \bar{W} will leave a trace $\bar{\gamma} = \pi(\bar{W})$ on ∂Q_{tr} when flown backwards in time; $\bar{\gamma}$ is a u -curve for the Poincaré section \mathcal{M}_{tr} and $\bar{W} = \{\bar{p}(s) = (\bar{q}(s), \bar{\tau}(s))\}$, with $\bar{q}(s)$ tracing $\bar{\gamma}$, and $\bar{\tau}$ C^1 with a uniformly Lipschitz continuous derivative.

Convention A.3. *In what follows we will work with the billiard table that has the additional transparent wall ∂Q_{tr} . The regularity parameters (cf. (2.1)) of this extended billiard configuration are denoted by $\mathcal{R}_{Q, \partial Q_{\text{tr}}}$. These depend, a priori, on ∂Q_{tr} . Nonetheless, by construction, and as ∂Q_{tr} is associated to a good u -curve, we have $\mathcal{R}_{Q, \partial Q_{\text{tr}}} = \mathcal{R}_Q \cup \mathcal{R}_u$. Note that \mathcal{R}_u includes the regularity parameter ε_0 , which is the minimum distance of the good u -curves from the scatterers.*

The proofs of Theorems 2.21, 2.22 and Theorem 2.23 are essentially the reduction to analogous statements for the billiard map, discussed in [12] as Theorem 5.67, Proposition 5.48 and Theorem 5.42, respectively. In fact, in the formulation of these statements in [12], the constants C depend on the billiard domain Q . In particular, a priori, in our setting this implies a dependence on the u -curve W_1^u via the choice of the transparent wall ∂Q_{tr} . However, a careful study of the relevant arguments in [12] reveals that the constants C depend only on the regularity parameters of the billiard domain (in the terminology of Section 2.2.1). To see this, we also refer to Appendix A.2. in [9] – in particular Extension 1 – where the analogous property of the constants is explicitly stated.⁸ Thus, in our case, we have $C = C(\mathcal{R}_{Q \cup \partial Q_{\text{tr}}})$ in the statements. By Convention A.3, this means $C = C(\mathcal{R}_Q, \mathcal{R}_u)$, so not only are the estimates uniform in W^u , but the constants only depend on Q through \mathcal{R}_Q , as desired by the theorems we are proving.

To reduce our statements to their discrete time analogues (with the help of the transparent wall ∂Q_{tr}), we first have to discuss the regularity of the projection π of W to $\gamma \subset \mathcal{M}_{\text{tr}}$.

Lemma A.4. *The map π is absolutely continuous and its Jacobian $J_{\gamma \rightarrow W}(s)$ satisfies*

$$C^{-1} < J_{\gamma \rightarrow W}(s) < C \quad \text{and} \quad |J_{\gamma \rightarrow W}(s) - J_{\gamma \rightarrow W}(s')| \leq C|s - s'|$$

for some uniform C .

Proof. Below we obtain an explicit formula for the Jacobian $J_{\gamma \rightarrow W}(s)$ in several steps. For any $q(s) \in \gamma$ and $p(s) = \pi^{-1}(q(s)) \in W$, we need to relate the line element ds along γ to the length of the corresponding tangent vector, $(d\xi, d\eta, d\omega)$, of W at $p(s)$. For brevity the dependence on s is omitted whenever there is no risk of ambiguity and the following notations are introduced:

- $m = \frac{d\varphi}{dr}$, the slope of γ in the Poincaré section.
- $K = K(r)$, the curvature of ∂Q_{tr} at the point of collision.

⁸The statement in [9] concerns the rate of correlation decay, yet the proof of exponential decay of correlations heavily relies on the regularity properties of the holonomy map.

- W_+ the post-collision front (or local orthogonal manifold) emerging from γ , and $(d\xi_+, 0, d\omega_+)$, its tangent vector.
- B_+ , the curvature of the post-collision front. In particular, $B_+ = \frac{d\omega_+}{d\xi_+}$.

The following relations are standard, see [12]:

$$ds = \sqrt{1 + m^2} \cdot dr; \quad d\xi_+ = \cos \varphi \cdot dr; \quad m = B_+ \cos \varphi - K.$$

By construction

- $\cos \varphi(r)$ is bounded away from 0,
- $m(r)$ is bounded from above, while $B_+(r)$ is bounded away from 0 and infinity,
- all these quantities depend on r in a Lipschitz continuous manner, with uniformly bounded Lipschitz constant,
- ds , dr and $d\xi_+$ are uniformly equivalent.

Hence $d\xi_+ = \frac{\cos \varphi(r)}{\sqrt{1+m(r)^2}} \cdot ds$, and as $d\omega_+ = B_+(r)d\xi_+$, the Jacobian $J_{\gamma \rightarrow W_+}$ is bounded away from 0 and infinity, and Lipschitz with uniformly bounded Lipschitz constant.

The free flight evolution of W_+ into W results in

$$d\xi = d\xi_+ + \tau d\omega_+ = (1 + \tau B_+)d\xi_+, \quad d\omega_+ = d\omega.$$

We still need to compute $d\eta$. By [12], section 3.3, we have

$$d\eta = \sin \varphi dr + d\tau = (\sin \varphi + \tau'(r))dr.$$

Now we are in the position to compute the Jacobian $J_{\gamma \rightarrow W} = \frac{\sqrt{d\xi^2 + d\eta^2 + d\omega^2}}{ds}$. As

$$\begin{aligned} \sqrt{d\xi^2 + d\eta^2 + d\omega^2} &= \sqrt{(1 + \tau B_+)^2 + B_+^2 + \frac{(\sin \varphi + \tau')^2}{\cos^2 \varphi}} \cdot \cos \varphi dr \\ &= \frac{\sqrt{(1 + \tau B_+)^2 \cos^2 \varphi + B_+^2 \cos^2 \varphi + (\sin \varphi + \tau')^2}}{\sqrt{1 + m^2}} \cdot ds \end{aligned} \quad (\text{A.1})$$

our analysis implies that the Jacobian $J_{\gamma \rightarrow W}(s)$ is uniformly bounded away from 0 and infinity, and depends on s in a Lipschitz continuous manner, with uniformly bounded Lipschitz constant. \square

Proof of Theorem 2.21. Let us mention first that it would be possible to prove Theorem 2.21 by discussing the flow directly, as in the proof of the related Proposition 6.10 from [12]. Yet, in accordance with the other arguments presented here, we prefer to deduce it from the corresponding statement for the map, i.e. Theorem 5.67 in [12].

Consider a good u -curve W , the associated transparent wall ∂Q_{tr} and the trace $\gamma = \pi(W)$. For $x \in W$, let $r^{c-s}(x)$ denote the inner radius of the homogeneous local central stable manifold centred at x , while for $p \in \gamma$, let $r^s(p)$ denote the inner radius of the homogeneous stable manifold (of the billiard map T with the new transparent wall) centred at p . Fix some $\varepsilon > 0$ and introduce

$$B_{W,\varepsilon} = \{x \in W | r^{c-s}(x) \leq \varepsilon\}; \quad B_{\gamma,\varepsilon} = \{x \in \gamma | r^s(x) \leq \varepsilon\}.$$

Now we refer to Theorem 5.67 from [12] which ensures that

$$m_\gamma(B_{\gamma,\varepsilon}) \leq C\varepsilon. \quad (\text{A.2})$$

On the other hand, Lemma A.4 ensures that

$$m_W(B_{W,\varepsilon}) \leq Cm_\gamma(\pi(B_{W,\varepsilon})). \quad (\text{A.3})$$

By construction of ∂Q_{tr} , the flow time between γ and W is uniformly bounded, and there exists some constant C_{15} such that if $r^s(\pi(x)) > C_{15}\varepsilon$, then $r^{c-s}(x) > \varepsilon$. Hence

$$\pi(B_{W,\varepsilon}) \subset B_{\gamma,C_{15}\varepsilon}. \quad (\text{A.4})$$

Theorem 2.21 follows by combining (A.2), (A.3) and (A.4). \square

Proof of Theorem 2.22. We use the notations introduced above and those of Theorem 2.22. Let W_1^u and W_2^u be two good u -curves, and let ∂Q_{tr} be the transparent wall constructed from W_1^u as discussed above. Let us assume that W_1^u and W_2^u are sufficiently close that both leave traces on ∂Q_{tr} , to be denoted by γ_1 and γ_2 , respectively. Let $H_1 \subset W_1^u$ denote the set of points for which the (homogeneous) central-stable manifold (of the flow) extends to W_2^u . Analogously, let $H_1^0 \subset \gamma_1$ denote the set of points for which the (homogeneous) stable manifold (of the map) extends to γ_2 . Let $h : H_1 \rightarrow W_2^u$ and $h_0 : H_1^0 \rightarrow \gamma_2$ denote the holonomy maps given by sliding along central-stable and stable manifolds, respectively, and let $H_2 = h(H_1)$ and $H_2^0 = h_0(H_1^0)$ denote the ranges of these holonomy maps, respectively. We have $H_i^0 = \pi(H_i)$ for $i = 1, 2$. For $p, p' \in H_1$ the separation times on H_1, H_2, H_1^0, H_2^0 are equal, i.e. $s^+(p, p') = s^+(\pi(p), \pi(p')) = s^+(h(p), h(p')) = s^+(h_0(\pi(p)), h_0(\pi(p')))$.

To express the Jacobian J_h , it is worth introducing the restrictions of the Jacobians $J_1 = J_{\gamma_1 \rightarrow W_1^u}|_{H_1^0}$ and $J_2 = J_{\gamma_2 \rightarrow W_2^u}|_{H_2^0}$. Then we have

$$J_h = (J_1)^{-1} \cdot J_{h_0} \cdot J_2. \quad (\text{A.5})$$

Theorem 2.22 follows from the following claim.

Claim. There exist uniform constants $C > 0$ and $\Theta < 1$ such that $C^{-1} < J(p) < C$ and $|\log J(p) - \log J(p')| \leq C\Theta^{s^+(p,p')}$, for $J = (J_1)^{-1}, J_2$ and J_{h_0} .

Concerning J_1 and J_2 first note that the analysis of Lemma A.4 applies to $J_{\gamma_i \rightarrow W_i^u}$ for $i = 1$ and 2 as well. Also, for $p, p' \in H_1$ let s and s' denote the arc length parameters of the projections $\pi(p), \pi(p') \in \gamma_i$, respectively. Then, by uniform expansion $|s - s'| \leq C\Theta^{s^+(p,p')}$. Hence Lemma A.4 implies the Claim for $(J_1)^{-1}$ and J_2 .

As for J_{h_0} , the Claim is explicitly stated in [12], Proposition 5.48. This completes the proof of the Claim, and hence Theorem 2.22 follows. \square

Proof of Theorem 2.23. Fix $p_1 \in W_1$. Introduce the transparent wall ∂Q_{tr} corresponding to W_1 as above. Throughout, we use the notations from the proof of Theorem 2.22. Note that $p_2 = h(p_1)$. Projections onto \mathcal{M}_{tr} are denoted as $q_i = \pi(p_i)$ and $\gamma_i = \pi(W_i)$; ($i = 1, 2$). For brevity let us introduce $J_i = J_i(q_i)$ ($i = 1, 2$) and $J_0 = J_{h_0}(q_1)$. As discussed in the proof of Theorem 2.22, $C^{-1} \leq J \leq C$ for $J = J_0, J_1, J_2$. Furthermore, the decomposition of Formula (A.5) applies and

$$|J_h(p) - 1| \leq |J_1^{-1} \cdot J_2| \cdot |J_0 - 1| + |J_1^{-1} \cdot J_2 - 1|.$$

Hence the following two claims imply the statement of Theorem 2.23.

Claim 1. $|J_2 - J_1| \leq C(\delta + \alpha)$.

Claim 2. $|J_0 - 1| \leq C(\delta + \alpha^{1/3})$.

Let $\tau_i = \tau(q_i)$, $\varphi_i = \varphi(q_i)$, ($i = 1, 2$). Let, furthermore, m_i denote the slope of γ_i at q_i , and let B_i denote the curvature of the corresponding outgoing front (B_+ in the proof of Lemma A.4).

We make the following observations.

- (i) As $d(p_1, p_2) = \delta$, we have $d(q_1, q_2) \leq C\delta$. Hence $|\tau_2 - \tau_1| \leq C\delta$ and $|\varphi_2 - \varphi_1| \leq C\delta$.
- (ii) Recall that α denotes the angle of the tangent vectors $v_1 = T_{p_1}W_1$ and $v_2 = T_{p_2}W_2$. Let $v_i = (d\xi_i, d\eta_i, d\omega_i)$, for $i = 1, 2$. As it only the directions of the vectors v_1 and v_2 that matter, we may fix the $d\xi_i$ to have unit length. $d\xi_1$ and $d\xi_2$ correspond to the flow directions at p_1 and p_2 , respectively, hence $|d\xi_1 - d\xi_2| \leq C\delta$. It follows that $\left| \frac{d\omega_1}{d\xi_1} - \frac{d\omega_2}{d\xi_2} \right| \leq C(\delta + \alpha)$ and $\left| \frac{d\eta_1}{d\xi_1} - \frac{d\eta_2}{d\xi_2} \right| \leq C(\delta + \alpha)$. On the other hand, by the analysis presented in the proof of Lemma A.4, $\frac{d\omega_i}{d\xi_i} = (\tau_i + B_i^{-1})^{-1}$ and $\frac{d\omega_i}{d\xi_i} = \frac{\sin \varphi_i + \tau'_i}{\cos \varphi_i (1 + \tau_i B_i)}$, for $i = 1, 2$.
- (iii) It follows that $|(\tau_2 + B_2^{-1})^{-1} - (\tau_1 + B_1^{-1})^{-1}| \leq C(\delta + \alpha)$. Furthermore, these quantities, just like the B_i ($i = 1, 2$), are bounded away from zero and infinity by some geometric constants. This, along with the previous items, implies $|B_2 - B_1| \leq C(\delta + \alpha)$ and hence $|m_2 - m_1| \leq C(\delta + \alpha)$. Similarly, $|(\sin \varphi_2 + \tau'(q_2)) - (\sin \varphi_1 + \tau'(q_1))| \leq C(\delta + \alpha)$ (and thus $|\tau'(q_2) - \tau'(q_1)| \leq C(\delta + \alpha)$).

To prove Claim 1, we use the explicit Formula (A.1) for both $J_1(q_1)$ and $J_2(q_2)$. Note first that in this formula both the numerator and the denominator are bounded away from zero and infinity by geometric constants. Hence Claim 1 follows from the above items.

To prove Claim 2, we refer to Theorem 5.42 in [12]. This Theorem implies that $|J_{h_0} - 1| \leq C(|\angle(\gamma_1(q_1), \gamma_2(q_2))| + d(q_1, q_2)^{1/3})$, where $d(q_1, q_2)$ is the distance of q_1 and q_2 , while $\angle(\gamma_1(q_1), \gamma_2(q_2))$ is the angle that the direction of γ_1 at q_1 makes with the direction of γ_2 at q_2 – which is bounded by $C|m_1 - m_2|$. Now, by the analysis above, $d(q_1, q_2) \leq C\delta$ while $\angle(\gamma_1(q_1), \gamma_2(q_2)) \leq C(\alpha + \delta)$. This implies Claim 2, and thus completes the proof of Theorem 2.23. \square

A.2 Exponential correlation decay for generalized Hölder observables

Here we present a simple extension of Theorem 1.5, which is from [1], to allow generalized Hölder observables. In the first step we only allow one of the two observables to be generalized Hölder, because this is the version we will use.

Theorem A.5. *A planar billiard flow with finite horizon and no corner points enjoys exponential correlation decay for a Hölder and a generalized Hölder observable. Quantitatively, let $0 < \alpha_F, \alpha_G \leq 1$. Then there exists an $a'' = a''(Q, \alpha_F, \alpha_G) > 0$ and a $\mathcal{C}_{BDL;g} = \mathcal{C}_{BDL;g}(Q, \alpha_F, \alpha_G) < \infty$ such that if $F : M \rightarrow \mathbb{R}$ is α_F -generalized Hölder and $G : M \rightarrow \mathbb{R}$ is α_G -Hölder, then for any $t \geq 0$*

$$\left| \int_M (F \circ \Phi^t) G d\mu - \int_M F d\mu \int_M G d\mu \right| \leq \mathcal{C}_{BDL;g} \text{var}_{\alpha_F}(F) \|G\|_{\alpha_G; H} e^{-a''t}.$$

(Here BDL stands for Baladi-Demers-Liverani.)

Moreover, $a''(Q, \alpha_F, \alpha_G)$ depends on Q and α_G only through $a'(Q, \alpha_G)$ from Theorem 1.5: actually, $a'' = \frac{\alpha_F}{\alpha_F + 1} a'(Q, \alpha_G)$. Similarly, $\mathcal{C}_{BDL;g}(Q, \alpha_F, \alpha_G)$ depends on Q only through \mathcal{R}_Q from (2.1) and $\mathcal{C}_{BDL}(Q, \alpha_G)$ from Theorem 1.5, so it has the form $\mathcal{C}_{BDL;g} = \mathcal{C}_{BDL;g}(\mathcal{R}_Q, \mathcal{C}_{BDL}(Q, \alpha_G), \alpha_F, \alpha_G)$.

Proof. Both the left and the right hand side remain unchanged if we add a constant to F , so it is common to assume – without loss of generality – that $\int_M F d\mu = 0$. However, the Hölder norm – which is used in Theorem 1.5 – does depend on additive constants. On the other hand, if F takes both positive and negative values, we have

$$\sup |F| \leq \sup F - \inf F \leq 2 \sup |F|,$$

hence in this case the Hölder norm can be controlled by $var_{\alpha_F}(F)$. For convenience, in the argument below we assume, without loss of generality, that

$$\sup |F| = \sup F = -\inf F = \frac{\sup F - \inf F}{2}. \quad (\text{A.6})$$

Note that if F is such that $\int_M F d\mu = 0$, and a constant is added to satisfy condition (A.6), $\sup |F|$ is affected at most by a factor 2, while both versions of F take both positive and negative values (unless F is identically zero).

To prove the statement, the natural idea is to choose some $r > 0$, replace F with the smoothed version $\tilde{F}(x) := \frac{1}{Leb(B_r(x))} \int_{B_r(x)} F dLeb$, and then apply Theorem 1.5 to \tilde{F} and G . This \tilde{F} would be Lipschitz continuous whenever F is (measurable and) bounded, if F were defined on \mathbb{R}^d or \mathbb{T}^d . The only thing we have to be careful about is that F is only defined on the phase space $M = Q \times \mathbb{T} \subset \mathbb{T}^3$. For that reason, we first extend F to the r -neighbourhood $B_r(M)$ of M in a fairly arbitrary way, say

$$\hat{F}(x) := \begin{cases} F(x) & \text{if } x \in M, \\ \inf\{F(y) \mid y \in B_r(x) \cap M\} & \text{if } x \in B_r(M) \setminus M. \end{cases}$$

Now for $x \in M$ we can define

$$\tilde{F}(x) := \frac{1}{Leb(B_r(x))} \int_{B_r(x)} \hat{F} dLeb = \frac{3}{4r^3\pi} \int_{B_r(x)} \hat{F} dLeb.$$

These definitions ensure that $\inf \hat{F} = \inf F$, $\sup \hat{F} = \sup F$, so for any $x, y \in M$

$$\begin{aligned} \tilde{F}(y) - \tilde{F}(x) &= \frac{3}{4r^3\pi} \left(\int_{B_r(y) \setminus B_r(x)} \hat{F} dLeb - \int_{B_r(x) \setminus B_r(y)} \hat{F} dLeb \right) \leq \\ &\leq \frac{3}{4r^3\pi} \left(\sup \hat{F} Leb(B_r(y) \setminus B_r(x)) - \inf \hat{F} Leb(B_r(x) \setminus B_r(y)) \right) \leq \\ &\leq \frac{3}{4r^3\pi} (\sup F - \inf F) r^2 \pi |y - x| = \frac{3}{4\pi} (\sup F - \inf F) \frac{1}{r} |y - x|, \end{aligned}$$

so \tilde{F} is Lipschitz continuous and

$$|\tilde{F}|_{1;H} \leq \frac{3}{4\pi} (\sup F - \inf F) \frac{1}{r}.$$

Now Lemma 2.11

ii says that \tilde{F} is also α_G -Hölder and with (A.6) we get

$$\|\tilde{F}\|_{\alpha_G;H} \leq \left(\frac{3}{4\pi} diam(M)^{1-\alpha_G} \frac{1}{r} + \frac{1}{2} \right) (\sup F - \inf F) \leq \left(\frac{3}{4\pi} diam(M)^{1-\alpha_G} \frac{1}{r} + \frac{1}{2} \right) var_{\alpha_F} F. \quad (\text{A.7})$$

On the other hand, the definitions imply that

$$\inf_{B_{2r}(x)} F \leq \tilde{F}(x) \leq \sup_{B_{2r}(x)} F,$$

so

$$|\tilde{F}(x) - F(x)| \leq \sup_{B_{2r}(x)} F - \inf_{B_{2r}(x)} F = (osc_{2r} F)(x),$$

which means that

$$\|\tilde{F} - F\|_{L^1(\mu)} \leq \int_M osc_{2r} F d\mu \leq |F|_{\alpha_F;gH}(2r)^{\alpha_F} \leq 2^{\alpha_F} var_{\alpha_F} F \cdot r^{\alpha_F}. \quad (\text{A.8})$$

To prove the theorem, we write

$$\left| \int_M (F \circ \Phi^t) G d\mu - \int_M F d\mu \int_M G d\mu \right| \leq \left| \int_M (\tilde{F} \circ \Phi^t) G d\mu - \int_M \tilde{F} d\mu \int_M G d\mu \right| + 2 \sup |G| \cdot \|\tilde{F} - F\|_{L^1(\mu)}.$$

We estimate the first term using Theorem 1.5 and (A.7), and the second term using (A.8) to get

$$\begin{aligned} & \left| \int_M (F \circ \Phi^t) G d\mu - \int_M F d\mu \int_M G d\mu \right| \leq \\ & \leq \mathcal{C}_{BDL}(Q, \alpha_G) \|\tilde{F}\|_{\alpha_G; H} \|G\|_{\alpha_G; H} e^{-a'(Q, \alpha_G)t} + 2 \|G\|_{\alpha_G; H} 2^{\alpha_F} \text{var}_{\alpha_F} F \cdot r^{\alpha_F} \leq \\ & \leq \text{var}_{\alpha_F} F \|G\|_{\alpha_G; H} \left(\frac{3\mathcal{C}_{BDL}}{4\pi} \text{diam}(M)^{1-\alpha_G} \frac{e^{-a'(Q, \alpha_G)t}}{r} + \frac{\mathcal{C}_{BDL}}{2} e^{-a'(Q, \alpha_G)t} + 2^{\alpha_F+1} r^{\alpha_F} \right). \end{aligned} \quad (\text{A.9})$$

Choosing $r = e^{-\frac{a'}{\alpha_F+1}t} \leq 1$ and

$$a'' = a''(Q, \alpha_F, \alpha_G) := \frac{\alpha_F}{\alpha_F + 1} a'(Q, \alpha_G)$$

we get $r^{\alpha_F} = \frac{e^{-a't}}{r} = e^{-a''t}$ and $e^{-a't} \leq e^{-a''t}$, so the statement follows with

$$\mathcal{C}_{BDL;g} = \mathcal{C}_{BDL;g}(Q, \alpha_F, \alpha_G) := 2^{\alpha_F+1} + \left(\frac{3}{4\pi} \text{diam}(M)^{1-\alpha_G} + \frac{1}{2} \right) \mathcal{C}_{BDL}(Q, \alpha_G).$$

□

For completeness we state the extension for the case when both observables are only generalized Hölder. We will not use this version in this paper.

Theorem A.6. *A planar billiard flow with finite horizon and no corner points enjoys exponential correlation decay for generalized Hölder observables. Quantitatively, let $0 < \alpha_F, \alpha_G \leq 1$. Then there exists an $a''' = a'''(Q, \alpha_F, \alpha_G) > 0$ and a $\mathcal{C}_{BDL;gg} = \mathcal{C}_{BDL;gg}(Q, \alpha_F, \alpha_G) < \infty$ such that if $F : M \rightarrow \mathbb{R}$ is α_F -generalized Hölder and $G : M \rightarrow \mathbb{R}$ is α_G -generalized Hölder, then for any $t \geq 0$*

$$\left| \int_M (F \circ \Phi^t) G d\mu - \int_M F d\mu \int_M G d\mu \right| \leq \mathcal{C}_{BDL;gg} \text{var}_{\alpha_F}(F) \text{var}_{\alpha_G}(G) e^{-a'''t}.$$

(Here BDL stands for Baladi-Demers-Liverani.)

The proof is completely analogous to the proof of Theorem A.5, and we omit it.

A.3 Extension of Hölder continuous functions

Lemma A.7. *Let (X, d) be a metric space, $\emptyset \neq D \subset X$, $z \in X$ and $f : D \rightarrow \mathbb{R}$ Hölder continuous with constants $0 \leq C < \infty$ and $0 < \alpha \leq 1$ meaning that*

$$|f(x) - f(y)| \leq Cd(x, y)^\alpha \text{ for any } x, y \in D. \quad (\text{A.10})$$

Then there exists a function $\tilde{f} : D \cup \{z\} \rightarrow \mathbb{R}$ such that $\tilde{f} = f$ on D , $\inf f \leq \tilde{f} \leq \sup f$ and \tilde{f} is Hölder continuous with the same constants as f :

$$|\tilde{f}(x) - \tilde{f}(y)| \leq Cd(x, y)^\alpha \text{ for any } x, y \in D \cup \{z\}. \quad (\text{A.11})$$

Proof. We must have $\tilde{f} = f$ on D , so (A.11) is trivial if $x, y \neq z$, and also if $x = y = z$. So we only need to choose $f(z) \in [\inf f, \sup f]$ so that (A.11) holds with $y = z$ for every $x \in D$. For every $x \in D$, define the numbers

$$a_x := f(x) - Cd(x, z)^\alpha \quad , \quad b_x := f(x) + Cd(x, z)^\alpha$$

and the interval $I_x := [a_x, b_x]$. If we can choose $f(z)$ such that $f(z) \in I_x \cap [\inf f, \sup f]$ for every $x \in D$, we are done because $f(z) \in I_x$ is equivalent to (A.11) with $y = z$. Set

$$a := \sup_{x \in D} a_x \quad , \quad b := \inf_{x \in D} b_x.$$

Clearly $a \leq \sup f$ and $b \geq \inf f$, so it enough to show that

$$a \leq b, \tag{A.12}$$

since this means not only that $\bigcap_{x \in D} I_x = [a, b] \neq \emptyset$, but also that

$$\bigcap_{x \in D} I_x \cap [\inf f, \sup f] = [a, b] \cap [\inf f, \sup f] \neq \emptyset.$$

First we show that $I_x \cap I_y \neq \emptyset$ for any $x, y \in D$. The function $g(u) := u^\alpha$ is concave and satisfies $g(0) \geq 0$. Such functions are known to be subadditive: $(u + v)^\alpha \leq u^\alpha + v^\alpha$ for any $u, v \geq 0$. So the triangle inequality $d(x, y) \leq d(x, z) + d(y, z)$ implies that

$$d(x, y)^\alpha \leq (d(x, z) + d(y, z))^\alpha \leq d(x, z)^\alpha + d(y, z)^\alpha,$$

so (A.11) implies that

$$f(y) - f(x) \leq Cd(x, z)^\alpha + Cd(y, z)^\alpha.$$

Rearranging this inequality gives

$$a_y = f(y) - Cd(y, z)^\alpha \leq f(x) + Cd(x, z)^\alpha = b_x.$$

Similarly, we can get $a_x \leq b_y$, so $I_x \cap I_y = [\max\{a_x, a_y\}, \min\{b_x, b_y\}] \neq \emptyset$.

Now if $b < a$ would hold, we would also have some x and y with $b_x < a_y$, a contradiction (we assumed $D \neq \emptyset$). So we have shown (A.12) and the proof is complete. \square

Lemma A.8. *Let (X, d) be a separable metric space, $\emptyset \neq D \subset X$ and $f : D \rightarrow \mathbb{R}$ Hölder continuous with constants $0 \leq C < \infty$ and $0 < \alpha \leq 1$ meaning that*

$$|f(x) - f(y)| \leq Cd(x, y)^\alpha \text{ for any } x, y \in D.$$

Then there exists a function $\tilde{f} : X \rightarrow \mathbb{R}$ such that $\tilde{f} = f$ on D , $\inf f \leq \tilde{f} \leq \sup f$ and \tilde{f} is Hölder continuous with the same constants as f :

$$|\tilde{f}(x) - \tilde{f}(y)| \leq Cd(x, y)^\alpha \text{ for any } x, y \in X.$$

Proof. Let $E = \{x_1, x_2, \dots\}$ be a countable dense subset in X . Let $D_0 := D$ and $D_n := D \cup \{x_1, \dots, x_n\}$ for $n = 1, 2, \dots$. We define \tilde{f} of each x_i inductively: If it is already defined with the required properties on D_{n-1} , then Lemma A.7 gives $\tilde{f}(x_n)$ by the extension to D_n . The function \tilde{f} defined on $D \cup E$ this way also has the required properties (for any $x, y \in D \cup E$, $\tilde{f}(x)$ and $\tilde{f}(y)$ are obtained in finitely many steps). Now we have a continuous function on a dense set, so the continuous extension to all of X exists and is unique with the trivial definition $\tilde{f}(x) := \lim_{D \cup E \ni y \rightarrow x} \tilde{f}(y)$,

and obviously preserves the regularity properties. \square

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