LIMIT THEOREMS FOR LOW DIMENSIONAL GENERALIZED T, T^{-1} TRANSFORMATIONS

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ABSTRACT. We consider generalized (T, T^{-1}) transformations such that the base map satisfies a multiple mixing local limit theorem and anticoncentration large deviation bounds and in the fiber we have $\mathbb{R}^d$ actions with $d = 1$ or 2 which are exponentially mixing of all orders. If the skewing cocycle has zero drift, we show that the ergodic sums satisfy the same limit theorems as the random walks in random scenery studied by Kesten and Spitzer (1979) and Bolthausen (1989). The proofs rely on the quenched CLT for the fiber action and the control of the quenched variance. This paper complements our previous work where the classical central limit theorem is obtained for a large class of generalized (T, T^{-1}) transformations.

1. INTRODUCTION

This work is a continuation of our study on generalized T, T^{-1} transformations, following previous work [11, 12, 13]. Our main innovation in this paper is to provide several limit theorems in the low dimensional setting, complementing to the higher dimensional case in [12].

1.1. **Results.** Let f be a smooth map of a manifold X preserving a measure μ, and G_t be an $\mathbb{R}^d$ action on a manifold Y preserving a measure ν.

Definition 1.1. G_t enjoys multiple exponential mixing of all orders if there is $\alpha > 0$ such that for each r there are constants $C, c > 0$ such that for all zero mean C^α functions $A_1, \ldots, A_r$, for all $t_1, \ldots t_r \in \mathbb{R}^d$

$$(1.1) \qquad \left| \nu \left(\prod_{j=1}^r A(G_{t_j} y) \right) \right| \leq C \left[\prod_{j=1}^r \|A\|_{C^\alpha} \right] e^{-c\mathfrak{l}}$$

where $\mathfrak{l}$ is the gap $\mathfrak{l} = \max_{i \neq j} \|t_i - t_j\|$.

Remark 1.2. A simple interpolation shows that if (1.1) holds for some α then it also holds for all α (see e.g. [14, Appendix A]).

In case $d = 1$, there are plenty of examples of multiply exponentially mixing systems see e.g. the discussion in [11]. In the case $d = 2$ our main example is the following: $Y = SL_3(\mathbb{R})/\Gamma$, where Γ is a cocompact lattice, $G_t : Y \to Y$ is the Cartan action on Y, and ν is the Haar measure. More generally one can consider subactions of Cartan actions on $\mathcal{G}/\Gamma$ where $\mathcal{G}$ is a semisimple Lie group with compact factors and Γ is a cocompact lattice. In particular, we can take $Y = SL_d(\mathbb{R})/\Gamma$, and let G_t be an action by a two dimensional subgroup of the group of diagonal matrices.

Let $\tau : X \to \mathbb{R}^d$ be a smooth map. We study the map $F : (X \times Y) \to (X \times Y)$ given by

$$(1.2) \qquad\qquad F(x,y) = (f(x), G_{\tau(x)}y).$$

Note that F preserves the measure $\zeta = \mu \times \nu$ and that

$$F^N(x,y) = (f^N x, G_{\tau_N(x)}y) \quad \text{where} \quad \tau_N(x) = \sum_{n=0}^{N-1} \tau(f^n x).$$

Let $H : X \times Y \to \mathbb{R}$ be a sufficiently smooth function with $\zeta(H) = 0$ and

$$S_N = \sum_{n=0}^{N-1} H(F^n(x,y)).$$

We want to study the distribution of S_N when the initial condition (x, y) is distributed according to ζ.

We shall assume that f is ergodic and satisfies the CLT for Hölder functions.

Definition 1.3. We say that τ satisfies the *mixing local limit theorem* (MLLT) if for any sequences $(\delta_n)_{n\in\mathbb{N}} \in \mathbb{R}$, with $\lim\limits_{n\to\infty} \delta_n = 0$ and $(z_n)_{n\in\mathbb{N}} \in \mathbb{R}^d$ such that $|\frac{z_n}{\sqrt{n}} - z| < \delta_n$ for any cube $\mathcal{C} \subset \mathbb{R}^d$ and any continuous functions $A_0, A_1 : X \to \mathbb{R}$

$$\lim_{n\to\infty} n^{d/2} \mu\Big(A_0(\cdot)A_1(f^n\cdot)\mathbb{1}_{\mathcal{C}}(\tau_n - z_n)\Big) = \mathfrak{g}(z)\mu(A_0)\mu(A_1)\mathrm{Vol}(\mathcal{C})$$

where $\mathfrak{g}(z)$ is a non-degenerate Gaussian density and the convergence is uniform once $(\delta_n)_{n\in\mathbb{N}}$ is fixed and A_0, A_1, z range over compact subsets of $C(X), C(X)$ and $\mathbb{R}^d$.

Definition 1.4. τ satisfies *multiple mixing local limit theorem* (MMLLT) if for each $m \in \mathbb{N}$ for any sequence $(\delta_n)_{n\in\mathbb{N}} \in \mathbb{R}$, with $\lim\limits_{n\to\infty} \delta_n = 0$, for any family of sequences $(z_n^{(1)}, \ldots, z_n^{(m)})_{n\in\mathbb{N}}$ with $|\frac{z_n^{(j)}}{\sqrt{n}} - z^{(j)}| < \delta_n$ for any cubes $\{\mathcal{C}_j\}_{j\le m} \subset \mathbb{R}^d$ and any continuous functions $A_0, A_1, \ldots A_m : X \to \mathbb{R}$ for any sequences $n_1 \ldots n_m \in \mathbb{N}$ such that $n_{j+1} - n_j \to \infty$ (with $n_0 = 0$),

$$\lim_{\min|n_j - n_{j'}|\to\infty} \left(\prod_{j=1}^m (n_j - n_{j-1})^{d/2}\right) \mu\left(\prod_{j=0}^m A_j(f^{n_j}\cdot)\prod_{j=1}^m \mathbb{1}_{\mathcal{C}_j}\left(\tau_{n_j} - z_{n_j}^{(j)}\right)\right)$$

$$= \prod_{j=0}^m \mu(A_j)\prod_{j=1}^m \mathfrak{g}\left(z^{(j)} - z^{(j-1)}\right)\prod_{j=1}^m \mathrm{Vol}(\mathcal{C}_j),$$

where $z^{(0)} = 0$. Moreover, the convergence is uniform once $(\delta_n)_{n \in \mathbb{N}}$ is fixed, $A_0, A_1, \ldots, A_m$ range over compact subsets of $C(X)$ and $z^{(j)}$ range over a compact subset of $\mathbb{R}^d$ for every $j \leq m$.

Definition 1.5. τ satisfies the *anticoncentration large deviation bound of order s* if there exist a constant K and a decreasing function Θ such that $\displaystyle\int_1^\infty \Theta(r) r^d \mathrm{d}r < \infty$ and for any numbers $n_1, n_2, \ldots, n_s$, for any unit cubes $C_1, C_2, \ldots, C_s$ centered at $c_1, c_2, \ldots, c_s$

$$(1.3) \qquad \mu\left(x : \tau_{n_j} \in C_j \text{ for } j = 1, \ldots, s\right) \leq$$

$$K \left(\prod_{j=1}^s (n_j - n_{j-1})^{-d/2} \right) \Theta\left(\max_j \frac{\|c_j - c_{j-1}\|}{\sqrt{n_j - n_{j-1}}} \right).$$

The class of maps satisfying the MMLLT and the large deviation anticoncentration bounds includes in particular the maps which admit Young towers with exponential tails, see [21, 19].

Theorem 1.6. Suppose that $d = 2$, G_t enjoys multiple exponential mixing of all orders, f satisfies the CLT for smooth observables and τ satisfies the MMLLT and the anticoncentration large deviation bound. Then there is Σ^2 such that $\dfrac{S_N}{\sqrt{N \ln N}}$ converges as $N \to \infty$ to the normal distribution with zero mean and variance Σ^2.

Theorem 1.7. Suppose that $d = 1$, G_t enjoys multiple exponential mixing of all orders, f satisfies the CLT for smooth observables and τ satisfies the MMLLT and the anticoncentration large deviation bound. Then there is a constant Σ such that

$$(1.4) \qquad \frac{S_N}{N^{3/4}} \quad \text{converges as } N \to \infty \text{ to a product} \quad \Sigma \mathcal{L} \mathcal{N},$$

where $\mathcal{L}$ and $\mathcal{N}$ are independent, $\mathcal{N}$ has standard normal distribution and

$$(1.5) \qquad \mathcal{L} = \sqrt{\int_{-\infty}^\infty \ell_x^2 \mathrm{d}x}$$

where ℓ_x is the local time of the standard Brownian Motion at time 1 and spatial location x.

Remark 1.8. If $d = 1$ then the exponential mixing condition can be weakened to sufficiently fast polynomial mixing, see Theorem 5.1 in Section 5. It seems more difficult to weaken the mixing assumption in two dimension. We do not pursue this topic since we do not know examples of smooth $\mathbb{R}^2$ actions which are mixing at a fast polynomial (rather than exponential) speed.

Remark 1.9. The asymptotic variance Σ^2 in Theorems 1.6 and 1.7 has similar form (see (3.10) and (4.5)). Namely let $\tilde{H}(y) = \displaystyle\int_X H(x, y) d\mu(x)$. Then

$$\Sigma^2 = c_d \int_{\mathbb{R}^d} \int_Y \tilde{H}(y) \tilde{H}(G_t y) d\nu(y) dt.$$

In dimension 1 if $\Sigma^2 = 0$ then there is an L^2 function $J(y)$ such that

$$\int_0^t \tilde{H}(G_s y) ds = J(G_t y) - J(y).$$

This was shown in the discrete case by Rudolph, [20] (Proposition 2 and Lemma 7), but the proof applies with no changes in the continuous case.

It is shown in [12, Theorem 6.4] that for $\mathbb{R}^1$ volume preserving exponentially mixing flows, the quadratic form $H \mapsto \Sigma^2(H)$ is not identically zero. Therefore the set of functions of vanishing asymptotic variance is a proper linear subspace. The proof of Theorem 6.4 in [12] works also for higher dimensional (not necessarily volume preserving actions) provided that

(i) there are constants $K_1, \beta_1, K_2, \beta_2$ such that for all y and r, $K_1 r^{\beta_1} \leq \nu(B(y, r)) \leq K_2 r^{\beta_2}$ and

(ii) there exists a *slowly recurrent point*, that is, a point $y \in Y$ such that for all positive constants K and A there exists $r_0(K, A)$ so that for all $r < r_0$ and for all t with $|t| < K$, $\nu(B(y, r) G_{-t} B(y, r)) \leq \dfrac{\nu(B(y, r))}{|\ln r|^A}$, where $B(y, r)$ is the ball of radius r centered at y.

(We note that for exponentially mixing volume preserving $\mathbb{R}$ actions, almost all points are slowly recurrent, see [14].)

We believe that in many cases the vanishing of the asymptotic variance entails that the ergodic sums of H satisfy the classical CLT but this will be a subject of a future work.

Remark 1.10. Results analogous to the above theorems can be proved in case G is an action of $\mathbb{Z}^d$ $(d = 1, 2)$ and $\tau : X \to \mathbb{Z}^d$ is a piecewise smooth map satisfying the appropriate assumptions such as the CLT, MMLLT, and anticoncentration large deviations bounds. Since the results as well as the proofs are virtually the same, we omit the case of discrete actions. One can also take X to be a subshift of finite type, see the discussion below.

1.2. **Discussion.** Here, we discuss previous results related to Theorems 1.6, 1.7.

The first results about T, T^{-1} transformations pertain to so called *random walks in random scenery*. In this model we are given a sequence $\{\xi_z\}_{z \in \mathbb{Z}^d}$ of i.i.d. random variables. Let τ_n be a simple random walk on $\mathbb{Z}^d$ independent of ξs. We are interested in $S_N = \displaystyle\sum_{n=1}^N \xi_{\tau_n}$. This model could be put in the present framework as follows. Let X be a set of sequences $\{v_n\}_{n \in \mathbb{Z}}$, where $v_n \in \{\pm e_1, \pm e_2, \cdots \pm e_d\}$ where e_j are basis vectors in $\mathbb{Z}^d$, μ is the Bernoulli measure with $\mathbb{P}(v = \pm e_j) = \frac{1}{2d}$ for all $j \in 1, \ldots, d$, Y is the space of sequences $\{\xi_z\}_{z \in \mathbb{Z}^d}$, ν is the product of distribution functions of ξ, f and G_t are shifts and $\tau(\{v\}) = v_0$. For random walks in random scenery, Theorem 1.7 is due to [15], and Theorem 1.6 are due to [4]. The results of [15] and [4] are extended to more general actions in the fiber (still assuming the random walk in the base) in [7, 8].

In the context of dynamical systems, Theorem 1.7 was proven in [17] under the assumption that we have a good rate of convergence in the Central Limit Theorem for

f. In the present paper we follow a method of [4] which seems more flexible and allows a larger class of base systems.

We also note that if $d \geq 3$ or τ has a drift, one has a classical CLT (that is, $\dfrac{S_N - \mathbb{E}(S_N)}{\sqrt{N}}$ converges to a Gaussian distributions). These results are proven in [11, 12]. The paper [11] also studies mixing properties of (T, T^{-1}) transformation defined by (1.2). In particular [11] shows that in many cases the mixing of the whole product F is exponential. We observe that in the case where F enjoys multiple exponential mixing, one can obtain the classical CLT by applying the results of [3]. The CLT of Björklund and Gorodnik [3] also plays a key role in our proof and we review it in Section 2.

In the case where the base map f is uniformly hyperbolic the skew product map F is partially hyperbolic. It is possible that generic partially hyperbolic maps enjoy strong statistical properties including the multiple exponential mixing and the Central Limit Thereom. However, such a result seems currently beyond reach (some special cases are considered in [1, 6]). In present (T, T^{-1}) setting the exotic limit theorems such as our Theorems 1.7 and 1.6 require the zero drift assumption, which is a codimension d condition.

2. Björklund - Gorodnik CLT

In order to prove our results we use the strategy of [4] replacing the Feller Lindenberg CLT for iid random variables by a CLT for exponentially mixing systems due to [3]. More precisely we need the following fact.

Proposition 2.1. Let $\mathfrak{m}_N$, $N \in \mathbb{N}$ be a sequence of measures on $\mathbb{R}^d$ and let $\{A_{t,N}\}_{t \in \mathbb{R}^d, N \in \mathbb{N}}$ be a family of real valued functions on Y so that $\|A_{t,N}\|_{C^1(Y)}$ is uniformly bounded and $\nu(A_{t,N}) \equiv 0$. Set $\mathcal{S}_N := \int_{\mathbb{R}^d} A_{t,N}(G_t y) \mathrm{d}\mathfrak{m}_N(t)$. Suppose that

(a) $\lim\limits_{N \to \infty} \|\mathfrak{m}_N\| = \infty$ where $\|\mathfrak{m}\| = \mathfrak{m}(\mathbb{R}^d)$.
(b) For each $r \in \mathbb{N}$, $r \geq 3$ for each K

$$\int \mathfrak{m}_N^{r-1}[B(t, K \ln \|\mathfrak{m}_N\|)] \mathrm{d}\mathfrak{m}_N(t) = 0,$$

where $B(t, R) \subset \mathbb{R}^d$ is the ball of radius R around $t \in \mathbb{R}^d$.
(c) $\lim\limits_{N \to \infty} V_N = \sigma^2$ where

$$V_N := \int \mathcal{S}_N^2(y) \mathrm{d}\nu(y) = \iiint A_{t_1,N}(G_{t_1} y) A_{t_2,N}(G_{t_2} y) \mathrm{d}\mathfrak{m}_N(t_1) \mathrm{d}\mathfrak{m}_N(t_2) \mathrm{d}\nu(y).$$

Then, as $N \to \infty$, $\mathcal{S}_N$ converges weakly to the normal distribution with zero mean and variance σ^2.

This proposition is proven in [3] in case $A_{t,T}$ does not depend on t and T, however the proof easily extends to the case of t, T-dependent A, see [13].

3. DIMENSION TWO

3.1. **Reduction to quenched CLT.** Here we prove Theorem 1.6.

Consider a function $\tilde{H}$ satisfying

$$(3.1) \qquad \int \tilde{H}(x,y)\mathrm{d}\nu(y) = 0$$

for all $x \in X$.

Given $x \in X$ define the measures $\mathfrak{m}_N$ on $\mathbb{R}^2$ by

$$(3.2) \qquad \mathfrak{m}_N(x) = \frac{1}{\sqrt{N \ln N}} \sum_{n=0}^{N-1} \delta_{\tau_n(x)}$$

and functions $A_{t,N,x}(y)$ by

$$(3.3) \qquad A_{t,N,x}(y) = \frac{1}{\mathrm{Card}(n \le N : \tau_n(x) = t)} \sum_{n \le N : \tau_n(x) = t} \tilde{H}(f^n x, y).$$

Proposition 3.1. Under the assumptions of Theorem 1.6, there exists σ^2 and subsets $X_N \subset X$ such that $\lim_{N \to \infty} \mu(X_N) = 1$ and for any sequence $x_N \in X_N$ the measures $\{\mathfrak{m}_N(x_N)\}$ satisfy the conditions of Proposition 2.1.

The proposition will be proven later. Now we shall show how to obtain Theorem 1.6 from the proposition.

Proof of Theorem 1.6 assuming Proposition 3.1. Split

$$(3.4) \qquad H(x,y) = \tilde{H}(x,y) + \bar{H}(x) \quad \text{where} \quad \bar{H}(x) = \int H(x,y)\mathrm{d}\nu(y).$$

Denote

$$\mathcal{S}_N(x,y) = \frac{1}{\sqrt{N \ln N}} \sum_{n=0}^{N-1} \tilde{H}(F^n(x,y)).$$

Note that $\tilde{H}$ satisfies (3.1) and hence, by Proposition 3.1, $\mathcal{S}_N(x,y)$ is asymptotically normal and is asymptotically independent from x. Finally the contribution of $\bar{H}$ is negligible. Indeed, the CLT for smooth observables gives $\dfrac{1}{\sqrt{N \ln N}} \sum_{n=0}^{N-1} \bar{H}(f^n(x)) \Rightarrow 0$. This completes the proof of the theorem (modulo Proposition 3.1). $\qquad\square$

The remaining part of this section is devoted to the proof of Proposition 3.1. It suffices to verify the conditions of Proposition 2.1.

Property (a) is clear, since $\|\mathfrak{m}_N(x)\| = \sqrt{N/\ln N}$.

Verifying properties (b) and (c) requires longer computations that are presented in §3.2 and §3.3.

3.2. **Property (b).** Let

$$(3.5) \qquad X_{K,N} = \left\{ x : \text{Card}(n : |n| < N \text{ and } \|\tau_n(x)\| \leq K \ln N) \geq N^{1/5} \right\}.$$

Lemma 3.2. If for each K, $\lim\limits_{N \to \infty} N\mu(X_{K,N}) = 0$, then there are sets $\hat{X}_N$ such that for all $x_N \in \hat{X}_N$ the measures $\mathfrak{m}_N(x_N)$ satisfy property (b) and $\mu(\hat{X}_N) \to 1$

Proof. Given K let $\hat{X}_N = \{ x : f^n x \notin X_{K,N} \text{ for } n < N \}$. By the assumption of the lemma, $\mu(\hat{X}_N) \to 1$. On the other hand, for $x \in \hat{X}_N$,

$$\int \mathfrak{m}_N^{r-1}(x)[B(t, K \ln N)]d\mathfrak{m}_N(x) = \frac{1}{(N \ln N)^{r/2}} \sum_{n=0}^{N-1} \text{Card}^{r-1}(j < N : \|\tau_j - \tau_n\| \leq K \ln N)$$

$$\leq N^{-\frac{r}{2} + 1 + \frac{r-1}{5}} \leq N^{-1/10},$$

where the first inequality holds since $x \in \hat{X}_N$ and the second one holds because $r \geq 3$. Since K is arbitrary, the result follows. $\qquad \square$

Let

$$(3.6) \qquad \ell(x, t, N) = \text{Card}(n \leq N : |\tau_n(x) - t| \leq 1).$$

Lemma 3.3. For each p there is a constant C_p such that for each $t \in \mathbb{R}^d$ and $N \in \mathbb{N}$,

$$(3.7) \qquad \mu\left(\ell^p(\cdot, t, N) \right) \leq C_p \ln^p N.$$

Proof. Since $\ell(x, t, N) = \sum\limits_{n < N} \mathbb{1}_{B(t,1)}(\tau_n(x))$ we have

$$\mu\left(\ell^p(\cdot, t, n) \right) = \sum_{n_1, \ldots n_p} \mu\left(x : |\tau_{n_j}(x) - t| \leq 1 \text{ for } j = 1, \ldots p \right)$$

$$\leq \sum_{n_1, \ldots n_p} K_p \frac{1}{n_1} \prod_{j=2}^{p} \left(\frac{1}{n_j - n_{j-1} + 1} \right),$$

where the last step uses the anticoncentration large deviation bound. $\qquad \square$

Lemma 3.3 and Markov inequality imply that for each ε, t, p we have

$$\mu\left(x : \ell(x, t, N) \geq \frac{N^{1/5}}{(K \ln N)^2} \right) \leq \frac{C_p(K \ln N)^{2p}}{N^{p/5}}$$

for N large enough. It follows that

$$\mu\left(x : \exists t : \|t\| \leq K \ln N \text{ and } \ell(x, t, N) \geq \frac{N^{1/5}}{(K \ln N)^2} \right) \leq \frac{C_p(K \ln N)^{2p+2}}{N^{p/5}}.$$

Taking $p = 6$ we verify the conditions of Lemma 3.2.

3.3. **Property (c).** Theorem 4.7 in [11] implies that

$$(3.8) \qquad \int \tilde{H}(x,y)\tilde{H}(F^k(x,y))\mathrm{d}\mu(x)\mathrm{d}\nu(y) = \frac{\mathfrak{g}(0)}{k}\varsigma_2^2 + o\left(\frac{1}{k}\right),$$

where

$$(3.9) \qquad \varsigma_2^2 = \iiint_{-\infty}^{\infty}\int \tilde{H}(x_1,y)\tilde{H}(x_2,G_t(y))\mathrm{d}\nu(y)\mathrm{d}t\mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2)$$

and $\mathfrak{g}$ is the limit Gaussian density of τ (that is, $\tau_N/\sqrt{N}$ converges as $N \to \infty$ to the normal distribution with density $\mathfrak{g}$). We note that the integral (3.9) converges by the exponential mixing of G and (3.1). Furthermore, $\varsigma_2^2 \geq 0$, which can be seen from the following formula (whose proof is standard and so is omitted)

$$\varsigma_2^2 = \lim_{T \to \infty}\frac{1}{T}\int\left[\int\int_0^T \tilde{H}(x,G_t(y))\mathrm{d}t\mathrm{d}\mu\right]^2 \mathrm{d}\nu.$$

Defining

$$V_N = \frac{1}{N\ln N}\int\left[\sum_{n=0}^{N-1}\tilde{H}(F^n(x,y))\right]^2 \mathrm{d}\nu(y),$$

we compute

$$V_N = \frac{1}{N\ln N}\sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1}\int \tilde{H}(F^{n_1}(x,y))\tilde{H}(F^{n_2}(x,y))\mathrm{d}\nu(y)$$

$$= \frac{1}{N\ln N}\sum_{k=0}^{N-1}(N-k)(1+\mathbb{1}_{k\neq 0})\int \tilde{H}(x,y)\tilde{H}(F^k(x,y))\mathrm{d}\nu(y),$$

whence from (3.8) we obtain

$$(3.10) \quad \Sigma^2 := \lim_{N\to\infty}\mu\left(V_N(x)\right) = 2\mathfrak{g}(0)\iiiint \tilde{H}(x,y)\tilde{H}(\bar{x},G_t y)\mathrm{d}\mu(x)\mathrm{d}\mu(\bar{x})\mathrm{d}\nu(y)\mathrm{d}t.$$

By Chebyshev's inequality, to establish *property (c)*, it suffices to show that

$$(3.11) \qquad \lim_{N\to\infty}\mu(V_N^2) = \Sigma^4.$$

Note that

$$\mu(V_N^2) = \frac{1}{N^2\ln^2 N}\sum_{n_1,n_2,n_3,n_4}\mu(\sigma_{n_1,n_2}\sigma_{n_3,n_4}),$$

where

$$\sigma_{n_1,n_2}(x) = \int H(f^{n_1}x,G_{\tau_{n_1}x}y)H(f^{n_2}x,G_{\tau_{n_2}(x)}y)\mathrm{d}\nu(y).$$

Fix a large L and let $\mathbf{k} = \mathbf{k}(n_1,n_2,n_3,n_4)$ be the number of indices j such that $|n_i-n_j| \geq L$ for all $i \neq j$. The number of terms where $\mathbf{k} \leq 1$ is $O(L^2N^2)$. Using the trivial bound $|\mu(\sigma_{n_1,n_2}\sigma_{n_3,n_4})| \leq \|\tilde{H}\|_\infty^2$, we see that the contribution of terms with $\mathbf{k} \leq 1$ is $O(N^2L^2)$ which is negligible.

Next, we consider the contribution of the terms with $\mathbf{k} = 4$. Without loss of generality, we will assume that

$$(3.12) \qquad n_1 < n_2, \quad n_3 < n_4, \quad n_1 < n_3.$$

We distniguish 3 cases based on the relative position of n_2 with respect to n_3 and n_4.

Case 1. We claim that for each $\varepsilon > 0$ there is L such that if

$$(3.13) \qquad n_1 < n_1 + L \le n_2 < n_2 + L \le n_3 < n_3 + L \le n_4,$$

then

$$(3.14) \qquad \left| \mu\left(\sigma_{n_1,n_2}\sigma_{n_3,n_4}\right) - \frac{\Sigma^4}{4|n_2 - n_1||n_4 - n_3|} \right| \le \frac{\varepsilon}{|n_2 - n_1||n_4 - n_3|}.$$

The proof of (3.14) is based on [11]. First, by [11, Lemma 3.3], we can assume that $\tilde{H}(x, y) = A(x)B(y)$ and without loss of generality we assume $\nu(B) = 0$. Then we have

$$\mu\left(\sigma_{n_1,n_2}\sigma_{n_3,n_4}\right) = \int \left[\prod_{i=1}^4 A(f^{n_i}x) \right] \rho(\tau_{n_2-n_1}(f^{n_1}x))\rho(\tau_{n_3-n_4}(f^{n_4}x))\mathrm{d}\mu(x),$$

where $\rho(t) = \int B(y)B(G_t y)\mathrm{d}\nu(y)$. Note that for $\tilde{H}$ as above (3.11) simplifies to

$$(3.15) \qquad \Sigma^2 = 2\mathfrak{g}(0)\mu(A)^2 \int \rho(t)\mathrm{d}t.$$

The remaining part of the proof of (3.14) closely follows the lines of [11, Theorems 4.6, 4.7] and so we only give a sketch. Decompose $\mathbb{R}^2$ into a countable disjoint family of small squares $\mathcal{C}_i$. Let z_i be the center of square $\mathcal{C}_i$. Then

$$\mu\left(\sigma_{n_1,n_2}\sigma_{n_3,n_4}\right) \approx \sum_{i,j} S_{i,j},$$

where

$$S_{i,j} = \rho(z_i)\rho(z_j) \int \prod_{k=1}^4 A(f^{n_k}x)\mathbb{1}_{\mathcal{C}_i}(\tau_{n_2-n_1}(f^{n_1}x))\mathbb{1}_{\mathcal{C}_j}(\tau_{n_3-n_4}(f^{n_4}x))\mathrm{d}\mu(x).$$

Fixing i, j, letting $L \to \infty$ and using the MMLLT and (3.13), we find

$$S_{i,j} \approx \frac{1}{|n_2 - n_1|}\frac{1}{|n_4 - n_3|} \int_{\mathcal{C}_i} \rho(t)\mathrm{d}t \int_{\mathcal{C}_j} \rho(t)\mathrm{d}t\,\mathfrak{g}(0)^2[\mu(A)]^4.$$

Summing over i, j and using (3.15), we obtain (3.14).

Case 2. We claim that

$$(3.16) \qquad \sum_{(n_1,\ldots,n_4):n_1<n_3<n_4<n_2} \mu\left(\sigma_{n_1,n_2}\sigma_{n_3,n_4}\right) \le CN^2 \ln N.$$

To prove (3.16) take $(n_1, ..., n_4)$ as in (3.16) and write

$$m_0 = 0, \quad m_1 = n_3 - n_1, \quad m_2 = n_4 - n_1, \quad m_3 = n_2 - n_1.$$

Then we have

$$\mu\left(\sigma_{n_1,n_2}\sigma_{n_3,n_4}\right) = \int \left[\prod_{i=0}^{3} A(f^{m_i}(x))\right] \rho(\tau_{m_3}(x))\rho(\tau_{m_2-m_1}(f^{m_1}x))\mathrm{d}\mu(x).$$

Decompose $\mathbb{R}^2$ into unit boxes $\mathcal{C}_i$ with center z_i. Then, we find

$$|\mu\left(\sigma_{n_1,n_2}\sigma_{n_3,n_4}\right)| \leq C \sum_{i_1,i_2,i_3} \tilde{\rho}(z_{i_3})\tilde{\rho}(z_{i_2} - z_{i_1}) \int \prod_{j=1}^{3} \mathbb{1}_{\tau_{m_j} \in \mathcal{C}_{i_j}} \mathrm{d}\mu(x),$$

where $\tilde{\rho}(t) = \sup\limits_{t':\|t'-t\|\leq 1} |\rho(t)|$. Applying (1.3), we find

$$(3.17) \quad |\mu\left(\sigma_{n_1,n_2}\sigma_{n_3,n_4}\right)| \leq C \sum_{i_1,i_2,i_3} \frac{\tilde{\rho}(z_{i_3})\tilde{\rho}(z_{i_2} - z_{i_1})}{m_1(m_2 - m_1)(m_3 - m_2)} \Theta\left(\max_{1\leq j\leq 3} \frac{\|z_{i_j} - z_{i_{j-1}}\|}{\sqrt{m_j - m_{j-1}}}\right)$$

$$=: \sum_{i_1,i_2,i_3} \mathcal{S}_{i_1,i_2,i_3}(m_1, m_2, m_3).$$

Let us consider the special case when $i_2 = i_1$ and $z_{i_3} = 0$ (without loss of generality we say that in this case $i_3 = 0$). Then we have

$$(3.18) \qquad\qquad \sum_{i_1} \mathcal{S}_{i_1,i_1,0}(m_1, m_2, m_3) \leq$$

$$C \sum_{i_1} \frac{1}{m_1} \frac{1}{m_2 - m_1} \frac{1}{m_3 - m_2} \Theta\left(\frac{\|z_{i_1}\|}{\min\{\sqrt{m_1}, \sqrt{m_3 - m_2}\}}\right).$$

Note that the last expression is symmetric in $m_1 = n_3 - n_1$ and $m_3 - m_2 = n_2 - n_4$. Thus without loss of generality we can assume $n_3 - n_1 \leq n_2 - n_4$ (indeed, the multiplier 2 can be incorporated into the constant C). Denoting $a = n_3 - n_1$, $b = n_4 - n_3$, $c = n_2 - n_4$, we obtain

$$\sum_{(n_1,...,n_4):n_1<n_3<n_4<n_2<N} \sum_{i_1} \mathcal{S}_{i_1,i_1,0}(m_1, m_2, m_3) \leq CN \sum_{i} \sum_{a=1}^{N} \sum_{b=1}^{N} \sum_{c=a}^{N} \frac{1}{abc} \Theta\left(\frac{\|z_i\|}{\sqrt{a}}\right).$$

Note that the multiplier N accounts for all choices of n_1. Thus

$$\sum_{(n_1,...,n_4):n_1<n_3<n_4<n_2<N} \sum_{i_1} \mathcal{S}_{i_1,i_1,0}(m_1, m_2, m_3) \leq CN \ln N \sum_{i} \sum_{a=1}^{N} \sum_{c=a}^{N} \frac{1}{ac} \Theta\left(\frac{\|z_i\|}{\sqrt{a}}\right)$$

$$\leq CN \ln N \int_{\mathbb{R}^2} \int_{1}^{N} \int_{a}^{N} \frac{1}{ac} \Theta\left(\frac{\|z\|}{\sqrt{a}}\right) \mathrm{d}c\mathrm{d}a\mathrm{d}z \leq CN \ln N \int_{1}^{N} \int_{a}^{N} \frac{1}{c} \mathrm{d}c\mathrm{d}a \leq CN^2 \ln N$$

where in the third inequality we used that $\int_{1}^{\infty} \Theta(r)r\,dr < \infty$ and the last inequality follows from direct integration. Thus we have verified that the terms corresponding to $i_1 = i_2$ and $i_3 = 0$ are in $O(N^2 \ln N)$.

To complete the proof of (3.16), we need to estimate the contribution of the sum corresponding to all i_2 and i_3. To this end, we will distinguish two cases based on whether the following assumption holds or not:

$$(3.19) \qquad \|z_{i_2} - z_{i_1}\| < \frac{1}{4}\|z_{i_1}\|, \text{ and } \|z_{i_3} - z_{i_2}\| < \frac{1}{4}\|z_{i_1}\|.$$

For any given fixed i_1, let $\widehat{\sum}_{i_2,i_3}$ denote the sum over i_2, i_3 that do not satisfy (3.19) and let $\widetilde{\sum}_{i_2,i_3}$ denote the sum for i_2, i_3 satisfying (3.19).

First, we assume that (3.19) fails. Then, we have

$$\sum_{i_1}\widehat{\sum_{i_2,i_3}}\mathcal{S}_{i_1,i_2,i_3}(m_1,m_2,m_3) \leq C\sum_{i_1}\widehat{\sum_{i_2,i_3}}\tilde{\rho}(z_{i_2}-z_{i_1})\tilde{\rho}(z_{i_3})\frac{1}{m_1}\frac{1}{m_2-m_1}\frac{1}{m_3-m_2}$$

$$\times\left[\Theta\left(\frac{\|z_{i_1}\|}{4\min\{\sqrt{m_1},\sqrt{m_2-m_1}\}}\right)+\Theta\left(\frac{\|z_{i_1}\|}{4\min\{\sqrt{m_1},\sqrt{m_3-m_2}\}}\right)\right].$$

Now we can perform the summation with respect to i_2, i_3 first. Denote $a = m_1$, $b = m_2 - m_1$, $c = m_3 - m_2$. By the exponential decay of $\tilde{\rho}$ we get

$$\sum_{i_1}\widehat{\sum_{i_2,i_3}}\mathcal{S}_{i_1,i_2,i_3}(m_1,m_2,m_3) \leq C\sum_{i_1}\frac{1}{abc}\left[\Theta\left(\frac{\|z_{i_1}\|}{4\min\{\sqrt{a},\sqrt{b}\}}\right)+\Theta\left(\frac{\|z_{i_1}\|}{4\min\{\sqrt{a},\sqrt{c}\}}\right)\right].$$

Summing the last displayed formula for n_1, n_2, n_3, n_4, we obtain a term $O(N^2 \ln N)$ exactly as in the estimate of (3.18).

Now we assume that (3.19) holds. Then observing that $\|z_{i_3}\| \geq \|z_{i_1}\|/2$, and using the exponential decay of $\tilde{\rho}$, we find that

$$\tilde{\rho}(z_{i_2}-z_{i_1})\tilde{\rho}(z_{i_3}) \leq Ce^{-c\|z_{i_1}\|}$$

with a fixed constant c. Thus we conclude

$$\sum_{m_1\leq m_2\leq m_3\leq N}\sum_{i_1}\widetilde{\sum_{i_2,i_3}}\mathcal{S}_{i_1,i_2,i_3}$$

$$\leq C\sum_{m_1\leq m_2\leq m_3\leq N}\frac{1}{m_1}\frac{1}{m_2-m_1}\frac{1}{m_3-m_2}\sum_{i_1}\widetilde{\sum_{i_2,i_3}}e^{-c\|z_{i_1}\|} \leq C\ln^3 N.$$

Summing over all choices of n_1, we obtain a term in $O(N\ln^3 N)$. This completes the proof of (3.16).

Case 3. We claim that

$$(3.20) \qquad \sum_{(n_1,\ldots,n_4):n_1<n_3<n_2<n_4}\mu\left(\sigma_{n_1,n_2}\sigma_{n_3,n_4}\right) \leq CN^2\ln N.$$

The proof of (3.20) is similar to that of (3.16). Indeed, the right hand side of (3.17) is now replaced by

$$C\sum_{i_1,i_2,i_3}\frac{\tilde{\rho}(z_{i_2})\tilde{\rho}(z_{i_3}-z_{i_1})}{m_1(m_2-m_1)(m_3-m_2)}\Theta\left(\max_j\frac{\|z_{i_j}-z_{i_{j-1}}\|}{\sqrt{m_j-m_{j-1}}}\right).$$

We consider the special case $i_2 = 0, i_3 = i_1$ first. Then (3.18) is replaced by

$$(3.21) \quad \sum_{i_1} \mathcal{S}_{i_1,0,i_1}(m_1, m_2, m_3)$$

$$\leq C \sum_{i_1} \frac{1}{m_1} \frac{1}{m_2 - m_1} \frac{1}{m_3 - m_2} \Theta \left(\frac{\|z_{i_1}\|}{\min\{\sqrt{m_1}, \sqrt{m_2 - m_1}, \sqrt{m_3 - m_2}\}} \right).$$

which is bounded by the right hand side of (3.18) (perhaps with a different constant C) and hence is in $O(N^2 \ln N)$. The contribution of general i_2, i_3 can be bounded as before. (3.20) follows.

The upshot of the above three cases is that the leading term only comes from indices $n_1, ..., n_4$ satisfying (3.13). Summing $\mu(\sigma_{n_1,n_2}\sigma_{n_3,n_4})$ for all indices $n_1, ..., n_4$ satisfying (3.13), we obtain $N^2 \ln^2 N[\Sigma^4/8 + o_{L\to\infty}(1)]$ where we have used (3.14) and the fact that $\sum_{n_1 < n_3 < N} \ln(n_3 - n_1) \ln(N - n_3) = \frac{N^2 \ln^2 N}{2}(1 + o_{N\to\infty}(1))$.

Next we claim that

$$(3.22) \quad \sum_{n_1,...,n_4:\mathbf{k}=4} \mu(\sigma_{n_1,n_2}\sigma_{n_3,n_4}) = [\Sigma^2 + o_{L\to\infty}(1)]N^2 \ln^2 N.$$

Indeed similarly to the case of (3.13) considered above we see that the main contribution to our sums comes from the terms where the pairs (n_1, n_2) and (n_3, n_4) are not intertwined. Note that if we choose a random permutation of (n_1, n_2, n_3, n_4) then the probability of non intertwining is $1/3$ (since the second element should come from the same pair as the first one). Therefore there are $24/3=8$ permutations [1] contributing to our sum which explains the additional factor of 8 in (3.22).

To complete the proof of property (c), it remains to verify that the contribution of terms with $\mathbf{k} = 2, 3$ is negligible. First note that $\mathbf{k} = 3$ is impossible by the definition of $\mathbf{k}$. Finally, if $\mathbf{k} = 2$, the we can repeat a simplified version of the proof for $\mathbf{k} = 4$ using the MLLT instead of the MMLLT. For example, let us consider the case $n_1 \leq n_2 \leq n_3 \leq n_4$. Since $\mathbf{k} = 2$, we must have $|n_2 - n_1| \geq L$ or $|n_4 - n_3| \geq L$. Without loss of generality assume $|n_4 - n_3| \geq L$. Then, using the fact that due to the exponential mixing of G, $\sigma_{n_3,n_4} \leq Ce^{-c\ell}$ on the set $\|\tau_{n_4} - \tau_{n_3}\| \in [\ell, \ell+1]$, we obtain from the MLLT that

$$\mu(\sigma_{n_1,n_2}\sigma_{n_3,n_4}) = O\left(\frac{1}{|n_4 - n_3|}\right).$$

Hence the total contribution terms with $\mathbf{k} = 2$ is in $O(LN^2 \ln N)$. This completes the proof of property (c). We have finished the proof of Proposition 3.1 and hence the proof of Theorem 1.6.

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[1]These eight permutations corresponds to all possible choices of signs ($* \in \{\leq, \geq\}$) in the inequalities

$$n_1 * n_2, \quad n_3 * n_4, \quad \min(n_1, n_2) * \min(n_3, n_4)$$

4. Dimension one

4.1. **Reduction to the limit theorem for the variance.** Here we prove Theorem 1.7.

We use the decomposition (3.4). Since f satisfies the CLT for smooth observables, $\frac{1}{\sqrt{N}} \sum_{n=1}^{N} \bar{H}(f^n(x))$ converges weakly, and so this sum is negligible after rescaling by $N^{3/4}$. It remains to study the ergodic sum of $\tilde{H}$.

Following the ideas of the previous section, we let

$$(4.1) \qquad \mathfrak{m}_N(x) = \frac{1}{\sqrt{V_N(x)}} \sum_{n=0}^{N-1} \delta_{\tau_n(x)},$$

$$A_{t,N,x}(y) = \frac{1}{\mathrm{Card}(n \le N : \tau_n(x) = t)} \sum_{n \le N : \tau_n(x) = t} \tilde{H}(f^n(x), y),$$

where

$$V_N(x) = \int S_N^2(x, y) \mathrm{d}\nu(y), \quad S_N(x, y) = \sum_{n=0}^{N-1} \tilde{H} \circ F^n(x, y).$$

Thus in the notation of Proposition 2.1, we have $\mathcal{S}_N = \frac{S_N}{\sqrt{V_N(x)}}$.

In contrast with Theorem 1.6, $V_N(x)$ does not satisfy a weak law of large numbers. Instead we have

Proposition 4.1. There is a constant Λ so that $\frac{V_N}{\Lambda N^{3/2}}$ converges in law as $N \to \infty$ to the random variable $\mathcal{L}^2$ given by (1.5).

Before proving Proposition 4.1, let us use it to derive Theorem 1.7. We start with the following analogue of Proposition 3.1:

Lemma 4.2. Under the assumptions of Theorems 1.7, there are subsets $X_N \subset X$ such that $\lim_{N \to \infty} \mu(X_N) = 1$ and for any sequence $x_N \in X_N$ the measures $\{\mathfrak{m}_N(x_N)\}$ satisfy the conditions of Proposition 2.1 with $\sigma = 1$.

Proof of Lemma 4.2 assuming Proposition 4.1. Clearly, $\|\mathfrak{m}_N\| \to \infty$ for a large set of x's and so (a) holds. Recalling that $\mathcal{S}_N(x, y) = S_N(x, y)/\sqrt{V_N(x)}$, property (c) holds trivially for all $x \in X$. It remains to show property (b).

Note that $\mathcal{L}^2$ is non-negative and non-atomic at zero, i.e. for any $\varepsilon > 0$ there is $\varepsilon' > 0$ so that $\mathbb{P}(\mathcal{L}^2 < \varepsilon') < \varepsilon$ and so by Proposition 4.1,

$$\lim_{N \to \infty} \mu(x : N^{1.49} < V_N(x) < N^{1.51}) = 1.$$

Thus we can assume that x is such that $N^{1.49} < V_N(x) < N^{1.51}$. Next, we define

$$\tilde{\ell}(x, t, N) = \frac{\mathrm{Card}(n \le N : |\tau_n(x) - t| < 1)}{\sqrt{N}}.$$

As in the proof of Lemma 3.3, we find that for every $p \in \mathbb{Z}_+$, every $t \in \mathbb{R}$, and every $N \in \mathbb{N}$ we have $\mu(\tilde{\ell}^p(., t, N)) < C_p$. Then choosing $p = 1000$, the Markov inequality

gives $\mu(x : \tilde{\ell}(x, t, N) > N^{1/100}) < CN^{-10}$. Thus we can assume that for all $t \in \mathbb{Z}$ with $|t| < N^{0.6}$, $\tilde{\ell}(x, t, N) < N^{1/100}$ holds. By the anticoncentration large deviation bounds, we can also assume that $\max_{n \leq N} |\tau_n(x)| \leq N^{0.6}$ since the set of points where this condition fails has negligible measure. In summary, the measure of the set of x's satisfying that

$$N^{1.49} < V_N(x) < N^{1.51} \quad \text{and} \quad \tilde{\ell}(x, t, N) \leq \begin{cases} N^{0.01} & |t| \leq N^{0.6}, \\ 0 & |t| \geq N^{0.6}. \end{cases}$$

tends to 1 as $N \to \infty$. For any such x, we have

$$\int \mathfrak{m}_N^{r-1}(x)(t, K \ln N) d\mathfrak{m}_N(x) = \frac{1}{(V_N(x))^{r/2}} \sum_{n=0}^{N-1} \text{Card}^{r-1}(j < N : |\tau_j - \tau_n| \leq K \ln \|\mathfrak{m}_N\|)$$

$$\leq \frac{1}{(N^{1.49})^{r/2}} \sum_{t \in \mathbb{Z}: |t| \leq N^{0.6}} \ell^r(x, t, N) \leq 2N^{0.51r+0.6-0.745r} = o(1).$$

if $r \geq 3$. Property (b) and the lemma follow. $\qquad\square$

Proof of Theorem 1.7 assuming Proposition 4.1. Recall that $\mathcal{S}_N(x, y) = \dfrac{S_N(x, y)}{\sqrt{V_N(x)}}$. By Lemma 4.2 and Proposition 2.1, there is a sequence of subsets $X_N \subset X$ with $\mu(X_N) \to 1$ such that for every sequence $x_N \in X_N$, the distribution of $\mathcal{S}_N(x, y)$ w.r.t. $\nu(y)$ converges to the standard normal (note that this time properties (a) and (c) are immediate).

In fact, we have the stronger statement that

$$(4.2) \qquad \left(\frac{V_N}{\Lambda N^{3/2}}, \mathcal{S}_N\right) \Rightarrow (\mathcal{L}^2, \mathcal{N}) \quad \text{as} \quad N \to \infty,$$

where $\mathcal{L}$ is given by (1.5), $\mathcal{N}$ is standard normal and $\mathcal{L}^2$ and $\mathcal{N}$ are independent. This follows from Proposition 4.1, Lemma 4.2 and the asymptotic independence of $\mathcal{S}_N$ and V_N which comes from the fact that V_N depends only on x while $\mathcal{S}_N$ is asymptotically independent of x. More precisely, let ϕ and ψ be two continuous compactly supported test functions. Then

$$\lim_{N \to \infty} \zeta\left(\phi\left(\frac{V_N}{\Lambda N^{3/2}}\right)\psi(\mathcal{S}_N)\right) = \lim_{N \to \infty} \zeta\left(\phi\left(\frac{V_N}{\Lambda N^{3/2}}\right)\psi(\mathcal{S}_N)\mathbb{1}_{X_N}\right)$$

$$= \lim_{N \to \infty} \int_{X_N} \left[\phi\left(\frac{V_N}{\Lambda N^{3/2}}\right)\int_Y \psi(\mathcal{S}_N)d\nu\right] d\mu = \mathbb{E}(\psi(\mathcal{N}))\lim_{N \to \infty}\int_{X_N}\phi\left(\frac{V_N}{\Lambda N^{3/2}}\right) d\mu$$

$$= \mathbb{E}(\psi(\mathcal{N}))\lim_{N \to \infty}\int_X \phi\left(\frac{V_N}{\Lambda N^{3/2}}\right) d\mu = \mathbb{E}(\psi(\mathcal{N}))\mathbb{E}(\phi(\mathcal{L}^2)),$$

where the first and the fourth equalities hold since $\mu(X_N) \to 1$, the third inequality holds by Proposition 2.1, and the last equality holds by Proposition 4.1. This proves (4.2).

Denote $\Sigma = \sqrt{\Lambda}$. By (4.2) and the continuous mapping theorem,

$$\frac{S_N}{\Sigma N^{3/4}} = \frac{\sqrt{V_N}}{\sqrt{\Lambda N^{3/2}}} \times \frac{\mathcal{S}_N}{\sqrt{V_N}} \Rightarrow \mathcal{L}\mathcal{N}$$

completing the proof of the Theorem 1.7. □

4.2. **Proof of Proposition 4.1.** We are going to use the following well known fact (see e.g. [2] Chapter 1.7, Problem 4). If $\mathcal{X}_n$ is a sequence of random variables so that for every $k \in \mathbb{Z}_+$

$$(4.3) \qquad J_k = \lim_{n \to \infty} \mathbb{E}(\mathcal{X}_n^k)$$

exists and

$$(4.4) \qquad \limsup_{k \to \infty} \left(\frac{J_k}{k!} \right)^{1/k} < \infty,$$

then $\mathcal{X}_n$ converges weakly to a random variable $\mathcal{X}$. Furthermore, $\mathbb{E}(\mathcal{X}^k) = J_k$ for every $k \in \mathbb{Z}_+$ and $\mathcal{X}$ is uniquely characterized by its moments.

Now we explain our strategy of proof of Proposition 4.1 (a similar strategy was used in [18]). We prove that there is a constant Λ depending on the (T, T^{-1}) transformation F and the observable H so that $\mathcal{X}_N = V_N/(\Lambda N^{3/2})$ satisfies (4.3) (with $\mathbb{E}$ meaning integral w.r.t. μ) and (4.4) with constants J_k that do not depend on F and H. Consequently, there is a random variable $\mathcal{X}$ so that $\mathcal{X}_N = V_N/(\Lambda N^{3/2})$ converges weakly to $\mathcal{X}$ for any (T, T^{-1}) transformation satisfying the assumptions of Theorem 1.7. Recall from §1.2 that one such example is the one dimensional random walk in random scenery, in which case by the result of [15], $\mathcal{X}_N = V_N/(\Lambda N^{3/2})$ converges weakly to $\mathcal{L}^2$. Thus $\mathcal{X} = \mathcal{L}^2$ and it has to be the limit for all (T, T^{-1}) transformation satisfying the assumptions of Theorem 1.7. It remains to check (4.3) and (4.4).

Recall that τ satisfies the MMLLT with a Gaussian density $\mathfrak{g}$. Let ς_1 be the standard deviation of this Gaussian random variable, that is $\mathfrak{g}(z) = \varphi(z/\varsigma_1)/\varsigma_1$, where φ is the standard Gaussian density.

Define

$$(4.5) \qquad \Lambda = \frac{\sqrt{\pi} \varsigma_2^2}{\varsigma_1} = \sqrt{2} \pi \varsigma_2^2 \mathfrak{g}(0)$$

where ς_2^2 is given by (3.9).

Lemma 4.3. For $k = 1$, (4.3) holds with

$$(4.6) \qquad J_1 = \frac{2}{\sqrt{\pi}} \int_{w \in \mathbb{R}} \int_{0 < t_1 < t_2 < 1} \frac{1}{\sqrt{t_1}} \frac{1}{\sqrt{t_2 - t_1}} \varphi \left(\frac{w}{\sqrt{t_1}} \right) \varphi(0) \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}w.$$

We note that the above integral can be performed explicitly. Namely $J_1 = \frac{4\sqrt{2}}{3\sqrt{\pi}}$. This shows that Λ was chosen correctly. Indeed, in case of the simple random walk in random scenery, formula (1.2) in [15] shows that $J_1 = \frac{4\sqrt{2}}{3\sqrt{\pi}}$ while by the main result of [15], the conclusion of Proposition 4.1 holds for simple random walk in random scenery.

Proof of Lemma 4.3. Fix some $\varepsilon > 0$. We need to show that for N large enough,

$$\left| N^{-3/2} \Lambda^{-1} \iint \left[\sum_{n=1}^{N} \tilde{H} \circ F^n(x, y) \right]^2 \mathrm{d}\mu \mathrm{d}\nu - J_1 \right| < \varepsilon.$$

In the following proof, we will choose a small $\eta = \eta(\varepsilon)$, a large $L = L(\varepsilon)$ a small $\eta'=\eta'(\varepsilon, \eta, L)<\eta$ and finally $N = N(\varepsilon, \eta, L, \eta')$. There will be finitely many restrictions on these parameters, we can choose the most strict one.

Choose a partition of X into sets $X_i, i \in I$ of diameter $< \eta$ and fix some elements $x_i \in X_i$. We have

$$\iint \left[\sum_{n=1}^{N} \tilde{H} \circ F^n(x, y)\right]^2 \mathrm{d}\mu \mathrm{d}\nu =$$

$$(4.7) \quad \sum_{n_1, n_2=1}^{N} \sum_{i_1, i_2 \in I} \sum_{z_1, z_2 \in \mathbb{Z}} \int p(x) \int \tilde{H}(f^{n_1}(x), G_{\tau_{n_1}(x)}(y)) \tilde{H}(f^{n_2}(x), G_{\tau_{n_2}(x)}(y)) \mathrm{d}\nu \mathrm{d}\mu,$$

where

$$p(x) = p_{\eta, n_1, n_2, i_1, i_2, z_1, z_2}(x) = \mathbb{1}_{f^{n_1}x \in X_{i_1}} \mathbb{1}_{f^{n_2}x \in X_{i_2}} \mathbb{1}_{\tau_{n_1}(x) \in [z_1, z_1+1]\eta} \mathbb{1}_{\tau_{n_2}(x) \in [z_2, z_2+1]\eta}.$$

Split the sum in (4.7) as $T_1 + T_2$, where T_1 corresponds to the terms satisfying

(A1) $\eta'N < \min\{n_1, n_2\} < \min\{n_1, n_2\} + \eta'N < \max\{n_1, n_2\}$
(A2) $|z_1| < L\sqrt{N}/\eta$
(A3) $|z_1 - z_2| < L/\eta$.

and T_2 stands for the terms where at least one of the conditions (A1)–(A3) is violated. Write $T_1 = \widetilde{\sum}(...)$.

We start by estimating T_1. Let us write $a_N \approx b_N$ if there are constants η, L and η' so that for N large enough, $|a_N - b_N| < \varepsilon \Lambda N^{3/2}/10$. We claim that

$$(4.8) \qquad T_1 \approx \widetilde{\sum} \int p(x) \mathrm{d}\mu \int \tilde{H}(x_{i_1}, G_{z_1\eta}(y)) \tilde{H}(x_{i_2}, G_{z_2\eta}(y)) \mathrm{d}\nu.$$

Indeed, by the continuity of H we can choose η so small that the difference between the LHS and the RHS of (4.8) does not exceed

$$\frac{\varepsilon \Lambda}{1000} \widetilde{\sum} \int p_{\eta, n_1, n_2, i_1, i_2, z_1, z_2}(x) \mathrm{d}\mu(x),$$

so (4.8) follows from the anticoncentration large deviation bound.

Next, writing $m_1 = \min\{n_1, n_2\}$, $m_2 = \max\{n_1, n_2\}$ and using the MMLLT, we find

$$(4.9) \quad T_1 \approx \frac{1}{\varsigma_1^2} \widetilde{\sum} \eta^2 \frac{1}{\sqrt{m_1}} \varphi\left(\frac{z_1\eta}{\varsigma_1\sqrt{m_1}}\right) \frac{1}{\sqrt{m_2 - m_1}} \varphi\left(\frac{(z_2 - z_1)\eta}{\varsigma_1\sqrt{m_2 - m_1}}\right) \mu(X_{i_1}) \mu(X_{i_2}) \times$$

$$\int \tilde{H}(x_{i_1}, G_{z_1\eta}(y)) \tilde{H}(x_{i_2}, G_{z_2\eta}(y)) \mathrm{d}\nu.$$

Noting that by (A1) and (A3), $(z_2 - z_1)\eta/[\varsigma_1\sqrt{m_2 - m_1}] = O(N^{-1/2})$, we see that $\varphi\left(\frac{(z_2 - z_1)\eta}{\varsigma_1\sqrt{m_2 - m_1}}\right)$ can be replaced by $\varphi(0)$ in (4.9) without invalidating $\approx$. Then the

only term depending on z_2 is under the integral with respect to ν. We can approximate the sum over z_2 by a Riemann integral:

$$(4.10) \quad \eta \sum_{z_2=z_1-L/\eta}^{z_1+L/\eta} \int \tilde{H}(x_{i_1}, G_{z_1\eta}(y))\tilde{H}(x_{i_2}, G_{z_2\eta}(y))\mathrm{d}\nu = \mathcal{I}_{i_1,i_2} + o_{\eta\to0,L\to\infty,N\to\infty}(1)$$

where

$$\mathcal{I}_{i_1,i_2} = \int_{\mathbb{R}} \int \tilde{H}(x_{i_1}, y)\tilde{H}(x_{i_2}, G_t(y))\mathrm{d}\nu\mathrm{d}t.$$

Combining (4.9) and (4.10), we conclude

$$T_1 \approx \frac{2}{\varsigma_1^2}\left[\sum_{i_1,i_2}\mu(X_{i_1})\mu(X_{i_2})\mathcal{I}_{i_1,i_2}\right]\times$$

$$\times\left[\sum_{\eta'N<m_1<m_1+\eta'N<m_2<N}\sum_{|z_1|<L\sqrt{N}/\eta}\eta\frac{1}{\sqrt{m_1}}\varphi\left(\frac{z_1\eta}{\varsigma_1\sqrt{m_1}}\right)\frac{1}{\sqrt{m_2-m_1}}\varphi(0)\right],$$

where the factor 2 comes from taking into account both $n_1 < n_2$ and $n_2 < n_1$. Further decreasing η and increasing L if necessary and recalling (3.9), we can replace the first sum by ς_2^2.

Now replacing two Riemann sums with the corresponding Riemann integrals, we obtain

$$T_1 \approx \frac{2\varsigma_2^2}{\varsigma_1^2}N^{3/2}\int_{|w|<L}\int_{\eta'<t_1<t_1+\eta'<t_2<1}\frac{1}{\sqrt{t_1}}\frac{1}{\sqrt{t_2-t_1}}\varphi\left(\frac{w}{\varsigma_1\sqrt{t_1}}\right)\varphi(0)\mathrm{d}t\mathrm{d}w \approx \frac{\varsigma_2^2}{\varsigma_1}\sqrt{\pi}N^{3/2}J_1.$$

In order to complete the proof of (4.6), it suffices to show that for L large and for η, η' small

$$(4.11) \qquad\qquad |T_2| < \varepsilon N^{3/2}/10.$$

We will estimate the contribution of the terms where exactly one of the assumptions (A1)–(A3) fail (the cases when more than one fail are similar and easier).

Suppose that (A1) fails and, moreover, $m_1 < \eta'N$ (the other cases are similar). Using the anticoncentration large deviation bounds instead of the MMLLT, we obtain that the corresponding sum is bounded by

$$\sum_{m_1=1}^{\eta'N}\sum_{m_2=0}^{N}\frac{1}{\sqrt{m_1}}\frac{1}{\sqrt{m_2+1}}\sum_{|z_1|<L\sqrt{N}/\eta}C(\eta,L) \leq \sqrt{\eta'N}C'(\eta,L)\sqrt{N} < \varepsilon N^{3/2}/10$$

if η' is small enough.

Let us assume that (A2) fails. Then again using the anticoncentration large deviation bounds instead of the MMLLT, we obtain that the corresponding sum is bounded by

$$\sum_{m_1=\eta'N}^{N}\sum_{m_2=m_1+\eta'N}^{N}\frac{1}{\sqrt{m_1}}\frac{1}{\sqrt{m_2-m_1}}\sum_{|z_1|\geq L\sqrt{N}/\eta}C\eta\Theta(|z_1|\eta/\sqrt{m_1})$$

$$\le CN \sum_{|z_1| \ge L\sqrt{N}/\eta} \eta \Theta(|z_1|\eta/\sqrt{N}) \le CN^{3/2} \int_{L-1}^{\infty} \Theta(r) \mathrm{d}r < \varepsilon N^{3/2}/10$$

for L large.

Finally assume that (A3) fails. Then we use the exponential mixing of G to derive

$$\eta \sum_{z_2:|z_2-z_1|>L/\eta} \int \tilde{H}(x_{i_1}, G_{z_1\eta}(y)) \tilde{H}(x_{i_2}, G_{z_2\eta}(y)) \mathrm{d}\nu$$

$$\le 2 \int_L^{\infty} \int \tilde{H}(x_{i_1}, y) \tilde{H}(x_{i_2}, G_t(y)) \mathrm{d}\nu \mathrm{d}t \le Ce^{-cL}.$$

Proceeding as in the case of T_1 we obtain that the sum corresponding to indices when (A3) is violated is bounded by $CN^{3/2}e^{-cL} < \varepsilon N^{3/2}/10$. This completes the proof of the lemma. $\qquad\square$

Lemma 4.4. For any $k \ge 2$, (4.3) holds with
(4.12)
$$J_k = \left[\frac{2}{\sqrt{\pi}}\right]^k \int_{w_1,\dots,w_k \in \mathbb{R}} \int_{0<t_1<\dots<t_{2k}<1} \prod_{j=1}^{2k} \frac{1}{\sqrt{t_j - t_{j-1}}} \sum_{v \in V} \prod_{j=1}^{2k} \varphi\left(\frac{w_{v(j)} - w_{v(j-1)}}{\sqrt{t_j - t_{j-1}}}\right) \mathrm{d}t \mathrm{d}w,$$

where V is the set of all two-to-one mappings from $\{1, 2, \dots, 2k\}$ to $\{1, 2, \dots, k\}$ (that is, $|v^{-1}(l)| = 2$ for all $l = 1, \dots, k$).

Proof of Lemma 4.4. We follow the strategy of the proof of Lemma 4.3.

Fix some $\varepsilon > 0$. We need to show that for N large enough,

$$\left| N^{-3k/2} \Lambda^{-k} \int \left(\int \left[\sum_{n=1}^N \tilde{H} \circ F^n(x, y) \right]^2 \mathrm{d}\nu \right)^k \mathrm{d}\mu - J_k \right| < \varepsilon.$$

Now we have

$$\int \left(\int \left[\sum_{n=1}^N \tilde{H} \circ F^n(x, y) \right]^2 \mathrm{d}\nu \right)^k \mathrm{d}\mu$$

$$(4.13) \quad = \sum_{n_1, n_2, \dots, n_{2k}=1}^N \sum_{i_1, i_2, \dots, i_{2k} \in I} \sum_{z_1, z_2, \dots, z_{2k} \in \mathbb{Z}}$$

$$\int p(x) \left[\prod_{j=1}^k \int \tilde{H}(f^{n_{2j-1}}(x), G_{\tau_{n_{2j-1}}(x)}(y)) \tilde{H}(f^{n_{2j}}(x), G_{\tau_{n_{2j}}(x)}(y)) \mathrm{d}\nu \right] \mathrm{d}\mu$$

where

$$p(x) = p_{\eta, \underline{n}, \underline{i}, \underline{z}}(x) = \left[\prod_{j=1}^{2k} \mathbb{1}_{f^{n_j} x \in X_{i_j}} \right] \left[\prod_{j=1}^{2k} \mathbb{1}_{\tau_{n_j}(x) \in [z_j, z_j+1]\eta} \right].$$

Let us order the numbers $n_1, \dots, n_{2k}$ as $m_1 \le m_2 \le \dots \le m_{2k}$ and denote $m_0 = 0$. As before, we write the sum in (4.13) as $T_1 + T_2$, where T_1 corresponds to the terms satisfying

(A1') for every $j = 1, ..., 2k$, $m_j - m_{j-1} > \eta' N$

(A2') $|z_{2j-1}| < L\sqrt{N}/\eta$ for all $j = 1, ..., k$

(A3') $|z_{2j-1} - z_{2j}| < L/\eta$ for all $j = 1, ..., k$

and write $T_1 = \widetilde{\sum}(...)$. As in (4.8),

$$(4.14) \qquad T_1 \approx \widetilde{\sum} \int p(x)\mathrm{d}\mu \prod_{j=1}^{k} \int \tilde{H}(x_{i_{2j-1}}, G_{z_{2j-1}\eta}(y))\tilde{H}(x_{i_{2j}}, G_{z_{2j}\eta}(y))\mathrm{d}\nu,$$

where $\approx$ now means that the difference between the LHS and the RHS an be made smaller than $\varepsilon\Lambda^k N^{3k/2}/10$ provided that η and η' are small enough and L and N are large enough. To compute $\int p(x)d\mu$, we use the MMLLT for times $m_1 < m_2 < ... < m_{2k}$. Note however that the range of summation for different z_j's are different and so it is important to keep track of the index i so that $n_i = m_j$. To this end, define to permutation ρ (uniquely defined by the tuple $(n_1, ..., n_{2k})$ if (A1') holds) so that $m_j = n_{\rho(j)}$. Writing $\rho(0) = 0$ and $z_0 = 0$ and using the MMLLT we obtain

$$(4.15) \quad T_1 \;\approx\; \frac{1}{\varsigma_1^{2k}}\widetilde{\sum}\eta^{2k}\left[\prod_{j=1}^{2k}\frac{1}{\sqrt{m_j - m_{j-1}}}\varphi\left(\frac{(z_{\rho(j)} - z_{\rho(j-1)})\eta}{\varsigma_1\sqrt{m_j - m_{j-1}}}\right)\right] \times$$

$$(4.16) \qquad \left[\prod_{j=1}^{2k}\mu(X_{i_j})\right]\left[\prod_{j=1}^{k}\int \tilde{H}(x_{i_{2j-1}}, G_{z_{2j-1}}(y))\tilde{H}(x_{i_{2j}}, G_{z_{2j}}(y))\mathrm{d}\nu\right].$$

By (A1') and (A3'), $\rho(l)$ can be replaced by $2\lceil\rho(l)/2\rceil - 1$ (the biggest odd integer not bigger than $\rho(j)$) for $l = j-1, j$ in the subscripts of z in (4.15). Consequently, z_{2j} only appears in (4.16). As before, we compute

$$\sum_{i_1,...,i_{2k}}\left[\prod_{j=1}^{2k}\mu(X_{i_j})\right]\eta^k \sum_{\substack{z_2,z_4,...,z_{2k}: \\ |z_{2j}-z_{2j-1}|<L/\eta}}\left[\prod_{j=1}^{k}\int \tilde{H}(x_{i_{2j-1}}, G_{z_{2j-1}}(y))\tilde{H}(x_{i_{2j}}, G_{z_{2j}}(y))\mathrm{d}\nu\right]$$

$$= \varsigma_2^{2k} + o(1).$$

Thus we arrive at

$$T_1 \approx \frac{\varsigma_2^{2k}}{\varsigma_1^{2k}}\widehat{\sum}\eta^k\left[\prod_{j=1}^{2k}\frac{1}{\sqrt{m_j - m_{j-1}}}\varphi\left(\frac{(z_{2\lceil\rho(j)/2\rceil-1} - z_{2\lceil\rho(j-1)/2\rceil-1})\eta}{\varsigma_1\sqrt{m_j - m_{j-1}}}\right)\right],$$

where $\widehat{\Sigma}$ refers to the sum for $n_1, ..., n_{2k}, z_1, z_3, ..., z_{2k-1}$, satisfying (A1'), (A3'). Next, observe that $u : \{1, ..., 2k\} \to \{1, 3..., 2k-1\}$ defined by $u(j) = 2\lceil\rho(j)/2\rceil - 1$ is a two-to-one mapping. Let U be the set of all such mappings. The summand in the last displayed formula only depends on $(n_1, ..., n_{2k})$ through $(m_1, ..., m_{2k})$ and u. Furthermore, for any given $(m_1, ..., m_{2k})$ and u, there are exactly 2^k corresponding tuples $(n_1, ..., n_{2k})$. Thus

$$T_1 \approx 2^k\frac{\varsigma_2^{2k}}{\varsigma_1^{2k}}\sum_{u\in U}\sum_{\substack{m_1,...m_{2k} \\ \text{satisfying (A1')}}}\sum_{\substack{z_1,..,z_{2k-1} \\ \text{satisfying (A3')}}}\eta^k\left[\prod_{j=1}^{2k}\frac{1}{\sqrt{m_j - m_{j-1}}}\varphi\left(\frac{(z_{u(j)} - z_{u(j-1)})\eta}{\varsigma_1\sqrt{m_j - m_{j-1}}}\right)\right].$$

Now we are ready to replace the last two sums (Riemann sums) by the corresponding Riemann integral with $t_j \sim m_j/N$ and $w_l \sim z_{2l-1}\eta/\sqrt{N}$. To simplify the notation a little, we denote $v(j) = (u(j)+1)/2$, thus v is a 2-to-1 mapping from $\{1,2,...,2k\}$ to $\{1,2,...,k\}$ (the set of all such mapping is denoted by V). We obtain

$$T_1 \approx 2^k \frac{\varsigma_2^{2k}}{\varsigma_1^{2k}} N^{3k/2} \sum_{v \in V}$$

$$\int_{|w_1|,...,|w_k|<L} \int_{\substack{0=t_0<t_1<...<t_{2k}<1: \\ t_j-t_{j-1}<\eta'}} \prod_{j=1}^{2k} \frac{1}{\sqrt{t_j - t_{j-1}}} \prod_{j=1}^{2k} \varphi\left(\frac{w_{v(j)} - w_{v(j-1)}}{\varsigma_1 \sqrt{t_j - t_{j-1}}}\right) \mathrm{d}t\mathrm{d}w.$$

Substituting w to w/ς_1 in the above integrals, we obtain

$$T_1 \approx \frac{\varsigma_2^{2k}}{\varsigma_1^k} \pi^{k/2} N^{3k/2} J_k.$$

As in Lemma 4.3, T_2 is negligible. This completes the proof of Lemma 4.4. □

To finish the proof of Proposition 4.1, it remains to verify (4.4). To this end, first note that since φ decays quickly, there is a constant M so that for every $v \in V$ and every $t_1, ..., t_{2k} \in (0,1]$,

$$\int_{\mathbb{R}^k} \prod_{j=1}^{2k} \varphi\left(\frac{w_{v(j)} - w_{v(j-1)}}{\varsigma_1 \sqrt{t_j - t_{j-1}}}\right) \mathrm{d}w < M^k.$$

Noting that $|V| = (2k)!/2^k$, we have

$$|J_k| \leq C^k (2k)! \int_{0<t_1<...<t_{2k}<1} \prod_{j=1}^{2k} \frac{1}{\sqrt{t_j - t_{j-1}}} \mathrm{d}t.$$

The above integral can be computed explicitly and is equal to $\dfrac{\Gamma(1/2)^{2k}}{\Gamma(k+1)} = \dfrac{\pi^k}{k!}$. Noting that $(2k)!/(k!) < C^k k!$, we find that $|J_k| \leq C^k k!$ whence (4.4) follows. This completes the proof of Proposition 4.1.

5. Polynomially mixing flows in dimension one.

In this section, we extend the result of Theorem 1.7 to some flows with polynomial mixing rates. We use the setting of [10]. Recall that a flow G_t is called *partially hyperbolic* if there is a G_t invariant splitting $TY = E_u \oplus E_{cs}$ and positive constants $C_1, C_2, \lambda_1, \lambda_2$ such that for each $t \geq 0$

(i) $\|dG_{-t}|E_u\| \leq C_1 e^{-\lambda_1 t}$;

(ii) For any unit vectors $v_u \in E_u$, $v_{cs} \in E_{cs}$ we have $\|dG_t|v_{cs}\| \leq C_2 e^{-\lambda_2 t}\|dG_t v_u\|$.

For partially hyperbolic flows the leaves of E_u are tangent to the leaves of an absolutely continuous foliation W^u.

Fix constants $R, \bar{v}, \bar{C}_1, \alpha_1$. Let $\mathfrak{A}(R, \bar{v}, \bar{C}_1, \alpha_1)$ denote the collection of sets D which belong to one leaf of W^u and satisfy

$$\mathrm{diam}(D) \leq R, \quad \mathrm{mes}(D) \geq \bar{v}, \quad \mathrm{mes}(\partial_\varepsilon D) \leq \bar{C}_1 \varepsilon^{\alpha_1}$$

for all $\varepsilon > 0$, where $\partial_\varepsilon D$ is the ε neighborhood of the boundary of D. We say that the sets from $\mathfrak{A}(R, \bar{v}, \bar{C}_1, \alpha_1)$ have *bounded geometry*.

Fix $\bar{C}_2, \alpha_2 > 0$. Let $\mathfrak{M}(R, \bar{v}, \bar{C}_1, \alpha_1, \bar{C}_2, \alpha_2)$ denote the set of linear functionals of the form

$$\mathbb{E}_{\ell_{D,\rho}}(A) = \int_D A(y)\rho(y)\mathrm{d}y,$$

where $D \in \mathfrak{A}(R, \bar{v}, \bar{C}_1, \alpha_1)$, ρ is a probability density on D, $\ln \rho \in C^{\alpha_2}(D)$, $\| \ln \rho \|_{C^r} \leq \bar{C}_2$. We shall often use the existence of *almost Markov decomposition* established in [10, Section 2]: if R, $\bar{C}_1$, and $\bar{C}_2$ are large enough and α_1, α_2 and $\bar{v}$ are small enough, then given $\ell \in \mathfrak{M}(R, \bar{v}, \bar{C}_1, \alpha_1, \bar{C}_2, \alpha_2)$ and $t > 0$ we can decompose

$$(5.1) \qquad \mathbb{E}_\ell(A \circ G_t) = \sum_s c_s \mathbb{E}_{\ell_s}(A) + O\left(\|A\|_{C^0}\theta^t\right)$$

with $\ell_s \in \mathfrak{M}(R, \bar{v}, \bar{C}_1, \alpha_1, \bar{C}_2, \alpha_2)$ and $\sum_s c_s = 1 + O(\theta^t)$ for some $\theta < 1$.

We say that a measure is *u-Gibbs* if its conditional measures on unstable leaves have smooth densities. The existence of almost Markov decomposition implies that u-Gibbs measures belong to the convex hull of $\mathfrak{M}(R, \bar{v}, \bar{C}_1, \alpha_1, \bar{C}_2, \alpha_2)$ for appropriate choice of $R, \bar{v}, \bar{C}_1, \alpha_1, \bar{C}_2, \alpha_2$ (see [5, §11.2]).

From now on we shall fix constants involved in the definition of $\mathfrak{M}(R, \bar{v}, \bar{C}_1, \alpha_1, \bar{C}_2, \alpha_2)$ which satisfies (5.1) and whose convex hull contains u-Gibbs measures. For the sake of simplicity, we will write $\mathfrak{M}$ instead of $\mathfrak{M}(R, \bar{v}, \bar{C}_1, \alpha_1, \bar{C}_2, \alpha_2)$.

We say that the *unstable leaves become equidistributed at rate $a(t)$* if there is $r > 0$ such that for all $\ell \in \mathfrak{M}$ we have

$$(5.2) \qquad |\mathbb{E}_\ell(A \circ G_t) - \nu(A)| \leq a(t)\|A\|_{C^r}$$

where ν is the reference invariant measure for G_t. We note that if (5.2) holds with $a(t) \to 0$, then ν belongs to the convex hull of $\mathfrak{M}$, whence it is a u-Gibbs measure (in fact, it is the unique SRB measure for G_t, see [10, Corollary 2]).

Theorem 5.1. Theorem 1.7 remains valid if the assumption that G enjoys exponential mixing of all orders is replaced by the assumption that G is partially hyperbolic and unstable leaves become equidistributed with rate $Kt^{-\gamma}$ for some $\gamma > 1$ and $K > 0$.

Corollary 5.2. Suppose that τ satisfies the assumption of Theorem 1.7, G_t is a topologically transitive Anosov flow whose stable and unstable foliations are not jointly integrable, and ν is the SRB measure. Then (1.4) holds for sufficiently smooth functions.

Indeed, according to [9, Theorem 3], for Anosov flows, unstable leaves are equidistributed on C^r at rate $Kt^{-\gamma(r)}$ where $\gamma(r) \to \infty$ as $r \to \infty$.

The proof of Theorem 5.1 requires a modification of Proposition 2.1. We shall use the same notation as in that proposition.

Proposition 5.3. Suppose that G_t is partially hyperbolic with unstable leaves equidistributed at rate $Kt^{-\gamma}$ for some $\gamma > 1$. Suppose that for some $0 < \delta < 1/20$ such that

$$(5.3) \qquad\qquad \gamma - 1 > \frac{16\delta}{1 - 20\delta}$$

and $\kappa > 0$ we have

(a) $\mathfrak{m}_N(t \in \mathbb{R} : |t| \leq N^{(1/2)+\delta}) \leq N^{1/4+\delta}$ and $\mathfrak{m}_N(t : |t| > N^{(1/2)+\delta}) = 0$;
(b) for every $t \in \mathbb{R}$, $\mathfrak{m}_N([t, t+1]) \leq N^{\delta-(1/4)}$;
(c) $\displaystyle\iint_{|t_1-t_2|>\sqrt{\ln N}} \frac{d\mathfrak{m}_N(t_1)d\mathfrak{m}_N(t_2)}{|t_1 - t_2|^\gamma} \leq \ln^{-\kappa} N$;
(d) $\displaystyle\lim_{N\to\infty} \nu(\mathcal{S}_N^2) = 1$.
Then $\mathcal{S}_N \Rightarrow \mathcal{N}(0,1)$ as $N \to \infty$.

Lemma 5.4. Under the assumptions of Theorems 5.1, there are subsets $X_N \subset X$ such that $\displaystyle\lim_{N\to\infty} \mu(X_N) = 1$ and for any sequence $x_N \in X_N$ the measures $\{\mathfrak{m}_N(x_N)\}$, defined by (4.1), satisfy the conditions of Proposition 5.3.

Proof. The fact that conditions (a) and (b) hold for arbitrary $\delta > 0$ are verified as before. Condition (d) is immediate from the definition of $\mathfrak{m}_N$. In order to verify condition (c) we note that by Proposition 4.1, for any $\kappa > 0$, $\mu(x : V_N(x) < N^{3/2} \ln^{-\kappa} N) \to 0$ as $N \to \infty$. Therefore it is enough to check that if δ and κ are sufficiently small, then

$$\mu(x : \mathcal{R}_N(x) < N^{3/2} \ln^{-2\kappa} N)$$

is close to 1, where

$$\mathcal{R}_N(x) := \sum_{n_1,n_2\in\mathbb{Z}:|n_1-n_2|>\sqrt{\ln N}} \frac{\ell(x, n_1, N)\ell(x, n_2, N)}{|n_1 - n_2|^\gamma}$$

and $\ell(x, t, N)$ is the local time defined by (3.6). Note that

$$\mu\left(\ell(n_1, N)\ell(n_2, N)\right) = \sum_{1\leq j_1,j_2\leq N} \mu(x : |\tau_{j_1}(x) - n_1| \leq 1, |\tau_{j_2}(x) - n_2| \leq 1).$$

Therefore, using the anticoncentration large deviation bound, we conclude that

$$\mu(\mathcal{R}_N) \leq C \sum_{1\leq j_1\leq j_2\leq N} \sum_{|n_1-n_2|\geq\sqrt{\ln N}} \frac{1}{\sqrt{j_1}\sqrt{j_2-j_1+1}}\Theta\left(\frac{n_1}{\sqrt{j_1}}\right)\frac{1}{|n_2-n_1|^\gamma}$$

$$\leq C\frac{\sqrt{N}}{\ln^{(\gamma-1)/2} N}\sum_{j_1}\sum_{n_1}\frac{1}{\sqrt{j_1}}\Theta\left(\frac{n_1}{\sqrt{j_1}}\right) \leq C\frac{\sqrt{N}}{\ln^{(\gamma-1)/2} N}\sum_{j_1}\int_0^\infty \Theta(r)\,dr \leq C\frac{N^{3/2}}{\ln^{(\gamma-1)/2} N}.$$

Now the result follows by the Markov inequality provided that $\gamma - 1 > 4\kappa$. $\qquad\square$

Thus in order to prove Theorem 5.1 it suffices to establish Proposition 5.3.

Proof of Proposition 5.3. We divide the interval $[-N^{1/2+\delta}, N^{1/2+\delta}]$ into big blocks of size $N^{\beta_1} > 0$ separated by small blocks of size N^{β_2} where the parameters $\beta_2 < \beta_1$ will be chosen later. Let J_j denote the j-th big block and L be the union of the small blocks.

Let $\mathcal{S}'_N = \int_{t \in L} A_t(G_t y) d\mathfrak{m}_N(y)$. Note that $\nu(\mathcal{S}'_N) = 0$. We claim that $\nu((\mathcal{S}'_N)^2) \to 0$ as $N \to \infty$ provided that

(5.4)
$$\beta_1 > \beta_2 + 3\delta.$$

Indeed

(5.5)
$$\nu\left((\mathcal{S}'_N)^2\right) \leq \iint_{L \times L} \nu(A_{t_1}(A_{t_2} \circ G_{t_2 - t_1})) d\mathfrak{m}_N(t_1) d\mathfrak{m}_N(t_2).$$

According to [10, Theorem 2], $\nu(A_{t_1}(A_{t_2} \circ G_{t_2 - t_1})) = O\left(|t_2 - t_1|^{-\gamma}\right)$. Therefore property (b) shows that the integral of the RHS of (5.5) with respect to t_2 is $O\left(N^{-(1/4)+\delta}\right)$. To integrate with respect to t_1 we divide L into unit intervals. Noting that the mass of each interval is $O\left(N^{\delta-(1/4)}\right)$ and the number of intervals is $O\left(N^{(1/2)+\delta-\beta_1+\beta_2}\right)$, we conclude that $\nu\left((\mathcal{S}'_N)^2\right) = O\left(N^{\beta_2+3\delta-\beta_1}\right)$ which tends to zero under (5.4).

Thus the main contribution to the variance comes from the big blocks. Let

$$\tau = N^{1/2+\delta}, \quad T_j = T_j(y) = \int_{J_j} A_t(G_{\tau+t} y) d\mathfrak{m}_N(t), \quad S_{N,j} = \sum_{k=1}^{j} T_k.$$

Fix any $\xi \in \mathbb{R}$. We will show that there is a sequence $\varepsilon_N \to 0$, depending only $|\xi|$ so that for each $\ell \in \mathfrak{M}$ we have

$$|\mathbb{E}_\ell\left(e^{i\xi S_{N,\bar{N}}}\right) - e^{-\xi^2/2}| \leq \varepsilon_N$$

where $\bar{N}$ is the number of big blocks. Let $\Phi_j(\xi) = \mathbb{E}_\ell\left(e^{i\xi S_{N,j}}\right)$ with $\Phi_0(\xi) = 1$.

Lemma 5.5. If

(5.6)
$$\beta_1 < 1/4 - 2\delta,$$

then

$$\Phi_j(\xi) = \Phi_{j-1}(\xi) \left[1 - \frac{\xi^2}{2} v_j\right] + O\left(v_j N^{-\delta} + w_j + \hat{\varepsilon}_N\right),$$

where

$$v_j = \nu(T_j^2), \quad w_j = \iint_{\substack{(t_1, t_2) \in J_j \\ |t_1 - t_2| \geq \sqrt{\ln N}}} \frac{d\mathfrak{m}_N(t_1) d\mathfrak{m}_N(t_2)}{|t_1 - t_2|^\gamma}, \quad \hat{\varepsilon}_N = N^{\delta-1/4-\beta_2(\gamma-1)}.$$

Note that by property (b) and (5.6), we have $T_j = O(N^{-\delta})$. Hence $v_j = O(N^{-2\delta})$ and so the estimate of Lemma 5.5 can be rewritten as

$$\Phi_j(\xi) = \Phi_{j-1}(\xi) e^{-\xi^2 v_j/2} + O\left(v_j N^{-\delta} + w_j + \hat{\varepsilon}_N\right).$$

Repeating this process, we obtain

(5.7)
$$\mathbb{E}_\ell\left(e^{i\xi S_{N,\bar{N}}}\right) = \Phi_{\bar{N}}(\xi) = \exp\left[-\frac{\xi^2}{2} \sum_{j=1}^{N} v_j\right] + O\left(\sum_j \left[v_j N^{-\delta} + w_j\right] + \bar{N} \hat{\varepsilon}_N\right).$$

Next we claim that

(5.8)
$$\sum_{j=1}^{\bar{N}} v_j = 1 + o_{N \to \infty}(1)$$

provided that

$$(5.9) \qquad\qquad \beta_2(\gamma - 1) > 2\delta.$$

Indeed, $1 + o(1) = \nu(S^2_{N,\bar{N}}) = \sum_{j=1}^{\bar{N}} v_j + \sum_{j_1 \neq j_2} \nu(T_{j_1} T_{j_2})$. The second term here is at most

$$C \sum_{n_1, n_2 : |n_1 - n_2| \geq N^{\beta_2}} \frac{\mathfrak{m}_N([n_1, n_1 + 1]) \mathfrak{m}_N([n_2, n_2 + 1])}{|n_1 - n_2|^\gamma}.$$

Summing over n_2 using assumption (b) we are left with

$$C \sum_{n_1} \mathfrak{m}_N([n_1, n_1 + 1]) N^{\delta - (1/4) - \beta_2(\gamma - 1)}$$

$$= C \mathfrak{m}_N([-N^{1/2+\delta}, N^{1/2+\delta}]) N^{\delta - (1/4) - \beta_2(\gamma - 1)} \leq \bar{C} N^{2\delta - \beta_2(\gamma - 1)} = o_{N \to \infty}(1),$$

where in the last inequality we also used assumption (a).

This proves that (5.9) implies (5.8). (5.8) shows in particular that $\sum_j v_j N^{-\delta} = O(N^{-\delta})$. Also $\sum_j w_j = o(1)$ due to assumption (c) of Proposition 5.3, while

$$\bar{N} \hat{\varepsilon} \leq N^{1/4 + 2\delta - \beta_1 - \beta_2(\gamma - 1)} = o(1)$$

provided that

$$(5.10) \qquad\qquad \beta_1 + \beta_2(\gamma - 1) > \frac{1}{4} + 2\delta.$$

Plugging these estimates into (5.7) we conclude that for all $\ell \in \mathfrak{M}$ we have

$$(5.11) \qquad\qquad \mathbb{E}_\ell\left(e^{i\xi S_{N,\bar{N}}}\right) = e^{-\xi^2/2} + o_{N \to \infty}(1)$$

if β_1 and β_2 satisfy (5.4), (5.6), (5.9), and (5.10). Thus we need β_1 and β_2 to satisfy

$$\beta_1 < \frac{1}{4} - 2\delta, \quad \beta_2 < \beta_1 - 3\delta, \quad \beta_1 + \beta_2(\gamma - 1) > \frac{1}{4} + 2\delta, \quad \beta_2(\gamma - 1) > 2\delta.$$

Since β_1 can be chosen arbitrary close to $\frac{1}{4} - 2\delta$ and β_2 can be chosen arbitrarily close to $\beta_1 - 3\delta$, the above inequalities are compatible if (5.3) holds. It then follows that (5.11) holds on the convex hull of $\mathfrak{M}$ which includes ν. This completes the proof of Proposition 5.3 modulo Lemma 5.5. $\qquad\square$

Proof of Lemma 5.5. Let $J_j = [n_j^-, n_j^+]$. Denote $m_j = \tau + \frac{n_{j-1}^+ + n_j^-}{2}$. We use the almost Markov decomposition (5.1):

$$\mathbb{E}_\ell(A \circ G_{m_j}) = \sum_s c_s \mathbb{E}_{\ell_s}(A) + O(\tilde{\varepsilon}_N)$$

where $\tilde{\varepsilon}_N = \theta^{N^{\beta_2}}$, $\ell_s = (D_s, \rho_s) \in \mathfrak{M}$ and $\sum_s c_s = 1 - O(\tilde{\varepsilon}_N)$.

Fix arbitrary $y_s \in G_{-m_j} D_s$. Then

$$\mathbb{E}_\ell\left(e^{i\xi S_{N,j}}\right) = \sum_s c_s \mathbb{E}_{\ell_s}\left(e^{i\xi[S_{N,j-1}(y_s) + \tilde{T}_j(y)]}\right) + O(\tilde{\varepsilon}_N)$$

where
$$\tilde{T}_j(y) = T_j(G_{-m_j}y) = \int_{J_j} A_t(G_{t-m_j}y)\mathrm{d}\mathfrak{m}_N(t).$$

Next
$$\mathbb{E}_{\ell_s}\left(e^{i\xi\tilde{T}_j}\right) = 1 + i\xi\mathbb{E}(\tilde{T}_j) - \frac{\xi^2}{2}\mathbb{E}_{\ell_s}(\tilde{T}_j^2) + O\left(\mathbb{E}_{\ell_s}\left(\left|\tilde{T}_j\right|^3\right)\right).$$

Note that $\mathbb{E}(\tilde{T}_j) = O\left(N^{\beta_2(\gamma-1)-(1/4)+\delta}\right)$ due to the equidistribution of unstable leaves.

To estimate $\mathbb{E}_{\ell_s}(\tilde{T}_j^2)$ we split
$$\mathbb{E}_{\ell_s}(\tilde{T}_j^2) = \iint_{J_j \times J_j} \mathbb{E}_{\ell_s}[A_{t_1}(G_{t_1-m_j}y)A_{t_2}(G_{t_2-m_j}y)]\mathrm{d}\mathfrak{m}_N(t_1)\mathrm{d}\mathfrak{m}_N(t_2) = I + I\!\!I$$

where I includes the terms where $|t_1 - t_2| \leq \sqrt{\ln N}$ and $I\!\!I$ includes the other terms.

According to [10, Theorem 2],
$$\mathbb{E}_{\ell_s}\left((A_{t_1} \circ G_{t_1-m_j})(A_{t_2} \circ G_{t_2-m_j})\right) = O\left(|t_2 - t_1|^{-\gamma}\right).$$

Therefore $I\!\!I = O(w_j)$.

To estimate I, we note that
$$\mathbb{E}_{\ell_s}\left((A_{t_1} \circ G_{t_1-m_j})(A_{t_2} \circ G_{t_2-m_j})\right) = \mathbb{E}_{\ell_s}\left((D_{t_1,t_2} \circ G_{t_1-m_j})\right)$$

where $D_{t_1,t_2} = A_{t_1}(A_{t_2} \circ G_{t_2-t_1})$ with $\|D_{t_1,t_2}\|_{C^r} \leq K^{t_2-t_1} \leq N^{\delta/2}$.

Hence using the equidistribution of unstable leaves, we obtain
$$\mathbb{E}_{\ell_s}\left((A_{t_1} \circ G_{t_1-m_j})(A_{t_2} \circ G_{t_2-m_j})\right) = \nu(D_{t_1,t_2}) + O\left(N^{\delta/2}(t_1 - m_j)^{-\gamma}\right)$$

$$= \nu(A_{t_1}(A_{t_2} \circ G_{t_2-t_1})) + O\left(N^{\delta/2}(t_1 - m_j)^{-\gamma}\right).$$

It follows that
$$I = \iint_{\substack{t_1,t_2 \in J_j \\ |t_1-t_2| \leq \sqrt{\ln N}}} \nu(A_{t_1}(A_{t_2}G_{t_2-t_1}))\mathrm{d}\mathfrak{m}_N(t_1)\mathrm{d}\mathfrak{m}_N(t_2) + O\left(N^{(5\delta/2)-(1/2)-\beta_2(\gamma-1)}\sqrt{\ln N}\right)$$

$$= v_j + O(w_j) + O\left(N^{3\delta-(1/2)-\beta_2(\gamma-1)}\right).$$

Finally
$$\mathbb{E}_{\ell_s}(|\tilde{T}_j|^3) \leq N^{-\delta}\mathbb{E}_{\ell_s}(|\tilde{T}_j|^2) = O\left([v_j + w_j]N^{-\delta} + N^{2\delta-(1/2)-\beta_2(\gamma-1)}\right).$$

Summing over s and using the fact that $\sum_s c_s\mathbb{E}_{\ell_s}\left(e^{i\xi S_{N,j-1}(y_s)}\right) = \mathbb{E}_\ell\left(e^{i\xi S_{N,j-1}}\right) + O(\tilde{\varepsilon}_N)$,

we obtain the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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