Recurrence properties of a special type of Heavy-Tailed Random Walk

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Abstract

In the proof of the invariance principle for locally perturbed periodic Lorentz process with finite horizon, a lot of delicate results were needed concerning the recurrence properties of its unperturbed version. These were analogous to the similar properties of Simple Symmetric Random Walk. However, in the case of Lorentz process with infinite horizon, the analogous results for the corresponding random walk are not known, either. In this paper, these properties are ascertained for the appropriate random walk (this happens to be in the non normal domain of attraction of the normal law). As a tool, an estimation of the remainder term in the local limit theorem for the corresponding random walk is computed.

1 Introduction

The appearance of the Brownian motion as a limit object in either stochastic or deterministic models is an extremely important and interesting phenomenon. The first result in this field is due to M. Donsker (see [8]) who proved that the diffusively scaled Simple Symmetric Random Walk (SSRW) converges to the Brownian Motion in each dimension. Later, D. Szász and A. Telcs in [19] proved that the local perturbation in the integer lattice of dimension at least two does not spoil the Brownian limit.

In the last decades, a more complex model, i.e. planar periodic Lorentz process was proven to have Brownian motion, as a limit object. Here, one considers periodically situated fixed strictly convex smooth scatterers, and a dimensionless point particle moving among the scatterers and bouncing off at the boundaries according to the classical law of mechanics (the angle of incidence coincides the angle of reflection). In the case of finite horizon (i.e. when the free flight vector of the particle is bounded), diffusive scaling produces Brownian motion (see [1] and [2]). In the case of infinite horizon (i.e. when the free flight vector of the particle is unbounded) a superdiffusive scaling is needed to obtain the non-trivial Brownian limit (see [20] and [4]). Again, the question of the effect of local perturbations naturally arises. This topic has a physical motivation as well, since Lorentz process can be thought of as the movement of a "classical" electron in a crystal, when local perturbation can be some impurities or some locally acting external force. The Brownian limit for diffusively scaled periodic Lorentz process with finite horizon and local perturbation was proven in [6] and [7]. Note that here a more involved investigation was needed than in the case of SSRW, namely, the wide treatment of recurrence properties in [6] was essential.

Recently, D. Paulin and D. Szász proved ([16]) that the random walk, which is very similar to the

Lorentz process with infinite horizon, with local impurities, enjoys the Brownian limit. However, they only treated some simplified local perturbation (see later), and did not consider the recurrence properties similar to the ones in [6], which are expected to be important in the case of infinite horizon, too. Here, we are going to focus on these recurrence properties.

This paper is organized as follows. In Section 2, basic definitions, statements are given and another motivation for our calculations (i.e. the proof of the polynomial decay of the velocity auto correlation function for some perturbed random walk) is provided. The quite well known local limit theorem for our specific type random walk will not be enough for our purposes, i.e. we need to estimate the remainder term of it. Section 3 is devoted to this computation. In Section 4, the desired recurrence properties are obtained, while in Section 5 we give a final remark, and indicate a possible direction of further research.

2 Preliminaries

Let us consider a Random Walk, the behavior of which is close to the one of the Lorentz process with infinite horizon. Namely, define independent random variables X_i, such that

$$\mathbb{P}\left(X_i = n\right) = c_1 |n|^{-3},$$

if $n \neq 0$, and E_i to be uniformly distributed on the 4 unit vectors in $\mathbb{Z}^2$. Now put $\xi_i = X_i E_i$. (Here, of course, $c_1 = \frac{1}{2\zeta(3)}$, but this will not be important for us.) Define the Heavy-Tailed Random Walk (HTRW) by $S_n := \sum_{i=1}^n \xi i$.

This distribution is the same, as the one of the free flight vector of the Lorentz process with infinite horizon (see [20]). However, one could think that our choice is rather special, as the walker can only step along the x and y axis. But this is not the case, as a particle performing Lorentz process can have arbitrary long steps only in finitely many directions, too. Here, we choose that two particular directions, but this is not essential.

Further, define the one dimensional HTRW as

$$Q_n := \sum_{i=1}^n X_i.$$

The quite well-known local limit theorem in one dimension states that

$$\mathbb{P}(Q_n = x) \sim \frac{1}{2\sqrt{\pi c_1 n \log n}} \exp\left(-\frac{x^2}{4c_1 n \log n}\right) \tag{1}$$

and in two dimensions that

$$\mathbb{P}(S_n = x) \sim \frac{1}{4\pi c_1 n \log n} \exp\left(-\frac{|x|^2}{4c_1 n \log n}\right). \tag{2}$$

These can found in [17]. Later, we will need estimations on the error terms in (1) and (2), and by computing them, a proof of (1) and (2) will be provided.

Further, we will use the notations

$$\begin{aligned} u_2(n) &= \mathbb{P}(S_n = (0,0)), \\ u_1(n) &= \mathbb{P}(Q_n = 0). \end{aligned}$$

In the case of billiards, a quite frequent strategy is to prove exponential decay of correlations (an interesting result for its own sake) and then to use this to prove convergence to the Brownian motion (see [3], for instance). As a motivation for our further calculations, we are going to illustrate that in the case of local perturbation, this does not seem to be a good strategy.

For this consider the simplest case: a perturbed SSRW (T_n) in $\mathbb{Z}^d$, where perturbation means that in the origin there is no scatterer, i.e. outside of the origin T_n behaves like an ordinary SSRW, while it flies through the origin. More precisely,

$$P\left(T_{n+1} = e_i | T_{n-1} = -e_i, T_n = 0\right) = 1, \tag{3}$$

where e_i is some neighboring point of the origin in $\mathbb{Z}^d$. The following Proposition is well known in the physics literature (see, for example [18]) but surprisingly, I was unable to find a mathematical proof for it.

Proposition 1 *The velocity autocorrelation function of T_n is $O\left(n^{-(d/2+1)}\right)$.*

Proof. First, suppose that $d = 1$ and $T_0 = 1$. We can identify our process with an unperturbed SSRW - U_n, say - by simply dropping the origin and the extra step from it. Formally, define $\tau(n) = \#\{1 \leq k < n : T_k = 0\}$. Now, if $T_n > 0$, then let $U(n - \tau(n)) = T_n$. If $T_n < 0$, then $U(n - \tau(n)) = T_n + 1$. Now, we have to show that

$$\begin{aligned} & \mathbb{P}(U(2n) = 0, U(2n+1) = 1) - \mathbb{P}(U(2n+1) = 1, U(2n+2) = 0) \\ = \quad & \frac{1}{2}\left[\mathbb{P}(U(2n) = 0) - \mathbb{P}(U(2n+1) = 1)\right] = O\left(n^{-3/2}\right), \end{aligned}$$

which is an elementary consequence of the well known Edgeworth expansion.

Now, suppose that $d > 1$ and $T_0 = (1, 0, ..., 0)$. It suffices to prove

$$\int_\Omega I_{\{T_n = T_0\}} - I_{\{T_n = -T_0\}} dP = O\left(n^{-(d/2+1)}\right). \tag{4}$$

Let V be the orthogonal complement space of T_0 and define

$$H = \{\omega : (V \setminus 0) \cap \{T_0, ..., T_n\} \neq \emptyset\} \subset \Omega.$$

Because of the reflection principle, the part of the integral in (4) over H is zero. The integral over $\Omega \setminus H$ can be treated similarly, as it was done in the one dimensional case.

■

3 Local Limit Theorem with Remainder Term

The aim of this section is to estimate remainder term in the limit theorem (2). To do this, first we have to deal with the one dimensional case. Similar calculations were done previously, see, for example [12] and [14]. However, in these articles only one dimensional, non-lattice distributions were considered. Fortunately, we do not need precise calculation of the remainder term, i.e. summability is enough for our purposes. As usual, we start with the computation of the characteristic function.

Lemma 1 *For the characteristic function ϕ of X_1*

$$\phi(t) = 1 - 2c_1 t^2 |\log|t|| + O\left(t^2\right),$$

as $t \to 0$.

Proof. Since the distribution is symmetric, it suffices to prove for $t > 0$. Fix $\varepsilon > 0$ such that $1-x^2-x^3 < \cos x < 1 - x^2 + x^3$ if $|x| < \varepsilon$. Now, let us consider the decomposition

$$\phi(t) = \mathbb{E}(\exp(itX)) = \sum_{n=1}^{\varepsilon\lfloor t^{-1}\rfloor} \frac{2c_1}{n^3}\cos(tn) + \sum_{n=\varepsilon\lfloor t^{-1}\rfloor+1}^{\infty} \frac{2c_1}{n^3}\cos(tn) =: S_1 + S_2.$$

It is easy to see that

$$S_2 = 2c_1 \int_{\varepsilon}^{\infty} \frac{\cos x}{x^3} dx t^2 + o\left(t^2\right) = O\left(t^2\right).$$

On the other hand, since

$$S_1 = 2c_1 \sum_{m=t, m\in t\mathbb{Z}}^{\varepsilon} t^3 m^{-3}\cos m,$$

we have

$$\left| \frac{S_1}{2c_1} - \sum_{m=t,m\in t\mathbb{Z}}^{\varepsilon} t^3 m^{-3} + \sum_{m=t,m\in t\mathbb{Z}}^{\varepsilon} t^3 m^{-1} \right| < \sum_{m=t,m\in t\mathbb{Z}}^{\varepsilon} t^3.$$

Now the estimations

$$\begin{aligned} \sum_{m=t,m\in t\mathbb{Z}}^{\varepsilon} t^3 m^{-3} &= \frac{1}{2c_1} + O\left(\frac{t^2}{\varepsilon^2}\right) \\ \sum_{m=t,m\in t\mathbb{Z}}^{\varepsilon} t^3 m^{-1} &= t^2 log\left(\frac{\varepsilon}{t}\right) + O\left(t^2\right) \end{aligned}$$

and

$$\sum_{m=t,m\in t\mathbb{Z}}^{\varepsilon} t^3 = O(t^2)$$

finish the proof.

■

Now, we turn to the estimation of the remainder term in the one dimensional local limit theorem.

Theorem 1 *For the one dimensional HTRW the following estimation holds uniformly in x*

$$\mathbb{P}(Q_n = x) - \frac{1}{\sqrt{2\pi}\sqrt{2c_1}\sqrt{n\log n}} \exp\left(-\frac{x^2}{4c_1 n\log n}\right) = O\left(\frac{\log\log n}{\sqrt{n\log^3 n}}\right)$$

Proof. Let g denote the probability density function of the standard Gaussian law. Then we have

$$g(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-izs - \frac{s^2}{2}\right) ds.$$

On the other hand, according to the Fourier inversion formula,

$$\mathbb{P}(Q_n = x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(-itx\right)\phi^n(t)dt.$$

By an elementary argument (see, for example, [13]) our result follows from the statement

$$\left|\sqrt{2c_1 n\log n}\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(-itx)\phi^n(t)dt - g\left(\frac{x}{\sqrt{2c_1 n\log n}}\right)\right| = O\left(\frac{\log\log n}{\log n}\right), \tag{5}$$

where the great order on the right hand side is uniform in x. As it is quite usual in the theory of limit theorems (see again [13]), we estimate the left hand side of (5) by the sum of several integrals

$$\begin{aligned}
&\int_{\frac{1}{\log n}<|s|<\log n}\left|\phi^n\left(\frac{s}{\sqrt{2c_1 n\log n}}\right) - \exp\left(-\frac{s^2}{2}\right)\right| ds \\
+ &\int_{|s|<\frac{1}{\log n}}\left|\phi^n\left(\frac{s}{\sqrt{2c_1 n\log n}}\right)\right| ds + \int_{|s|<\frac{1}{\log n}}\left|\exp\left(-\frac{s^2}{2}\right)\right| ds \\
+ &\int_{\log n<|s|<\gamma\sqrt{2c_1 n\log n}}\left|\phi^n\left(\frac{s}{\sqrt{2c_1 n\log n}}\right)\right| ds \\
+ &\int_{\gamma\sqrt{2c_1 n\log n}<|s|<\pi\sqrt{2c_1 n\log n}}\left|\phi^n\left(\frac{s}{\sqrt{2c_1 n\log n}}\right)\right| ds \\
+ &\int_{\log n<|s|}\left|\exp\left(-\frac{s^2}{2}\right)\right| ds =: I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}$$

So it suffices to prove that $I_j = O\left(\frac{\log\log n}{\log n}\right)$, for $j \in \{1,2,3,4,5,6\}$.
For the estimation of I_1, observe that for $\frac{1}{\log n} < |s| < \log n$ Lemma 1 yields

$$\phi^n\left(\frac{s}{\sqrt{2c_1 n\log n}}\right) = \exp\left(-\frac{s^2}{2}\right)\left[1 + O\left(\frac{s^2\log\log n}{\log n}\right)\right],$$

where the great order on the right hand side is uniform in s. Hence

$$I_1 < \int_{\frac{1}{\log n}<|s|<\log n} s^2\exp\left(-\frac{s^2}{2}\right) dsO\left(\frac{\log\log n}{\log n}\right) = O\left(\frac{\log\log n}{\log n}\right).$$

Further, $|\phi(t)| \leq 1$ yields $I_2 = O\left(\frac{\log\log n}{\log n}\right)$ and $I_3 = O\left(\frac{\log\log n}{\log n}\right)$ is trivial. It can be proven (see Theorem 4.2.1. in [13]) that there exists $\gamma > 0$ such that

$$\phi^n\left(\frac{s}{\sqrt{2c_1 n\log n}}\right) < \exp\left(-C|s|\right),$$

with an appropriate C if $|t| < \gamma$. This estimation implies $I_4 < O\left(\frac{\log\log n}{\log n}\right)$. Observe that $|\phi(t)| \leq 1$ and $|\phi(t)| = 1$ holds if and only if $t \in 2\pi\mathbb{Z}$. As $|\phi(t)|$ is continuous in t, there exists some $C' < 1$ such that $|\phi(t)| < C'$ for $t \in [\gamma, \pi]$. It follows that $I_5 < O\left(\frac{\log\log n}{\log n}\right)$. Finally, $I_6 < O\left(\frac{\log\log n}{\log n}\right)$ by elementary computation. Hence the statement.

■

Now, we turn to the two dimensional case. Define the two dimensional characteristic function $\phi_2 : \mathbb{R}^2 \to \mathbb{C}$, $\phi_2(t) = \mathbb{E}(\exp(it'\xi_1))$, where $'$ stands for transpose, and write $t = (t_1, t_2)'$, $s = (s_1, s_2)'$. Lemma 1 implies that

$$\phi_2(t) = 1 - c_1 t_1^2|\log|t_1|| - c_1 t_2^2|\log|t_2|| + O\left(|t|^2\right),$$

as $|t| \to 0$. Similarly to the one dimensional case, the local limit theorem with remainder term reads as follows.

Theorem 2 *For the two dimensional HTRW the following estimation holds uniformly for* $x \in \mathbb{R}^2$

$$\mathbb{P}(S_n = x) - \frac{1}{2\pi 2c_1 n \log n} \exp\left(-\frac{|x|^2}{4c_1 n \log n}\right) = O\left(\frac{\log\log n}{n \log^2 n}\right)$$

Proof. The proof is similar to the proof of Theorem 1. Let g denote the probability density function of the two dimensional standard Gaussian law. Then we have

$$g(z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-it'z - \frac{t't}{2}\right) dt.$$

On the other hand, according to the Fourier inversion formula

$$\mathbb{P}(S_n = x) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \exp\left(-it'x\right) \phi_2^n(t) dt.$$

Just like previously, it is enough to prove that

$$\left| 2c_1 n \log n \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \exp(-it'x) \phi_2^n(t) dt - g\left(\frac{x}{\sqrt{2c_1 n \log n}}\right) \right| \qquad (6)$$

is in $O\left(\frac{\log\log n}{\log n}\right)$. The analogue of the previous decomposition in the present case is

$$\begin{aligned}
& \int_{\frac{1}{\log^3 n} < |s_1|, |s_2| < \log n} \left| \phi_2^n\left(\frac{s}{\sqrt{2c_1 n \log n}}\right) - \exp\left(-\frac{s's}{2}\right) \right| ds \\
+ \quad & 2 \int_{|s_1| < \frac{1}{\log^3 n} \& |s_2| < \log n} \left| \phi_2^n\left(\frac{s}{\sqrt{2c_1 n \log n}}\right) \right| ds \\
+ \quad & \int_{|s_1| < \frac{1}{\log^3 n} \& |s_2| < \log n} \left| \exp\left(-\frac{s's}{2}\right) \right| ds \\
+ \quad & \int_{\log n < |s| < \gamma\sqrt{2c_1 n \log n}} \left| \phi_2^n\left(\frac{s}{\sqrt{2c_1 n \log n}}\right) \right| ds \\
+ \quad & \int_{\gamma\sqrt{2c_1 n \log n} < |s| < \pi\sqrt{2c_1 n \log n}} \left| \phi_2^n\left(\frac{s}{\sqrt{2c_1 n \log n}}\right) \right| ds \\
+ \quad & \int_{\log n < |s|} \left| \exp\left(-\frac{s's}{2}\right) \right| ds =: I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}$$

So it suffices to prove that $I_j = O\left(\frac{\log\log n}{\log n}\right)$, for $j \in \{1, 2, 3, 4, 5, 6\}$.
All the above integrals can be estimated as it was done in the proof of Theorem 1 except for I_4. For the latter, we adapt the argument of Rvaceva (see [17]). It is easy to see that

$$\frac{\Re \log \phi_2(at)}{\Re \log \phi_2(t)} \to a^2$$

as $|t| \to 0$ (here $\Re$ denotes real part). Hence, for γ small enough,

$$\Re \log \phi_2(t) > e \Re \log \phi_2(t/e)$$

holds for $|t| < \gamma$. Now, pick $k \in \mathbb{N}$ such that $\exp(k) \leq \gamma\sqrt{2c_1 n \log n} < \exp(k+1)$ and write

$$\begin{aligned} I_4 &\leq \sum_{m=\log\log n}^{k} \int_{\exp(m)<|s|<\exp(m+1)} \left| \phi_2^n \left(\frac{s}{\sqrt{2c_1 n \log n}} \right) \right| ds \\ &< \sum_{m=\log\log n}^{k} \exp(2m) \int_{1<|s|<e} \exp\left(n \exp(m) \Re \log \phi_2 \left(\frac{s}{\sqrt{2c_1 n \log n}} \right) \right) ds. \end{aligned}$$

The argument used in the estimation of I_1 implies that

$$n \Re \log \phi_2 \left(\frac{s}{\sqrt{2c_1 n \log n}} \right) = -\frac{|s|^2}{2} + o(1)$$

holds uniformly for $s \in [1, e]$, whence for some $C' < 1$

$$I_4 < \sum_{m=\log\log n}^{k} \exp(2m)(e-1) C'^{\exp(m)}.$$

So we proved $I_4 = O(\frac{1}{\log n})$, hence the statement.

■

4 Recurrence properties

In this section we discuss the recurrence properties of S_n and Q_n that are supposed to be important in the case of billiards, too (note that these are analogous to the ones considered in [6]). For SSRW, these kind of results were proven in [10] and [5]. We begin with the two dimensional case.

Definition 1 *Let τ_2 be the first return to the origin in two dimensions, i.e.*

$$\tau_2 = \min\{n > 0 : S_n = (0,0)\}$$

Theorem 3 $\mathbb{P}(\tau_2 > n) \sim \frac{4\pi c_1}{\log\log n}$

Theorem 4 *Let $N_2^n = \#\{k \leq n : S_k = (0,0)\}$. Then*

$$\frac{N_2^n}{\log\log n}$$

converges to an exponential random variable with expected value $\frac{1}{4\pi c_1}$.

Theorem 3 and Theorem 4 can be easily proven combining the original proofs (see [9] and [10]) with (2).

Definition 2 *Let t_v be the hitting time of the origin, starting from the site $v \in \mathbb{Z}^2$, i.e.*

$$t_v = \min\{k \geq 0 : S_k = (0,0) | S_0 = v\}.$$

The following recurrence property is less known but is of crucial importance in the argument of [7].

Theorem 5

$$\frac{\log\log t_v}{\log\log |v|} \Rightarrow \frac{1}{U}$$

as $|v| \to \infty$*, where* U *is uniformly distributed on* $[0,1]$ *and* $\Rightarrow$ *stands for weak convergence.*

Proof. We adapt the proof of [10]. Let

$$\zeta(x,n) = \#\{1 \le k \le n : S_k = x\}$$

be the local time of the walk at site x up to time n and

$$\gamma(n) = \mathbb{P}\left(\tau_2 > n\right).$$

Further, we will need the estimation on the remainder term of the local limit theorem. More precisely, we will use the following estimation

$$\mathbb{P}\left(S_n = y\right) = \frac{1}{4\pi c_1 n \log n} - |y|^2 O\left(\frac{1}{n^2 \log^2 n}\right) + O\left(\frac{\log\log n}{n \log^2 n}\right), \tag{7}$$

where the great orders are uniform in $\{y : |y| < \sqrt{n \log n}\}$. Note that (7) is a consequence of Theorem 2. We are going to prove the following assertion.
If we choose $x_n \in \mathbb{Z}^2$ such that

$$|x_n| \sim \exp\left(\frac{1}{2}\log^\delta n\right)$$

for some fix $0 < \delta < 1$, then

$$\mathbb{P}\left(\zeta\left(x_n, n\right) = 0\right) \to \delta, \tag{8}$$

as $n \to \infty$. It is easy to see that (8) implies the statement of the theorem.
As in [10], we consider the identities

$$\sum_{i=0}^{n} u_2(i)\gamma(n-i) = 1 \tag{9}$$

and

$$\mathbb{P}\left(\zeta\left(x_n, n\right) = 0\right) + \sum_{i=1}^{n} \mathbb{P}\left(S_i = x_n\right)\gamma(n-i) = 1. \tag{10}$$

Combining (9) and (10) we obtain

$$\mathbb{P}\left(\zeta\left(x_n, n\right) = 0\right) - \gamma(n) = \sum_{i=1}^{n} \left(u_2(i) - \mathbb{P}\left(S_i = x_n\right)\right)\gamma(n-i). \tag{11}$$

Using the fact that γ is monotonic, Theorem 3 and the estimation (7) we conclude that the right hand side of (11) is smaller than

$$
\begin{aligned}
& \frac{4\pi c_1 + o(1)}{\log\log n} \sum_{k=1}^{\exp\left(\log^\delta n\right)} \frac{1}{4\pi c_1 k \log k} \\
+ & \frac{4\pi c_1 + o(1)}{\delta \log\log n} \sum_{k=\exp\left(\log^\delta n\right)}^{\sqrt{n}} |x_n|^2 O\left(\frac{1}{k^2 \log^2 k}\right) + \sum_{k=\sqrt{n}}^{n} |x_n|^2 O\left(\frac{1}{k^2 \log^2 k}\right) \\
+ & \sum_{k=\exp\left(\log^\delta n\right)}^{\infty} O\left(\frac{\log\log k}{k \log^2 k}\right) = \delta + o(1).
\end{aligned}
$$

So we arrived at the upper bound. For the lower bound define

$$
k_1 = \frac{\exp\left(\log^\delta n\right)}{\log n}.
$$

Theorem 3 and Theorem 2 imply that the right hand side of (11) is bigger than

$$
\begin{aligned}
& \gamma(n) \sum_{k=1}^{k_1} \left[u_2(k) - \mathbb{P}(S_k = x_n)\right] \geq \\
& \frac{4\pi c_1 + o(1)}{\log\log n} \sum_{k=1}^{k_1} \Big[\frac{1}{4\pi c_1 k \log k} + O\left(\frac{\log\log k}{k \log^2 k}\right)\Big] \\
& + \frac{4\pi c_1 + o(1)}{\log\log n} \sum_{k=1}^{k_1} \Big[-\frac{1}{4\pi c_1 k \log k} \exp\left(-\frac{|x_n|^2}{4 c_1 k \log k}\right)\Big] \\
> \; & \delta + o(1) + \frac{O(1)}{\log\log n} - O\left(\frac{1}{\log\log n}\right) \exp\left(-\frac{|x_n|^2}{k_1 \log k_1}\right) \sum_{k=1}^{k_1} \frac{1}{k \log k} \\
> \; & \delta + o(1).
\end{aligned}
$$

Thus we have proved (8). The statement follows. ■

Remark 1 *Note that for the adaptation of the Erdős-Taylor type argument for our setting, the summability of the remainder term in the local limit theorem - i.e. Theorem 2 - was essential. The situation was basically the same in [15], however, in a different context.*

It would be interesting to find an intuitive reason for the appearance of the exponential and the uniform distributions as limit laws. However, neither Erdős and Taylor gave explanation in [10], nor the present author can give any. Now, we turn to the one dimensional case.

Definition 3 *Let τ_1 be the first return to the origin in one dimension, i.e.*

$$
\tau_1 = \min\{n > 0 : Q_n = 0\}
$$

Theorem 6 $\mathbb{P}(\tau_1 > n) \sim \frac{2\sqrt{c_1}}{\sqrt{\pi}} \sqrt{\frac{\log n}{n}}$

Proof. Theorem 6 can be easily proven by the usual way. One has to consider the renewal equation

$$\sum_{k=0}^{n} u_1(k)\mathbb{P}(\tau_1 > n-k) = 1,$$

and the identity

$$U(x)V(x) = \frac{1}{1-x},$$

where

$$\begin{aligned} U(x) &= \sum_{k=0}^{\infty} u_1(k)x^k \\ V(x) &= \sum_{k=0}^{\infty} \mathbb{P}(\tau_1 > k)x^k. \end{aligned}$$

Now, the well known Tauberian theorem (Theorem XIII.5. in [11]) implies that

$$U(x) \sim \frac{1}{\sqrt{1-x}} \frac{1}{\sqrt{\pi c_1}} \Gamma\left(\frac{3}{2}\right) \frac{1}{\sqrt{\log \frac{1}{1-x}}}$$

as $x \to 1$, thus

$$V(x) \sim \frac{1}{\sqrt{1-x}} \frac{\sqrt{\pi c_1}}{\Gamma\left(\frac{3}{2}\right)} \sqrt{\log \frac{1}{1-x}}$$

as $x \to 1$. Since $\mathbb{P}(\tau_1 > n)$ is monotonic in n, the previous Tauberian theorem infers the statement. ∎

Theorem 7 *Let* $N_1^n = \#\{k \le n : Q_k = 0\}$*. Then*

$$\frac{N_1^n \sqrt{\log n}}{\sqrt{n}}$$

converges to a Mittag-Leffler distribution with parameters $1/2$ *and* $(2\sqrt{c_1})^{-1}$*, i.e. to the distribution, the* k^{th} *moment of which is*

$$\frac{1}{(2\sqrt{c_1})^k} \frac{k!}{\Gamma\left(\frac{k}{2}+1\right)}.$$

Proof. As in the case of [6], it suffices to prove that for k fix:

$$\sum_{n_i \ge 3, n_1+n_2+\ldots+n_k \le n} \prod_j \frac{1}{\sqrt{n_j \log n_j}} \sim \frac{n^{k/2}}{\log^{k/2} n} \frac{\Gamma(1/2)^k}{\Gamma(k/2+1)}. \tag{12}$$

Note that $\Gamma(1/2) = \sqrt{\pi}$. Elementary calculations show that (12) holds for $k = 1$. For $k > 1$ define

$$\begin{aligned} \mathcal{H}_1 &= \{n_i \ge \frac{n}{\log n}, n_1 + n_2 + \ldots + n_k \le n\} \\ \mathcal{H}_2 &= \{n_i \ge \frac{\sqrt{n}}{\log n}, \exists j : n_j < \frac{n}{\log n}, n_1 + n_2 + \ldots + n_k \le n\} \\ \mathcal{H}_3 &= \{n_i \ge 3, \exists j : n_j < \frac{\sqrt{n}}{\log n}, n_1 + n_2 + \ldots + n_k \le n\} \end{aligned}$$

Now, split the sum in (12) into three parts, sums over $\mathcal{H}_i$'s, $1 \le i \le 3$.
Define $s_j = n_j/n$ and observe that

$$\frac{1}{\sqrt{n_j \log n_j}} = \frac{1}{\sqrt{s_j}} \frac{1}{\sqrt{n}} \frac{1}{\sqrt{\log s_j + \log n}}.$$

Since $\log s_j + \log n = (1 + o(1)) \log n$ uniformly in $\mathcal{H}_1$, it is not difficult to deduce that

$$\begin{aligned} & \sum_{(n_1,n_2,\ldots,n_k)\in\mathcal{H}_1} \prod_j \frac{1}{\sqrt{n_j \log n_j}} \\ \sim \ & \frac{n^{k/2}}{\log^{k/2} n} \int \ldots \int_{0<t_1<t_2<\ldots<t_k<1} \frac{1}{\sqrt{t_1}} \frac{1}{\sqrt{t_2 - t_1}} \ldots \frac{1}{\sqrt{t_k - t_{k-1}}} dt_1 \ldots dt_k \\ = \ & \frac{n^{k/2}}{\log^{k/2} n} \frac{\Gamma(1/2)^k}{\Gamma(k/2+1)}. \end{aligned}$$

For the sum over $\mathcal{H}_2$, consider the case when $\frac{\sqrt{n}}{\log n} < n_1 < \frac{n}{\log n}$ and $n_i > \frac{n}{\log n}$ for $2 \le i$ (other cases can be treated similarly). Now, $\log s_1 + \log n > (1/2 + o(1)) \log n$ and $\log s_i + \log n = (1 + o(1)) \log n$ for $2 \le i$, uniformly. Thus,

$$\begin{aligned} & \sum_{\frac{\sqrt{n}}{\log n} < n_1 < \frac{n}{\log n}, n_i > \frac{n}{\log n} : 2 \le i} \prod_j \frac{1}{\sqrt{n_j \log n_j}} \\ < \ & (\sqrt{2} + o(1)) \sum_{\frac{\sqrt{n}}{\log n} < n_1 < \frac{n}{\log n}, n_i > \frac{n}{\log n} : 2 \le i} \prod_j \frac{1}{\sqrt{s_j n \log n}} < 2 \frac{n^{k/2}}{\log^{k/2} n} o(1). \end{aligned}$$

For the third sum, the proof goes by induction on k. Assuming that (12) holds for $k-1$, one has

$$\sum_{(n_1,n_2,\ldots,n_k)\in\mathcal{H}_3} \prod_{j=1}^{k} \frac{1}{\sqrt{n_j \log n_j}} < k \frac{\sqrt{n}}{\log n} \sum_{n_1+n_2+\ldots+n_{k-1} \le n} \prod_{j=1}^{k-1} \frac{1}{\sqrt{n_j \log n_j}},$$

which is $o\left(\frac{n^{k/2}}{\log^{k/2} n}\right)$. (12) follows.

■

5 Final remark

As it was mentioned in the Introduction, in the case of Lorentz process with infinite horizon, another type of 'recurrence' can happen. Namely, if a scatterer is moved into a corridor (here corridor means infinite trajectories without collision), then there are arbitrary long flights where in the periodic Lorentz process there would not be collision, while in the perturbed one there are some. In the random walk context, it can happen that the unperturbed walk would fly over the origin, while the perturbed one has to stop. Note that this phenomenon is evitable if one considers finite horizon, or in the case of infinite horizon just shrinks one of the scatterers as a perturbation. However, the same behavior (i.e. the Brownian limit with the same scaling) is conjectured in this general perturbation, as well. The aim of the following computation is to give some reason for this conjecture. As the constants do not play important role in the sequel, they will not be computed and every appearance of C may denote different constant.

Define

$$a_n = \mathbb{P}((0,0) \in \overline{S_n, S_{n+1}}, (0,0) \neq S_n)$$

to be the probability of the event that step $n+1$ flies over the origin. Observe that

$$a_n = \frac{1}{2}\mathbb{P}\left((S_n)_1 = 0, |X_{n+1}| \geq |(S_n)_2|\right),$$

where $(S_n)_i$ denotes the i^{th} coordinate of S_n. The local limit theorem implies $a_n < C\frac{1}{\sqrt{n \log n}} b_n$, where

$$b_n = \mathbb{P}\left(|X_{n+1}| \geq |(S_n)_2|\right).$$

For the estimation of b_n observe that if $|(S_n)_2| > d_n$, then b_n is bounded by $C\sum_{k=d_n}^{\infty} k^{-3} = O(d_n^{-2})$. On the other hand, the probability of $|(S_n)_2|$ being smaller than d_n is roughly estimated by $O(d_n \frac{1}{\sqrt{n \log n}})$. Thus

$$b_n = O(d_n^{-2}) + O(d_n \frac{1}{\sqrt{n \log n}}) = O\left((n \log n)^{-1/3}\right),$$

whence

$$a_n = O\left((n \log n)^{-5/6}\right).$$

If ρ_n denotes the number of jumps over the origin up to time n and $\theta_n = \mathbb{E}(\rho_n)$, then we have just proved

$$\theta_n = o(n^{1/6}).$$

Note that in the case of [19] and [16] the key observation was that the time spent at the perturbed area up to n is much smaller than $\sqrt{n}$. That is why it is reasonable to expect the same Brownian limit in the case of such perturbation, where we introduce some nice further step at the time of flying over the origin, too. Here nice means that presumably the step distribution should have some finite moment of order ε. This could be subject of future research.

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