Lorentz Process with shrinking holes in a wall

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Abstract

We ascertain the diffusively scaled limit of a periodic Lorentz process in a strip with an almost reflecting wall at the origin. Here, almost reflecting means that the wall contains a small hole waning in time. The limiting process is a quasi-reflected Brownian motion, which is Markovian but not strong Markovian. Local time results for the periodic Lorentz process, having independent interest, are also found and used.

The periodic Lorentz process is a fascinating non-linear, chaotic model that has been deeply investigated in the last decades. The model is very simple: a massless point particle moves freely in the plane (or in our case, in a strip) until it hits one of the periodically situated smooth convex scatterers, when it is reflected. The limit of the diffusively scaled trajectory of the particle is known to be the Brownian motion. Further, if the particle is restricted to a half strip, then the scaling limit is going to be the so-called reflected Brownian motion. Here we introduce a *time-dependent scatterer configuration* **(by adding a vertical wall with a shrinking hole) that almost confines the particle to the half strip in such a way that the scaling limit is a** *quasi-reflected Brownian motion*, **a natural generalization of both the Brownian motion and the reflected Brownian motion.**

1 Introduction

Lorentz models were introduced by H. Lorentz in 1905 for understanding the motion of a classical electron in a crystal (cf. [10]). In the last decade after a broad and thorough study of Sinai billiards - or equivalently of periodic Lorentz processes - the non-homogeneous case got also widely examined. Here, *non-homogeneity* may appear either in time (cf. [5] as to a mechanical model of Brownian motion or [9], [6], [12] as to models of Fermi acceleration) or in space (cf. [8] as to local perturbations of periodic Lorentz processes). In the present work we investigate a question with non-homogeneity in time. Consider a periodic Lorentz process with a finite horizon given in a horizontal strip, where the scatterer configuration is assumed to be symmetric with respect to a vertical axis - through the origin, say. Now, put a vertical wall at the symmetry axis and a tiny hole onto the wall. The hole is getting smaller and smaller with time, thus giving the particle less and less chance to cross the wall. It is an intriguing question at which speed the hole should shrink to result a non trivial scaling limit of the trajectory of the particle (if such a speed exists at all). Here, non trivial means that it is neither Brownian motion (BM), nor reflected Brownian motion (RBM) since, if the hole was of full size or absent, then these two processes would appear in the limit (see [8]).

Indeed, if one takes the hole arbitrarily small, but fixed of size $\varepsilon > 0$, then the limiting process is a BM whereas if the hole is empty, then it is a RBM. The essence of this observation is that the limiting process does not change continuously as $\varepsilon \to 0$ and our goal is precisely to understand the situation when the limit is taken in a more delicate, time-dependent way.

To be more precise, let the configuration space in the absence of the wall be $\mathcal{D} := (\mathbb{R} \times [0,1]) \setminus \cup_{i=1}^{\infty} O_i$. Here, $\{O_i\}_i$ is a $\mathbb{Z}$-periodic extension of a finite scatterer configuration in the unit square, which consists of strictly convex, pairwise disjoint scatterers, with C^3 smooth boundaries, whose curvatures are bounded from below by a positive constant. Further, we assume that $\cup_{i=1}^{\infty} O_i$ is symmetric with respect to the y-axis. The wall without the hole is $W_\infty = \{(x,y) \in \mathcal{D} | \, x = 0\} = \cup_{k=1}^{K} [\mathcal{J}_{k,l}, \mathcal{J}_{k,r}]$ where the subintervals of the y-axis, denoted by $[\mathcal{J}_{k,l}, \mathcal{J}_{k,r}]$, are the connected components of W_∞. For later reference, put

$$c_1 = \sum_{k=1}^{K} (\mathcal{J}_{k,r} - \mathcal{J}_{k,l}).$$

The *holes* will be subintervals $I_n \subset W_\infty$, thus we will be considering a sequence $\{W_n = W_\infty \setminus I_n\}_n$ of walls. Now, the n-th configuration space of the *billiard flow* is $\mathcal{D}_n := (\mathbb{R} \times [0,1]) \setminus (W_n \cup \cup_{i=1}^{\infty} O_i)$. A massless point particle moves inside $\mathcal{D}_n$ (at time $t = 0$ the first hole is present, i.e. $n = 1$) with unit speed until it hits the boundary $\partial \mathcal{D}_n$. Then it is reflected by the classical laws of mechanics (the angle of incidence equals to the angle of reflection) and continues free movement (or free flight) in $\mathcal{D}_{n+1}$. Thus, at the time instant of each reflection, the hole is replaced by an other one (meaning that the shrinking rate of the hole corresponds to real time). We also mention that the reflections on the horizontal boundaries of the strip does not play any role in our study. Thus one could define the horizontal direction to be periodic (formally replace $[0,1]$ by S^1 in the definitions of $\mathcal{D}$ and $\mathcal{D}_n$) yielding the same results (with some different limiting variance).
Since we change the configuration space in the moment of the reflection, it is more convenient to use the discretized version of the billiard flow (the usual Poincaré section, which is often called billiard ball map). Thus define the *phase spaces*

$$\mathcal{M}_n = \{x = (q,v), q \in \partial \mathcal{D}_n, v \in S^1, \langle v, u \rangle \geq 0 \text{ if } q \in \partial \mathcal{D} \},$$

where u denotes the inward unit normal vector to $\partial \mathcal{D}$ at the point $q \in \partial \mathcal{D}$. Here,

q denotes the position of the particle at a collision and v is the postcollisional velocity vector. If $q \in \partial\mathcal{D}$, v can be naturally parametrized by the angle between u and v which is in the interval $[-\pi/2, \pi/2]$. If $q \in \partial W_n = W_n$, one can parametrize v by its angle to the horizontal axis. Thus, if this angle is in the interval $[-\pi/2, \pi/2]$, then the particle is on the right-hand side of the wall, while it is on the left-hand side if this angle is either in the interval $[\pi/2, \pi]$ or in $(-\pi, -\pi/2]$.
Thus, the discretized version of the previously described billiard flow can be defined by the *billiard ball maps* $\mathcal{F}_n : \mathcal{M}_n \to \mathcal{M}_{n+1}$. Further, denote by $\kappa_n : \mathcal{M}_n \to \mathbb{R}$ the projection to the horizontal direction of the *free flight* vector from $\mathcal{M}_n$ to $\mathcal{M}_{n+1}$ (that is, if $x = (q, v) \in \mathcal{M}_n$ and $\mathcal{F}_n(x) = (\tilde{q}, \tilde{v})$, then $\kappa_n(x)$ is the projection to the horizontal axis of the vector $\tilde{q} - q$). We also assume that the billiard has *finite horizon*, meaning that, in the $\mathbb{Z}^2$-periodic extension of the scatterer configuration, there is no infinite line on the plane that would be disjoint to all the scatterers. Further, write $\mathcal{I}_n = \{I_k\}_{1 \le k \le n}$ for the collection of the first n holes, and

$$S_n(x, \mathcal{I}_n) = S_n(x) = \sum_{k=1}^{n} \kappa_k \mathcal{F}_{k-1} \dots \mathcal{F}_1(x),$$

where $x \in \mathcal{M}_1$.
What remains is the definition of the holes I_n. For this, fix some sequence $\underline{\alpha} = (\alpha_n)_{n \ge 1}$ with $\alpha_n \to 0$ and, independently of each other, choose uniformly distributed points ξ_n, $n \ge 1$ on $\cup_{i=1}^{K} [\mathcal{J}_{i,l}, \mathcal{J}_{i,r}]$. We will use the following three special choices:

1. Assume that $\xi_n \in [\mathcal{J}_{i,l}, \mathcal{J}_{i,r}]$, and denote $l_n = \mathcal{J}_{i,r} - \xi_n$. If $l_n > \alpha_n$, then put $I_n = (\xi_n, \xi_n + \alpha_n)$, otherwise put $I_n = (\xi_n, \mathcal{J}_{i,r}) \cup (\mathcal{J}_{i,l}, \mathcal{J}_{i,l} + \alpha_n - l_n)$, which is a subset of W_∞ for n large enough. With this particular choice, write
$$S_n^{\searrow}(x, \underline{\alpha}) = S_n^{\searrow}(x) = S_n(x, \mathcal{I}_n)$$
and
$$\mathcal{F}_n^{\searrow} = \mathcal{F}_n.$$

2. For each $1 \le k \le n$, let the random variables $\xi_n^{(k)}$ be independent and distributed like ξ_n. Assume that $\xi_n^{(k)} \in [\mathcal{J}_{i,l}, \mathcal{J}_{i,r}]$, and denote $l_n^{(k)} = \mathcal{J}_{i,r} - \xi_n^{(k)}$. If $l_n^{(k)} > \alpha_n$, then put $I_n^{(k)} = (\xi_n^{(k)}, \xi_n^{(k)} + \alpha_n)$, otherwise put $I_n^{(k)} = (\xi_n^{(k)}, \mathcal{J}_{i,r}) \cup (\mathcal{J}_{i,l}, \mathcal{J}_{i,l} + \alpha_n - l_n^{(k)})$, and finally $\mathcal{I}_n = (I_n^{(k)})_{1 \le k \le n}$. With this particular choice, write
$$S_n^{\equiv}(x, \underline{\alpha}) = S_n^{\equiv}(x) = S_n(x, \mathcal{I}_n).$$

3. Let $I_n = W_\infty$. With this particular choice, write
$$S_n^{(per)}(x) = S_n(x, \mathcal{I}_n),$$

and for a fixed x, define $S_t^{(per)}(x)$ for $t \geq 0$ as the piecewise linear, continuous extension of $S_n^{(per)}(x)$. Finally, write

$$\begin{aligned} \mathcal{F}^{(per)} &= \mathcal{F}_1, \\ \mathcal{M}^{(per)} &= \mathcal{M}_1. \end{aligned}$$

Here the first choice - the only really time dependent - is the most interesting one. In the second case, one has to redefine the whole trajectory segment $S_1^{\equiv}, \dots S_n^{\equiv}$ for each n, thus we have a sequence of billiards (in other words, the increments of $S_n^{\equiv}$ form a double array), while the third one is just a usual periodic Lorentz process.
There is a natural measure - the projection of the *Liouville measure* of the periodic billiard flow - on $\mathcal{M}^{(per)}$ which is invariant under $\mathcal{F}^{(per)}$. Denote the restriction of this measure to the two neighboring tori to the origin by $\mathbf{P}$. Note that $\mathbf{P}$ is finite, so normalize it to be a probability measure.
Finally, define $\mathcal{J} \subset \mathcal{M}^{(per)}$ as such points on the discrete phase space without any wall, from which before the forthcoming collision, the particle crosses $\cup_{i=1}^{K}(\mathcal{J}_{i,l}, \mathcal{J}_{i,r})$. Note that the finite horizon condition implies that $\mathcal{J}$ is bounded.

Now we proceed to the definition of the limiting processes. (The intuition behind their appearance in our result and in its proof as well will be explained after the formulation of the theorem.) Since we are going to have two very similar processes, we call both quasi-reflected Brownian motions and distinguish between them only in the abbreviation.
Consider a BM $\mathfrak{B} = (\mathfrak{B}_t)_{t\in[0,1]}$ with parameter σ on $[0, 1]$. Its local time at the origin is denoted by $\mathfrak{L} = (\mathfrak{L}_t)_{t\in[0,1]}$. That is,

$$\mathfrak{L}_t = \lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{\{|\mathfrak{B}_S|<\varepsilon\}} ds.$$

Now, given $\mathfrak{B}$, consider a Poisson Point Process Π with intensity measure $cd\mathfrak{L}$ with some positive constant c. The intuition behind this process is roughly speaking the following: since the local time describes the relative time process a Brownian motion spends in an infinitesimal neighborhood of a point, in our case of the origin, it can also be interpreted as telling us the density process of number of visits of the origin by the Lorentz process. Out of them only those visits are successful, i. e. resulting in getting to the other side of the wall, when the particle hits the hole, and these instants of time are precisely given by a Poisson process – according to the Poisson limit law. With probability one, the support of the measure $c(d\mathfrak{L})$ is $\mathfrak{Z}$, where $\mathfrak{Z} = \{s : 0 \leq s \leq 1 : \mathfrak{B}_s = 0\}$ is the zero set of $\mathfrak{B}$. Denote the points of Π by $P_1, P_2, ...$ in decreasing order. In fact, Π has finitely many points. If it has m points, then put $P_{m+1} = P_{m+2} = ... = 0$. Further, write $P_0 = 1$ and introduce a Bernoulli distributed random variable η with parameter $1/2$ (where the parameter means the probability of being equal to 1) which is independent of $\mathfrak{B}$ and Π.

Now, the process $\mathfrak{Q} = (\mathfrak{Q}_t)_{t\in[0,1]}$ with $\mathfrak{Q}_0 = 0$ and

$$\mathfrak{Q}_t = \begin{cases} (-1)^{\eta}|\mathfrak{B}_t| & \text{if } \exists n \in \mathbb{Z}_+ \cup \{0\} : t \in (P_{2n+1}, P_{2n}] \\ (-1)^{1-\eta}|\mathfrak{B}_t| & \text{otherwise} \end{cases}$$

is called the quasi-reflected Brownian motion with parameters c and σ, and denoted by qRBM(c,σ).
The definition of QRBM is similar to that of qRBM. The difference is that $c(d\mathfrak{L})$ now should be replaced by $c\frac{1}{\sqrt{t}}(d\mathfrak{L}_t)$. As a result, the Poisson process will have infinitely many points, which accumulate only at the origin. Now, denote by $P_1, P_2, \ldots$ these points in decreasing order (N. B.: there is no smallest one among them), put $P_0 = 1$ and define η and QRBM(c,σ) as before.

Remark 1. *One can easily check the following statements. The qRBM(c,σ) is almost surely continuous on $[0,1]$, homogeneous Markovian but not strong Markovian (think of the stopping time $T = \min\{t > 1/2 : \mathfrak{Q}_t = 0\} \wedge 1$) and $\mathfrak{Q}_t$ has Gaussian distribution with mean zero and variance $t\sigma^2$.*
The QRBM, similarly to the qRBM, is continuous, Markovian (however not time homogeneous), not strong Markovian, and has the same one dimensional distributions as qRBM. Contrary to the qRBM, the QRBM is self similar in the following sense: if $\mathfrak{Q}_t$ is a QRBM, then

$$(\mathfrak{Q}_t)_{t\in[0,1/p]} \stackrel{d}{=} \left(\frac{1}{\sqrt{p}}\mathfrak{Q}_{pt}\right)_{t\in[0,1/p]},$$

where $1 < p$.
Further, one can easily extend the definition of both processes to $\mathbb{R}_+$.

As usual, $C[0,1]$ will denote the space of continuous functions and $D[0,1]$ the Skorokhod space over $[0,1]$ (for the definition of the latter, we refer to [1]). We will also use evident modifications, for instance, $D_{\mathbb{R}^2}[t_0,1]$ will denote the Skorokhod space of $\mathbb{R}^2$-valued functions over an interval $[t_0,1]$.
Let the function $\mathbf{W}_n^{\searrow}$ be the following: $\mathbf{W}_n^{\searrow}(k/n) = S_k^{\searrow}/\sqrt{n}$ for $0 \le k \le n$ and define $\mathbf{W}_n^{\searrow}(t)$ for $t \in [0,1]$ as its piecewise linear, continuous extension. Let $\mu_n^{\searrow}$ denote the measure on $C[0,1]$ induced by $\mathbf{W}_n^{\searrow}$, where the initial distribution, i.e. the distribution of x, is given by $\mathbf{P}$. Analogously, define $\mu_n^{\equiv}$ with $\mathbf{W}_n^{\equiv}$, where $\mathbf{W}_n^{\equiv}(k/n) = S_k^{\equiv}/\sqrt{n}$.
Now, we can formulate our main result.

Theorem 1. *There are positive constants σ and c_2 depending only on the periodic scatterer configuration, such that*

1. *if $\exists c > 0 : \alpha_n\sqrt{n} \to c$, then $\mu_n^{\searrow}$ converges weakly to the measure induced by $QRBM(c_2 c, \sigma)$.*
2. *if $\exists c > 0 : \alpha_n\sqrt{n} \to c$, then $\mu_n^{\equiv}$ converges weakly to the measure induced by $qRBM(c_2 c, \sigma)$.*

3. *if* $\alpha_n\sqrt{n} \to 0$, *then both* $\mu_n^{\searrow}$ *and* $\mu_n^{\equiv}$ *converge weakly to the convex combination of the measures induced by RBM and -RBM with weights* $1/2$.

4. *if* $\alpha_n\sqrt{n} \to \infty$, *then both* $\mu_n^{\searrow}$ *and* $\mu_n^{\equiv}$ *converge weakly to the Wiener measure.*

Returning to the intuitive picture provided at the introduction of the process $qRBM(c_2c, \sigma)$, it, indeed, explains statement 2 of the theorem. Since, in the setup of the definition $\mu_n^{\searrow}$, the holes are not uniformly small, but are only decreasing as of order $\frac{1}{\sqrt{n}}$, the chances to get over the wall are larger but also decreasing as in the definition of $QRBM(c_2c, \sigma)$.

Instead of introducing the holes on the wall one could think about the wall as a *trapdoor*, i.e. sometimes it is open and then the particle crosses it without collisions, other times it is closed. If one opens the door randomly with probability α_n/c_1, then obtains the same result.

The analogue of Theorem 1 for random walks is, of course, easy to formulate in the following way. Define the stochastic process $\mathfrak{S}_n$ by: $Prob(\mathfrak{S}_0 = 1) = Prob(\mathfrak{S}_0 = -1) = 1/2$ and for $k > 0$:

$$Prob(\mathfrak{S}_{k+1} = \mathfrak{S}_k + 1 | \mathfrak{S}_k \neq 0) = Prob(\mathfrak{S}_{k+1} = \mathfrak{S}_k - 1 | \mathfrak{S}_k \neq 0) = 1/2,$$

and

$$Prob(\mathfrak{S}_{k+1} = \mathfrak{S}_{k-1} | \mathfrak{S}_k = 0) = 1 - \epsilon, \tag{1}$$

$$Prob(\mathfrak{S}_{k+1} = -\mathfrak{S}_{k-1} | \mathfrak{S}_k = 0) = \epsilon. \tag{2}$$

Here - and also in the sequel - $Prob$ stands for some abstract probability measure.

In the definition of $\mathfrak{S}_k$ put first $\epsilon = \alpha_k$ and denote by $\nu_n^{\searrow}$ the measure on $C[0, 1]$ induced by $\mathbf{W}_n$, where $\mathbf{W}_n(k/n) = \mathfrak{S}_k/\sqrt{n}$ for $0 \leq k \leq n$ and is linearly interpolated in between. Analogously, define $\nu_n^{\equiv}$ for each n with the choice $\epsilon = \alpha_n$. Then, if we replace each μ with ν in Theorem 1, then the statement remains true (with $\sigma = c_2 = 1$), and can be proven the same way as we prove Theorem 1.

In the next section, we discuss some results concerning the periodic Lorentz process, that are necessary for proving Theorem 1. Finally, Section 3 contains the actual proof of Theorem 1.

2 Limit theorems for the periodic Lorentz Process

In this section, we present some facts about the periodic Lorentz process in a strip. Whereas Proposition 1 is simply a strengthening of Theorem 4.2 of [17], Proposition 3 is a completely new statement interesting in itself. For later reference, we need to introduce some abstract stochastic processes.
As before, $\mathfrak{B} = (\mathfrak{B}_t)_{t\in[0,1]}$ denotes a BM with parameter σ (to be specified

later) and $\mathfrak{L} = (\mathfrak{L}_t)_{t\in[0,1]}$ is its local time at the origin. We also use the notation $\mathfrak{B}^{a,t_0} = (\mathfrak{B}^{a,t_0}_t)_{t\in[t_0,1]}$ for a BM with parameter σ starting from a at time t_0; and $\mathfrak{L}^{a,t_0} = (\mathfrak{L}^{a,t_0}_t)_{t\in[t_0,1]}$ denotes its local time at the origin. Finally, $\mathfrak{B}^{a,t_0\rightsquigarrow b,t_1} = (\mathfrak{B}^{a,t_0\rightsquigarrow b,t_1}_t)_{t\in[t_0,t_1]}$ stands for a Brownian bridge with parameter σ starting from a at time t_0 and arriving at b at time t_1 (that is heuristically a BM with pinned down endpoints), and $\mathfrak{L}^{a,t_0\rightsquigarrow b,t_1} = (\mathfrak{L}^{a,t_0\rightsquigarrow b,t_1}_t)_{t\in[t_0,t_1]}$ is the local time of $\mathfrak{B}^{a,t_0\rightsquigarrow b,t_1}$ at the origin. For a thorough description of all these processes, see [15].
Similarly to the previous notations, denote by $L_{nt},\ t \in [0,1]$ the number of visits to $\mathcal{J}$ in the time interval $[1, \lfloor nt\rfloor]$, and L_H is the number of visits to $\mathcal{J}$ in the time interval H.

The first statement is a local limit theorem, formulated in a fashion tailored to our purposes. For this, let ϕ denote the density of the standard normal law. Now, the assertion reads as follows.

Proposition 1. *Fix some positive integer k and a subset $\mathcal{Z}$ of the set $\{1,2,\dots k\}$. For all $i \in \{1,2,\dots k\}\setminus\mathcal{Z}$, let $t_i \in [0,1]$, $b^{(i)} \in \mathbb{R}$ be real numbers such that if $i < j$ with $i,j \in \mathcal{Z}$, then $t_i < t_j$. Write $b^{(i)}_n := \lfloor b^{(i)}\sqrt{n}\rfloor$ and $n_i = \lfloor nt_i\rfloor$ for any positive integer n. Define $n_0 = b^{(0)}_n = 0$. For $i' \in \mathcal{Z}$, write $b^{(i')}_n = 0$ and choose some sequences $n_{i'}$ such that for any $i,j \in \{1,2,\dots k\}$ with $i < j$, $n_i \le n_j$ holds. Then*

$$\begin{aligned}&\mathbf{P}\quad\left(\forall i \in \{1,2,\dots k\}\setminus\mathcal{Z} : \lfloor S^{(per)}_{n_i}(x)\rfloor = b^{(i)}_n; \forall i' \in \mathcal{Z} : \left(\mathcal{F}^{(per)}\right)^{n_{i'}}(x) \in \mathcal{J}\right)\\ &= \quad c_0^{|\mathcal{Z}|}\prod_{i=1}^{k}\frac{\phi\left(\frac{b^{(i)}_n - b^{(i-1)}_n}{\sigma\sqrt{n_i-n_{i-1}}}\right) + o_i(1)}{\sigma\sqrt{n_i-n_{i-1}}},\end{aligned}$$

with some constants σ and c_0 depending only on the periodic scatterer configuration. Further, there exist a sequence $\varsigma(n) \to 0$, such that $|o_i(1)| < \varsigma(n_i - n_{i-1})$ for all $i \in \{1,2,\dots k\}$.

Proposition 1 is an extension of Theorem 4.2 in [17] in two aspects. On the one hand, it is formulated for k-tuples, while in [17] it is only stated for $k = 1,2$. On the other hand, the error term is claimed to be uniform in the choice of n_i (it is, in fact, uniform in more general choices of $b^{(i)}_n$, but we only use it for $b^{(i)}_n$ of the form presented in Proposition 1). Both generalizations follow from the proof presented in [17], thus we do not provide a formal proof here. We also note that Proposition 1 is an extension of Proposition 3.6 in [7], too. From now on, all stochastic processes derived from the BM will have parameter σ of Proposition 1.
The next important fact is the weak invariance principle for the position, which was first proven in [3] and [4].

Proposition 2.

$$\left(\frac{S^{(per)}_{nt}}{\sqrt{n}} \right)_{t\in[0,1]} \Rightarrow (\mathfrak{B}_t)_{t\in[0,1]},$$

where $\Rightarrow$ stands for weak convergence in the space $C[0,1]$.

The novelty of this section is in fact the following statement. The position of the particle and its local time at $\mathcal{J}$ jointly converge to a BM and its local time at the origin (the latter being multiplied by a constant). Formally,

Proposition 3.

$$\left(\frac{S^{(per)}_{nt}}{\sqrt{n}}, \frac{L_{nt}}{\sqrt{n}} \right)_{t\in[0,1]} \Rightarrow (\mathfrak{B}_t, c_0\mathfrak{L}_t)_{t\in[0,1]},$$

as $n \to \infty$ where the left hand side is understood as a random variable with respect to the probability measure $\mathbf{P}$, and $\Rightarrow$ stands for weak convergence in the Skorokhod space $D_{\mathbb{R}^2}[0,1]$.

Proof of Proposition 3. As usual, one has to check the convergence of finite dimensional distributions and the tightness (see [1] for conditions implying weak convergence on some function spaces).
First, we prove the convergence of the finite dimensional distributions. Note that the convergence of the first coordinate follows from Proposition 2 (and even from Proposition 1), while the convergence of the second coordinate follows from an extended version of the proof of Theorem 9 in [7]. But the joint convergence is a stronger statement then the convergence of the individual coordinates, and it requires a formal proof.
To obtain the joint convergence, first observe that the convergence of the first coordinate (the rescaled position) is well known, even in the local sense (eg. Proposition 1). Thus we are going to prove that under the condition that the rescaled position is close to some specific number, the second coordinate converges to the desired limit. In order to do this computation, we need to define some new measures on $\mathcal{M}^{(per)}$.
First, choose $0 < t_0 < 1$, $a \in \mathbb{R}$ and write $a_n = \lfloor \sqrt{n} a \rfloor$. Restrict the measure $\mathbf{P}$ to such points x where $\lfloor S^{(per)}_{\lfloor nt_0 \rfloor}(x) \rfloor = a_n$ and rescale it to obtain a probability measure. The resulting measure is denoted by $\mathbf{P_n}$. Thus, with the notation

$$\mathcal{A}_1 = \mathcal{A}_1(n) = \{x : \lfloor S^{(per)}_{\lfloor nt_0 \rfloor}(x) \rfloor = a_n\} \subset \mathcal{M}^{(per)},$$

for $M \subset \mathcal{M}^{(per)}$ measurable sets, $\mathbf{P_n}(M) = \mathbf{P}(M \cap \mathcal{A}_1)/\mathbf{P}(\mathcal{A}_1)$. Then, choose $t_0 < t_1 < 1$, $b \in \mathbb{R}$ and write $b_n = \lfloor \sqrt{n} b \rfloor$. Define $\mathbf{Q_n}$ as the conditional measure of $\mathbf{P_n}$ on such points x, where $\lfloor S^{(per)}_{\lfloor nt_1 \rfloor}(x) \rfloor = b_n$. That is, with the notation

$$\mathcal{A}_2 = \mathcal{A}_2(n) = \{x : \lfloor S^{(per)}_{\lfloor nt_1 \rfloor}(x) \rfloor = b_n\} \subset \mathcal{M}^{(per)},$$

for $M \subset \mathcal{M}^{(per)}$ measurable sets, $\mathbf{Q_n}(M) = \mathbf{P_n}(M \cap \mathcal{A}_2)/\mathbf{P_n}(\mathcal{A}_2)$.
Now, we prove the following lemma.

Lemma 1.

$$L_{[nt_0,nt_1]}/(c_0\sqrt{n}) \Rightarrow \mathfrak{L}_{t_1}^{a,t_0 \rightsquigarrow b,t_1},$$

where $L_{[nt_0,nt_1]}/(c_0\sqrt{n})$ *is understood as a random variable with respect to* $\mathbf{Q_n}$. *Similarly,*

$$L_{nt_0}/(c_0\sqrt{n}) \Rightarrow \mathfrak{L}_{t_0}^{0,0 \rightsquigarrow a,t_0},$$

where $L_{t_0}/(c_0\sqrt{n})$ *is understood as a random variable with respect to* $\mathbf{P_n}$.

Proof of Lemma 1. We prove only the first statement, since the second one can be proven analogously. Similarly to the proof of Theorem 9 in [7], we are going to use the method of moments (see [1], Chapter 1.7, Problem 4., for instance). That is, we are going to estimate

$$\mathbb{I}_n^k := \int (L_{[nt_0,nt_1]})^k d\mathbf{Q_n}.$$

For some fixed positive integer k and for $\lfloor nt_0 \rfloor = n_0 < n_1 < n_2 < ... < n_k < n_{k+1} = \lfloor nt_1 \rfloor$, define the set

$$\mathcal{A}_3 = \mathcal{A}_3(n_1, \dots, n_k) = \{x : \{\left(\mathcal{F}^{(per)}\right)^{n_i} x, 1 \le i \le k\} \subset \mathcal{J}\} \subset \mathcal{M}^{(per)}.$$

Representing $L_{[nt_0,nt_1]}$ as a sum of $\lfloor nt_1 \rfloor - \lfloor nt_0 \rfloor + 1$ indicator variables, one concludes

$$\mathbb{I}_n^k \sim k! \sum_{\lfloor nt_0 \rfloor = n_0 < n_1 < n_2 < \dots < n_k < n_{k+1} = \lfloor nt_1 \rfloor} \mathbf{Q_n}(\mathcal{A}_3(n_1, \dots n_k)). \tag{3}$$

In fact, there should be $k-1$ similar sums, for $n_1 < \cdots < n_l$, $1 \le l \le k-1$, respectively, but the contribution of them is of smaller order of magnitude, as we will see in the forthcoming computation. Thus, we need to estimate $\mathbf{Q_n}(\mathcal{A}_3(n_1, \dots n_k))$. By definition,

$$\mathbf{Q_n}(\mathcal{A}_3) = \frac{\mathbf{P}(\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3)}{\mathbf{P}(\mathcal{A}_1 \cap \mathcal{A}_2)}. \tag{4}$$

Using Proposition 1, one obtains the asymptotic equalities

$$\begin{aligned} &\frac{\mathbf{P}(\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3)}{\mathbf{P}(\mathcal{A}_1)} \\ \sim\; &\frac{c_0^k}{\sigma^{k+1}(2\pi)^{\frac{k-1}{2}}} \phi\left(\frac{a_n}{\sigma\sqrt{n_1 - n_0}}\right) \phi\left(\frac{b_n}{\sigma\sqrt{n_{k+1} - n_k}}\right) \prod_{i=1}^{k+1} \frac{1}{\sqrt{n_i - n_{i-1}}}, \end{aligned} \tag{5}$$

and

$$\frac{\mathbf{P}(\mathcal{A}_1 \cap \mathcal{A}_2)}{\mathbf{P}(\mathcal{A}_1)} \sim \frac{\phi\left(\frac{b_n - a_n}{\sigma\sqrt{n(t_1 - t_0)}}\right)}{\sigma\sqrt{n(t_1 - t_0)}}. \tag{6}$$

Next, we substitute (4) by the product of the right hand sides of (5) and (6) in the sum of (3). The resulting sum is a Riemann sum which is asymptotically equal to the following Riemann integral

$$n^{\frac{k}{2}} c_0^k k! \sigma^{-k} (2\pi)^{-\frac{k-1}{2}} \sqrt{t_1 - t_0} \left[\phi \left(\frac{b-a}{\sigma \sqrt{t_1 - t_0}} \right) \right]^{-1} \tag{7}$$
$$\int \cdots \int_{0 < s_1 < s_2 < \ldots < s_k < t_1 - t_0} d\underline{s}$$
$$\phi \left(\frac{a}{\sigma \sqrt{s_1}} \right) \frac{1}{\sqrt{s_1}} \frac{1}{\sqrt{s_2 - s_1}} \cdots \frac{1}{\sqrt{s_k - s_{k-1}}} \frac{1}{\sqrt{t_1 - t_0 - s_k}} \phi \left(\frac{b}{\sigma \sqrt{t_1 - t_0 - s_k}} \right),$$

where $\underline{s} = (s_1, \ldots, s_k)$. Note that when we substituted (4) by the product of the right hand sides of (5) and (6), we made an error. Due to Proposition 1, this error is bounded by

$$C \sqrt{n} \varsigma (\min_i \{n_i - n_{i-1}\}) \prod_{j=1}^{k+1} \frac{1}{\sqrt{n_j - n_{j-1}}},$$

with some constant C. Thus, in order to see that (7) is asymptotically equal to $\mathbb{I}_n^k$, it remains to prove that

$$\sum_{\lfloor nt_0 \rfloor = n_0 < n_1 < n_2 < \ldots < n_k < n_{k+1} = \lfloor nt_1 \rfloor} \sqrt{n} \varsigma (\min_i \{n_i - n_{i-1}\}) \prod_{j=1}^{k+1} \frac{1}{\sqrt{n_j - n_{j-1}}} \tag{8}$$

is in $o\left(n^{\frac{k}{2}}\right)$. To prove this, pick $\varepsilon > 0$ small and K such that $\varsigma(K) < \varepsilon$. The sum over indices $n_1, \ldots n_k$, where all $n_i - n_{i-1}$ is larger then K, is asymptotically bounded by

$$\varepsilon n^{\frac{k}{2}} \int \cdots \int_{0 < s_1 < s_2 < \ldots < s_k < t_1 - t_0} d\underline{s}$$
$$\frac{1}{\sqrt{s_1}} \frac{1}{\sqrt{s_2 - s_1}} \cdots \frac{1}{\sqrt{s_k - s_{k-1}}} \frac{1}{\sqrt{t_1 - t_0 - s_k}}.$$

Now, choose a subset H of the set $\{1, 2 \ldots k+1\}$, with $|H| = l \geq 1$. Then the sum over indices $n_1, \ldots n_k$, where $n_i - n_{i-1} \leq K$ for $i \in H$, and $n_i - n_{i-1} > K$ otherwise, is asymptotically bounded by $K^l n^{\frac{k-l}{2}}$ multiplied by an integral similar to the previous one. Thus, we have verified that (8) is in $o\left(n^{\frac{k}{2}}\right)$, which implies that $\mathbb{I}_n^k$ is asymptotically equal to (7). One can compute explicitly the integrals not involving the function ϕ. Namely, use the identity

$$\int_C^{t_1 - t_0} \frac{(t_1 - t_0 - x)^l}{\sqrt{x - C}} dx = (t_1 - t_0 - C)^{l + 1/2} \frac{\Gamma(l+1)\Gamma(1/2)}{\Gamma(l + 3/2)}$$

$k-2$ times, to deduce the following formula from (7):

$$\mathbb{I}_n^k \sim n^{\frac{k}{2}} c_0^k k! \sigma^{-k} (2\pi)^{-\frac{k-1}{2}} \sqrt{t_1 - t_0} \left[\phi\left(\frac{b-a}{\sigma\sqrt{t_1-t_0}}\right)\right]^{-1} \left[\Gamma\left(\frac{1}{2}\right)\right]^{k-1} \frac{1}{\Gamma\left(\frac{k-1}{2}\right)}$$
$$\iint_{0<s_1<s_2<t_1-t_0} ds_1 ds_2 \frac{\phi\left(\frac{a}{\sigma\sqrt{s_1}}\right)}{\sqrt{s_1}} \frac{\phi\left(\frac{b}{\sigma\sqrt{s_2-s_1}}\right)}{\sqrt{s_2-s_1}} (t_1-t_0-s_2)^{\frac{k}{2}-\frac{3}{2}} \tag{9}$$

for $k \geq 2$ (for $k = 1$ a simpler formula holds). Finally, one can slightly simplify the formula (9), since $\Gamma(1/2) = \sqrt{\pi}$. In order to complete the method of moments, on the one hand, one needs to prove that

$$\lim_{n\to\infty} \mathbb{I}_n^k c_0^{-k} n^{-k/2} = \mathbb{J}^k, \tag{10}$$

where $\mathbb{J}^k$ is the k-th moment of $\mathfrak{L}_{t_1}^{a,t_0 \rightsquigarrow b,t_1}$. It is easy to derive from the formulas computed in [2] and [14] that

$$Prob\left(\mathfrak{L}_{t_1}^{a,t_0\rightsquigarrow b,t_1} > y\right) = \exp\left[-\frac{1}{2\sigma^2(t_1-t_0)}\left(\left(|a|+|b|+\sigma^2 y\right)^2 - (b-a)^2\right)\right],$$

whence $\mathbb{J}^k$ can be expressed with an integral. On the other hand, one needs to prove that

$$\limsup_{k\to\infty} \left(\frac{\lim_{n\to\infty} \mathbb{I}_n^k c_0^{-k} n^{-k/2}}{k!}\right)^{1/k} < \infty \tag{11}$$

so as to verify that the limit distribution is uniquely determined. Observe that (9) immediately implies (11), but proving (10) turns out to be a nontrivial computation.

That is why we need to argue in a slightly different way. Namely, we are going to prove the first statement of the Lemma for random walks and then - since the moments for the random walk have the same asymptotic behavior, and these moments do converge - we arrive at the original statement.

To be more precise, pick a one dimensional simple symmetric random walk that starts from a_n and denote its position after $\lfloor n(t_1-t_0)\rfloor$ steps by Y_n. Similarly, its total number of visits to the origin until $\lfloor n(t_1-t_0)\rfloor$ is denoted by Z_n. Write f_1 for the probability density function of $\mathfrak{B}_{t_1}^{a,t_0}$ with σ being replaced by 1 (that is, $f_1(y) = \frac{1}{\sqrt{t_1-t_0}}\phi(\frac{y-a}{\sqrt{t_1-t_0}})$). Similarly, $F_{2|1}(z|y)$ stands for the conditional cumulative distribution function of $\mathfrak{L}_{t_1}^{a,t_0}$ under the condition $\mathfrak{B}_{t_1}^{a,t_0} = y$, again with σ replaced by 1. Note that $F_{2|1}(z|y)$ is the cumulative distribution function of $\mathfrak{L}_{t_1}^{a,t_0\rightsquigarrow y,t_1}$. Let y be a real number and $y_n = \lfloor y\sqrt{n}\rfloor$.

The following two statements are well known for random walks (see for example [2] and [13]):

$$Prob\left(Y_n < y_n, \frac{Z_n}{\sqrt{n}} < z\right) \to Prob\left(\mathfrak{B}_{t_1}^{a,t_0} < y, \mathfrak{L}_{t_1}^{a,t_0} < z\right), \tag{12}$$

and

$$\frac{\sqrt{n}}{2} Prob\left(Y_n \in \{y_n, y_n+1\}\right) \to f_1(y). \tag{13}$$

We want to prove that

$$\frac{\sqrt{n}}{2} Prob\left(Y_n \in \{y_n, y_n+1\}, \frac{Z_n}{\sqrt{n}} < z\right) =: p_n(y,z) \to f_1(y) F_{2|1}(z|y). \tag{14}$$

Note that in (13) and (14) the division by 2 is needed because of the periodicity of the random walk (i.e. it can return to the origin only in even number of steps). Also notice that using (13), one easily sees that (14) is equivalent to the first statement of the Lemma for simple symmetric random walks. Further, we mention that (14) is proved in [16] for the case $y = a$. The well known local limit theorem for random walks, and our previous computation yield that the k-th moment of $Z_n/\sqrt{n}$, under the condition $Y_n \in \{y_n, y_n+1\}$, have the same asymptotics as $\mathbb{I}_n^k c_0^{-k} n^{-k/2}$, with b replaced by y, and $\sigma = 1$. Thus (11), the method of moments and (13) imply that the distribution of $Z_n/\sqrt{n}$ - under the condition $Y_n \in \{y_n, y_n+1\}$ - weakly converges to a uniquely determined limit distribution. Whence, $p(y,z) := \lim_{n\to\infty} p_n(y,z)$ exists. Now suppose that there exist some y_0, z_0 such that $p(y_0,z_0) \neq f_1(y_0)F_{2|1}(z_0|y_0)$. For this fixed z_0, $f_1(y)F_{2|1}(z_0|y)$ is clearly continuous in y. Further, since the integral representation in (9) is continuous in b, the method of moments imply that the limit distribution, as $n \to \infty$, of $Z_n/\sqrt{n}$ - under the condition $Y_n \in \{y_n, y_n+1\}$ -, continuously depends on y (with respect to the weak topology). Hence, $p(y,z_0)$ is also continuous in y. Thus one can find an interval I containing y_0 such that $\int_I p(y,z_0)dy \neq \int_I f_1(y)F_{2|1}(z_0|y)dy$, which is a contradiction to (12). So we have verified (14). But (14) together with (11) implies (10) and the first assertion of the Lemma. $\square$

Now, we prove of the convergence of one dimensional distributions by a standard argument. That is, we need that for any open intervals A, B,

$$\mathbf{P}\left(\frac{S_{nt_0}^{(per)}}{\sqrt{n}} \in A, \frac{L_{nt_0}}{\sqrt{n}} \in B\right) \to Prob\left(\mathfrak{B}_{t_0} \in A, c_0\mathfrak{L}_{t_0} \in B\right). \tag{15}$$

The second statement of Lemma 1 implies the local version of (15) in the first coordinate, namely

$$\begin{aligned} & \sqrt{n}\mathbf{P}\left(S_{\lfloor nt_0 \rfloor}^{(per)} = \lfloor x\sqrt{n} \rfloor, \frac{L_{nt_0}}{\sqrt{n}} \in B\right) \qquad (16) \\ = \quad & \frac{1}{\sigma\sqrt{t_0}} \phi\left(\frac{x}{\sigma\sqrt{t_0}}\right) Prob\left(c_0 \mathfrak{L}_{t_0}^{0,0\rightsquigarrow x,t_0} \in B\right) + o(1). \end{aligned}$$

Now, define the real function φ_n, by setting $\varphi_n(x)$ to be equal to (16). Note that for fix n, φ_n is constant on the intervals $[k/\sqrt{n}, (k+1)/\sqrt{n})$ for any integer k. We have for any x,

$$\varphi_n(x) \to \frac{1}{\sigma\sqrt{t_0}} \phi\left(\frac{x}{\sigma\sqrt{t_0}}\right) Prob\left(c_0 \mathfrak{L}_{t_0}^{0,0\rightsquigarrow x,t_0} \in B\right) =: \varphi(x).$$

Thus, by Fatou's lemma,

$$\liminf_n \int_A \varphi_n(x)dx \geq \int_A \varphi(x)dx = Prob\left(\mathfrak{B}_{t_0} \in A, c_0\mathfrak{L}_{t_0} \in B\right). \qquad (17)$$

Analogously,

$$\liminf_n \int_{A^c} \varphi_n(x)dx \geq \int_{A^c} \varphi(x)dx = Prob\left(\mathfrak{B}_{t_0} \in A^c, c_0\mathfrak{L}_{t_0} \in B\right). \qquad (18)$$

As it was already mentioned in the beginning of the proof of Lemma 1, the rescaled local times converge to the appropriate limit. Thus,

$$\int_A \varphi_n(x)dx + \int_{A^c} \varphi_n(x)dx = \mathbf{P}\left(\frac{L_{nt_0}}{\sqrt{n}} \in B\right) \to$$
$$Prob\left(c_0\mathfrak{L}_{t_0} \in B\right) = \int_A \varphi(x)dx + \int_{A^c} \varphi(x)dx. \qquad (19)$$

Now, using (17), (18) and (19), we conclude that the inequalities in (17) and (18) are, in fact, equalities and the lim inf can be replaced by lim. This, together with the observation that the difference of $\int_A \varphi_n(x)dx$ and the left hand side of (15) is bounded by a constant times $n^{-1/2}$, implies (15).
The convergence of any finite dimensional marginals can be proven analogously, as we proved the one dimensional ones. The only main difference is that one needs a multiple version of the statements of Lemma 1, but its proof is also analogous.

Now we turn to the proof of tightness. Proposition 2 implies that the first coordinate converges weakly to the desired limit (in $C[t_0, 1]$ thus in $D[t_0, 1]$ as well), hence is tight, too. We are going to establish the tightness of the local times. Then it will follow that

$$\left(\frac{S_{nt}^{(per)}}{\sqrt{n}}, \frac{L_{nt}}{\sqrt{n}}\right)_{t\in[0,1]}$$

is tight, by definition.
Since the process L_{nt} is nondecreasing in t, tightness, in fact, can be deduced from the convergence of finite dimensional distributions. Namely, Theorem 15.2 in [1] yields that we only have to verify the following two statements:

1. For each $\eta > 0$ there is a $d \in \mathbb{R}$ such that
$$\mathbf{P}\left(\frac{L_n}{\sqrt{n}} > d\right) < \eta, \ n \geq 1. \qquad (20)$$

2. For each positive η and ε there is a δ, $0 < \delta < 1$ and an integer n_0 such that
$$\mathbf{P}\left(w_{\frac{L_{nt}}{\sqrt{n}}}(\delta) \geq \varepsilon\right) \leq \eta, \ n \geq n_0. \qquad (21)$$

Here,

$$w_\psi(\delta) = \inf_{\{t_i\}} \max_{0<i\le r} (\lim_{\tau \nearrow t_i} \psi(\tau) - \psi(t_{i-1})),$$

where the infimum is taken over finite sets $\{t_i\}$, for which $0 < t_1 < ... < t_r = 1$, $t_i - t_{i-1} > \delta$ for all i.

Since we have just verified that $L_n/\sqrt{n}$ converges weakly, (20) follows.
Again, the convergence of finite dimensional distributions implies that for fix $\eta > 0$ and $\varepsilon > 0$ one can find $\delta > 0$ and n_0 such that for all $n > n_0$, $0 \le k_1 \le \lfloor 1/\delta \rfloor$

$$\mathbf{P}\left(\frac{\#\{k : nk_1\delta < k < n(k_1+1)\delta, S_k^{(per)} \in \mathcal{J}\}}{\sqrt{n\delta}} > \frac{\varepsilon}{\sqrt{\delta}}\right) < \eta\delta,$$

Now the equidistant partition $\{t_i\}$ is enough to verify (21). Thus we have finished the proof of Proposition 3.

□

3 Proof of Theorem 1

Note that, though in its spirit our statement is very close to the results of [8], their proof cannot be applied here since the limiting process is not strong Markovian (see Remark 1) thus leaving no chance to apply the martingale method. Thus we need to argue in a more direct way, using Proposition 3. In Subsection 3.1 we prove the first statement of the theorem. That proof with trivial modifications is easily applicable to cases 2 and 3. The only non trivial modification is needed in case 4, which is treated in Subsection 3.2. Everywhere in this Section, we also use the notations introduced in Section 2.

3.1 Proof of case 1

In order to prove the statement, we need some technical lemmas.

Lemma 2. *Let E and F be any Polish spaces, X, X_n any random variables taking values in the space E such that $X_n \Rightarrow X$. Then for any continuous function $f : E \to F$ one has $(X_n, f(X_n)) \Rightarrow (X, f(X))$ in the product topology.*

Proof. Pick any $U \subset E \times F$ open set and define $V = \{x \in E : (x, f(x)) \in U\}$. If $x \in V$, then one can find an open product set $U_x = E_x \times F_x \subset U$ containing $(x, f(x))$. Since $f^{-1}(F_x)$ is open, $x \in E_x \cap f^{-1}(F_x)$ is also open. Now $x \in E_x \cap f^{-1}(F_x) \subset V$ implies that V is open, too. Thus

$$Prob((X_n, f(X_n)) \in U) = Prob(X_n \in V)$$

and the Portmanteau Theorem (see [1], for instance) yield the statement. □

Next, we prove the following Lemma which is an extension of Proposition 3.

Lemma 3.

$$\left(\frac{S_{nt}^{(per)}}{\sqrt{n}}, \frac{L_{nt}}{\sqrt{n}}, \int_{\tau=t_0}^{t} \frac{1}{\sqrt{\tau}} d\frac{L_{n\tau}}{\sqrt{n}}\right)_{t\in[t_0,1]} \Rightarrow \left(\mathfrak{B}_t, c_0\mathfrak{L}_t, c_0\int_{\tau=t_0}^{t} \frac{1}{\sqrt{\tau}} d\mathfrak{L}_\tau\right)_{t\in[t_0,1]},$$

where the left hand side is understood as a random variable with respect to the probability measure $\mathbf{P}$ *and* $\Rightarrow$ *stands for weak convergence in the Skorokhod space* $D_{\mathbb{R}^3}[t_0,1]$.

Proof. Use Proposition 3 and Lemma 2 with the choice

$$\begin{aligned} E &= \{\psi = (\psi_1,\psi_2) \in D_{\mathbb{R}^2}[t_0,1] : \psi_2 \text{ is non decreasing}\}, \\ F &= D[t_0,1] \\ f((\psi_1,\psi_2)) &= \left(\int_{\tau=t_0}^{t} \frac{1}{\sqrt{\tau}} d\psi_2\right)_{t\in[t_0,1]} \end{aligned}$$

to infer Lemma 3. □

Note that we needed to restrict the processes of Proposition 3 to $D_{\mathbb{R}^2}[t_0,1]$ in order to the above stochastic integrals be finite. (This technical difficulty can be avoided in the proof of case 2.) Finally, we will need Le Cam's famous inequality which was proven in [11].

Lemma 4. *Assume* Σ_m *is the sum of* m *independent, non-identically distributed Bernoulli random variables* ε_j; $1 \le j \le m$ *such that* $Prob(\varepsilon_j = 1) = p_j$. *Then*

$$\sum_{k=0}^{\infty} \left| Prob(\Sigma_m = k) - e^{-\lambda}\lambda^k/k! \right| \le 2\sum_{j=1}^{m} p_j^2,$$

where $\lambda = p_1 + \cdots + p_m$.

Now, we can proceed to the proof of case 1 of Theorem 1. First, we are going to prove a simplified version of the assertion, namely, the convergence of the measures $\mu_n^{\searrow}$ restricted to $C[t_0,1]$. Then the statement of the first part of the Theorem will follow easily.
Note that one can think about our model as having two sources of randomness. The first one is the choice of x and the second is the choice of ξ's. In Section 2, we were only dealing with the first source, but now we are going to treat the second one, as well.
It would be more convenient to consider $S_n^{\searrow}$ as if the time instants of the reflections on the wall W_k $(1 \le k \le n)$ were not computed. Since Proposition 3 and the sctterer configuration being symmetric to the y-axis imply that $|\{i \le n : \mathcal{F}_i^{\searrow} \dots \mathcal{F}_1^{\searrow} \in \mathcal{J}\}|$ is asymptotically of order $\sqrt{n}$, the diffusively scaled limits of $S_n^{\searrow}$ and of this "modified $S_n^{\searrow}$" (i.e. when we do not count the reflections on the wall) have the same limit. Thus it is sufficient to prove our

statement for the "modified $S_n^{\searrow}$" - which will also be denoted by $S_n^{\searrow}$ in the sequel.
Note that the assumption of the periodic scatterer configuration being symmetric implies

$$|S_n^{\searrow}| = |S_n^{(per)}|. \tag{22}$$

Now for fix x, define $p(nt)$ as the probability, generated by the choice of ξ_n's, of the event that $S^{\searrow}_{\lfloor nt \rfloor}(x) S^{\searrow}_{\lfloor nt \rfloor + 1}(x) < 0$, i.e. after step number $\lfloor nt \rfloor$, the particle crosses the hole. Lemma 3 implies that - by Skorokhod's representation theorem, cf. [1] - there exists a probability space $(\Omega, \mathbb{Q})$ together with random variables $(\tilde{X}_n, \tilde{Y}_n, \tilde{Z}_n)$ having the same joint distribution as

$$\left(\left(\frac{S_{nt}^{(per)}}{\sqrt{n}} \right)_{t \in [t_0,1]}, \left(\frac{c}{c_1} \int_{\tau = t_0}^{t} \frac{1}{\sqrt{\tau}} d \frac{L_{n\tau}}{\sqrt{n}} \right)_{t \in [t_0,1]}, \left(\int_{\tau = t_0}^{t} p(n\tau) dL_{n\tau} \right)_{t \in [t_0,1]} \right)$$

with respect to $\mathbf{P}$, and also with random variables $(\tilde{X}, \tilde{Y})$ having the same joint distribution as

$$\left((\mathfrak{B}_t)_{t \in [t_0,1]}, \left(\frac{cc_0}{c_1} \int_{\tau = t_0}^{t} \frac{1}{\sqrt{\tau}} d\mathfrak{L}_\tau \right)_{t \in [t_0,1]} \right),$$

such that $(\tilde{X}_n, \tilde{Y}_n) \to (\tilde{X}, \tilde{Y})$ $\mathbb{Q}$-almost surely. Here, $c = \lim_n \alpha_n \sqrt{n}$ Now, for $\mathbb{Q}$-almost all $\omega \in \Omega$ we define the measures $\nu(\omega)$, $\nu_n(\omega), \lambda_n(\omega)$ on $C[t_0, 1]$ in the following way. Consider the modulus of $\tilde{X}(\omega)$, i.e. $|\tilde{X}(\omega)| \in C[t_0, 1]$ (if $\tilde{X}(t_0)(\omega) > 0$; otherwise consider $-|\tilde{X}(\omega)|$), pick a Poisson point process - on some abstract probability space $(\Omega_\omega, \mathbb{Q}_\omega)$ - with intensity measure $d\tilde{Y}(\omega)$, and denote its point by $P_1 < P_2 < ...$ N.b. there are finitely many points. If it has m points, put $P_{m+1} = 1$. Now reflect the subgraph of $|\tilde{X}(\omega)| [P_{2i+1}, P_{2i+2}]$ to the origin for each i (if $\tilde{X}(t_0)(\omega) > 0$; otherwise reflect $-|\tilde{X}(\omega)|$). The distribution of the resulting random function - with respect to $\mathbb{Q}_\omega$ - generates a measure on $C[t_0, 1]$ which we denote by $\nu(\omega)$. The construction of $\nu_n(\omega)$ is similar, with two differences. The first is that one should replace $\tilde{X}$ and $\tilde{Y}$ by $\tilde{X}_n$ and $\tilde{Y}_n$ and the second is that instead of the Poisson point process, one introduces independent Bernoulli random variables for each discontinuity of the function $\tilde{Y}_n(\omega)$ with parameters being equal to the size of jump of $\tilde{Y}_n(\omega)$ at the corresponding discontinuity. Then denote by $P_1 < P_2 < \dots$ the positions, where the Bernoulli random variables are equal to 1. Finally, $\lambda_n(\omega)$ is defined the same way as $\nu_n(\omega)$ with $\tilde{Y}_n$ being replaced by $\tilde{Z}_n$.
Using Lemma 4, one can infer that for $\mathbb{Q}$-almost all ω, $\nu_n(\omega) \Rightarrow \nu(\omega)$ on $C[t_0, 1]$. Further, $\alpha_n \sqrt{n} \to c$ implies that for any fixed x and $\varepsilon > 0$, if $L_{\lfloor n\tau \rfloor - 1} < L_{\lfloor n\tau \rfloor}$, i.e. in $\lfloor n\tau \rfloor$ steps the particle arrives to $\mathcal{J}$, then $|p(\lfloor n\tau \rfloor) - c/(c_1 \sqrt{\lfloor n\tau \rfloor})| < \varepsilon/\sqrt{n}$ assuming that n is large enough. Whence, one can naturally couple the Bernoulli distributed random variables used by the definition of ν_n and λ_n in such a way that the resulting random functions in $C[t_0, 1]$ coincide on a subset of Ω_ω, whose $\mathbb{Q}_\omega$ measure tends to 1 as $n \to \infty$. Consequently, $\lambda_n(\omega) \Rightarrow \nu(\omega)$ on $C[t_0, 1]$ for

$\mathbb{Q}$-almost all ω, too.
Define the measures ϱ and ϱ_n on $C[t_0,1]$ by

$$\begin{aligned}\varrho(A) &= \int_\Omega \nu(\omega)(A)d\mathbb{Q}(\omega),\\ \varrho_n(A) &= \int_\Omega \lambda_n(\omega)(A)d\mathbb{Q}(\omega).\end{aligned}$$

Using that $\lambda_n(\omega)\Rightarrow\nu(\omega)$ for $\mathbb{Q}$-almost all ω, Fatou's lemma and the Portmanteau theorem, we obtain for any $A\subset C[t_0,1]$ open set:

$$\begin{aligned}\liminf_n \varrho_n(A) &= \liminf_n \int_\Omega \lambda_n(\omega)(A)d\mathbb{Q}(\omega)\\ &\geq \int_\Omega \liminf_n \lambda_n(\omega)(A)d\mathbb{Q}(\omega) \geq \int_\Omega \nu(\omega)(A)d\mathbb{Q}(\omega)=\varrho(A).\end{aligned}$$

Whence - by the Portmanteau theorem, again - $\varrho_n\Rightarrow\varrho$ on $C[t_0,1]$.
Observe that by construction, ϱ is the measure on $C[t_0,1]$ generated by a QRBM$(cc_0/c_1,\sigma)$. Similarly, ϱ_n is the restriction of $\mathbf{W}_n^{\searrow}$ to $C[t_0,1]$.
Now, one can easily prove the first part of the Theorem. Since the choice of t_0 was arbitrary, a limit theorem of any finite dimensional distributions is implied by the above computation. The tightness is also easy since the moduli of the random functions are tight. Thus we have finished the proof of the first part of the Theorem (in fact, with the constant $c_2=c_0/c_1$).

3.2 Proof of case 4

As in the previous subsection, the tightness is trivial since the moduli of the random functions are tight. The convergence of one dimensional distributions follows from symmetry and Proposition 1. Here, we are only going to prove the convergence of two dimensional marginals since the convergence of any finite dimensional ones can be proven similarly.
The idea of the proof is that we know the convergence of $(|S^{\searrow}_{\lfloor nt_0\rfloor}|/\sqrt{n},|S^{\searrow}_{\lfloor nt_1\rfloor}|/\sqrt{n})$ to the desired limit, thus we only need to care about the sign. For the latter, assume that $S^{\searrow}_{\lfloor nt_0\rfloor}/\sqrt{n}$ is in a fixed positive interval, while $|S^{\searrow}_{\lfloor nt_1\rfloor}|/\sqrt{n}$ is in another fixed positive interval. Using Proposition 3, and the results of the previous subsection, we can estimate the asymptotic probability of the local time being zero under the above condition. If the local time is zero, then the trajectory avoids the origin, hence $S^{\searrow}_{\lfloor nt_1\rfloor}>0$. If not, then the particle arrives at the origin eventually in $[nt_0,nt_1]$, and we need to verify that at time nt_1, it will end up in the positive half line with probability 1/2. The heuristic reason for this is that once it is near the origin, since $\alpha_n\sqrt{n}$ is large, it will cross the holes many times, and thus forget that it came from the positive half-line. This argument will imply that the weak limit must be the two dimensional marginal of the BM. Let us make the above argument precise. To do so, we will use the notations

of the previous subsection. Especially, introduce the modification of $S_n^{\searrow}$ as in the previous subsection. Thus (22) still holds. Fix $0 < t_0 < t_1 \leq 1$ and J_0, J_1 compact subintervals of $\mathbb{R}_+ \cup \{0\}$. Our aim is to prove that

$$\mathbf{P}\left(\frac{S^{\searrow}_{\lfloor nt_0 \rfloor}}{\sqrt{n}} \in J_0, \frac{S^{\searrow}_{\lfloor nt_1 \rfloor}}{\sqrt{n}} \in J_1\right) \to Prob\left(\mathfrak{B}_{t_0} \in J_0, \mathfrak{B}_{t_1} \in J_1\right), \tag{23}$$

and

$$\mathbf{P}\left(\frac{S^{\searrow}_{\lfloor nt_0 \rfloor}}{\sqrt{n}} \in J_0, \frac{S^{\searrow}_{\lfloor nt_1 \lfloor}}{\sqrt{n}} \in -J_1\right) \to Prob\left(\mathfrak{B}_{t_0} \in J_0, \mathfrak{B}_{t_1} \in -J_1\right), \tag{24}$$

as $n \to \infty$. Once we verify (23) and (24), by symmetry, they will also hold true for J_0 being a compact interval in $\mathbb{R}_- \cup \{0\}$, and hence the convergence of two dimensional marginals will follow.
Define the probabilities

$$p_{J_0,J_1} = Prob\left(\forall s : t_0 < s < t_1 : \mathfrak{B}_s > 0 | \mathfrak{B}_{t_0} \in J_0, |\mathfrak{B}_{t_1}| \in J_1\right).$$

Now let A be the set of functions ψ in $C[0,1]$ for which $\psi(t_0) \in J_0$, $\forall s : t_0 < s < t_1 : \psi(s) > 0$, and $\psi(t_1) \in J_1$. Then the Wiener measure of ∂A is zero, thus Proposition 2 implies that

$$\mathbf{P}\left(\forall t_0 < s < t_1 : S^{(per)}_{ns} > 0 | \frac{S^{(per)}_{\lfloor nt_0 \rfloor}}{\sqrt{n}} \in J_0, \frac{|S^{(per)}_{\lfloor nt_1 \rfloor}|}{\sqrt{n}} \in J_1\right) \to p_{J_0,J_1}, \tag{25}$$

as $n \to \infty$. On the other hand, the strong Markov property of the BM obviously implies

$$Prob\left(\mathfrak{B}_{t_1} \in J_1 | \mathfrak{B}_{t_0} \in J_0, \exists s : t_0 < s < t_1 : \mathfrak{B}_s = 0, |\mathfrak{B}_{t_1}| \in J_1\right) = \frac{1}{2}. \tag{26}$$

Now, with the notation

$$\mathcal{A}_4(n) = \{x : \frac{S^{\searrow}_{\lfloor nt_0 \rfloor}(x)}{\sqrt{n}} \in J_0, \exists t_0 < s < t_1 : S^{(per)}_{ns}(x) = 0, \frac{|S^{\searrow}_{\lfloor nt_1 \rfloor}(x)|}{\sqrt{n}} \in J_1\},$$

we want to prove that

$$\mathbf{P}\left(\frac{S^{\searrow}_{\lfloor nt_1 \rfloor}}{\sqrt{n}} \in J_1 | \mathcal{A}_4(n)\right) \to \frac{1}{2}. \tag{27}$$

Note that combining Proposition 2, (22), (25), (26) and (27), one can deduce (23) and (24). Thus it remains to prove (27).
To do so, first observe that by Proposition 3, for every $\varepsilon > 0$ there exists $\delta > 0$ and N such that for all $n > N$,

$$\mathbf{P}\left(L_{[nt_0,nt_1]} > \delta\sqrt{n} | \mathcal{A}_4(n)\right) > 1 - \varepsilon. \tag{28}$$

Now consider the Markov transition matrices

$$A_p = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}$$

on the space $\{+,-\}$ and a time dependent Markov chain M_k such that $M_0 = +$ and the transition between k and $k+1$ is described by A_{p_k} with some numbers p_k. Now, for a fixed $x \in \mathcal{M}^{(per)}$ and n, define $D_k(x)$ as the k-th leftmost discontinuity of the function $s \to L_{\lfloor ns \rfloor}(x)$ on $s \in [t_0, t_1)$. With the choice $p_k(x) = \alpha_{nD_k(x)}/c_1$, $1 \le k \le L_{[\lfloor nt_0 \rfloor, \lfloor nt_1 \rfloor - 1]}(x)$ for each x, one easily sees that for n large enough, the probability in (27) is equal to

$$\frac{1}{\mathbf{P}(\mathcal{A}_4(n))} \int_{\mathcal{A}_4(n)} Prob(M_{L_{[\lfloor nt_0 \rfloor, \lfloor nt_1 \rfloor -1]}(x)} = +) d\mathbf{P}(x). \tag{29}$$

If fact, this is only true for the case of $\mu_n^{\searrow}$, while in the case of $\mu_n^{\equiv}$, one needs to set $p_k(x) = \alpha_n/c_1$.
On the other hand, elementary computations show that if one selects sequences $B(n) \to \infty, m(n) \to \infty$ and non-negative numbers $p_{k,n}, 1 \le n, 1 \le k \le m(n)$, then with the transition matrices corresponding to $p_{1,n}, ..., p_{m(n),n}$,

$$\begin{aligned} & Prob(M_{m(n)} = +) \\ = \;& \begin{pmatrix} 1 & 0 \end{pmatrix} A_{p_{1,n}} \dots A_{p_{m(n),n}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ = \;& \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \prod_{k=1}^{m(n)} (1 - 2p_{k,n}) \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \\ = \;& 1/2 + o(1), \end{aligned}$$

as $n \to \infty$. Further, $o(1)$ converges to zero uniformly in m and p_k if $m(n) > \delta\sqrt{n}$ and $\min_{1 \le k \le m(n)} \{p_{k,n}\sqrt{n}\} > B(n)$. Now, choose $B(n) = \min_{N \ge \lfloor nt_0 \rfloor} \{\alpha_N \sqrt{N}/c_1\}$. This estimation together with (28) and (29) yields (27).

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