Asymptotic properties of the Lorentz process and some closely related models

PhD Thesis

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Chapter 1

Introduction

1.1 Background

Chaotic, "stochastic" behavior of deterministic systems is much interesting from both theoretical and applied points of view. An archetype of such systems is the *Sinai billiard* - or equivalently, its periodic extension, the *periodic Lorentz process*. The motivation for studying these models is multiple. In the physics literature, Hendrik Lorentz [L05] introduced Lorentz gas as a model of motion of electrons in a metal. By considering the dynamics of just one classical electron in a crystal, one obtains the (periodic) Lorentz process. Nowadays, a central problem in statistical physics is to derive macroscopic laws from microscopic dynamics. In the optimal case, the microscopic dynamics are Newtonian which makes the model more realistic. The two model families, where such rigorous results are available, are mathematical billiards and oscillators.

On the other hand, Sinai was also interested in physically relevant examples, where the mathematically precise notions of chaotic dynamical systems can be verified. Most importantly, this means the precise formulation and proof of *Boltzmann's Ergodic Hypothesis* which is roughly speaking the following: in any large enough physical system, the time average (as time tends to infinity) of a physical quantity coincides with its spatial average (with respect to the equilibrium). Sinai [S63] conjectured that this is true, more precisely *ergodicity* holds, for hard ball systems - hard balls moving on a flat torus and colliding with each other solely - once the trivially conserved quantities (total momentum, center of mass, total kinetic energy) are fixed. This statement is now called *Boltzmann-Sinai Ergodic Hypothesis*. The motion of any number of hard balls in any dimension can be encoded into the motion of a point particle on a possibly higher dimensional torus with some restricted regions (scatterers). In particular, this led to the definition of Sinai billiard (or *dispersing billiard* following Sinai's terminology) in the celebrated paper [S70]. Here Sinai also proved the Boltzmann-Sinai Ergodic Hypothesis in the simplest case (two balls in two dimensions) and other properties of the planar Sinai billiard (hyperbolicity, K-mixing) which reflect strongly chaotic nature.

The definition of *semi-dispersing billiard* is the following. Fix some convex subsets $B_1, \dots B_k$ of the

d dimensional torus, whose boundaries fulfill some regularity conditions. These sets are thought of as scatterers. The continuous dynamics, called the *billiard flow* is the free flight of a point particle among the scatterers and its specular reflection on their boundaries. The speed of the particle is constant (equals to one, say), thus the phase space of the billiard flow consists of a spatial component (d dimensional torus minus the scatterers) and a velocity vector, which is an element of the $d-1$ dimensional unit sphere. Lebesgue measure is a natural invariant measure in both coordinates. The same motion is described by the *billiard ball map*, which is the Poincaré section of the flow on the boundaries of the scatterers. Hence its phase space has spatial dimension is $d-1$ and full dimension $2d-2$. The simplest case is of course the planar one ($d=2$), where the phase space of the billiard ball map is two dimensional.
If the scatterers $B_1, \dots B_k$ are *strictly convex*, then the billiard is called dispersing, or Sinai billiard. If one encodes the motion of hard balls into semi-dispersing billiards, then the scatterers are going to be strictly convex if and only if the number of hard balls is two. Since in the case of more hard balls, some cylindrical scatterers are also needed, proving the Boltzmann-Sinai Ergodic Hypothesis is even much harder in this case (see [S09] and references therein).
Some spectacular results for the Sinai billiards were the convergence to equilibrium in [KSz83a], the linear Boltzmann equation in the case of small scatterer size limits, [BBS83], [MS10] and the (super)diffusion in the planar case. In the sequel, we focus on the latter.
The development of the theory of planar Sinai billiard in the last decades is miraculous (consult [CM06] for lot of details). Besides ergodicity and hyperbolicity, the most interesting statistical properties are the decay of correlation and the central limit theorem (CLT), or diffusion. For an abstract dynamical system $(\mathcal{M}, \mathcal{F}, \mu)$, the former means that $\int f(g \circ \mathcal{F}^n) d\mu$ is exponentially small in n if $\int f d\mu = \int g d\mu = 0$ and f and g are chosen from a nice set of functions (definitely containing the free flight function for the billiard ball map). With this terminology, CLT means that $\frac{1}{\sqrt{n}} \sum_{k=1}^{n} f \circ \mathcal{F}^k$, as a random variable with respect to μ, weakly converges to a Gaussian distribution.
The classical method of proving CLT in hyperbolic dynamical systems consists of the construction of Markov partition, switching to the symbolic space and applying functional analytic methods to the Perron-Frobenius operator. Although in the case of Sinai billiards this was basically done in [BS81] and [BSCh91] thus providing the first proof of the CLT, this method is cumbersome (for instance, the Markov partition is not finite), and does not provide exponential decay of correlation. It is also a crucial fact, that a geometric assumption in needed for the above results, namely, that the free flight function should be bounded.

Definition 1. *We say that a Sinai billiard (or periodic Lorentz process) has* finite horizon*, if the free flight function is bounded. Equivalently, it has* infinite horizon*, if there is an infinite line which is disjoint from the interior of all scatterers.*

The following breakthrough in this theory was the paper of Young [Y98]. She introduced the tower technique which was strong enough to prove exponential decay of correlation for the billiard ball map and also provided a new, transparent proof of the CLT. Her method was successfully applied by Szász and Varjú in the case of infinite horizon [SzV07]. According to their most interesting result (also conjectured by Bleher [B92] much before), the presence of infinite horizon yields a slightly super-diffusive behavior.

The displacement of the particle in n steps, rescaled by $\sqrt{n \log n}$, converges to some Gaussian distribution. In fact, they proved a local version of this limit theorem and also that of the CLT in case of finite horizon [SzV04].
Chernov and Dolgopyat managed to further simplify the proof of the CLT with their method of "standard pairs". This method allowed them to treat a model of two moving particles, which is in some sense a parametric family of billiards [ChD09b] (note that this model is much more complicated then the case of just one moving particle). The standard pair technique, which is in fact the state-of-the-art method, is also applicable to many other problems. Besides the proof of the limit theorem in both finite and infinite horizons ([Ch06b, ChD09a]), it also yielded more delicate statistical properties (e.g. convergence to Brownian motion, law of iterated logarithm) and also limit theorems for related models (e.g. for billiards under external fields). Further, several arguments from probability theory has been successfully reapplied to Sinai billiards, thanks to this technique. Dolgopyat, Szász and Varjú [DSzV08] proved delicate recurrence properties of the periodic Lorentz process with finite horizon. These properties are all the same, as if the billiard particle was substituted by a random walker (of course, the proof is much more involved). They also managed to prove CLT for a non-periodic Lorentz process with finite horizon [DSzV09], where periodicity is spoiled in a compact domain. A related conjecture is the following.

Conjecture 1.1. *Modify the scatterer configuration of a periodic Lorentz process with infinite horizon on a compact subset (the modification still satisfies the assumptions of the Sinai billiard). Then, the super-diffusive limit theorem remains valid.*

Going back to the motivation by the work of Lorentz [L05], one sees that these kind of problems are also physically motivated (crystals often have impurities). In the last few years, some other non-homogeneous modifications of the periodic Lorentz process were also considered, see for instance [SYZ12] for a very recent one. As both the delicate statistical properties of the periodic Lorentz process and the basic statistical properties of some non-homogeneous versions are current active research fields, there are plenty of interesting, challenging questions, a few of which we are going to address in this thesis.

1.2 Structure of the thesis

This thesis consists of six more or less self-contained chapters. Chapters 2 3 4 and 5 contain (almost verbatim) the articles [N11a, N11b, NSz12, NSzV12a], respectively. Chapter 6 is the preprint [NSzV12b], while Chapter 7 is an unpublished work, also joint with Domokos Szász and Tamás Varjú. I would also like to remark that Chapter 2 heavily overlaps with my MSc thesis. At several points - mainly in the introductions -, the Chapters may overlap (by not much, though).
The high level logic of the thesis is the following: Chapters 2 and 3 are about some stochastic models (random walks) that are motivated by the periodic Lorentz processes. Chapter 4 is about a specific type of inhomogeneity (both in space and time) in Lorentz process. On the technical level, Chapters 2-4 require ideas almost exclusively from Probability theory. Chapter 5 suggests an approach to study general time inhomogeneity in dynamical systems (at its present state, not strong enough to treat two dimensional dynamics, though). Chapter 6 deals with Lorentz processes with infinite horizon in dimension

$d \geq 3$, while Chapter 7 is roughly speaking a new proof for the convergence to the Brownian motion in the plane, again, in the infinite horizon case. On the technical level, Chapters 5-7 require ideas primarily from the theory of Dynamical systems and elementary geometry, although Probability theory is still an important ingredient. We also mention that the motivation of Chapters 3 and 7 is mainly (but not exclusively) is the hope that they might be useful at attacking Conjecture 1.1. Each Chapter starts with an introductory Section and some of them has some remarks in the end pointing out some possible extensions and open questions. In the rest of this Section, we introduce each Chapter in some more details.

Chapter 2 is about the number of distinct sites visited by a random walker that has a finite memory (internal state). In fact, we extend the results of Dvoretzky and Erdős [DE51] for Simple Symmetric Random Walk. These are the asymptotics for the expected value and the variance of the number of visited distinct sites, and both weak and strong laws of large numbers. In this problem, the "intermittent dimension" is 2, thus the computation in the plane is much more delicate that that of $d = 1$ and $d \geq 3$. As a tool for these results, the error term of the local limit theorem of [KSz83b] is also estimated. Sinai's motivation behind this model of Random Walk with Internal States (see[S81]) is the better understanding of the periodic Lorentz process with finite horizon. We also point out the fact that in the case of planar Lorentz process (with finite horizon), the same questions has been answered by Pène [P09a].
Chapter 3 is about some delicate recurrence properties of a random walk, the step distribution of which has the same tail asymptotics as the planar Lorentz process with infinite horizon. We address exactly the same questions Dolgopyat, Szász and Varjú were dealing with in [DSzV08] (which were also important by the proof of the convergence to the Brownian motion in locally perturbed periodic Lorentz process with finite horizon [DSzV09]). These questions include the tail asymptotics of the distribution of the first return time to the starting position (origin), limit theorem for the local time at the origin and for the hitting time of the origin as started from far away. The consequence is that in case of the infinite horizon, the recurrence properties are weaker, as expected - for instance, the local time up to n is scaled by $\log \log n$ in compare to $\log n$ in finite horizon. Some of these results can be proven to the Lorentz process too, but some of them are open. As in Chapter 2, a refinement of the local limit theorem is also needed here.
In Chapter 4, we consider a periodic planar Lorentz process with finite horizon, restricted to a horizontal strip. In this setting, the diffusively rescaled trajectory converges to the Brownian motion, which is an easy consequence of the same statement in the plane. Now, if one puts a vertical wall to the zeroth cell, then the trajectory converges to the reflected Brownian motion, but if there is a hole on the wall - thus the particle eventually get through it -, then the limit is again the Brownian motion (see [DSzV09]). In Chapter 4, we prove that if one puts a hole of decreasing size to the wall, then the limit is the so-called quasi reflected Brownian motion, a joint generalization of Brownian motion and reflected Brownian motion. It is worth mentioning that this is a stochastic process which is Markovian but not strong Markovian. The most important ingredient of the proof is the local limit theorem for planar periodic Lorentz process with finite horizon, see [SzV04].
In Chapter 5, we prove functional central limit theorem for deterministic time-dependent dynamical

systems. The result itself is applicable only in restricted settings - mainly for one dimensional expanding maps - but the time inhomogeneity is general. The latter means that instead of proving the central limit theorem for a typical sequence of some mappings, we can prove it for fixed sequences under some conditions. These are connected to the zero-cohomology condition in the autonomous case.

Chapter 6 is about periodic Lorentz process in dimension $d \geq 3$ and with infinite horizon. Note that the high dimensional case is much more difficult than the planar one. Even for finite horizon, much less is known than in $d = 2$, see [BT08] for the present state of the theory. The infinite horizon also makes the picture more difficult. Recall that in the planar case, the scaling of the trajectory is slightly super-diffusive. In $d \geq 3$, the first step is to ascertain the tail asymptotics of the free flight function. If it is the same as the one in the planar case, then it is reasonable to expect the same super-diffusive behavior. In [D12], Dettmann formulated conjectures, which provide the tail asymptotics in quite a generality. The essence of the conjectures is that super-diffusion is expected if and only if there is a horizon of maximal possible dimension. In Chapter 6, we prove the first two conjectures of Dettmann. It is worth mentioning that our proof uses results from the theory of the small scatterer size limit of Lorentz processes, see [MS10].

Chapter 7 has the closest connection to Conjecture 1.1. In the proof of the CLT for locally modified periodic Lorentz process with finite horizon, [DSzV09] uses the "martingale method" of Stroock and Varadhan. Hence it is reasonable to expect that this method could be useful by attacking Conjecture 1.1, too. In Chapter 7, we prove that the only possible limit point of the super-diffusively rescaled trajectory is the Brownian motion with the appropriate covariance matrix by a combination of the standard pair and the martingale methods. Chernov and Dolgopyat proved the same in [ChD09a] by combining the standard pair technique with Bernstein's big block-small block method.

Chapter 2

Number of Distinct Sites Visited by a Random Walk with Internal States

2.1 Introduction

The model of a random walk with internal states (or, alternatively, random walk with internal degrees of freedom; briefly RWwIS) was introduced by Sinai in 1981 in his Kyoto talk [S81]. His aim was to get an efficient tool for examining the Lorentz process (in this context, internal states would represent the elements of the Markov partition or of a Markov sieve). For this kind of argument see, for instance, [PSz12]. Beside the Lorentz process, however, several other motivations and applications have appeared, among others, in some models of queueing systems, cf. [H95] as for an extensive treatment of other motivations. Nevertheless, the investigation of this model is important for its own sake, as it is a manifest generalization of a gem of probability theory: the simple symmetric random walk. Let us begin with the definition of RWwIS with the notation in [KSz83b] and [KSz84] (or of [KSSz86], where RWwIS served as a model of Fourier law of heat conduction).

Definition 2. *Let E be a finite set. On the set $H = \mathbb{Z}^d \times E$ $(d = 1, 2, ...)$, the Markov chain $\xi_n = (\eta_n, \varepsilon_n)$ is a random walk with internal states (RWwIS), if for $\forall x_n, x_{n+1} \in \mathbb{Z}^d, \quad j_n, j_{n+1} \in E$*

$$P(\xi_{n+1} = (x_{n+1}, j_{n+1})|\xi_n = (x_n, j_n)) = p_{x_{n+1}-x_n, j_n, j_{n+1}}.$$

In fact, E could be countable, as well, but we will consider only the finite case. We will denote $s = \#E$.

There are some basic assumptions which will throughout be supposed. These are the following:

(i) $(\varepsilon_0, \varepsilon_1, ...)$ - obviously a Markov chain - is irreducible and aperiodic (its stationary distribution will be denoted by μ)

(ii) the arithmetics are trivial, with the notation in [KSz83b], $L = \mathbb{Z}^d$

(iii) the expectation of one step is zero provided that ε_0 is distributed according to its unique stationary measure

(iv) the covariance matrix, which is exactly defined in Section 2.2, exists and is nonsingular.

In general, we will assume that $\eta_0 = 0$. Let $L_d(n)$ denote the number of distinct sites visited by $(\eta_k)_k$ up to n steps. The expectation of $L_d(n)$ is $E_d(n)$, and the variance is $V_d(n)$. $\{e_j\}_{j=1,...,s}$ is the standard basis in $\mathbb{R}^s$, and $\underline{1} = (1, 1, ..., 1)^T$. Our aim is to find asymptotics of $E_d(n)$, further, by using bounds on $V_d(n)$, we want to prove weak and strong laws of large numbers. Similar results in terms of simple symmetric random walks (which will later on be referred to as SSRW) are found in [DE51]. Recently, in the case of two dimensional Lorenz process, Pène discussed the same question in [P09a]. There are numerous fairly new papers on $L_d(n)$ for random walks with independent steps (see [BR05] and references therein).

This Chapter is organized as follows: in Section 2.2 the main theorem of [KSz83b] is generalized. Namely, a remainder term of the local limit theorem is computed, as it will be necessary for estimating $E_2(n)$. A further refinement of the local limit theorem will also be given as it will be useful when proving the strong law of large numbers in the plane. Although these results are used in the forthcoming Sections, they can be interesting in their own rights. In Section 2.3, the number of visited points in the high dimensional case, i.e. when $d \geq 3$, is dealt with. We prove asymptotics for $E_d(n)$, and estimate $V_d(n)$, from which we can prove both the weak and strong laws of large numbers. In this Section, we will not use the result of Section 2.2, Theorem 5.2. in [KSz83b] will be enough for our purposes. In Section 2.4 the $d = 2$ case is discussed. For $E_2(n)$, same asymptotics ($const\frac{n}{\log n}$) is found as in [DE51], but with some different constant. $V_2(n)$ is also estimated, and the weak law of large numbers is also proved. The proof of the strong law in the plane is a little bit cumbersome calculation, so it is postponed to Section 2.5. In Section 2.6, the one dimensional settings are considered. This case requires a little bit different approach from the previous ones (and is not treated in [DE51]), so the application of a Tauberian theorem will be very useful. Section 2.7 gives some remarks.

2.2 Preliminaries

2.2.1 Local limit theorem with remainder term

In this subsection, we calculate a remainder term for Theorem 5.2. in [KSz83b]. Furthermore, another refinement of this theorem will be proved, as it will be used when proving the strong law of large numbers in the plane. First, we reformulate the mentioned theorem. We have to start with some definitions. Denote

$$\begin{aligned} A_y &= (p_{y,j,k})_{j,k=1,\dots,s} : \mathbb{C}^s \to \mathbb{C}^s, \\ Q &= \sum_{y\in\mathbb{Z}^d} A_y, \\ M_l &= \sum_{y\in\mathbb{Z}^d} y_l A_y, \\ \Sigma_{l,m} &= \sum_{y\in\mathbb{Z}^d} y_l y_m A_y. \end{aligned}$$

So, the transition matrix of the Markov chain $(\varepsilon_0, \varepsilon_1, ...)$ is Q and its unique stationary measure is μ.

Theorem 2.1. *(Krámli-Szász [KSz83b]) Consider a RWwIS in $\mathbb{Z}^d$ and assume that the matrix $\sigma = (\sigma_{l,m})_{1\le l,m\le d}$ whose elements are*

$$\sigma_{l,m} = \langle \mu, \Sigma_{l,m}\underline{1}\rangle - \left\langle \mu, M_l (Q-1)^{-1} M_m \underline{1}\right\rangle - \left\langle \mu, M_m (Q-1)^{-1} M_l \underline{1}\right\rangle$$

(which can be called a covariance matrix) is positive definite, then

$$\sum_{(x,k)\in H} \left| P\left(\xi_n = (x,k) | \xi_0 = (0,j)\right) - n^{-d/2} \mu_k g_\sigma \left(\frac{x}{\sqrt{n}}\right)\right| \to 0$$

as $n \to \infty$, where $g_\sigma(x)$ denotes the density of a Gaussian distribution with mean 0 and covariance matrix σ.

Of course, the condition concerning the positive definiteness of the matrix in one dimension means $\sigma > 0$. We omit the proof, it can be found in [DE51]. In fact, there is a typo in [DE51] as they write $n^{-1/2}$ instead of $n^{-d/2}$ but it is easy to correct it even in the proof.

Our calculation will be similar to the one of [KSz83b]. The main point is that while in [KSz83b] it is sufficient to consider the Taylor expansion of the largest eigenvalue up to the quadratic term, now, we have to calculate the third term, as well.

Define the Fourier transform

$$\alpha(t) = \sum_{y\in\mathbb{Z}^d} \exp\left(i\langle t, y\rangle\right) A_y, t \in [-\pi,\pi]^d.$$

Now, we have to consider the Taylor expansion of the largest eigenvalue of $\alpha(t)$, which is denoted by $\lambda(t)$, up to the third term.

Let us first assume that $d = 1$. From our basic assumptions it follows that $M = \sum_{y\in\mathbb{Z}} yA_y$ and $\Sigma = \sum_{y\in\mathbb{Z}} y^2 A_y$ are convergent series. But now, we also suppose the absolute convergence of

$$\Xi = \sum_{y\in\mathbb{Z}} y^3 A_y. \tag{2.1}$$

The existence of M, Σ and Ξ implies

$$\alpha(t) = Q + itM - \frac{t^2}{2}\Sigma - \frac{it^3}{6}\Xi + o(t^3) \quad (t \to 0). \tag{2.2}$$

Now, by perturbation theoretic means (i.e. the straightforward extension of Theorem 5.11. of Chapter II. in [K80]) it can be easily proved that

$$\lambda(t) = 1 + r_1 t + \frac{r_2}{2}t^2 + \frac{r_3}{6}t^3 + o(t^3) \quad (t \to 0). \tag{2.3}$$

From [KSz83b] we know that $r_1 = 0$ and $r_2 = -\langle \Sigma \underline{1}, \mu \rangle + 2\langle M(Q-1)^{-1}M, \mu \rangle$.

Using the notation $\sigma^2 = -r_2$ we can now formulate our theorem:

Theorem 2.2. *For a one dimensional RWwIS the existence of (2.1) imply*

$$\begin{aligned} P(\xi_n &= (x,k)|\xi_0 = (0,j)) - \\ &- \mu_k \frac{1}{\sqrt{2\pi n}\sigma} \exp\left(-\frac{x^2}{2n\sigma^2}\right)\left[1 - \frac{ir_3}{6}x\left(3\sigma^2 n - x^2\right)\frac{1}{\sigma^6}\frac{1}{n^2}\right] = o\left(\frac{1}{n}\right), \end{aligned}$$

where the small order is uniform in x.

Proof. The proof is similar to the one of Theorem 2.1. in [KSz83b]. In the neighborhood of the origin, we have $\alpha^n(t) = \lambda^n(t)p(t) + b_n(t)$, where p is the projector to the eigenspace associated to $\lambda(t)$, and $b_n(t)$ is the contribution of the other eigenvalues. The term $b_n(t)$ is in $O(\alpha^n)$ for some $\alpha \in (0,1)$.

Because of (2.3) we have

$$\alpha^n(t) = \left(\underline{1}\mu^T + tp'(0) + O(t^2)\right)\left(1 - \frac{\sigma^2 t^2}{2} + \frac{r_3}{6}t^3 + o\left(t^3\right)\right)^n + b_n(t). \tag{2.4}$$

Elementary calculations show that

$$\left(1 - \frac{\sigma^2 s^2}{2n} + \frac{r_3}{6}\frac{s^3}{n^{\frac{3}{2}}} + o\left(\frac{s^3}{n^{\frac{3}{2}}}\right)\right)^n = \exp\left(-\frac{\sigma^2 s^2}{2}\right)\left(1 + \frac{r_3}{6}s^3\frac{1}{\sqrt{n}} + o\left(\frac{s^3}{\sqrt{n}}\right)\right) \tag{2.5}$$

holds uniformly for $|s| < n^{\varepsilon}$ with $0 < \varepsilon < 1/6$. In order to prove the statement, we use the Fourier transforms and the usual estimations

$$\begin{aligned} &\Bigg\| \sqrt{n}\int_{-\pi}^{\pi} \exp(-ixt)\, e_j^T \alpha^n(t)\, dt \\ &-\mu^T \frac{\sqrt{2\pi}}{\sigma}\exp\left(-\frac{x^2}{2n\sigma^2}\right)\left[1 - \frac{ir_3}{6}x\left(3\sigma^2 n - x^2\right)\frac{1}{\sigma^6}\frac{1}{n^2}\right]\Bigg\| \\ \le\ & \int_{|s|<n^{\varepsilon}} \left\| e_j^T p(0)\lambda^n\left(\frac{s}{\sqrt{n}}\right) - \mu^T \exp(-\frac{\sigma^2 s^2}{2})\left(1 + \frac{r_3}{6}\frac{s^3}{\sqrt{n}}\right)\right\| ds + o\left(\frac{1}{\sqrt{n}}\right) \\ &+ c\,\|\mu\| \int_{|s|>n^{\varepsilon}} (1+s^3)\exp(-\frac{\sigma^2 s^2}{2})ds + \int_{n^{\varepsilon}<|s|<\gamma\sqrt{n}} \left\| e_j^T \alpha^n\left(\frac{s}{\sqrt{n}}\right)\right\| ds \\ &+ \int_{\gamma\sqrt{n}<|s|<\pi\sqrt{n}} \left\| e_j^T \alpha^n\left(\frac{s}{\sqrt{n}}\right)\right\| ds \\ =\ & I_1 + o\left(\frac{1}{\sqrt{n}}\right) + I_2 + I_3 + I_4, \end{aligned}$$

where $0 < \varepsilon < \frac{1}{6}$ is arbitrary. The term $o\left(\frac{1}{\sqrt{n}}\right)$ is the contribution of the terms $\frac{s}{\sqrt{n}}p'(0) + O(\frac{s^2}{n})$ in (2.4), as we can see that

$$\int\limits_{|s|<n^\varepsilon} \lambda^n \left(\frac{s}{\sqrt{n}}\right) \frac{s}{\sqrt{n}} p'(0) ds$$
$$= \int\limits_{|s|<n^\varepsilon} \exp\left(-\frac{\sigma^2 s^2}{2}\right) \frac{s}{\sqrt{n}} p'(0) ds + O\left(\int\limits_{|s|<n^\varepsilon} \exp\left(-\frac{\sigma^2 s^2}{2}\right) \frac{s^3}{\sqrt{n}} \frac{s}{\sqrt{n}} p'(0) ds \right),$$

which is $0 + o\left(\frac{1}{\sqrt{n}}\right)$, and

$$\int\limits_{|s|<n^\varepsilon} \left|\lambda^n \left(\frac{s}{\sqrt{n}}\right)\right| \frac{s^2}{n} ds = o\left(\frac{1}{\sqrt{n}}\right).$$

It is clear that proving $I_j = o\left(\frac{1}{\sqrt{n}}\right), \quad j = 1, 2, 3, 4$ is enough for our purposes. (2.5) yields that the integrand in I_1 is equal to $\frac{\delta(n)}{n^{1/2}} s^3 \exp\left(-\frac{\sigma^2 s^2}{2}\right)$, where $\delta(n) \to 0$ uniformly in s. Thus we have $I_1 = o\left(\frac{1}{\sqrt{n}}\right)$. It is clear that $I_2 = o\left(\frac{1}{\sqrt{n}}\right)$, and I_4 converges exponentially fast to zero. Finally, if $\gamma > 0$ is small enough, then

$$I_3 = \int\limits_{n^\varepsilon < |s| < \gamma\sqrt{n}} \left\| e_j^T \alpha^n \left(\frac{s}{\sqrt{n}}\right) \right\| ds \le \int\limits_{n^\varepsilon < |s| < \gamma\sqrt{n}} \exp\left(-\frac{\sigma^2 s^2}{4}\right) ds.$$

So we have $I_3 = o\left(\frac{1}{\sqrt{n}}\right)$, too. □

Remark 2.3. *In Theorem 2.2 for the expression subtracted from the appropriate probability we have:*

$$\mu_k \frac{1}{\sqrt{2\pi n}\sigma} \exp\left(-\frac{x^2}{2n\sigma^2}\right) \left[1 - \frac{ir_3}{6} x \left(3\sigma^2 n - x^2\right) \frac{1}{\sigma^6} \frac{1}{n^2}\right]$$
$$= \mu_k \frac{1}{\sqrt{2\pi n}\sigma} \exp\left(-\frac{y^2}{2}\right) + \mu_k \frac{1}{\sqrt{n}} \frac{1}{\sigma} \frac{q_1(y)}{\sqrt{n}},$$

where $y = \frac{x}{\sqrt{n}\sigma}$, and the $q_1(y)$ is the function defined in [P75], Chapter VI. (1.14). In this sense, the local limit theorem concerning RWwIS is analogous to the one of Simple Symmetric Random Walk (see [P75] Chapter VII. Theorem 13).

The extension of Theorem 2.2 to the multidimensional case is straightforward. Analogously to (2.3), we have:

$$\lambda(t) = 1 - \frac{1}{2} t^T \sigma t + f(t) + o(|t|^3) \quad (|t| \to 0),$$

where $f(t) = \sum_{i=1}^{d}\sum_{j=1}^{d}\sum_{k=1}^{d} r_{3,i,j,k} t_i t_j t_k$ is the third term of the Taylor expansion. Denote

$$\Omega = n^{d/2} P(\xi_n = (x, .)|\xi_0 = (0, j)) = \frac{n^{d/2}}{(2\pi)^d} \int_{-\pi}^{\pi} ... \int_{-\pi}^{\pi} \exp\left(-i\langle x, t\rangle\right) e_j^T \alpha^n(t)\, dt.$$

So the analogue of the expression subtracted from the appropriate probability in Theorem 2.2 (multiplied by $\frac{n^{d/2}}{(2\pi)^d}$) is

$$I^{(n)} := \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \exp\left(-\frac{s\sigma s}{2} - i\left\langle x, \frac{s}{\sqrt{n}}\right\rangle\right) \frac{f(s)}{\sqrt{n}} ds.$$

Using Lebesgue's Theorem, it is easy to see that $I^{(n)} = O\left(n^{-1/2}\right)$. One can estimate I_1, I_2, I_3, I_4 the same way, as it was done in the proof of Theorem 2.2 (see [KSz83b] Section 5. for more details). So we have arrived at

Proposition 2.4. *Supposing that (2.1) exists, for a d dimensional RWwIS*

$$P(\xi_n = (x,k)|\xi_0 = (0,j)) = \frac{1}{n^{d/2}}\mu_k g_\sigma\left(\frac{x}{\sqrt{n}}\right) + O\left(n^{-(d+1)/2}\right)$$

holds, where $g_\sigma(x)$ denotes the density of a Gaussian distribution with mean 0 and covariance matrix σ and the great order is uniform is x.

A further refinement of the local limit theorem will be useful in the sequel. Now, we would like to go further in the asymptotic expansion, and apply our techniques in the two dimensional case. Nevertheless, we are interested only in an estimation, not in the exact result which will simplify the calculation. Just like previously, let us begin with the one dimensional case. Assume the convergence of the series

$$\Upsilon = \sum_{y \in \mathbb{Z}} y^4 A_y. \tag{2.6}$$

Now, just like previously, we may write

$$\alpha(t) = Q + itM - \frac{t^2}{2}\Sigma - \frac{it^3}{6}\Xi + \frac{t^4}{24}\Upsilon + o(t^4) \quad (t \to 0)$$

for the Fourier transform, and

$$\lambda(t) = 1 + r_1 t - \frac{\sigma^2}{2}t^2 + \frac{r_3}{6}t^3 + O(t^4) \quad (t \to 0) \tag{2.7}$$

for the largest eigenvalue of $\alpha(t)$. As previously, we have

$$\begin{aligned} &\left(1 - \frac{\sigma^2 s^2}{2n} + \frac{r_3}{6}\frac{s^3}{n^{\frac{3}{2}}} + O\left(\frac{s^4}{n^2}\right)\right)^n \\ = \quad &\exp\left(-\frac{\sigma^2 s^2}{2}\right)\left(1 + \frac{r_3}{6}s^3\frac{1}{\sqrt{n}} + O\left(\frac{s^4 + s^6}{n}\right)\right) \end{aligned}$$

uniformly for $|s| < n^\varepsilon$. A very similar argument to the previous one (with $I_j = o\left(\frac{1}{n}\right), \quad j = 1,2,3,4$) leads to

$$\begin{aligned} &P(\xi_n = (x,k)|\xi_0 = (0,j)) \\ = \quad &\mu_k \frac{1}{\sqrt{2\pi n}\sigma}\exp\left(-\frac{x^2}{2n\sigma^2}\right)\left[1 - \frac{ir_3}{6}x\left(3\sigma^2 n - x^2\right)\frac{1}{\sigma^6}\frac{1}{n^2}\right] + O\left(\frac{1}{n^{3/2}}\right), \end{aligned} \tag{2.8}$$

where the great order on the right hand side is uniform in x.

Now our aim is to formulate an assertion similar to (2.8) in two dimensions. Applying the one dimensional proof to the two dimensional case it is easily seen that

$$P(\xi_n = (x,k)|\xi_0 = (0,j)) - \mu_k \frac{1}{n} g_\sigma\left(\frac{x}{\sqrt{n}}\right) + F(x,n) = O\left(\frac{1}{n^2}\right),$$

where the great order is again uniform in x, and $F(x,n)$ is equal to

$$\frac{1}{2\pi n^{3/2}} \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{s^T\sigma s}{2} - i\left\langle x, \frac{s}{\sqrt{n}}\right\rangle\right) r_{3,i_1,i_2,i_3} s_{i_1} s_{i_2} s_{i_3} ds.$$

We estimate $F(x,n)$ just like it was done in [P09b]. Observe that with the notation

$$\Psi(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{s^T\sigma s}{2} - i\langle x, s\rangle\right) ds = \frac{2\pi}{\sqrt{|\sigma|}} \exp\left(-\frac{x^T\sigma^{-1}x}{2}\right),$$

we have with an appropriate C_1 constant

$$|F(x,n)| < C_1 \frac{1}{n^{3/2}} \max_{i_1,i_2,i_3} \left|\frac{\partial^3\Psi}{\partial x_{i_1}\partial x_{i_2}\partial x_{i_3}}\left(\frac{x}{\sqrt{n}}\right)\right|.$$

Further, observe that

$$\left|\frac{\partial^3\Psi}{\partial x_{i_1}\partial x_{i_2}\partial x_{i_3}}(x)\right| < C_2\left(\|x\| + \|x\|^3\right) \exp\left(-\frac{x^T\sigma^{-1}x}{2}\right).$$

So we have arrived at

Proposition 2.5. *Assume that for a two dimensional RWwIS (2.6) exists. Then there is a C constant, such that for every $x \in \mathbb{R}^2$ and for every $1 \le j,k \le s$ the following estimation holds*

$$\begin{aligned} &\left|P(\xi_n = (x,k)|\xi_0 = (0,j)) - \mu_k \frac{1}{n} g_\sigma\left(\frac{x}{\sqrt{n}}\right)\right| \\ \le\ & C\left(\frac{1}{n^{3/2}}\left(\frac{\|x\|}{n^{1/2}} + \frac{\|x\|^3}{n^{3/2}}\right) \exp\left(-\frac{x^T\sigma^{-1}x}{2n}\right) + \frac{1}{n^2}\right). \end{aligned}$$

By an elementary argument (see, for instance in [IL65] Theorem 4.2.2), using Proposition 2.5 one can easily deduce

Corollary 2.6. *Under the conditions of Proposition 2.5*

$$\sum_{x\in\mathbb{Z}^2} \left|P(\eta_n = x|\eta_0 = 0) - \frac{1}{n} g_\sigma\left(\frac{x}{\sqrt{n}}\right)\right| = O\left(n^{-1/4}\right).$$

2.2.2 Reversed walks

The so-called reversed walk will be important in the sequel. If a RWwIS is given with the appropriate $(p_{y,i,j})$ probabilities, then we define the $(q_{y,i,j})$ reversed random walk for which

$$q_{y,i,j} = \frac{\mu_j p_{-y,j,i}}{\mu_i}. \tag{2.9}$$

Obviously, the stationary measure of the reversed walk is also μ. As we would like to apply the local limit theorem for the reversed walk, we need

Proposition 2.7. *If the primary RWwIS fulfills our basic assumptions, then the reversed walk fulfills them as well. Furthermore, the so-called covariance matrix of the reversed walk is the same as the one of the primary walk.*

Proof. Basic assumptions (i)-(iii) are fulfilled obviously. So it suffices to prove the second statement. Let us introduce some notations

$$\begin{aligned}\widetilde{A}_y &= \left(q_{y,j,k}\right)_{j,k=1,\ldots,s}, \\ \widetilde{Q} &= \sum_{y\in\mathbb{Z}^d}\widetilde{A}_y, \\ \widetilde{M}_l &= \sum_{y\in\mathbb{Z}^d} y_l\widetilde{A}_y, \\ \widetilde{\Sigma}_{l,m} &= \sum_{y\in\mathbb{Z}^d} y_l y_m\widetilde{A}_y,\end{aligned}$$

and a new inner product

$$\begin{aligned}(,) &: \mathbb{R}^s\times\mathbb{R}^s\to\mathbb{R}, \\ (u,v) &= \sum_{i=1}^{s}\mu_i u_i v_i.\end{aligned}$$

Let us denote by A^* the adjoint of the linear operator A, i.e. $(u, Av) = (A^*u, v)$ for all $u, v \in \mathbb{R}^s$. Elementary calculations show that $\widetilde{Q} = Q^*$, $\widetilde{A}_y = (A_{-y})^*$, $\widetilde{M}_l = -(M_l)^*$, $\widetilde{\Sigma}_{l,m} = (\Sigma_{l,m})^*$ for all $y \in \mathbb{Z}^d, 1 \le l, m \le s$. Now, for an arbitrary element $\widetilde{\sigma}_{l,m}$ of the "covariance matrix" defined for the reversed walk

$$\begin{aligned}\widetilde{\sigma}_{l,m} &= \left(1,\widetilde{\Sigma}_{l,m}1\right)-\left(1,\widetilde{M}_l\left(\widetilde{Q}-1\right)^{-1}\widetilde{M}_m1\right)-\left(1,\widetilde{M}_m\left(\widetilde{Q}-1\right)^{-1}\widetilde{M}_l1\right) \\ &= \left(\Sigma_{l,m}1,1\right)-\left(M_m\left(Q-1\right)^{-1}M_l1,1\right)-\left(M_l\left(Q-1\right)^{-1}M_m1,1\right) \\ &= \sigma_{l,m}.\end{aligned}$$

Hence the statement. □

2.3 Visited points in high dimensions

In the high dimensional case, we find that $E_d(n)$ grows fast, i.e. linearly in n, as we could have conjectured it from the transiency of the RWwIS. In Theorem 2.8 we prove this fact and compute remainder terms, too. Our approach is based on the one of [DE51], but there are some main differences. First, we have to consider the reversed random walk which is trivial in the case of [DE51]. After it, the renewal equation is written with matrices and vectors, which is more technical than in the case of [DE51]. Moreover, there will be a technical difficulty, namely we will have to consider the case, when the distribution of ε_0 is arbitrary. This will be treated separately in Proposition 2.9. After it, we will be able to estimate $V_d(n)$. In fact, $o\left(n^2\right)$ is enough for proving weak law of large numbers, and $O\left(n^{2-\delta}\right)$

for strong law of large numbers, but our estimations will be sharper. Nevertheless, these estimations are weaker than the ones of [DE51] because a symmetry argument, used in [DE51], fails here. That is why the computation is longer and it uses Proposition 2.9, too. Let us see the details.

Theorem 2.8. *Let $d \geq 3$. Assuming that ε_0 is distributed according to its unique stationary measure, we have*

$$\begin{aligned} E_3(n) &= n\gamma_3 + O(\sqrt{n}) \\ E_4(n) &= n\gamma_4 + O(\log n) \\ E_d(n) &= n\gamma_d + \beta_d + O(n^{2-d/2}) \quad \textit{for } d \geq 5 \end{aligned}$$

with some constants γ_d, β_d, depending on the RWwIS.

Proof. Fix some dimension $d \geq 3$. For the sake of simplicity, we skip the index d and denote $E_d(n) = \sum_{k=1}^{n} \gamma(k)$. Consider an $\{\xi_k = (\eta_k, \varepsilon_k), 0 \leq k\}$ RWwIS fulfilling our assumptions. Let $\left\{\widetilde{\xi}_k = (\widetilde{\eta}_k, \widetilde{\varepsilon}_k), 0 \leq k\right\}$ be the reversed walk, i.e. for which the transition probabilities are defined by (2.9). Put $\eta_0 = 0$, $\gamma(0) = 1$ and define

$$\gamma(n) = P(\eta_n \notin \{\eta_0, ..., \eta_{n-1}\})$$

which is just the probability that the walk visits a new point at step n. Obviously

$$\begin{aligned} \gamma(n) &= P(\eta_i \neq \eta_n \quad i = 0, ..n-1) \\ &= P(\eta_n - \eta_i \neq 0 \quad i = 0, ..n-1) \\ &= P(\widetilde{\eta}_{n-i} \neq 0 \quad i = 0, ..n-1) \\ &= P(\widetilde{\eta}_j \neq 0 \quad j = 1, ..n). \end{aligned}$$

It is clear that we have to examine the reversed walk.

Define $U_k \in \mathbb{R}^{s \times s}$ with

$$(U_k)_{i,j} = P\left(\widetilde{\xi}_k = (0, j) | \widetilde{\xi}_0 = (0, i)\right)$$

and $R_k \in \mathbb{R}^s$ with

$$(R_k)_j = P(0 \notin \{\widetilde{\eta}_1, ..., \widetilde{\eta}_k\} | \widetilde{\xi}_0 = (0, j)).$$

Obviously, we have:

$$\sum_{k=0}^{n} U_k \cdot R_{n-k} = \underline{1}.$$

We are interested in $\langle R_n, \mu \rangle = \gamma(n)$. From the definition of R_k, for $n_1 > n_2$ we have $R_{n_2} - R_{n_1} \geq \underline{0}$, which means that all the components of the vector are non-negative.

We know from Proposition 2.7 and [KSz83b] Theorem 5.2. that $(U_k)_{i,j} = c_j k^{-\frac{d}{2}} + o_{i,j}(k^{-\frac{d}{2}})$. Here we have $c_j = c\mu_j$, but this fact will not be used. So we have

$$\left(\sum_{k=0}^{n} U_k\right)_{i,j} = \widetilde{c}_{i,j} + O\left(n^{1-\frac{d}{2}}\right).$$

Using the monotonicity of R_k we infer

$$\underline{1} \geq \left(\sum_{k=0}^{n} U_k\right) \cdot R_n.$$

Defining $\widehat{c}_j$ the following way

$$\left(\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{k=0}^{n} U_k\right)\right)_j = \frac{1}{s}\sum_{i=1}^{s}\left(\widetilde{c}_{i,j} + O\left(n^{1-\frac{d}{2}}\right)\right) = \widehat{c}_j + O\left(n^{1-\frac{d}{2}}\right),$$

we have

$$1 \geq \left\langle \left(\widehat{c}_1 + O\left(n^{1-\frac{d}{2}}\right), ..., \widehat{c}_s + O\left(n^{1-\frac{d}{2}}\right)\right), R_n \right\rangle. \tag{2.10}$$

For all j, $(R_n)_j$ has a limit in n, being a decreasing non-negative sequence. So write $(R_n)_j = R^j + a_n^j$, where $a_n^j \searrow 0$. It will be enough to estimate the order of a_n^j, because $\gamma(n) = \sum_{j=1}^{s} \mu_j \left(R^j + a_n^j\right)$.

For the estimation of the other direction let $k < n$. We have:

$$\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{i=0}^{k} U_i\right) \cdot R_{n-k} + \left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{i=k+1}^{n} U_i\right) \cdot \underline{1} \geq 1.$$

Since $(U_k)_{i,j} \geq 0$ for all k, i, j, we have $\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{i=0}^{k} U_i\right) \leq (\widehat{c}_1, ..., \widehat{c}_s)$. On the other hand, $\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{i=k+1}^{n} U_i\right) \cdot \underline{1} = o(1)$, as $k \to \infty$, thus

$$\left\langle (\widehat{c}_1, ..., \widehat{c}_s), R_{n-k} \right\rangle \geq 1 + o(1). \tag{2.11}$$

So if we let $n \to \infty$, $k \to \infty$, $n - k \to \infty$, (2.11) together with (2.10) yields

$$\widehat{c}_1 R^1 + ... + \widehat{c}_s R^s = 1.$$

Substituting to (2.10) we have:

$$\sum_{j=1}^{s} \left[\widehat{c}_j a_n^j + O\left(n^{1-\frac{d}{2}}\right) R^j + O\left(n^{1-\frac{d}{2}}\right) a_n^j\right] \leq 0,$$

whence

$$\sum_{j=1}^{s} \widehat{c}_j a_n^j \leq O\left(n^{1-\frac{d}{2}}\right).$$

Since $\widehat{c}_j > 0$ and $a_n^j \geq 0$, we conclude that $a_n^j = O\left(n^{1-\frac{d}{2}}\right)$ for $1 \leq j \leq s$. This yields $\gamma(n) = \sum_{j=1}^{s} \mu_j \left(R^j + a_n^j\right) = \gamma + O\left(n^{1-\frac{d}{2}}\right)$. Hence the statement (just like in [DE51]). □

Proposition 2.9. *The assertion of Theorem 2.8 remains true when the distribution of ε_0 is arbitrary.*

Proof. With the notation $\gamma(n) = \gamma + h(n)$ we already know that $h(n) = O\left(n^{1-\frac{d}{2}}\right)$. Define $\gamma^{e_j}(n) = P\left(\eta_n \notin \{\eta_0, ..., \eta_{n-1}\} | \varepsilon_0 = j\right)$ and $\gamma^{e_j}(n) = \gamma + h^j(n)$ for $j = 1, ..., s$. As in the previous proof, it would be sufficient to prove $h^j(n) = O\left(n^{1-\frac{d}{2}}\right)$ for all j.

For the present, let K be a fixed, great natural number, and

$$\mu_k + b_k^j(K) = P\left(\varepsilon_K = k | \varepsilon_0 = j\right) \quad j, k = 1, ...s.$$

We know from the ergodic theorem of Markov chains that $b_k^j(K)$ tends to zero exponentially fast in K.

Denote by $p(K, n)$ the probability of visiting such a site at time n that was visited during the first K steps, but was not visited in the following $(n - K - 1)$ steps, provided that $\varepsilon_0 = j$. We know from [KSz83b] Theorem 5.2. that $p(K, n) = O\left(K \cdot (n - K)^{-\frac{d}{2}}\right)$, whence

$$\gamma^{e_j}(n) = \sum_{k=1}^{s} \left[\left(\mu_k + b_k^j(K)\right) \gamma^{e_k}(n - K)\right] + O\left(K \cdot (n - K)^{-\frac{d}{2}}\right). \tag{2.12}$$

Recall $\gamma^{e_j}(n) = \gamma + h^j(n)$ to infer that $h^j(n)$ is equal to

$$\sum_{k=1}^{s} \mu_k h^k(n - K) + \sum_{k=1}^{s} b_k^j(K) h^k(n - K) + O\left(K \cdot (n - K)^{-\frac{d}{2}}\right)$$
$$=: I + II + III. \tag{2.13}$$

Now, put $K = K(n) = \lfloor n^\alpha \rfloor$ with arbitrary $0 < \alpha < 1$. It is clear that I is equal to $h(n - K)$, so the proof of Theorem 2.8 yields $I = O\left((n - n^\alpha)^{1-\frac{d}{2}}\right) \leq O\left(n^{1-\frac{d}{2}}\right)$. Since $b_k^j(K)$ tends to zero exponentially fast in K we have $II \leq O\left(n^{1-\frac{d}{2}}\right)$. Finally, $III = O\left(n^\alpha (n - n^\alpha)^{-\frac{d}{2}}\right) \leq O\left(n^{1-\frac{d}{2}}\right)$. Hence the statement. □

Now, let us see the estimation of $V_d(n)$.

Theorem 2.10. *For $d \geq 3$ assuming that $\varepsilon_0 \sim \mu$ we have*

$$V_d(n) = O\left(n^{1+\frac{2}{d}}\right).$$

Proof. Let $\gamma(n, m)$ denote the probability that the RWwIS visits new points in both the n^{th} and the m^{th} step under the condition that $\varepsilon_0 \sim \mu$, and let $A = \{\eta_i \neq \eta_m, \quad i = 0, ..., m - 1\}$. Obviously, $\gamma_d(n, m) = \gamma_d(m, n)$, so, when estimating $\gamma(n, m)$ one can assume $n > m$.

$$\begin{aligned}
\gamma(m, n) &= P\left(A \,\&\, \eta_j \neq \eta_n, \quad j = 0, ..., n - 1\right) \\
&\leq P\left(A \,\&\, \eta_j \neq \eta_n, \quad j = m, ..., n - 1\right) \\
&= \gamma(n) P\left(\eta_j \neq \eta_n, \quad j = m, ..., n - 1 \quad | \quad A\right).
\end{aligned}$$

Here, $P\left(\eta_j \neq \eta_n, \quad i = m, ..., n - 1 \quad | \quad A\right)$ is the probability that the RWwIS visits a new point in the $(n - m)^{th}$ step, assuming that the distribution of ε_0 is some $\mu(n)$. So the condition A is involved in $\mu(n)$, and because of the Markov property, it has no other contribution. The probability of this event is denoted by $\gamma_d^{\mu(n)}(n-m)$. Because of Proposition 2.9 we know that $\gamma_d^{\mu(n)}(n-m) \to \gamma_d$, as $(n-m) \to \infty$, and it is easy to see that this convergence is uniform in $\mu(n)$. So we know that for $\forall \delta > 0 \ \exists\ N = N(\delta)$, such that for $\forall n-m > N$ the following estimation holds.

$$\gamma_d^{\mu(n)}(n-m) = \sum_{j=1}^{s} \mu(n)_j \gamma_d^{e_j}(n-m) < (1+\delta)\gamma_d(n-m).$$

In addition, using Proposition 2.9, one can estimate $N(\delta)$, which will be done a little bit later. Now, let us see the estimation of $V_d(n)$

$$\begin{aligned} V_d(n) &= \sum_{i,j=0}^{n} \gamma_d(i,j) - \sum_{i=0}^{n} \gamma_d(i) \sum_{j=0}^{n} \gamma_d(j) \\ &\leq 2 \sum_{0 \leq i \leq j \leq n} \left(\gamma_d(i,j) - \gamma_d(i)\gamma_d(j)\right) \\ &\leq 2 \sum_{0 \leq i < i+K \leq j \leq n} \left(\gamma_d(i,j) - \gamma_d(i)\gamma_d(j)\right) + 2 \sum_{\substack{0 \leq i \leq n \\ i \leq j < i+K}} \gamma_d(i,j) \\ &=: S_1 + S_2. \end{aligned}$$

Let K be big enough, such that for $n-m > K$ one would have $\gamma_d^{\nu}(n-m) < (1+\delta)\gamma_d(n-m)$ for arbitrary ν. Estimating S_1 and S_2 separately, we get

$$\begin{aligned} \frac{S_1}{2} &= \sum_{i=0}^{n-K} \sum_{j=i+K}^{n} \gamma_d(i,j) - \sum_{i=0}^{n-K} \sum_{j=i}^{n} \gamma_d(i)\gamma_d(j) + \sum_{i=0}^{n-K} \sum_{j=i}^{i+K} \gamma_d(i)\gamma_d(j) \\ &\leq \sum_{i=0}^{n-K} \gamma_d(i) \max_{0 \leq i \leq n-K} \left(\sum_{j=i}^{n} (1+\delta)\gamma_d(j-i) - \sum_{j=i}^{n} \gamma_d(j) \right) \\ &\quad + \sum_{i=0}^{n-K} \gamma_d(i) \sum_{j=i}^{i+K} \gamma_d(j), \end{aligned}$$

which can be bounded by

$$\begin{aligned} &\leq \sum_{i=0}^{n-K} \gamma_d(i) \left[\delta E_d(n) + E_d(n - \left\lfloor \frac{n}{2} \right\rfloor) - E_d(n) + E_d(\left\lfloor \frac{n}{2} \right\rfloor) \right] \\ &\quad + \sum_{i=0}^{n-K} \gamma_d(i) K. \end{aligned}$$

On the other hand,

$$S_2 \leq 2 \sum_{\substack{0 \leq i \leq n \\ i \leq j < i+K}} \gamma(i) \leq 2K E_d(n).$$

From the proof of Proposition 2.9, one can easily deduce that for k large enough

$$\gamma_d^{\nu}(k) < \left(1 + O(k^{1-\frac{d}{2}})\right) \gamma_d(k),$$

uniformly in ν. So replacing K to $K(n)$ in the above argument, one can change δ to $O\left(K(n)^{1-\frac{d}{2}}\right)$, thus

$$\begin{aligned} V_3(n) &\le O(n)\left[O\left(K(n)^{1-\frac{d}{2}}\right)O(n)+O\left(\sqrt{n}\right)\right]+K(n)\,O(n) \\ V_4(n) &\le O(n)\left[O\left(K(n)^{1-\frac{d}{2}}\right)O(n)+O(\log n)\right]+K(n)\,O(n) \\ V_d(n) &\le O(n)\left[O\left(K(n)^{1-\frac{d}{2}}\right)O(n)+O(1)\right]+K(n)\,O(n) \quad d\ge 5. \end{aligned}$$

Now, the choice $K(n)=\left\lfloor n^{\frac{2}{d}}\right\rfloor$ completes the proof. □

Proposition 2.11. *The assertion of Theorem 2.10 remains true when the distribution of ε_0 is some arbitrary ν. Moreover, the great order is uniform in ν.*

Proof. Let us introduce the notation $E^{\nu}[.]$ for the expectation when $\varepsilon_0\sim\nu$. For convenience, we also write $E_d^{\nu}(n)$ and $V_d^{\nu}(n)$ for the expectation and variance of $L_d(n)$ when $\varepsilon_0\sim\nu$. Obviously,

$$V_d^{\nu}(n)=E^{\nu}\left[(L_d(n))^2\right]-\left(E_d^{\nu}(n)\right)^2. \tag{2.14}$$

On the other hand,

$$\sum_{j=1}^{s}\nu_j V_d^{e_j}(n)=\sum_{j=1}^{s}\nu_j E^{e_j}\left[(L_d(n))^2\right]-\sum_{j=1}^{s}\nu_j\left(E_d^{e_j}(n)\right)^2. \tag{2.15}$$

Since $E^{\nu}\left[(L_d(n))^2\right]=\sum\limits_{j=1}^{s}\nu_j E^{e_j}\left[(L_d(n))^2\right]$, subtracting (2.15) from (2.14), we conclude

$$V_3^{\nu}(n)-\sum_{j=1}^{s}\nu_j V_3^{e_j}(n) = O\left(n^{3/2}\right), \tag{2.16}$$

$$V_d^{\nu}(n)-\sum_{j=1}^{s}\nu_j V_d^{e_j}(n) = O(n\log n) \quad d\ge 4. \tag{2.17}$$

It is clear that the great order on the right hand side is uniform in ν. In the sense of (2.16) and (2.17) it is enough to prove the statement for $\nu=e_j,(j=1,...,s)$. To do so, substitute $\mu=\nu$ to (2.16) and (2.17) and use Theorem 2.10 to infer

$$\sum_{j=1}^{s}\mu_j V_3^{e_j}(n)=O\left(n^{1+\frac{d}{2}}\right),\quad d\ge 3.$$

Since for all d,j and n μ_j and $V_d^{e_j}(n)$ are non negative, we have proved the statement for all e_j. □

Corollary 2.12. *For RWwIS in $d\ge 3$ the weak law of large numbers holds, namely*

$$P\left(|L_d(n)-E_d(n)|>\varepsilon E_d(n)\right)\to 0$$

for $\forall\varepsilon>0$.

Proof. Since $V_d(n)=o\left(n^2\right)$, Chebyshev's inequality applies (just like in [DE51]). □

From Theorem 2.10 one can deduce even strong law of large numbers:

Theorem 2.13. *For RWwIS in $d \geq 3$ strong law of large numbers holds, namely*

$$P\left(\lim_{n\to\infty}\frac{L_d(n)}{E_d(n)} = 1\right) = 1.$$

Theorem 2.13 can be proved almost the same way as it was done in [DE51]. The difference is that if we have $V_d(n) = O(n^\tau)$ with some $\tau < 2$, then we have to choose parameters α and β to fulfill

$$\begin{aligned} \frac{1+\tau}{3} &< \alpha < 1 \\ \frac{1}{2\alpha-\tau} &< \beta < \frac{1}{1-\alpha}. \end{aligned}$$

After it, the argument of [DE51] works. So the main point is that we should have some $\tau < 2$ such that $V_d(n) = O(n^\tau)$ as it was mentioned at the beginning of the Section.

Identifying the constant γ is an interesting question, though we cannot give a closed formula in the general case.

We only know that for the constant γ we have

$$\gamma = P(\eta_k \neq 0 : k \geq 1 | \varepsilon_0 \sim \mu). \tag{2.18}$$

To see this, first, observe that the constant γ is the same for the primary and the reversed walk. We have seen that

$$\gamma(n) = P(\widetilde{\eta}_j \neq 0 \quad j = 1, ..n).$$

Taking $n \to \infty$, (2.18) follows.

2.4 Visited points in two dimensions

In this section we calculate $E_2(n)$ and estimate $V_2(n)$. The arguments (assuming that $\varepsilon_0 \sim \mu$) are similar to the ones of Theorem 2.8 and 2.10, or [DE51] Theorem 1 and Theorem 2. The computations are longer than in [DE51]. We have to write the renewal equation in terms of vectors and matrices, which is a new idea, and we use the above proved Proposition 2.4 because it is essential that the remainder term of the probability of returning to the origin should be summable, which was trivial in the case of [DE51]. We have to consider the case of arbitrary initial distribution, separately, just like in Section 2.3. In this case, we formulate the fact that after some steps the distribution of ε will be very close to μ.

Theorem 2.14. *Let $d = 2$. Assuming that $\varepsilon_0 \sim \mu$ and that (2.1) exists, we have*

$$E_2(n) = \frac{2\pi\sqrt{|\sigma|}n}{\log n} + O\left(\frac{n\log\log n}{\log^2 n}\right).$$

Proof. As in the proof of Theorem 2.8, we examine the reversed RWwIS and write the renewal equation

$$\sum_{k=0}^{n} U_k \cdot R_{n-k} = \underline{1}. \tag{2.19}$$

Proposition 2.4 yields

$$(U_k)_{i,j} = \frac{1}{2\pi\sqrt{|\sigma|}} \mu_j \frac{1}{k} + O\left(k^{-3/2}\right),$$

thus

$$\left(\sum_{k=0}^{n} U_k\right)_{i,j} = \frac{1}{2\pi\sqrt{|\sigma|}} \mu_j \log\left(c_{i,j} n\right) + O\left(n^{-1/2}\right). \tag{2.20}$$

Our purpose is to estimate $\langle R_n, \mu \rangle = \gamma(n)$. Exactly as in the high dimensional case, R_n is decreasing, so (2.19) yields

$$\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{l=0}^{k} U_l\right) \cdot R_{n-k} + \left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{l=k+1}^{n} U_l\right) \cdot \underline{1} \geq 1. \tag{2.21}$$

Let $k \to \infty$, $n \to \infty$. The relation between k and n will be fixed later. From (2.20) it follows that

$$\left[\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{l=0}^{k} U_l\right)\right]_j = \frac{1}{2\pi\sqrt{|\sigma|}} \mu_j \log\left(\widehat{c}_j k\right) + O\left(k^{-1/2}\right) \tag{2.22}$$

for some $\widehat{c}_j$. So we have for $k < n$

$$\left[\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{l=k+1}^{n} U_l\right)\right]_j = \frac{1}{2\pi\sqrt{|\sigma|}} \mu_j \log\frac{n}{k} + O\left(k^{-1/2}\right). \tag{2.23}$$

Substituting (2.22) and (2.23) to the left hand side of (2.21) we get

$$\begin{aligned} &\sum_{j=1}^{s} \left[\frac{1}{2\pi\sqrt{|\sigma|}} \mu_j \log\left(\widehat{c}_j k\right) + O\left(k^{-1/2}\right)\right] \left(R_{n-k}\right)_j \\ + \quad &\sum_{j=1}^{s} \frac{1}{2\pi\sqrt{|\sigma|}} \mu_j \log\frac{n}{k} + O\left(k^{-1/2}\right). \end{aligned} \tag{2.24}$$

Put $k = \left\lfloor n - \frac{n}{\log n} \right\rfloor$. This yields $\log k \sim \log(n-k)$. Using the fact $\gamma(n-k) = \sum_{j=1}^{s} \mu_j \left(R_{n-k}\right)_j$, (2.24) can be written as

$$\begin{aligned} &\gamma\left(n-k\right)\left[\frac{1}{2\pi\sqrt{|\sigma|}} \log k\right] + \\ &\sum_{j=1}^{s} \left[\frac{1}{2\pi\sqrt{|\sigma|}} \mu_j \log \widehat{c}_j + O\left(k^{-1/2}\right)\right] \left(R_{n-k}\right)_j + C \log\frac{n}{k} + O\left(k^{-1/2}\right). \end{aligned} \tag{2.25}$$

Since $\log \frac{n}{k} \to 0$, and $\left(R_{n-k}\right)_j \to 0$, as $n-k \to \infty$ (the latter is the recurrence property of the two dimensional RWwIS, which is proved in [T83]), it follows that

$$\gamma\left(n-k\right) \geq \frac{2\pi\sqrt{|\sigma|}}{\log k} + o\left(\frac{1}{\log k}\right). \tag{2.26}$$

Hence, by the choice of k,

$$\gamma\left(n-k\right) \geq \frac{2\pi\sqrt{|\sigma|}}{\log\left(n-k\right)} + o\left(\frac{1}{\log\left(n-k\right)}\right). \tag{2.27}$$

Now let us give an upper estimation to $\gamma(n)$. From (2.19) it follows that

$$\left(\sum_{k=0}^{n} U_k\right) \cdot R_n \leq \underline{1}.$$

Multiplying by the vector $\frac{1}{s}\underline{1}$, we get

$$\sum_{j=1}^{s}\left[\frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \log\left(\widehat{c}_j n\right) + O\left(n^{-1/2}\right)\right]\left(R_n\right)_j \leq 1,$$

thus

$$\begin{aligned} S_1 \quad + \quad S_2 + S_3 &:= \frac{1}{2\pi\sqrt{|\sigma|}}\sum_{j=1}^{s}\mu_j \left(R_n\right)_j \log n \\ &+ \quad \frac{1}{2\pi\sqrt{|\sigma|}}\sum_{j=1}^{s}\mu_j \left(R_n\right)_j \log \widehat{c}_j + \sum_{j=1}^{s} O\left(n^{-1/2}\right)\left(R_n\right)_j \leq 1. \end{aligned}$$

Since $\left(R_n\right)_j \to 0$, it follows that $S_2 + S_3 = o(1)$. So we have the upper estimation

$$\gamma\left(n\right) \leq \frac{2\pi\sqrt{|\sigma|}}{\log n} + o\left(\frac{1}{\log n}\right). \tag{2.28}$$

From (2.27) and (2.28) we get

$$\gamma\left(n\right) = \frac{2\pi\sqrt{|\sigma|}}{\log n} + o\left(\frac{1}{\log n}\right). \tag{2.29}$$

Unfortunately, the estimation (2.29) is not good enough for our purposes (but observe that we have not really used (2.4) yet). Now, (2.28) yields $\left(R_n\right)_j = O\left(\frac{1}{\log n}\right)$ for all $1 \leq j \leq s$. Hence, with the previous notation, $S_2 = O\left(\frac{1}{\log n}\right)$. Obviously $S_3 = O\left(\frac{1}{\log n}\right)$. Thus we arrived at

$$\gamma(n) \leq \frac{2\pi\sqrt{|\sigma|}}{\log n} + O\left(\frac{1}{\log^2 n}\right). \tag{2.30}$$

This estimation will be sharp enough.

Now, we have to improve our lower estimation. From (2.29) and (2.25) it follows that

$$\gamma\left(n-k\right)\left[\frac{1}{2\pi\sqrt{|\sigma|}}\log k + O\left(1\right)\right] + C\log\frac{n}{k} + O\left(k^{-\frac{1}{2}}\right) \geq 1,$$

thus

$$\gamma\left(n-k\right)\log\left(n-k\right) \geq \left(2\pi\sqrt{|\sigma|} - C2\pi\sqrt{|\sigma|}\log\frac{n}{k} + O\left(k^{-\frac{1}{2}}\right)\right)\frac{\log\left(n-k\right)}{\log k + O(1)}.$$

Now, similarly to the case of [DE51], it follows that

$$\gamma\left(n\right) = \frac{2\pi\sqrt{|\sigma|}}{\log n} + O\left(\frac{\log\log n}{\log^2 n}\right). \tag{2.31}$$

Now, an elementary calculation completes the proof. □

As in the high dimensional case, the initial distribution does not influence the asymptotic behavior. More precisely

Proposition 2.15. *The assertion of Theorem 2.14 remains true when the distribution of ε_0 is arbitrary.*

Proof. The proof is very similar to the one of Proposition 2.9. We know that

$$\gamma(n) = \frac{2\pi\sqrt{|\sigma|}}{\log n} + O\left(\frac{\log\log n}{\log^2 n}\right).$$

With the notation $\gamma^{e_j}(n) = \frac{2\pi\sqrt{|\sigma|}}{\log n} + h^j(n)$ our aim is to prove $h^j(n) = O\left(\frac{\log\log n}{\log^2 n}\right)$. The analogue of (2.12) is

$$\begin{aligned} &\frac{2\pi\sqrt{|\sigma|}}{\log n} + h^j(n) \\ = &\sum_{k=1}^{s}\left[\left(\mu_k + b_k^j(K)\right)\left(\frac{2\pi\sqrt{|\sigma|}}{\log(n-K)} + h^k(n-K)\right)\right] + O\left(K\cdot(n-K)^{-1}\right), \end{aligned}$$

and the analogue of (2.13) is

$$\begin{aligned} h^j(n) &= \sum_{k=1}^{s}\mu_k h^k(n-K) + \sum_{k=1}^{s} b_k^j(K)h^k(n-K) + O\left(K\cdot(n-K)^{-1}\right) \\ &\quad + \left(\frac{2\pi\sqrt{|\sigma|}}{\log(n-K)} - \frac{2\pi\sqrt{|\sigma|}}{\log n}\right) \\ &= : I + II + III + IV. \end{aligned}$$

With the choice $K(n) = \lfloor\sqrt{n}\rfloor$ elementary calculations show that $I + II + III + IV \le O\left(\frac{\log\log n}{\log^2 n}\right)$. □

Now let us see the estimation of the variance.

Theorem 2.16. *If (2.1) exists, then we have with arbitrary ν distribution of ε_0*

$$V_2(n) = O\left(\frac{n^2\log\log n}{\log^3 n}\right).$$

Moreover, the great order is uniform in ν.

Proof. First, suppose $\varepsilon_0 \sim \mu$. The beginning of the proof of this case is the same as in Theorem 2.10. The difference is that when we change K to $K(n)$, we can write $O\left(\frac{\log\log K(n)}{\log K(n)}\right)$ instead of δ in the sense of Proposition 2.15. From now, just like in the proof of Theorem 2.10, it is not difficult to deduce that

$$O\left(\frac{n}{\log n}\right)\left[\frac{\log\log K(n)}{\log K(n)}O\left(\frac{n}{\log n}\right) + O\left(\frac{n\log\log n}{\log^2 n}\right)\right] + K(n)\,O\left(\frac{n}{\log n}\right)$$

is an upper bound for $V_2(n)$. Taking $K(n) = \left\lfloor\frac{n}{\log^2 n}\right\rfloor$ proves the statement. For the case of arbitrary initial distribution, one can repeat the proof of Proposition 2.11. □

Corollary 2.17. *For a RWwIS in $d = 2$ dimension weak law of large numbers holds.*

Proof. Since $O\left(\frac{n^2\log\log n}{\log^3 n}\right) < O\left(\frac{n^2}{\log^2 n}\right)$, Chebyshev's inequality applies. □

The proof of the strong law of large numbers is quite complicated, so we treat it in a different Section.

2.5 Law of large numbers in the plane

This Section is dedicated to the strong law in $d = 2$.

Theorem 2.18. *For any RWwIS in $d = 2$, for which (2.6) exists, strong law of large numbers holds, namely*

$$P\left(\lim_{n\to\infty}\frac{L_2(n)}{E_2(n)} = 1\right) = 1.$$

Almost the whole proof in [DE51] can be easily generalized to our case with the observation that since our estimations for $E_2(n)$ and $V_2(n)$ are uniform in the initial distribution, the computations, used in [DE51], can be repeated. That is why we write here the only non-trivial part (i.e. formulae corresponding to (5.13) and (5.15) in [DE51]) of the generalization. In fact, there is apparently a gap in the argument in [DE51], as it was already remarked in [JP70]. What we represent here is a simplified version of a proof in [P09a]. For the other parts of the proof the reader is referred to [DE51].

Proof. Denote

$$K = \lfloor \log\log n \rfloor ,$$

and let M_{ij} $(1 \le i, j \le K)$ be the number of lattice points which are common in path parts M_i and M_j, where M_i denotes the set of points which are visited between $\lfloor (i-1)n/K\rfloor + 1$ and $\lfloor in/K\rfloor$ $(1 \le i \le K)$. First, we would like to prove the formula corresponding to [DE51] (5.13):

$$\sup_{i<j} E\left(M_{ij}\right) = O\left(\frac{n\log\log n}{K\log^2 n}\right). \tag{2.32}$$

If it is done, then for every ϑ with $0 < \vartheta < 1$ we will have

$$\sup_{i<j} P\left(M_{ij} > \frac{n\log\log n}{K\log^{1+\vartheta} n}\right) = O\left(\frac{1}{\log^{1-\vartheta} n}\right). \tag{2.33}$$

Let C_{ij} denote the event whose probability is estimated in (2.33). As (2.32) yields

$$\sup_{j} E\left(M_{1j}\right) = O\left(\frac{n\log\log n}{K\log^2 n}\right)$$

for arbitrary ν initial distribution of internal states, and under the condition C_{ij} the probability of $C_{i'j'}$ with $1 \le i < j < i' < j' \le K$ is only affected via the distribution of $\varepsilon_{i'}$, we conclude

$$\sup_{1\le i<j<i'<j'\le K} P(C_{i,j} \cap C_{i',j'}) = O\left(\frac{1}{\log^{2-2\vartheta} n}\right). \tag{2.34}$$

If we were able to prove

$$\sup_{i,j,i',j'} E\left(M_{ij}M_{i'j'}\right) = O\left(\frac{n^2\log^2\log n}{K^2\log^4 n}\right), \tag{2.35}$$

where the supremum is taken over indices for which $\#\{i,j,i',j'\} = 4$ and either $1 \le i < i' < j' < j \le K$ or $1 \le i < i' < j < j' \le K$ holds, then using

$$P\left(M_{ij} > \frac{n\log\log n}{K\log^{1+\vartheta} n}, M_{i'j'} > \frac{n\log\log n}{K\log^{1+\vartheta} n}\right) < P\left(M_{ij}M_{i'j'} > \frac{n^2\log^2\log n}{K^2\log^{2+2\vartheta} n}\right)$$

and (2.34) we could infer that the probability that two events C_{ij} and $C_{i'j'}$ with $\#\{i,j,i',j'\}=4$ occur is

$$O\left(\frac{K^4}{\log^{2-2\vartheta} n}\right), \tag{2.36}$$

which is the formula corresponding to [DE51] (5.15).

So our aim is to prove (2.32) and (2.35). The idea of [P09a] is that in order to prove (2.32) and (2.35) it is useful to cut down the points of M_i which are visited in the extreme $\lceil n/\log^2 n\rceil$ steps. The number of these points can be roughly estimated, while the others are visited in steps quite far from each other and this will be enough for us. However, the precise arguments need some awkward computations.

Proof of (2.32) We introduce the notations

$$\alpha_{a,b} = \{\forall t = a, ..., b-1 : \eta_t \neq \eta_b\} \text{ and } \beta_{a,b} = \{\forall t = a+1, ..., b : \eta_a \neq \eta_t\}$$

which will be useful in the sequel. Following [P09a], we define

$$n_{(i,-)} = \lfloor (i-1)n/K\rfloor + \lceil n/\log^2 n\rceil \text{ and } n_{(i,+)} = \lfloor in/K\rfloor - \lceil n/\log^2 n\rceil .$$

A point, which is common in the paths $\eta_{n_{(i,-)}}, ..., \eta_{n_{(i,+)}}$ and $\eta_{n_{(j,-)}}, ..., \eta_{n_{(j,+)}}$ and not visited in the extreme $\lceil n/\log^2 n\rceil$ steps of M_i and M_j, has a pair of indices (k,l), $k \in n_{(i,-)}, ..., n_{(i,+)}$, $l \in n_{(j,-)}, ..., n_{(j,+)}$, such that it is visited at steps k and l, and it is not visited during steps $\lfloor (i-1)n/K\rfloor + 1, ..., k-1$, and steps $l+1, ..., \lfloor jn/K\rfloor$. So we have

$$\begin{aligned} E(M_{ij}) &\leq \frac{3n}{\log^2 n} + \sum_{k=n_{(i,-)}}^{n_{(i,+)}} \sum_{l=n_{(j,-)}}^{n_{(j,+)}} P\left(\alpha_{\lfloor (i-1)n/K\rfloor+1,k} \cap \{\eta_k = \eta_l\} \cap \beta_{l,\lfloor jn/K\rfloor}\right) \\ &\leq \frac{3n}{\log^2 n} \\ &\quad + C_1 \sum_{k=n_{(i,-)}}^{n_{(i,+)}} \sum_{l=n_{(j,-)}}^{n_{(j,+)}} \frac{1}{\log\left(k - \lfloor (i-1)n/K\rfloor\right)} \frac{1}{l-k} \frac{1}{\log\left(\lfloor jn/K\rfloor - l\right)} \\ &\leq \frac{3n}{\log^2 n} + C_2 \frac{n}{K} \frac{\log n - \log\left(n/\log^2 n\right)}{\log^2 n} = O\left(\frac{n \log\log n}{K \log^2 n}\right). \end{aligned}$$

Note that we have used our estimations for the probability of avoiding the origin in some steps, visiting a new point, and returning to the origin, and these estimations are uniform in the initial distribution (with an appropriate C_1). Because the events whose intersection's probability is estimated above are dependent only via the internal states, it is obvious that the great order is uniform in i and j. So we arrived at (2.32).

Proof of (2.35) Let us prove

$$\sup_{1\leq i<i'<j'<j\leq K} E\left(M_{ij} M_{i'j'}\right) = O\left(\frac{n^2 \left(\log\log n\right)^2}{K^2 \log^4 n}\right).$$

Let us introduce the notation $\mathcal{L}$ for the set of (k, k', l', l) such that

$$\begin{aligned} &n_{(i,-)} \leq k \leq n_{(i,+)},\ n_{(i',-)} \leq k' \leq n_{(i',+)}, \\ &n_{(j',-)} \leq l' \leq n_{(j',+)},\ n_{(j,-)} \leq l \leq n_{(j,+)}. \end{aligned}$$

As it was mentioned before, we estimate the number of pair of points one of which is visited in either extreme $\lceil n/\log^2 n\rceil$ steps of M_i, M_j, $M_{i'}$ or $M_{j'}$ in a very obvious manner. The other pairs of lattice points (x and y, say) have a (k,k',l',l) element of $\mathcal{L}$, such that x is visited at step k but not visited during $\lfloor (i-1)n/K\rfloor+1,...,k-1$, and it is visited again at step l but not visited during $l+1,...,\lfloor jn/K\rfloor$; while y is visited at step k' but not visited during $k'+1,...,\lfloor i'n/K\rfloor$, and it is visited again at step l' but not visited during $\lfloor (j'-1)n/K\rfloor+1,...,l'-1$. So we have

$$E\left(M_{ij}M_{i'j'}\right)=O\left(\frac{n^2\log\log n}{K\log^4 n}\right)+\sum_{M\in\mathbb{Z}^2}\sum_{(k,k',l',l)\in\mathcal{L}}P\left(\mathcal{A}\right), \tag{2.37}$$

where

$$\begin{aligned}\mathcal{A} \;&=\; \alpha_{\lfloor (i-1)n/K\rfloor+1,k}\cap\{\eta_{k'}-\eta_k=M\}\cap\beta_{k',\lfloor i'n/K\rfloor}\cap\{\eta_{l'}-\eta_{k'}=(0,0)\}\\ &\cap\alpha_{\lfloor (j'-1)n/K\rfloor+1,l'}\cap\{\eta_l-\eta_{l'}=-M\}\cap\beta_{l,\lfloor jn/K\rfloor},\end{aligned}$$

Denote the seven events, whose intersection is $\mathcal{A}$, by $\mathcal{A}_1,...,\mathcal{A}_7$. Observe that for every $2\le m\le 7$ the probability of $\mathcal{A}_m$ under the condition $\mathcal{A}_1\cap...\cap\mathcal{A}_{m-1}$ is just the probability of $\mathcal{A}_m$ with an appropriate initial distribution of ε. As we have uniform estimations in the initial distribution, we will be able to use them.

In the first step, let us estimate the part of the sum in (2.37) corresponding to $M\in[-n,n]^2$. Proposition 2.5 yields the existence of $a>0$ (which depends only on the RWwIS), such that

$$\begin{aligned}P\left(\eta_k-\eta_0=M\right) \;&<\; C_3\exp\left(-\frac{a}{2k}M^TM\right)\left[\frac{1}{k}+\frac{1}{k^{3/2}}\right]+\frac{C_3}{k^2}\\ &<\; C_4\left(\frac{1}{k}\exp\left(-\frac{a}{2k}M^TM\right)+\frac{1}{k^2}\right).\end{aligned}$$

So the formula

$$C_5\frac{1}{\log n}\left(\frac{1}{k'-k}\exp\left(-\frac{a}{2(k'-k)}M^TM\right)+\frac{1}{(k'-k)^2}\right), \tag{2.38}$$

is an upper bound for $P(\mathcal{A}_1\cap\mathcal{A}_2)$, and the formula

$$C_5\frac{1}{\log n}\left(\frac{1}{l-l'}\exp\left(-\frac{a}{2(l-l')}M^TM\right)+\frac{1}{(l-l')^2}\right). \tag{2.39}$$

is an upper bound for $P(\mathcal{A}_6\cap\mathcal{A}_7|\mathcal{A}_1\cap...\cap\mathcal{A}_5)$.
Consider the following factorization

$$P(\mathcal{A})=P(\mathcal{A}_1\cap\mathcal{A}_2)P(\mathcal{A}_3\cap\mathcal{A}_4\cap\mathcal{A}_5|\mathcal{A}_1\cap\mathcal{A}_2)P(\mathcal{A}_6\cap\mathcal{A}_7|\mathcal{A}_1\cap...\cap\mathcal{A}_5), \tag{2.40}$$

and observe that

$$\sum_{k',l'}P(\mathcal{A}_3\cap\mathcal{A}_4\cap\mathcal{A}_5|\mathcal{A}_1\cap\mathcal{A}_2)<C_6E(M_{i'j'})=O\left(\frac{n\log\log n}{K\log^2 n}\right). \tag{2.41}$$

So we have to take the product of the expressions in (2.38), (2.39) and $P(\mathcal{A}_3\cap\mathcal{A}_4\cap\mathcal{A}_5|\mathcal{A}_1\cap\mathcal{A}_2)$ and sum them up in all of the four indices to estimate (2.37). First, let us consider the product of the first terms in (2.38) and (2.39). We have to estimate

$$\sum\sum\exp\left(-\frac{a}{2}\left(\frac{1}{k'-k}+\frac{1}{l-l'}\right)M^TM\right)\frac{P(\mathcal{A}_3\cap\mathcal{A}_4\cap\mathcal{A}_5|\mathcal{A}_1\cap\mathcal{A}_2)}{(k'-k)(l-l')\log^2 n},$$

where the to sums are taken over $M \in [-n,n]^2$ and $(k,k',l',l) \in \mathcal{L}$, respectively. Using the fact

$$\sup_{d>a} \frac{1}{d} \sum_{M \in \mathbb{Z}^2} \exp\left(-\frac{a}{2d} M^T M\right) < +\infty \tag{2.42}$$

it suffices to estimate

$$\begin{aligned} &\sum_{(k,k',l',l)\in\mathcal{L}} \frac{(k'-k)(l-l')}{(l-l')+(k'-k)} \frac{P(\mathcal{A}_3 \cap \mathcal{A}_4 \cap \mathcal{A}_5 | \mathcal{A}_1 \cap \mathcal{A}_2)}{(k'-k)(l-l')\log^2 n} \\ \leq\ & \frac{1}{\log^2 n} \sum_{(k,k',l',l)\in\mathcal{L}} \frac{P(\mathcal{A}_3 \cap \mathcal{A}_4 \cap \mathcal{A}_5 | \mathcal{A}_1 \cap \mathcal{A}_2)}{(l-n_{(j',+)})+(n_{(i',-)}-k)}. \end{aligned}$$

Using (2.41), it remains to estimate

$$\frac{1}{\log^2 n} E\left(M_{ij}\right) \sum_{k,l} \frac{1}{(l-n_{(j,-)})+(n_{(i,+)}-k)+2n/\log^2 n - 1}$$

and it is just

$$O\left(\frac{n^2 \left(\log\log n\right)^2}{K^2 \log^4 n}\right)$$

uniformly in i and j, by an elementary computation.

Now, let us consider the product of the first term in (2.38) and the second term in (2.39) (the product of the second term in (2.38) and the first term in (2.39) can be estimated equivalently). In this case the easier estimation

$$P(\mathcal{A}_3 \cap \mathcal{A}_4 \cap \mathcal{A}_5 | \mathcal{A}_1 \cap \mathcal{A}_2) < C_8 \frac{1}{l'-k'} \tag{2.43}$$

will be enough. Thus our aim is to estimate

$$\frac{1}{\log^2 n} \sum_{(k,k',l',l)\in\mathcal{L}} \sum_{M\in[-n,n]^2} \exp\left(-\frac{a}{2\left(k'-k\right)} M^T M\right) \frac{1}{(k'-k)(l-l')^2(l'-k')}.$$

As above, we use (2.42) to handle the exponential terms. So the following estimation is enough for our purposes

$$\begin{aligned} \frac{1}{\log^2 n} \sum_{(k,k',l',l)\in\mathcal{L}} \frac{1}{(l-l')^2(l'-k')} &\leq \frac{1}{\log^2 n} \frac{n^4}{K^4} \frac{\log^4 n}{n^2} \frac{\log^2 n}{n} \\ &= O\left(\frac{n^2 \left(\log\log n\right)^2}{K^2 \log^4 n}\right). \end{aligned}$$

Our last task is to estimate the product of the second term in (2.38) and (2.39). The previous estimation (2.43) and

$$\begin{aligned} &\frac{1}{\log^2 n} \sum_{(k,k',l',l)\in\mathcal{L}} \sum_{M\in[-n,n]^2} \frac{1}{(k'-k)^2(l-l')^2(l'-k')} \\ \leq\ & n^6 \frac{1}{\log^2 n} \frac{\log^4 n}{n^2} \frac{\log^4 n}{n^2} \frac{\log^2 n}{n} = O\left(\frac{n^2 \left(\log\log n\right)^2}{K^2 \log^4 n}\right) \end{aligned}$$

yield the required estimation.

In the second step, we estimate the part of the sum in (2.37) corresponding to $M \in \mathbb{Z}^2 \setminus [-n,n]^2$. Corollary 2.6 implies that

$$\sup_{(k,k',l,l')\in\mathcal{L}} \sum_{M\in\mathbb{Z}^2\setminus[-n,n]^2} P(\mathcal{A}_6|\mathcal{A}_1\cap\ldots\cap\mathcal{A}_5) < \frac{1}{n^{1/4-\varepsilon}}$$

for small $\varepsilon > 0$. Thus

$$\begin{aligned} &\sup_{k',l'}\sum_{k,l}\sup_{M\in\mathbb{Z}^2\setminus[-n,n]^2} P(\mathcal{A}_1\cap\mathcal{A}_2) \sum_{M\in\mathbb{Z}^2\setminus[-n,n]^2} P(\mathcal{A}_6\cap\mathcal{A}_7|\mathcal{A}_1\cap\ldots\cap\mathcal{A}_5) \\ < \quad & C_7 n^2 \frac{1}{\log n}\frac{\log^2 n}{n}\frac{1}{\log n}\frac{1}{n^{1/4-\varepsilon}}. \end{aligned}$$

The above estimation together with (2.41) yield the required error term.

A modified version of the proof presented above can be repeated for indices $1 \le i < i' < j < j' \le K$. So we have finished the proof of formula (2.35). □

2.6 Visited points in one dimension

Investigating the one dimensional case is not as important as the higher dimensions, as Lorentz processes used to be examined mainly in higher dimensions. However, one dimension is also interesting, as we will see some new features. We need some different means from the previous ones to prove asymptotics for $E_1(n)$, namely Tauberian arguments. Let us see the details.

Proposition 2.19. *For a one dimensional RWwIS with $\varepsilon_0 \sim \mu$ we have*

$$\gamma_1(n) \sim \sqrt{\frac{2|\sigma|}{\pi}} n^{-1/2}$$

Proof. Just like in the higher dimensional cases we consider the renewal equation for the reversed walk

$$\sum_{k=0}^{n} U_k \cdot R_{n-k} = \underline{1}.$$

Now, from row i we obtain

$$\sum_{j=1}^{s}\sum_{k=0}^{n} (U_k)_{i,j}\, x^k\, (R_{n-k})_j\, x^{n-k} = x^n. \tag{2.44}$$

Let us introduce the notations

$$\begin{aligned} \sum_{k=0}^{\infty} (U_k)_{i,j}\, x^k &= \alpha_{ij}(x) \\ \sum_{k=0}^{\infty} (R_k)_j\, x^k &= \beta_j(x) \\ \sum_{k=0}^{\infty} x^k &= \omega(x). \end{aligned}$$

Obviously, these power series are convergent for $0 \le x < 1$. In these terms, (2.44) means

$$\sum_{j=1}^{s} \alpha_{ij}\beta_j = \omega. \tag{2.45}$$

In order to obtain the order of the coefficients of $\gamma_1(n) = \sum_{j=1}^{s} \mu_j (R_n)_j$ we use a Tauberian theorem which may be found in [F71] (Theorem 5 of XIII.5). According to this we have

$$\omega(x) \sim \frac{1}{1-x}, \qquad x \to 1- . \tag{2.46}$$

For the coefficients of α_{ij}

$$\sum_{k=0}^{n} (U_k)_{i,j} \sim 2\frac{1}{\sqrt{2\pi|\sigma|}}\mu_j n^{1/2}.$$

So, using the Tauberian theorem, we infer

$$\alpha_{ij}(x) \sim 2\frac{1}{\sqrt{2\pi|\sigma|}}\mu_j \Gamma\left(\frac{3}{2}\right)\frac{1}{(1-x)^{1/2}}, \qquad x \to 1- . \tag{2.47}$$

From (2.45) we obtain

$$\sum_{j=1}^{s} \frac{\alpha_{ij}}{\alpha_{ii}}\beta_j = \frac{\omega}{\alpha_{ii}}. \tag{2.48}$$

Now, (2.47) yields

$$\frac{\alpha_{ij}(x)}{\alpha_{ii}(x)} \to \frac{\mu_j}{\mu_i}, \qquad x \to 1- . \tag{2.49}$$

Whence

$$\sum_{j=1}^{s} \mu_j \beta_j(x) \sim \frac{\sqrt{2\pi|\sigma|}}{2\Gamma\left(\frac{3}{2}\right)}\frac{1}{(1-x)^{1/2}}, \qquad x \to 1- .$$

Since $\sum_{j=1}^{s} \mu_j (R_k)_j$ is monotonic in k, using the mentioned Tauberian theorem we conclude

$$\gamma_1(n) = \sum_{j=1}^{s} \mu_j (R_k)_j \sim \frac{\sqrt{2\pi|\sigma|}}{2\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)} n^{-1/2} = \sqrt{\frac{2|\sigma|}{\pi}} n^{-1/2}$$

□

Proposition 2.20. *With arbitrary distribution of ε_0 the following holds*

$$E_1(n) \sim \sqrt{\frac{8|\sigma|}{\pi}} n^{1/2}.$$

Proof. From Proposition 2.19 the assertion immediately follows in the case of $\varepsilon_0 \sim \mu$. However, the case of arbitrary initial distribution requires a little care. Analogously to (2.12), we have

$$\gamma^{e_j}(n) = \sum_{k=1}^{s} \mu_k \gamma^{e_k}(n-K) + \sum_{k=1}^{s} b_k^j(K)\gamma^{e_k}(n-K) + p(K,n). \tag{2.50}$$

But now, the rough estimation of $p(K,n)$ used in higher dimensions is not enough, as the local limit theorem provides a term of order $n^{-1/2}$ and our aim is to prove $o\left(n^{-1/2}\right)$. Nevertheless, because of the definition of $p(K,n)$, we have to estimate the probability of the first return to some place after m steps. In particular, if we proved that this probability is $O(m^{-3/2})$, then taking $K = \lfloor\sqrt{n}\rfloor$ and multiplying (2.50) by $\sqrt{n}$ we would find that the right hand side converges to $\sqrt{\frac{2|\sigma|}{\pi}}$ as $n \to \infty$. So, in order to finish our proof, we need the following lemma.

□

Lemma 2.21. *For a one dimensional RWwIS fulfilling our basic assumptions with arbitrary ν distribution of ε_0*

$$f^{\nu}(n) = O\left(n^{-3/2}\right), \tag{2.51}$$

where $f^{\nu}(n)$ denotes the probability of the event that the random walker starting from the origin with $\varepsilon_0 \sim \nu$ returns to the origin at time n for the first time.

Proof. First of all, observe that proving the statement for $\nu = \mu$ would be enough as since our basic assumption (i) all component of μ are positive. In the proof, we generalize an argument in [BLPW04]. Define

$$Q^n(x,i,y,j) = P\left(\xi_n = (y,j), \eta_k \neq 0, \forall 1 \le k < n | \xi_0 = (x,i)\right).$$

Let $n = 3m$ and $1 \le i \le m$. The cases $n = 3m \pm 1$ can be treated the same way.

$$\begin{aligned} f^{e_i}(n) = \sum_{l=1}^{s} Q^n(0,i,0,l) &= \sum_{y,z\neq 0} \sum_{j,k,l=1}^{s} Q^m(0,i,y,j) Q^m_d(y,j,z,k) Q^m(z,k,0,l) \\ &\le \sup_{y,z,j,k} Q^m(y,j,z,k) P(\eta_k \neq 0, \forall 1 \le k < m | \xi_0 = (0,i)) \sum_{z\neq 0} \sum_{k,l=1}^{s} Q^m(z,k,0,l) \end{aligned}$$

From the local limit theorem it follows that

$$\sup_{y,z,j,k} Q^m(y,j,z,k) = O(m^{-1/2}).$$

Proposition 2.19 yields $P(\eta_k \neq 0, \forall 1 \le k < m | \xi_0 = (0,i)) = O(m^{-1/2})$. So, it suffices to prove

$$\sum_{z\neq 0} \sum_{k,l=1}^{s} Q^m(z,k,0,l) = O(m^{-1/2}). \tag{2.52}$$

In order to prove (2.52) we use the reversed walk, again. (2.9) yields that for all $((0,i_1),(y_1,i_2),(y_1+y_2,i_3),...,(y_1+y_2+...+y_{m-1},i_m))$ trajectories

$$\begin{aligned} & \mu_{i_1} p_{y_1,i_1,i_2} \mu_{i_2} p_{y_2,i_2,i_3} \cdots \mu_{i_{m-1}} p_{y_{m-1},i_{m-1},i_m} \\ = \; & \mu_{i_2} q_{-y_1,i_2,i_1} \mu_{i_3} q_{-y_2,i_3,i_2} \cdots \mu_{i_m} q_{-y_{m-1},i_m,i_{m-1}}, \end{aligned}$$

where the factors $\mu_{i_2},...,\mu_{i_{m-1}}$ drop out. Thus

$$\sum_{z\neq 0} \sum_{k,l=1}^{s} Q^m(z,k,0,l) \le \max_{1\le i,j\le s} \frac{\mu_i}{\mu_j} \sum_{z\neq 0} \sum_{k,l=1}^{s} \widetilde{Q}^m(0,l,z,k), \tag{2.53}$$

where $\widetilde{Q}$ is the same object as Q defined for the reversed walk. The right hand side of (2.53) can be bounded by some constant times the probability of the event that the stationary reversed walk does not return to the origin in the first m steps, which is $O(m^{-1/2})$. Thus we arrived at (2.52).

□

So, we have ascertained the asymptotic behavior of $E_d(n)$ in each dimension. While strong law of large numbers holds in $d \geq 2$, even the weak law of large numbers for one dimensional SSRW fails to hold, which is a consequence of, for instance, Theorem 1 in [Ch06a].

2.7 Final remarks

1. Our asymptotic investigations show that RWwIS behaves like the simple symmetric random walk in an asymptotic sense. The main features are very similar, only the involved constants differ. The results showing that the asymptotic behavior is independent from the initial distribution on the internal states (e.g. Proposition 2.9 and 2.15) are intuitively trivial as after some steps ε will be very close to μ. Nevertheless, these assertions need formal proofs as well, especially as they are used in the sequel. Of course, this similarity to the simple symmetric random walk could change if the generalization were carried further, for instance, if a countable set of internal states was allowed. This model is not yet discussed, it must need some more involved techniques.

2. Our basic assumption (ii) is not essential. The above theorems could be generalized to the case of dropping basic assumption (ii), as the limit theorem in [KSz83b] is proved for this case, as well. Only the computations would become longer. The other three assumptions are essential.

Chapter 3

Recurrence properties of a special type of Heavy-Tailed Random Walk

3.1 Introduction

The appearance of the Brownian motion as a limit object in either stochastic or deterministic models is an extremely important and interesting phenomenon. The first result in this field is due to M. Donsker (see [D51]) who proved that the diffusively scaled Simple Symmetric Random Walk (SSRW) converges to the Brownian Motion in each dimension. Later, D. Szász and A. Telcs in [SzT81] proved that the local perturbation in the integer lattice of dimension at least two does not spoil the Brownian limit.
In the case of Lorentz process, the first such result was the appearance of the Brownian motion in the diffusive limit in finite horizon (see [BS81] and [BSCh91]). Analogous result with slightly super-diffusive scaling in infinite horizon is proven in [SzV07] and [ChD09a]. Again, the question of the effect of local perturbations naturally arises. This topic has a physical motivation as well, since Lorentz process can be thought of as the movement of a "classical" electron in a crystal, when local perturbation can be some impurities or some locally acting external force. The Brownian limit for diffusively scaled periodic Lorentz process with finite horizon and local perturbation was proven in [DSzV08] and [DSzV09]. Note that here a more involved investigation was needed than in the case of SSRW, namely, the wide treatment of recurrence properties in [DSzV08] was essential.
Recently, D. Paulin and D. Szász proved ([PSz10]) that the random walk, which is very similar to the Lorentz process with infinite horizon, with local impurities, enjoys the Brownian limit. However, they only treated some simplified local perturbation (see later), and did not consider the recurrence properties similar to the ones in [DSzV08], which are expected to be important in the case of infinite horizon (that is, Conjecture 1.1), too. Here, we are going to focus on these recurrence properties.
This Chapter is organized as follows. In Section 3.2, basic definitions, statements are given and another motivation for our calculations (i.e. the proof of the polynomial decay of the velocity auto correlation function for some perturbed random walk) is provided. The quite well known local limit theorem for our

specific type random walk will not be enough for our purposes, i.e. we need to estimate the remainder term of it. Section 3.3 is devoted to this computation. In Section 3.4, the desired recurrence properties are obtained, while in Section 3.5 we give final remarks indicating possible directions of further research.

3.2 Preliminaries

Let us consider a Random Walk, the behavior of which is close to the one of the Lorentz process with infinite horizon. Namely, define independent random variables X_i, such that

$$\mathbb{P}\left(X_i = n\right) = c_1 |n|^{-3},$$

if $n \neq 0$, and E_i to be uniformly distributed on the 4 unit vectors in $\mathbb{Z}^2$. Now put $\xi_i = X_i E_i$. (Here, of course, $c_1 = \frac{1}{2\zeta(3)}$, but this will not be important for us.) Define the Heavy-Tailed Random Walk (HTRW) by $S_n := \sum_{i=1}^n \xi i$.

This distribution is the same, as the one of the free flight vector of the Lorentz process with infinite horizon (see [SzV07]). However, one could think that our choice is rather special, as the walker can only step along the x and y axis. But this is not the case, as a particle performing Lorentz process can have arbitrary long steps only in finitely many directions, too. Here, we choose that two particular directions, but this is not essential.

Further, define the one dimensional HTRW as

$$Q_n := \sum_{i=1}^n X_i.$$

The quite well-known local limit theorem in one dimension states that

$$\mathbb{P}(Q_n = x) \sim \frac{1}{2\sqrt{\pi c_1 n \log n}} \exp\left(-\frac{x^2}{4c_1 n \log n}\right) \tag{3.1}$$

and in two dimensions that

$$\mathbb{P}(S_n = x) \sim \frac{1}{4\pi c_1 n \log n} \exp\left(-\frac{|x|^2}{4c_1 n \log n}\right). \tag{3.2}$$

These can be found in [R62]. Later, we will need estimations on the error terms in (3.1) and (3.2), and by computing them, a proof of (3.1) and (3.2) will be provided.

Further, we will use the notations

$$\begin{aligned} u_2(n) &= \mathbb{P}(S_n = (0,0)), \\ u_1(n) &= \mathbb{P}(Q_n = 0). \end{aligned}$$

In the case of billiards, a quite frequent strategy is to prove exponential decay of correlations (an interesting result for its own sake) and then to use this to prove convergence to the Brownian motion (see [Ch06b], for instance). As a motivation for our further calculations, we are going to illustrate that in the case of local perturbation, this does not seem to be a good strategy.

For this consider the simplest case: a perturbed SSRW (T_n) in $\mathbb{Z}^d$, where perturbation means that in the origin there is no scatterer, i.e. outside of the origin T_n behaves like an ordinary SSRW, while it flies through the origin. More precisely,

$$P\left(T_{n+1} = e_i | T_{n-1} = -e_i, T_n = 0\right) = 1, \tag{3.3}$$

where e_i is some neighboring point of the origin in $\mathbb{Z}^d$. The following Proposition is well known in the physics literature (see, for example [S80]) but surprisingly, I was unable to find a mathematical proof for it.

Proposition 3.1. *The velocity autocorrelation function of T_n is $O\left(n^{-(d/2+1)}\right)$.*

Proof. First, suppose that $d = 1$ and $T_0 = 1$. We can identify our process with an unperturbed SSRW - U_n, say - by simply dropping the origin and the extra step from it. Formally, define $\tau(n) = \#\{1 \le k < n : T_k = 0\}$. Now, if $T_n > 0$, then let $U(n - \tau(n)) = T_n$. If $T_n < 0$, then $U(n - \tau(n)) = T_n + 1$. Now, we have to show that

$$\begin{aligned} & \mathbb{P}(U(2n) = 0, U(2n+1) = 1) - \mathbb{P}(U(2n+1) = 1, U(2n+2) = 0) \\ = \; & \frac{1}{2}\left[\mathbb{P}(U(2n) = 0) - \mathbb{P}(U(2n+1) = 1)\right] = O\left(n^{-3/2}\right), \end{aligned}$$

which is an elementary consequence of the well known Edgeworth expansion.
Now, suppose that $d > 1$ and $T_0 = (1, 0, ..., 0)$. It suffices to prove

$$\int_\Omega I_{\{T_n = T_0\}} - I_{\{T_n = -T_0\}} dP = O\left(n^{-(d/2+1)}\right). \tag{3.4}$$

Let V be the orthogonal complement space of T_0 and define

$$H = \{\omega : (V \setminus 0) \cap \{T_0, ..., T_n\} \neq \emptyset\} \subset \Omega.$$

Because of the reflection principle, the part of the integral in (3.4) over H is zero. The integral over $\Omega \setminus H$ can be treated similarly, as it was done in the one dimensional case.

□

3.3 Local limit theorem with remainder term

The aim of this section is to estimate remainder term in the limit theorem (3.2). To do this, first we have to deal with the one dimensional case. Similar calculations were done previously, see, for example [dHP97] and [JP98]. However, in these articles only one dimensional, non-lattice distributions were considered. Fortunately, we do not need precise calculation of the remainder term, i.e. summability is enough for our purposes. As usual, we start with the computation of the characteristic function.

Lemma 3.2. *For the characteristic function ϕ of X_1*

$$\phi(t) = 1 - 2c_1 t^2 |\log|t|| + O\left(t^2\right),$$

as $t \to 0$.

Proof. Since the distribution is symmetric, it suffices to prove for $t > 0$. Fix $\varepsilon > 0$ such that $1-x^2-x^3 < \cos x < 1 - x^2 + x^3$ if $|x| < \varepsilon$. Now, let us consider the decomposition

$$\phi(t) = \mathbb{E}(\exp(itX)) = \sum_{n=1}^{\varepsilon\lfloor t^{-1}\rfloor} \frac{2c_1}{n^3}\cos(tn) + \sum_{n=\varepsilon\lfloor t^{-1}\rfloor+1}^{\infty} \frac{2c_1}{n^3}\cos(tn) =: S_1 + S_2.$$

It is easy to see that

$$S_2 = 2c_1 \int_{\varepsilon}^{\infty} \frac{\cos x}{x^3} dx t^2 + o\left(t^2\right) = O\left(t^2\right).$$

On the other hand, since

$$S_1 = 2c_1 \sum_{m=t, m\in t\mathbb{Z}}^{\varepsilon} t^3 m^{-3}\cos m,$$

we have

$$\left| \frac{S_1}{2c_1} - \sum_{m=t,m\in t\mathbb{Z}}^{\varepsilon} t^3 m^{-3} + \sum_{m=t,m\in t\mathbb{Z}}^{\varepsilon} t^3 m^{-1} \right| < \sum_{m=t,m\in t\mathbb{Z}}^{\varepsilon} t^3. \tag{3.5}$$

Now the estimations

$$\begin{aligned} \sum_{m=t,m\in t\mathbb{Z}}^{\varepsilon} t^3 m^{-3} &= \frac{1}{2c_1} + O\left(\frac{t^2}{\varepsilon^2}\right) \\ \sum_{m=t,m\in t\mathbb{Z}}^{\varepsilon} t^3 m^{-1} &= t^2 log\left(\frac{\varepsilon}{t}\right) + O\left(t^2\right) \end{aligned}$$

and

$$\sum_{m=t,m\in t\mathbb{Z}}^{\varepsilon} t^3 = O(t^2)$$

finish the proof.

□

Now, we turn to the estimation of the remainder term in the one dimensional local limit theorem.

Theorem 3.3. *For the one dimensional HTRW the following estimation holds uniformly in x*

$$\mathbb{P}(Q_n = x) - \frac{1}{\sqrt{2\pi}\sqrt{2c_1}\sqrt{n\log n}} \exp\left(-\frac{x^2}{4c_1 n \log n}\right) = O\left(\frac{\log\log n}{\sqrt{n\log^3 n}}\right)$$

Proof. Let g denote the probability density function of the standard Gaussian law. Then we have

$$g(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-izs - \frac{s^2}{2}\right) ds.$$

On the other hand, according to the Fourier inversion formula,

$$\mathbb{P}(Q_n = x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(-itx\right)\phi^n(t) dt.$$

By an elementary argument (see, for example, [IL71]) our result follows from the statement

$$\left| \sqrt{2c_1 n \log n}\frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(-itx)\phi^n(t)dt - g\left(\frac{x}{\sqrt{2c_1 n\log n}}\right)\right| = O\left(\frac{\log\log n}{\log n}\right), \tag{3.6}$$

where the great order on the right hand side is uniform in x. As it is quite usual in the theory of limit theorems (see again [IL71]), we estimate the left hand side of (3.6) by the sum of several integrals

$$\int_{|s|<\log^{1/3} n} \left| \phi^n\left(\frac{s}{\sqrt{2c_1 n \log n}}\right) - \exp\left(-\frac{s^2}{2}\right)\right| ds$$
$$+ \int_{\log^{1/3} n<|s|<\gamma\sqrt{2c_1 n \log n}} \left| \phi^n\left(\frac{s}{\sqrt{2c_1 n \log n}}\right)\right| ds$$
$$+ \int_{\gamma\sqrt{2c_1 n \log n}<|s|<\pi\sqrt{2c_1 n \log n}} \left| \phi^n\left(\frac{s}{\sqrt{2c_1 n \log n}}\right)\right| ds$$
$$+ \int_{\log^{1/3} n<|s|} \left|\exp\left(-\frac{s^2}{2}\right)\right| ds =: I_1 + I_2 + I_3 + I_4.$$

So it suffices to prove that $I_j = O\left(\frac{\log\log n}{\log n}\right)$, for $j \in \{1,2,3,4\}$.
For the estimation of I_1, observe that for $|s| < \log^{1/3} n$ Lemma 3.2 yields

$$\phi^n\left(\frac{s}{\sqrt{2c_1 n \log n}}\right) = \exp\left(-\frac{s^2}{2}\right)\left[1 + O\left(\frac{(s^2+1)\log\log n}{\log n}\right)\right],$$

where the great order on the right hand side is uniform in s. Hence

$$I_1 < \int_{|s|<\log^{1/3} n} (s^2+1)\exp\left(-\frac{s^2}{2}\right) ds O\left(\frac{\log\log n}{\log n}\right) = O\left(\frac{\log\log n}{\log n}\right).$$

It can be proven (see Theorem 4.2.1. in [IL71]) that there exists $\gamma > 0$ such that

$$\phi^n\left(\frac{s}{\sqrt{2c_1 n \log n}}\right) < \exp\left(-C|s|\right),$$

with an appropriate C if $|t| < \gamma$. This estimation implies $I_2 < O\left(\frac{\log\log n}{\log n}\right)$. Observe that $|\phi(t)| \le 1$ and $|\phi(t)| = 1$ holds if and only if $t \in 2\pi\mathbb{Z}$. As $|\phi(t)|$ is continuous in t, there exists some $C' < 1$ such that $|\phi(t)| < C'$ for $t \in [\gamma, \pi]$. It follows that $I_3 < O\left(\frac{\log\log n}{\log n}\right)$. Finally, $I_4 < O\left(\frac{\log\log n}{\log n}\right)$ by elementary computation. Hence the statement.

□

Now, we turn to the two dimensional case. Define the two dimensional characteristic function $\phi_2 : \mathbb{R}^2 \to \mathbb{C}$, $\phi_2(t) = \mathbb{E}(\exp(it'\xi_1))$, where $'$ stands for transpose, and write $t = (t_1, t_2)'$, $s = (s_1, s_2)'$. Lemma 3.2 implies that

$$\phi_2(t) = 1 - c_1 t_1^2 |\log|t_1|| - c_1 t_2^2 |\log|t_2|| + O\left(|t|^2\right),$$

as $|t| \to 0$. Similarly to the one dimensional case, the local limit theorem with remainder term reads as follows.

Theorem 3.4. *For the two dimensional HTRW the following estimation holds uniformly for $x \in \mathbb{R}^2$*

$$\mathbb{P}(S_n = x) - \frac{1}{2\pi 2c_1 n \log n}\exp\left(-\frac{|x|^2}{4c_1 n \log n}\right) = O\left(\frac{\log\log n}{n \log^2 n}\right)$$

Proof. The proof is similar to the proof of Theorem 3.3. Let g denote the probability density function of the two dimensional standard Gaussian law. Then we have

$$g(z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-it'z - \frac{t't}{2}\right) dt.$$

On the other hand, according to the Fourier inversion formula

$$\mathbb{P}(S_n = x) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \exp\left(-it'x\right) \phi_2^n(t) dt.$$

Just like previously, it is enough to prove that

$$\left| 2c_1 n \log n \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \exp(-it'x)\phi_2^n(t)dt - g\left(\frac{x}{\sqrt{2c_1 n \log n}}\right) \right| \tag{3.7}$$

is in $O\left(\frac{\log\log n}{\log n}\right)$. The analogue of the previous decomposition in the present case is

$$\begin{aligned}
& \int_{|s|<\log^{1/3} n} \left| \phi_2^n\left(\frac{s}{\sqrt{2c_1 n \log n}}\right) - \exp\left(-\frac{s's}{2}\right) \right| ds \\
+ & \int_{\log^{1/3} n<|s|<\gamma\sqrt{2c_1 n \log n}} \left| \phi_2^n\left(\frac{s}{\sqrt{2c_1 n \log n}}\right) \right| ds \\
+ & \int_{\gamma\sqrt{2c_1 n \log n}<|s|<\pi\sqrt{2c_1 n \log n}} \left| \phi_2^n\left(\frac{s}{\sqrt{2c_1 n \log n}}\right) \right| ds \\
+ & \int_{\log^{1/3} n<|s|} \left| \exp\left(-\frac{s's}{2}\right) \right| ds =: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

So it suffices to prove that $I_j = O\left(\frac{\log\log n}{\log n}\right)$, for $j \in \{1, 2, 3, 4\}$.
All the above integrals can be estimated as it was done in the proof of Theorem 3.3 except for I_2. For the latter, we adapt the argument of Rvaceva (see [R62]). It is easy to see that

$$\frac{\Re \log \phi_2(at)}{\Re \log \phi_2(t)} \to a^2$$

as $|t| \to 0$ (here $\Re$ denotes real part). Hence, for γ small enough,

$$\Re \log \phi_2(t) > e \Re \log \phi_2(t/e)$$

holds for $|t| < \gamma$. Now, pick $k \in \mathbb{N}$ such that $\exp(k) \le \gamma\sqrt{2c_1 n \log n} < \exp(k+1)$ and write

$$\begin{aligned}
I_2 &\le \sum_{m=\frac{1}{3}\log\log n}^{k} \int_{\exp(m)<|s|<\exp(m+1)} \left| \phi_2^n\left(\frac{s}{\sqrt{2c_1 n \log n}}\right) \right| ds \\
&< \sum_{m=\frac{1}{3}\log\log n}^{k} \exp(2m) \int_{1<|s|<e} \exp\left(n \exp(m) \Re \log \phi_2\left(\frac{s}{\sqrt{2c_1 n \log n}}\right)\right) ds.
\end{aligned}$$

The argument used in the estimation of I_1 implies that

$$n \Re \log \phi_2\left(\frac{s}{\sqrt{2c_1 n \log n}}\right) = -\frac{|s|^2}{2} + o(1)$$

holds uniformly for $s \in [1, e]$, whence for some $C' < 1$

$$I_4 < \sum_{m=\frac{1}{3}\log\log n}^{k} \exp(2m)(e-1)C'^{\exp(m)}.$$

So we proved $I_2 = O(\frac{1}{\log n})$, hence the statement.

□

3.4 Recurrence properties

In this section we discuss the recurrence properties of S_n and Q_n that are supposed to be important in the case of billiards, too (note that these are analogous to the ones considered in [DSzV08]). For SSRW, these kind of results were proven in [ET60] and [DK57]. We begin with the two dimensional case.

Definition 3. *Let τ_2 be the first return to the origin in two dimensions, i.e.*

$$\tau_2 = \min\{n > 0 : S_n = (0,0)\}$$

Theorem 3.5. $\mathbb{P}(\tau_2 > n) \sim \frac{4\pi c_1}{\log\log n}$

Theorem 3.6. *Let $N_2^n = \#\{k \leq n : S_k = (0,0)\}$. Then*

$$\frac{N_2^n}{\log\log n}$$

converges to an exponential random variable with expected value $\frac{1}{4\pi c_1}$.

Theorem 3.5 and Theorem 3.6 can be easily proven combining the original proofs (see [DE51] and [ET60]) with (3.2).

Definition 4. *Let t_v be the hitting time of the origin, starting from the site $v \in \mathbb{Z}^2$, i.e.*

$$t_v = \min\{k \geq 0 : S_k = (0,0) | S_0 = v\}.$$

The following recurrence property is less known but is of crucial importance in the argument of [DSzV09].

Theorem 3.7.

$$\frac{\log\log t_v}{\log\log |v|} \Rightarrow \frac{1}{U}$$

as $|v| \to \infty$, where U is uniformly distributed on $[0,1]$ and $\Rightarrow$ stands for weak convergence.

Proof. We adapt the proof of [ET60]. Let

$$\zeta(x, n) = \#\{1 \leq k \leq n : S_k = x\}$$

be the local time of the walk at site x up to time n and

$$\gamma(n) = \mathbb{P}\left(\tau_2 > n\right).$$

Further, we will need the estimation on the remainder term of the local limit theorem. More precisely, we will use the following estimation

$$\mathbb{P}\left(S_n = y\right) = \frac{1}{4\pi c_1 n \log n} - |y|^2 O\left(\frac{1}{n^2 \log^2 n}\right) + O\left(\frac{\log\log n}{n \log^2 n}\right), \tag{3.8}$$

where the great orders are uniform in $\{y : |y| < \sqrt{n \log n}\}$. Note that (3.8) is a consequence of Theorem 3.4. We are going to prove the following assertion.
If we choose $x_n \in \mathbb{Z}^2$ such that

$$|x_n| \sim \exp\left(\frac{1}{2} \log^\delta n\right)$$

for some fix $0 < \delta < 1$, then

$$\mathbb{P}\left(\zeta\left(x_n, n\right) = 0\right) \to \delta, \tag{3.9}$$

as $n \to \infty$. It is easy to see that (3.9) implies the statement of the theorem.
As in [ET60], we consider the identities

$$\sum_{i=0}^{n} u_2(i)\gamma(n-i) = 1 \tag{3.10}$$

and

$$\mathbb{P}\left(\zeta\left(x_n, n\right) = 0\right) + \sum_{i=1}^{n} \mathbb{P}\left(S_i = x_n\right)\gamma(n-i) = 1. \tag{3.11}$$

Combining (3.10) and (3.11) we obtain

$$\mathbb{P}\left(\zeta\left(x_n, n\right) = 0\right) - \gamma(n) = \sum_{i=1}^{n} \left(u_2(i) - \mathbb{P}\left(S_i = x_n\right)\right)\gamma(n-i). \tag{3.12}$$

Using the fact that γ is monotonic, Theorem 3.5 and the estimation (3.8) we conclude that the right hand side of (3.12) is smaller than

$$\begin{aligned}
& \frac{4\pi c_1 + o(1)}{\log\log n} \sum_{k=1}^{\exp\left(\log^\delta n\right)} \frac{1}{4\pi c_1 k \log k} \\
+ \; & \frac{4\pi c_1 + o(1)}{\delta \log\log n} \sum_{k=\exp\left(\log^\delta n\right)}^{\sqrt{n}} |x_n|^2 O\left(\frac{1}{k^2 \log^2 k}\right) + \sum_{k=\sqrt{n}}^{n} |x_n|^2 O\left(\frac{1}{k^2 \log^2 k}\right) \\
+ \; & \sum_{k=\exp\left(\log^\delta n\right)}^{\infty} O\left(\frac{\log\log k}{k \log^2 k}\right) = \delta + o(1).
\end{aligned}$$

So we arrived at the upper bound. For the lower bound define

$$k_1 = \frac{\exp\left(\log^\delta n\right)}{\log n}.$$

Theorem 3.5 and Theorem 3.4 imply that the right hand side of (3.12) is bigger than

$$\begin{aligned}
& \gamma(n)\sum_{k=1}^{k_1}[u_2(k)-\mathbb{P}(S_k=x_n)] \geq \\
& \frac{4\pi c_1+o(1)}{\log\log n}\sum_{k=1}^{k_1}\Big[\frac{1}{4\pi c_1 k\log k}+O\left(\frac{\log\log k}{k\log^2 k}\right)\Big] \\
& +\frac{4\pi c_1+o(1)}{\log\log n}\sum_{k=1}^{k_1}\Big[-\frac{1}{4\pi c_1 k\log k}\exp\left(-\frac{|x_n|^2}{4c_1 k\log k}\right)\Big] \\
> \; & \delta+o(1)+\frac{O(1)}{\log\log n}-O\left(\frac{1}{\log\log n}\right)\exp\left(-\frac{|x_n|^2}{k_1\log k_1}\right)\sum_{k=1}^{k_1}\frac{1}{k\log k} \\
> \; & \delta+o(1).
\end{aligned}$$

Thus we have proved (3.9). The statement follows. □

Remark 3.8. *Note that for the adaptation of the Erdős-Taylor type argument for our setting, the summability of the remainder term in the local limit theorem - i.e. Theorem 3.4 - was essential. The situation was basically the same in Chapter 2, however, in a different context.*

It would be interesting to find an intuitive reason for the appearance of the exponential and the uniform distributions as limit laws. However, neither Erdős and Taylor gave explanation in [ET60], nor the present author can give any. Now, we turn to the one dimensional case.

Definition 5. *Let τ_1 be the first return to the origin in one dimension, i.e.*

$$\tau_1=\min\{n>0: Q_n=0\}$$

Theorem 3.9. $\mathbb{P}(\tau_1>n)\sim\frac{2\sqrt{c_1}}{\sqrt{\pi}}\sqrt{\frac{\log n}{n}}$

Proof. Theorem 3.9 can be easily proven by the usual way. One has to consider the renewal equation

$$\sum_{k=0}^{n}u_1(k)\mathbb{P}(\tau_1>n-k)=1,$$

and the identity

$$U(x)V(x)=\frac{1}{1-x},$$

where

$$\begin{aligned}
U(x) &= \sum_{k=0}^{\infty}u_1(k)x^k \\
V(x) &= \sum_{k=0}^{\infty}\mathbb{P}(\tau_1>k)x^k.
\end{aligned}$$

Now, the well known Tauberian theorem (Theorem XIII.5. in [F71]) implies that

$$U(x)\sim\frac{1}{\sqrt{1-x}}\frac{1}{\sqrt{\pi c_1}}\Gamma\left(\frac{3}{2}\right)\frac{1}{\sqrt{\log\frac{1}{1-x}}}$$

as $x \to 1$, thus

$$V(x) \sim \frac{1}{\sqrt{1-x}} \frac{\sqrt{\pi c_1}}{\Gamma\left(\frac{3}{2}\right)} \sqrt{\log \frac{1}{1-x}}$$

as $x \to 1$. Since $\mathbb{P}(\tau_1 > n)$ is monotonic in n, the previous Tauberian theorem infers the statement. □

Theorem 3.10. *Let* $N_1^n = \#\{k \le n : Q_k = 0\}$. *Then*

$$\frac{N_1^n \sqrt{\log n}}{\sqrt{n}}$$

converges to a Mittag-Leffler distribution with parameters $1/2$ *and* $(2\sqrt{c_1})^{-1}$, *i.e. to the distribution, the* k^{th} *moment of which is*

$$\frac{1}{(2\sqrt{c_1})^k} \frac{k!}{\Gamma\left(\frac{k}{2}+1\right)}.$$

Proof. As in the case of [DSzV08], it suffices to prove that for k fix:

$$\sum_{n_i \ge 3, n_1+n_2+\ldots+n_k \le n} \prod_j \frac{1}{\sqrt{n_j \log n_j}} \sim \frac{n^{k/2}}{\log^{k/2} n} \frac{\Gamma(1/2)^k}{\Gamma(k/2+1)}. \tag{3.13}$$

Note that $\Gamma(1/2) = \sqrt{\pi}$. Elementary calculations show that (3.13) holds for $k = 1$. For $k > 1$ define

$$\begin{aligned}
\mathcal{H}_1 &= \{n_i \ge \frac{n}{\log n}, n_1 + n_2 + \ldots + n_k \le n\} \\
\mathcal{H}_2 &= \{n_i \ge \frac{\sqrt{n}}{\log n}, \exists j : n_j < \frac{n}{\log n}, n_1 + n_2 + \ldots + n_k \le n\} \\
\mathcal{H}_3 &= \{n_i \ge 3, \exists j : n_j < \frac{\sqrt{n}}{\log n}, n_1 + n_2 + \ldots + n_k \le n\}
\end{aligned}$$

Now, split the sum in (3.13) into three parts, sums over $\mathcal{H}_i$'s, $1 \le i \le 3$.
Define $s_j = n_j/n$ and observe that

$$\frac{1}{\sqrt{n_j \log n_j}} = \frac{1}{\sqrt{s_j}} \frac{1}{\sqrt{n}} \frac{1}{\sqrt{\log s_j + \log n}}.$$

Since $\log s_j + \log n = (1 + o(1)) \log n$ uniformly in $\mathcal{H}_1$, it is not difficult to deduce that

$$\begin{aligned}
&\sum_{(n_1,n_2,\ldots,n_k) \in \mathcal{H}_1} \prod_j \frac{1}{\sqrt{n_j \log n_j}} \\
\sim\ & \frac{n^{k/2}}{\log^{k/2} n} \int \ldots \int_{0<t_1<t_2<\ldots<t_k<1} \frac{1}{\sqrt{t_1}} \frac{1}{\sqrt{t_2 - t_1}} \ldots \frac{1}{\sqrt{t_k - t_{k-1}}} dt_1 \ldots dt_k \\
=\ & \frac{n^{k/2}}{\log^{k/2} n} \frac{\Gamma(1/2)^k}{\Gamma(k/2+1)}.
\end{aligned}$$

For the sum over $\mathcal{H}_2$, consider the case when $\frac{\sqrt{n}}{\log n} < n_1 < \frac{n}{\log n}$ and $n_i > \frac{n}{\log n}$ for $2 \le i$ (other cases can be treated similarly). Now, $\log s_1 + \log n > (1/2 + o(1)) \log n$ and $\log s_i + \log n = (1 + o(1)) \log n$ for $2 \le i$, uniformly. Thus,

$$\sum_{\frac{\sqrt{n}}{\log n}<n_1<\frac{n}{\log n}, n_i>\frac{n}{\log n}:2\le i} \prod_j \frac{1}{\sqrt{n_j \log n_j}}$$

$$< \quad (\sqrt{2}+o(1)) \sum_{\frac{\sqrt{n}}{\log n}<n_1<\frac{n}{\log n}, n_i>\frac{n}{\log n}:2\le i} \prod_j \frac{1}{\sqrt{s_j n \log n}} < 2\frac{n^{k/2}}{\log^{k/2} n} o(1).$$

For the third sum, the proof goes by induction on k. Assuming that (3.13) holds for $k-1$, one has

$$\sum_{(n_1,n_2,\dots,n_k)\in\mathcal{H}_3} \prod_{j=1}^{k} \frac{1}{\sqrt{n_j \log n_j}} < k\frac{\sqrt{n}}{\log n} \sum_{n_1+n_2+\dots+n_{k-1}\le n} \prod_{j=1}^{k-1} \frac{1}{\sqrt{n_j \log n_j}},$$

which is $o\left(\frac{n^{k/2}}{\log^{k/2} n}\right)$. (3.13) follows.

□

3.5 Final remarks

1. It would be beneficial to extend the results of this Chapter to a more general class of random walks. Namely, choose $p(|n|) = \mathbb{P}(X_i = n)$ with

$$p(n)\frac{n^3}{c_1} - 1 = \delta(n), \text{ and } |\delta(n)| < Cn^{-\xi}$$

with some $\xi > 0$ (e.g. for planar Lorentz process with infinite horizon, $\xi = 2$).

In this case, we need to adjust the estimations of Section 3.3, since a nontrivial modification is needed in the proof of Lemma 3.2. Namely, the second sum in the formula (3.5) is now

$$\sum_{n=1,2,\dots\lfloor\varepsilon/t\rfloor} \frac{p(n)}{c_1} n^2 t^2 = \sum_{n=1,2,\dots K} \frac{p(n)}{c_1} n^2 t^2 + \sum_{n=(K+1)\dots\lfloor\varepsilon/t\rfloor} \frac{t^2}{n} + \sum_{n=(K+1)\dots\lfloor\varepsilon/t\rfloor} \frac{t^2}{n}\delta(n)$$

Now, the first sum on the right hand side is bounded by K^2t^2. The second sum is $t^2(\log(\varepsilon/t) - \log K + O(1))$, while the modulus of the last one is bounded by $t^2K^{-\xi}(\log(\varepsilon/t) - \log K + O(1))$. Now choose $K = |\log|t||^{\frac{1}{2+\xi}}$ to conclude that

$$\phi(t) = 1 - 2c_1 t^2 |\log|t|| + O\left(t^2 |\log|t||^{\frac{2}{2+\xi}}\right).$$

The error term in the local limit theorem should also be modified accordingly, namely the right hand side of the formula in Theorem 3.3 is now

$$O\left(\frac{\log^{\frac{2}{2+\xi}} n}{\sqrt{n \log^3 n}}\right)$$

and in Theorem 3.4 is

$$O\left(\frac{\log^{\frac{2}{2+\xi}} n}{n \log^2 n}\right).$$

Since the last expression is summable in n, the recurrence properties in Section 3.4 follow exactly the same way, as before.

2. Extending the results presented in this Chapter to periodic Lorentz process with infinite horizon is an important, challenging question. We claim that the extension of Theorems 3.6 and 3.10 is straightforward. Indeed, the local limit theorem for periodic Lorentz process with infinite horizon states that the recurrence to the origin (more precisely, zeroth cell) has the same asymptotic probability (more precisely, Liouville measure), as our above models: see [SzV07] (and its adapted version to the Lorentz process is a strip in the spirit of Proposition 3.6 of [DSzV08]). Thus the same argument, presented here, is applicable. However, the extension of Theorems 3.5, 3.7 and 3.9 is not that obvious. The most challenging is probably Theorem 3.7, since in our argument, the error term of the local limit theorem was used, which is expected to be difficult for Lorentz processes.

3. As it was mentioned in Section 3.1, in the case of Lorentz process with infinite horizon, another type of 'recurrence' can happen. Namely, if a scatterer is moved into a corridor (here corridor means infinite trajectories without collision), then there are arbitrary long flights where in the periodic Lorentz process there would not be collision, while in the perturbed one there are some. In the random walk context, it can happen that the unperturbed walk would fly over the origin, while the perturbed one has to stop. Note that this phenomenon is evitable if one considers finite horizon, or in the case of infinite horizon just shrinks one of the scatterers as a perturbation. However, the same behavior (i.e. the Brownian limit with the same scaling) is conjectured in this general perturbation, as well. The aim of the following computation is to give some reason for this conjecture. As the constants do not play important role in the sequel, they will not be computed and every appearance of C may denote different constant.
Define

$$a_n = \mathbb{P}((0,0) \in \overline{S_n, S_{n+1}}, (0,0) \neq S_n)$$

to be the probability of the event that step $n+1$ flies over the origin. Observe that

$$a_n = \frac{1}{2}\mathbb{P}\left((S_n)_1 = 0, |X_{n+1}| \geq |(S_n)_2|\right),$$

where $(S_n)_i$ denotes the i^{th} coordinate of S_n. The local limit theorem implies $a_n < C\frac{1}{\sqrt{n\log n}}b_n$, where

$$b_n = \mathbb{P}\left(|X_{n+1}| \geq |(S_n)_2|\right).$$

For the estimation of b_n observe that if $|(S_n)_2| > d_n$, then b_n is bounded by $C\sum_{k=d_n}^{\infty} k^{-3} = O(d_n^{-2})$. On the other hand, the probability of $|(S_n)_2|$ being smaller than d_n is roughly estimated by $O(d_n\frac{1}{\sqrt{n\log n}})$. Thus

$$b_n = O(d_n^{-2}) + O(d_n\frac{1}{\sqrt{n\log n}}) = O\left((n\log n)^{-1/3}\right),$$

whence

$$a_n = O\left((n\log n)^{-5/6}\right).$$

If ρ_n denotes the number of jumps over the origin up to time n and $\theta_n = \mathbb{E}(\rho_n)$, then we have just proved

$$\theta_n = o(n^{1/6}).$$

Note that in the case of [SzT81] and [PSz10] the key observation was that the time spent at the perturbed area up to n is much smaller than $\sqrt{n}$. That is why it is reasonable to expect the same Brownian limit in the case of such perturbation, where we introduce some nice further step at the time of flying over the origin, too. Here nice means that presumably the step distribution should have some finite moment of order ε. This could be subject of future research.

Chapter 4

Lorentz Process with shrinking holes in a wall

4.1 Introduction

In the last decade after a broad and thorough study of Sinai billiards - or equivalently of periodic Lorentz processes - the non-homogeneous case got also widely examined. Here, *non-homogeneity* may appear either in time (cf. [ChD09b] as to a mechanical model of Brownian motion or [GR-KST11], [DK09], [LChK10] as to models of Fermi acceleration) or in space (cf. [DSzV09] as to local perturbations of periodic Lorentz processes). In the present work we investigate a question where non-homogeneity is present both in time and space. Consider a periodic Lorentz process with a finite horizon given in a horizontal strip, where the scatterer configuration is assumed to be symmetric with respect to a vertical axis - through the origin, say. Now, put a vertical wall at the symmetry axis and a tiny hole onto the wall. The hole is getting smaller and smaller with time, thus giving the particle less and less chance to cross the wall. It is an intriguing question at which speed the hole should shrink to result a non trivial scaling limit of the trajectory of the particle (if such a speed exists at all). Here, non trivial means that it is neither Brownian motion (BM), nor reflected Brownian motion (RBM) since, if the hole was of full size or absent, then these two processes would appear in the limit (see [DSzV09]).

Indeed, if one takes the hole arbitrarily small, but fixed of size $\varepsilon > 0$, then the limiting process is a BM whereas if the hole is empty, then it is a RBM. The essence of this observation is that the limiting process does not change continuously as $\varepsilon \to 0$ and our goal is precisely to understand the situation when the limit is taken in a more delicate, time-dependent way.

To be more precise, let the configuration space in the absence of the wall be $\mathcal{D} := (\mathbb{R} \times [0,1]) \setminus \cup_{i=1}^{\infty} O_i$. Here, $\{O_i\}_i$ is a $\mathbb{Z}$-periodic extension of a finite scatterer configuration in the unit square, which consists of strictly convex, pairwise disjoint scatterers, with C^3 smooth boundaries, whose curvatures are bounded from below by a positive constant. Further, we assume that $\cup_{i=1}^{\infty} O_i$ is symmetric with respect to the y-axis. The wall without the hole is $W_\infty = \{(x, y) \in \mathcal{D} | \, x = 0\} = \cup_{k=1}^{K} [\mathcal{J}_{k,l}, \mathcal{J}_{k,r}]$ where the subintervals of the y-axis, denoted by $[\mathcal{J}_{k,l}, \mathcal{J}_{k,r}]$, are the connected components of W_∞. For later reference, put

$$c_1 = \sum_{k=1}^{K} (\mathcal{J}_{k,r} - \mathcal{J}_{k,l}).$$

The *holes* will be subintervals $I_n \subset W_\infty$, thus we will be considering a sequence $\{W_n = W_\infty \setminus I_n\}_n$ of walls. Now, the n-th configuration space of the *billiard flow* is $\mathcal{D}_n := (\mathbb{R} \times [0, 1]) \setminus (W_n \cup \cup_{i=1}^{\infty} O_i)$. A massless point particle moves inside $\mathcal{D}_n$ (at time $t = 0$ the first hole is present, i.e. $n = 1$) with unit speed until it hits the boundary $\partial\mathcal{D}_n$. Then it is reflected by the classical laws of mechanics (the angle of incidence equals to the angle of reflection) and continues free movement (or free flight) in $\mathcal{D}_{n+1}$. Thus, at the time instant of each reflection, the hole is replaced by an other one (meaning that the shrinking rate of the hole corresponds to real time). We also mention that the reflections on the horizontal boundaries of the strip does not play any role in our study. Thus one could define the horizontal direction to be periodic (formally replace $[0, 1]$ by S^1 in the definitions of $\mathcal{D}$ and $\mathcal{D}_n$) yielding the same results (with some different limiting variance).

Since we change the configuration space in the moment of the reflection, it is more convenient to use the discretized version of the billiard flow (the usual Poincaré section, which is often called billiard ball map). Thus define the *phase spaces*

$$\mathcal{M}_n = \{x = (q, v), q \in \partial\mathcal{D}_n, v \in S^1, \langle v, u\rangle \geq 0 \text{ if } q \in \partial\mathcal{D}\ \},$$

where u denotes the inward unit normal vector to $\partial\mathcal{D}$ at the point $q \in \partial\mathcal{D}$. Here, q denotes the position of the particle at a collision and v is the post-collisional velocity vector. If $q \in \partial\mathcal{D}$, v can be naturally parameterized by the angle between u and v which is in the interval $[-\pi/2, \pi/2]$. If $q \in \partial W_n = W_n$, one can parameterize v by its angle to the horizontal axis. Thus, if this angle is in the interval $[-\pi/2, \pi/2]$, then the particle is on the right-hand side of the wall, while it is on the left-hand side if this angle is either in the interval $[\pi/2, \pi]$ or in $(-\pi, -\pi/2]$.

Thus, the discretized version of the previously described billiard flow can be defined by the *billiard ball maps* $\mathcal{F}_n : \mathcal{M}_n \to \mathcal{M}_{n+1}$. Further, denote by $\kappa_n : \mathcal{M}_n \to \mathbb{R}$ the projection to the horizontal direction of the *free flight* vector from $\mathcal{M}_n$ to $\mathcal{M}_{n+1}$ (that is, if $x = (q, v) \in \mathcal{M}_n$ and $\mathcal{F}_n(x) = (\tilde{q}, \tilde{v})$, then $\kappa_n(x)$ is the projection to the horizontal axis of the vector $\tilde{q} - q$). We also assume that the billiard has *finite horizon*, meaning that, in the $\mathbb{Z}^2$-periodic extension of the scatterer configuration, there is no infinite line on the plane that would be disjoint to all the scatterers. Further, write $\mathcal{I}_n = \{I_k\}_{1\leq k\leq n}$ for the collection of the first n holes, and

$$S_n(x, \mathcal{I}_n) = S_n(x) = \sum_{k=1}^{n} \kappa_k \mathcal{F}_{k-1} \ldots \mathcal{F}_1(x),$$

where $x \in \mathcal{M}_1$.

What remains is the definition of the holes I_n. For this, fix some sequence $\underline{\alpha} = (\alpha_n)_{n\geq 1}$ with $\alpha_n \to 0$ and, independently of each other, choose uniformly distributed points ξ_n, $n \geq 1$ on $\cup_{i=1}^{K} [\mathcal{J}_{i,l}, \mathcal{J}_{i,r}]$. We will use the following three special choices:

1. Assume that $\xi_n \in [\mathcal{J}_{i,l}, \mathcal{J}_{i,r}]$, and denote $l_n = \mathcal{J}_{i,r} - \xi_n$. If $l_n > \alpha_n$, then put $I_n = (\xi_n, \xi_n + \alpha_n)$, otherwise put $I_n = (\xi_n, \mathcal{J}_{i,r}) \cup (\mathcal{J}_{i,l}, \mathcal{J}_{i,l} + \alpha_n - l_n)$, which is a subset of W_∞ for n large enough. With this particular choice, write

$$S_n^{\searrow}(x, \underline{\alpha}) = S_n^{\searrow}(x) = S_n(x, \mathcal{I}_n)$$

and

$$\mathcal{F}_n^{\searrow} = \mathcal{F}_n.$$

2. For each $1 \le k \le n$, let the random variables $\xi_n^{(k)}$ be independent and distributed like ξ_n. Assume that $\xi_n^{(k)} \in [\mathcal{J}_{i,l}, \mathcal{J}_{i,r}]$, and denote $l_n^{(k)} = \mathcal{J}_{i,r} - \xi_n^{(k)}$. If $l_n^{(k)} > \alpha_n$, then put $I_n^{(k)} = (\xi_n^{(k)}, \xi_n^{(k)} + \alpha_n)$, otherwise put $I_n^{(k)} = (\xi_n^{(k)}, \mathcal{J}_{i,r}) \cup (\mathcal{J}_{i,l}, \mathcal{J}_{i,l} + \alpha_n - l_n^{(k)})$, and finally $\mathcal{I}_n = (I_n^{(k)})_{1 \le k \le n}$. With this particular choice, write

$$S_n^{\equiv}(x, \underline{\alpha}) = S_n^{\equiv}(x) = S_n(x, \mathcal{I}_n).$$

3. Let $I_n = W_\infty$. With this particular choice, write

$$S_n^{(per)}(x) = S_n(x, \mathcal{I}_n),$$

and for a fixed x, define $S_t^{(per)}(x)$ for $t \ge 0$ as the piecewise linear, continuous extension of $S_n^{(per)}(x)$. Finally, write

$$\begin{aligned} \mathcal{F}^{(per)} &= \mathcal{F}_1, \\ \mathcal{M}^{(per)} &= \mathcal{M}_1. \end{aligned}$$

Here the first choice - the only really time dependent - is the most interesting one. In the second case, one has to redefine the whole trajectory segment $S_1^{\equiv}, \dots S_n^{\equiv}$ for each n, thus we have a sequence of billiards (in other words, the increments of $S_n^{\equiv}$ form a double array), while the third one is just a usual periodic Lorentz process.

There is a natural measure - the projection of the *Liouville measure* of the periodic billiard flow - on $\mathcal{M}^{(per)}$ which is invariant under $\mathcal{F}^{(per)}$. Denote the restriction of this measure to the two neighboring tori to the origin by $\mathbf{P}$. Note that $\mathbf{P}$ is finite, so normalize it to be a probability measure.

Finally, define $\mathcal{J} \subset \mathcal{M}^{(per)}$ as such points on the discrete phase space without any wall, from which before the forthcoming collision, the particle crosses $\cup_{i=1}^{K} (\mathcal{J}_{i,l}, \mathcal{J}_{i,r})$. Note that the finite horizon condition implies that $\mathcal{J}$ is bounded.

Now we proceed to the definition of the limiting processes. (The intuition behind their appearance in our result and in its proof as well will be explained after the formulation of the theorem.) Since we are going to have two very similar processes, we call both quasi-reflected Brownian motions and distinguish between them only in the abbreviation.

Consider a BM $\mathfrak{B} = (\mathfrak{B}_t)_{t \in [0,1]}$ with parameter σ on $[0, 1]$. Its local time at the origin is denoted by $\mathfrak{L} = (\mathfrak{L}_t)_{t \in [0,1]}$. That is,

$$\mathfrak{L}_t = \lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{\{|\mathfrak{B}_S| < \varepsilon\}} ds.$$

Now, given $\mathfrak{B}$, consider a Poisson Point Process Π with intensity measure $cd\mathfrak{L}$ with some positive constant c. The intuition behind this process is roughly speaking the following: since the local time describes the relative time process a Brownian motion spends in an infinitesimal neighborhood of a point, in our case of the origin, it can also be interpreted as telling us the density process of number of visits of the origin by the Lorentz process. Out of them only those visits are successful, i. e. resulting in getting to the other side of the wall, when the particle hits the hole, and these instants of time are precisely given by a Poisson process – according to the Poisson limit law. With probability one, the support of the measure $c(d\mathfrak{L})$ is $\mathfrak{Z}$, where $\mathfrak{Z} = \{s : 0 \le s \le 1 : \mathfrak{B}_s = 0\}$ is the zero set of $\mathfrak{B}$. Denote the points of Π by $P_1, P_2, ...$ in decreasing order. In fact, Π has finitely many points. If it has m points, then put $P_{m+1} = P_{m+2} = ... = 0$. Further, write $P_0 = 1$ and introduce a Bernoulli distributed random variable η with parameter $1/2$ (where the parameter means the probability of being equal to 1) which is independent of $\mathfrak{B}$ and Π.
Now, the process $\mathfrak{Q} = (\mathfrak{Q}_t)_{t\in[0,1]}$ with $\mathfrak{Q}_0 = 0$ and

$$\mathfrak{Q}_t = \begin{cases} (-1)^{\eta}|\mathfrak{B}_t| & \text{if } \exists n \in \mathbb{Z}_+ \cup \{0\} : t \in (P_{2n+1}, P_{2n}] \\ (-1)^{1-\eta}|\mathfrak{B}_t| & \text{otherwise} \end{cases}$$

is called the quasi-reflected Brownian motion with parameters c and σ, and denoted by qRBM(c,σ).
The definition of QRBM is similar to that of qRBM. The difference is that $c(d\mathfrak{L})$ now should be replaced by $c\frac{1}{\sqrt{t}}(d\mathfrak{L}_t)$. As a result, the Poisson process will have infinitely many points, which accumulate only at the origin. Now, denote by $P_1, P_2, \ldots$ these points in decreasing order (N. B.: there is no smallest one among them), put $P_0 = 1$ and define η and QRBM(c,σ) as before.

Remark 4.1. *One can easily check the following statements. The qRBM(c,σ) is almost surely continuous on $[0,1]$, homogeneous Markovian but not strong Markovian (think of the stopping time $T = \min\{t > 1/2 : \mathfrak{Q}_t = 0\} \wedge 1$) and $\mathfrak{Q}_t$ has Gaussian distribution with mean zero and variance $t\sigma^2$.*
The QRBM, similarly to the qRBM, is continuous, Markovian (however not time homogeneous), not strong Markovian, and has the same one dimensional distributions as qRBM. Contrary to the qRBM, the QRBM is self similar in the following sense: if $\mathfrak{Q}_t$ is a QRBM, then

$$(\mathfrak{Q}_t)_{t\in[0,1/p]} \stackrel{d}{=} \left(\frac{1}{\sqrt{p}}\mathfrak{Q}_{pt}\right)_{t\in[0,1/p]},$$

where $1 < p$.
Further, one can easily extend the definition of both processes to $\mathbb{R}_+$.

As usual, $C[0,1]$ will denote the space of continuous functions and $D[0,1]$ the Skorokhod space over $[0,1]$ (for the definition of the latter, we refer to [B68]). We will also use evident modifications, for instance, $D_{\mathbb{R}^2}[t_0,1]$ will denote the Skorokhod space of $\mathbb{R}^2$-valued functions over an interval $[t_0,1]$.
Let the function $\mathbf{W}_n^{\searrow}$ be the following: $\mathbf{W}_n^{\searrow}(k/n) = S_k^{\searrow}/\sqrt{n}$ for $0 \le k \le n$ and define $\mathbf{W}_n^{\searrow}(t)$ for $t \in [0,1]$ as its piecewise linear, continuous extension. Let $\mu_n^{\searrow}$ denote the measure on $C[0,1]$ induced by $\mathbf{W}_n^{\searrow}$, where the initial distribution, i.e. the distribution of x, is given by $\mathbf{P}$. Analogously, define $\mu_n^{\equiv}$ with $\mathbf{W}_n^{\equiv}$, where $\mathbf{W}_n^{\equiv}(k/n) = S_k^{\equiv}/\sqrt{n}$.
Now, we can formulate our main result.

Theorem 4.2. *There are positive constants σ and c_2 depending only on the periodic scatterer configuration, such that*

1. *if $\exists c > 0 : \alpha_n\sqrt{n} \to c$, then $\mu_n^{\searrow}$ converges weakly to the measure induced by $QRBM(c_2c, \sigma)$.*

2. *if $\exists c > 0 : \alpha_n\sqrt{n} \to c$, then $\mu_n^{\equiv}$ converges weakly to the measure induced by $qRBM(c_2c, \sigma)$.*

3. *if $\alpha_n\sqrt{n} \to 0$, then both $\mu_n^{\searrow}$ and $\mu_n^{\equiv}$ converge weakly to the convex combination of the measures induced by RBM and -RBM with weights 1/2.*

4. *if $\alpha_n\sqrt{n} \to \infty$, then both $\mu_n^{\searrow}$ and $\mu_n^{\equiv}$ converge weakly to the Wiener measure.*

Returning to the intuitive picture provided at the introduction of the process $qRBM(c_2c, \sigma)$, it, indeed, explains statement 2 of the theorem. Since, in the setup of the definition $\mu_n^{\searrow}$, the holes are not uniformly small, but are only decreasing as of order $\frac{1}{\sqrt{n}}$, the chances to get over the wall are larger but also decreasing as in the definition of $QRBM(c_2c, \sigma)$.

Instead of introducing the holes on the wall one could think about the wall as a *trapdoor*, i.e. sometimes it is open and then the particle crosses it without collisions, other times it is closed. If one opens the door randomly with probability α_n/c_1, then obtains the same result.

The analogue of Theorem 4.2 for random walks is, of course, easy to formulate in the following way. Define the stochastic process $\mathfrak{S}_n$ by: $Prob(\mathfrak{S}_0 = 1) = Prob(\mathfrak{S}_0 = -1) = 1/2$ and for $k > 0$:

$$Prob(\mathfrak{S}_{k+1} = \mathfrak{S}_k + 1|\mathfrak{S}_k \neq 0) = Prob(\mathfrak{S}_{k+1} = \mathfrak{S}_k - 1|\mathfrak{S}_k \neq 0) = 1/2,$$

and

$$Prob(\mathfrak{S}_{k+1} = \mathfrak{S}_{k-1}|\mathfrak{S}_k = 0) = 1 - \epsilon, \tag{4.1}$$

$$Prob(\mathfrak{S}_{k+1} = -\mathfrak{S}_{k-1}|\mathfrak{S}_k = 0) = \epsilon. \tag{4.2}$$

Here - and also in the sequel - $Prob$ stands for some abstract probability measure.

In the definition of $\mathfrak{S}_k$ put first $\epsilon = \alpha_k$ and denote by $\nu_n^{\searrow}$ the measure on $C[0,1]$ induced by $\mathbf{W}_n$, where $\mathbf{W}_n(k/n) = \mathfrak{S}_k/\sqrt{n}$ for $0 \leq k \leq n$ and is linearly interpolated in between. Analogously, define $\nu_n^{\equiv}$ for each n with the choice $\epsilon = \alpha_n$. Then, if we replace each μ with ν in Theorem 4.2, then the statement remains true (with $\sigma = c_2 = 1$), and can be proven the same way as we prove Theorem 4.2.

In the next section, we discuss some results concerning the periodic Lorentz process, that are necessary for proving Theorem 4.2. Finally, Section 3 contains the actual proof of Theorem 4.2.

4.2 Limit theorems for the periodic Lorentz Process

In this section, we present some facts about the periodic Lorentz process in a strip. Whereas Proposition 1 is simply a strengthening of Theorem 4.2 of [SzV04], Proposition 3 is a completely new statement interesting in itself. For later reference, we need to introduce some abstract stochastic processes.
As before, $\mathfrak{B} = (\mathfrak{B}_t)_{t\in[0,1]}$ denotes a BM with parameter σ (to be specified later) and $\mathfrak{L} = (\mathfrak{L}_t)_{t\in[0,1]}$

is its local time at the origin. We also use the notation $\mathfrak{B}^{a,t_0} = (\mathfrak{B}^{a,t_0}_t)_{t\in[t_0,1]}$ for a BM with parameter σ starting from a at time t_0; and $\mathfrak{L}^{a,t_0} = (\mathfrak{L}^{a,t_0}_t)_{t\in[t_0,1]}$ denotes its local time at the origin. Finally, $\mathfrak{B}^{a,t_0\rightsquigarrow b,t_1} = (\mathfrak{B}^{a,t_0\rightsquigarrow b,t_1}_t)_{t\in[t_0,t_1]}$ stands for a Brownian bridge with parameter σ starting from a at time t_0 and arriving at b at time t_1 (that is heuristically a BM with pinned down endpoints), and $\mathfrak{L}^{a,t_0\rightsquigarrow b,t_1} = (\mathfrak{L}^{a,t_0\rightsquigarrow b,t_1}_t)_{t\in[t_0,t_1]}$ is the local time of $\mathfrak{B}^{a,t_0\rightsquigarrow b,t_1}$ at the origin. For a thorough description of all these processes, see [RY91].
Similarly to the previous notations, denote by $L_{nt},\ t \in [0,1]$ the number of visits to $\mathcal{J}$ in the time interval $[1, \lfloor nt \rfloor]$, and L_H is the number of visits to $\mathcal{J}$ in the time interval H.

The first statement is a local limit theorem, formulated in a fashion tailored to our purposes. For this, let ϕ denote the density of the standard normal law. Now, the assertion reads as follows.

Proposition 4.3. *Fix some positive integer k and a subset $\mathcal{Z}$ of the set $\{1,2,\dots k\}$. For all $i \in \{1,2,\dots k\} \setminus \mathcal{Z}$, let $t_i \in [0,1]$, $b^{(i)} \in \mathbb{R}$ be real numbers such that if $i < j$, then $t_i < t_j$. Write $b_n^{(i)} := \lfloor b^{(i)}\sqrt{n} \rfloor$ and $n_i = \lfloor nt_i \rfloor$ for any positive integer n. Define $n_0 = b_n^{(0)} = 0$. For $i' \in \mathcal{Z}$, write $b_n^{(i')} = 0$ and choose some sequences $n_{i'}$ such that for any $i,j \in \{1,2,\dots k\}$ with $i<j$, $n_i \le n_j$ holds. Then*

$$\begin{aligned} &\mathbf{P}\quad \left(\forall i \in \{1,2,\dots k\} \setminus \mathcal{Z} : \lfloor S^{(per)}_{n_i}(x) \rfloor = b_n^{(i)}; \forall i' \in \mathcal{Z} : \left(\mathcal{F}^{(per)}\right)^{n_{i'}}(x) \in \mathcal{J}\right) \\ &= \quad c_0^{|\mathcal{Z}|} \prod_{i=1}^{k} \frac{\phi\left(\frac{b_n^{(i)} - b_n^{(i-1)}}{\sigma\sqrt{n_i - n_{i-1}}}\right) + o_i(1)}{\sigma\sqrt{n_i - n_{i-1}}}, \end{aligned}$$

with some constants σ and c_0 depending only on the periodic scatterer configuration. Further, there exist a sequence $\varsigma(n) \to 0$, such that $|o_i(1)| < \varsigma(n_i - n_{i-1})$ for all $i \in \{1,2,\dots k\}$.

Proposition 4.3 is an extension of Theorem 4.2 in [SzV04] in two aspects. On the one hand, it is formulated for k-tuples, while in [SzV04] it is only stated for $k = 1,2$. On the other hand, the error term is claimed to be uniform in the choice of n_i (it is, in fact, uniform in more general choices of $b_n^{(i)}$, but we only use it for $b_n^{(i)}$ of the form presented in Proposition 4.3). Both generalizations follow from the proof presented in [SzV04], thus we do not provide a formal proof here. We also note that Proposition 4.3 is an extension of Proposition 3.6 in [DSzV08], too. From now on, all stochastic processes derived from the BM will have parameter σ of Proposition 4.3.
The next important fact is the weak invariance principle for the position, which was first proven in [BS81] and [BSCh91].

Proposition 4.4.

$$\left(\frac{S^{(per)}_{nt}}{\sqrt{n}}\right)_{t\in[0,1]} \Rightarrow (\mathfrak{B}_t)_{t\in[0,1]},$$

where $\Rightarrow$ stands for weak convergence in the space $C[0,1]$.

The novelty of this section is in fact the following statement. The position of the particle and its local time at $\mathcal{J}$ jointly converge to a BM and its local time at the origin (the latter being multiplied by a constant). Formally,

Proposition 4.5.
$$\left(\frac{S_{nt}^{(per)}}{\sqrt{n}}, \frac{L_{nt}}{\sqrt{n}}\right)_{t\in[0,1]} \Rightarrow (\mathfrak{B}_t, c_0\mathfrak{L}_t)_{t\in[0,1]},$$
as $n \to \infty$ where the left hand side is understood as a random variable with respect to the probability measure $\mathbf{P}$, and $\Rightarrow$ stands for weak convergence in the Skorokhod space $D_{\mathbb{R}^2}[0,1]$.

Proof of Proposition 4.5. As usual, one has to check the convergence of finite dimensional distributions and the tightness (see [B68] for conditions implying weak convergence on some function spaces).
First, we prove the convergence of the finite dimensional distributions. Note that the convergence of the first coordinate follows from Proposition 4.4 (and even from Proposition 4.3), while the convergence of the second coordinate follows from an extended version of the proof of Theorem 9 in [DSzV08]. But the joint convergence is a stronger statement then the convergence of the individual coordinates, and it requires a formal proof.
To obtain the joint convergence, first observe that the convergence of the first coordinate (the rescaled position) is well known, even in the local sense (eg. Proposition 4.3). Thus we are going to prove that under the condition that the rescaled position is close to some specific number, the second coordinate converges to the desired limit. In order to do this computation, we need to define some new measures on $\mathcal{M}^{(per)}$.
First, choose $0 < t_0 < 1$, $a \in \mathbb{R}$ and write $a_n = \lfloor \sqrt{n}a \rfloor$. Restrict the measure $\mathbf{P}$ to such points x where $\lfloor S_{\lfloor nt_0 \rfloor}^{(per)}(x) \rfloor = a_n$ and rescale it to obtain a probability measure. The resulting measure is denoted by $\mathbf{P_n}$. Thus, with the notation

$$\mathcal{A}_1 = \mathcal{A}_1(n) = \{x : \lfloor S_{\lfloor nt_0 \rfloor}^{(per)}(x) \rfloor = a_n\} \subset \mathcal{M}^{(per)},$$

for $M \subset \mathcal{M}^{(per)}$ measurable sets, $\mathbf{P_n}(M) = \mathbf{P}(M \cap \mathcal{A}_1)/\mathbf{P}(\mathcal{A}_1)$. Then, choose $t_0 < t_1 < 1$, $b \in \mathbb{R}$ and write $b_n = \lfloor \sqrt{n}b \rfloor$. Define $\mathbf{Q_n}$ as the conditional measure of $\mathbf{P_n}$ on such points x, where $\lfloor S_{\lfloor nt_1 \rfloor}^{(per)}(x) \rfloor = b_n$. That is, with the notation

$$\mathcal{A}_2 = \mathcal{A}_2(n) = \{x : \lfloor S_{\lfloor nt_1 \rfloor}^{(per)}(x) \rfloor = b_n\} \subset \mathcal{M}^{(per)},$$

for $M \subset \mathcal{M}^{(per)}$ measurable sets, $\mathbf{Q_n}(M) = \mathbf{P_n}(M \cap \mathcal{A}_2)/\mathbf{P_n}(\mathcal{A}_2)$.
Now, we prove the following lemma.

Lemma 4.6.
$$L_{[nt_0,nt_1]}/(c_0\sqrt{n}) \Rightarrow \mathfrak{L}_{t_1}^{a,t_0 \rightsquigarrow b,t_1},$$
where $L_{[nt_0,nt_1]}/(c_0\sqrt{n})$ is understood as a random variable with respect to $\mathbf{Q_n}$. Similarly,
$$L_{nt_0}/(c_0\sqrt{n}) \Rightarrow \mathfrak{L}_{t_0}^{0,0 \rightsquigarrow a,t_0},$$
where $L_{t_0}/(c_0\sqrt{n})$ is understood as a random variable with respect to $\mathbf{P_n}$.

Proof of Lemma 4.6. We prove only the first statement, since the second one can be proven analogously. Similarly to the proof of Theorem 9 in [DSzV08], we are going to use the method of moments (see [B68], Chapter 1.7, Problem 4., for instance). That is, we are going to estimate

$$\mathbb{I}_n^k := \int (L_{[nt_0,nt_1]})^k d\mathbf{Q_n}.$$

For some fixed positive integer k and for $\lfloor nt_0 \rfloor = n_0 < n_1 < n_2 < ... < n_k < n_{k+1} = \lfloor nt_1 \rfloor$, define the set

$$\mathcal{A}_3 = \mathcal{A}_3(n_1, \ldots, n_k) = \{x : \{\left(\mathcal{F}^{(per)}\right)^{n_i} x, 1 \le i \le k\} \subset \mathcal{J}\} \subset \mathcal{M}^{(per)}.$$

Representing $L_{[nt_0,nt_1]}$ as a sum of $\lfloor nt_1 \rfloor - \lfloor nt_0 \rfloor + 1$ indicator variables, one concludes

$$\mathbb{I}_n^k \sim k! \sum_{\lfloor nt_0 \rfloor = n_0 < n_1 < n_2 < \ldots < n_k < n_{k+1} = \lfloor nt_1 \rfloor} \mathbf{Q_n}(\mathcal{A}_3(n_1, \ldots n_k)). \tag{4.3}$$

In fact, there should be $k-1$ similar sums, for $n_1 < \cdots < n_l$, $1 \le l \le k-1$, respectively, but the contribution of them is of smaller order of magnitude, as we will see in the forthcoming computation. Thus, we need to estimate $\mathbf{Q_n}(\mathcal{A}_3(n_1, \ldots n_k))$. By definition,

$$\mathbf{Q_n}(\mathcal{A}_3) = \frac{\mathbf{P}(\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3)}{\mathbf{P}(\mathcal{A}_1 \cap \mathcal{A}_2)}. \tag{4.4}$$

Using Proposition 4.3, one obtains the asymptotic equalities

$$\begin{aligned} &\frac{\mathbf{P}(\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3)}{\mathbf{P}(\mathcal{A}_1)} \\ \sim \quad &\frac{c_0^k}{\sigma^{k+1}(2\pi)^{\frac{k-1}{2}}} \phi\left(\frac{a_n}{\sigma\sqrt{n_1 - n_0}}\right) \phi\left(\frac{b_n}{\sigma\sqrt{n_{k+1} - n_k}}\right) \prod_{i=1}^{k+1} \frac{1}{\sqrt{n_i - n_{i-1}}}, \end{aligned} \tag{4.5}$$

and

$$\frac{\mathbf{P}(\mathcal{A}_1 \cap \mathcal{A}_2)}{\mathbf{P}(\mathcal{A}_1)} \sim \frac{\phi\left(\frac{b_n - a_n}{\sigma\sqrt{n(t_1 - t_0)}}\right)}{\sigma\sqrt{n(t_1 - t_0)}}. \tag{4.6}$$

Next, we substitute (4.4) by the product of the right hand sides of (4.5) and (4.6) in the sum of (4.3). The resulting sum is a Riemann sum which is asymptotically equal to the following Riemann integral

$$\begin{aligned} &n^{\frac{k}{2}} c_0^k k! \sigma^{-k} (2\pi)^{-\frac{k-1}{2}} \sqrt{t_1 - t_0} \left[\phi\left(\frac{b-a}{\sigma\sqrt{t_1 - t_0}}\right)\right]^{-1} \\ &\int \cdots \int_{0 < s_1 < s_2 < \ldots < s_k < t_1 - t_0} d\underline{s} \\ &\phi\left(\frac{a}{\sigma\sqrt{s_1}}\right) \frac{1}{\sqrt{s_1}} \frac{1}{\sqrt{s_2 - s_1}} \cdots \frac{1}{\sqrt{s_k - s_{k-1}}} \frac{1}{\sqrt{t_1 - t_0 - s_k}} \phi\left(\frac{b}{\sigma\sqrt{t_1 - t_0 - s_k}}\right), \end{aligned} \tag{4.7}$$

where $\underline{s} = (s_1, \ldots, s_k)$. Note that when we substituted (4.4) by the product of the right hand sides of (4.5) and (4.6), we made an error. Due to Proposition 4.3, this error is bounded by

$$C\sqrt{n}\varsigma(\min_i\{n_i - n_{i-1}\}) \prod_{j=1}^{k+1} \frac{1}{\sqrt{n_j - n_{j-1}}},$$

with some constant C. Thus, in order to see that (4.7) is asymptotically equal to $\mathbb{I}_n^k$, it remains to prove that

$$\sum_{\lfloor nt_0\rfloor=n_0<n_1<n_2<\ldots<n_k<n_{k+1}=\lfloor nt_1\rfloor} \sqrt{n}\varsigma(\min_i\{n_i-n_{i-1}\})\prod_{j=1}^{k+1}\frac{1}{\sqrt{n_j-n_{j-1}}} \tag{4.8}$$

is in $o\left(n^{\frac{k}{2}}\right)$. To prove this, pick $\varepsilon>0$ small and K such that $\varsigma(K)<\varepsilon$. The sum over indices $n_1,\ldots n_k$, where all n_i-n_{i-1} is larger then K, is asymptotically bounded by

$$\varepsilon n^{\frac{k}{2}}\int\cdots\int_{0<s_1<s_2<\ldots<s_k<t_1-t_0} d\underline{s}$$
$$\frac{1}{\sqrt{s_1}}\frac{1}{\sqrt{s_2-s_1}}\cdots\frac{1}{\sqrt{s_k-s_{k-1}}}\frac{1}{\sqrt{t_1-t_0-s_k}}.$$

Now, choose a subset H of the set $\{1,2\ldots k+1\}$, with $|H|=l\geq 1$. Then the sum over indices $n_1,\ldots n_k$, where $n_i-n_{i-1}\leq K$ for $i\in H$, and $n_i-n_{i-1}>K$ otherwise, is asymptotically bounded by $K^l n^{\frac{k-l}{2}}$ multiplied by an integral similar to the previous one. Thus, we have verified that (4.8) is in $o\left(n^{\frac{k}{2}}\right)$, which implies that $\mathbb{I}_n^k$ is asymptotically equal to (4.7). One can compute explicitly the integrals not involving the function ϕ. Namely, use the identity

$$\int_C^{t_1-t_0}\frac{(t_1-t_0-x)^l}{\sqrt{x-C}}dx=(t_1-t_0-C)^{l+1/2}\frac{\Gamma(l+1)\Gamma(1/2)}{\Gamma(l+3/2)}$$

$k-2$ times, to deduce the following formula from (4.7):

$$\mathbb{I}_n^k\sim n^{\frac{k}{2}}c_0^k k!\sigma^{-k}(2\pi)^{-\frac{k-1}{2}}\sqrt{t_1-t_0}\left[\phi\left(\frac{b-a}{\sigma\sqrt{t_1-t_0}}\right)\right]^{-1}\left[\Gamma\left(\frac{1}{2}\right)\right]^{k-1}\frac{1}{\Gamma\left(\frac{k-1}{2}\right)}$$
$$\iint_{0<s_1<s_2<t_1-t_0} ds_1ds_2\frac{\phi\left(\frac{a}{\sigma\sqrt{s_1}}\right)}{\sqrt{s_1}}\frac{\phi\left(\frac{b}{\sigma\sqrt{s_2-s_1}}\right)}{\sqrt{s_2-s_1}}(t_1-t_0-s_2)^{\frac{k}{2}-\frac{3}{2}} \tag{4.9}$$

for $k\geq 2$ (for $k=1$ a simpler formula holds). Finally, one can slightly simplify the formula (4.9), since $\Gamma(1/2)=\sqrt{\pi}$. In order to complete the method of moments, on the one hand, one needs to prove that

$$\lim_{n\to\infty}\mathbb{I}_n^k c_0^{-k}n^{-k/2}=\mathbb{J}^k, \tag{4.10}$$

where $\mathbb{J}^k$ is the k-th moment of $\mathfrak{L}_{t_1}^{a,t_0\rightsquigarrow b,t_1}$. It is easy to derive from the formulas computed in [B89] and [P99] that

$$Prob\left(\mathfrak{L}_{t_1}^{a,t_0\rightsquigarrow b,t_1}>y\right)=\exp\left[-\frac{1}{2\sigma^2(t_1-t_0)}\left(\left(|a|+|b|+\sigma^2y\right)^2-(b-a)^2\right)\right],$$

whence $\mathbb{J}^k$ can be expressed with an integral. On the other hand, one needs to prove that

$$\limsup_{k\to\infty}\left(\frac{\lim_{n\to\infty}\mathbb{I}_n^k c_0^{-k}n^{-k/2}}{k!}\right)^{1/k}<\infty \tag{4.11}$$

so as to verify that the limit distribution is uniquely determined. Observe that (4.9) immediately implies (4.11), but proving (4.10) turns out to be a nontrivial computation.

That is why we need to argue in a slightly different way. Namely, we are going to prove the first statement of the Lemma for random walks and then - since the moments for the random walk have the same asymptotic behavior, and these moments do converge - we arrive at the original statement.
To be more precise, pick a one dimensional simple symmetric random walk that starts from a_n and denote its position after $\lfloor n(t_1 - t_0) \rfloor$ steps by Y_n. Similarly, its total number of visits to the origin until $\lfloor n(t_1 - t_0) \rfloor$ is denoted by Z_n. Write f_1 for the probability density function of $\mathfrak{B}_{t_1}^{a,t_0}$ with σ being replaced by 1 (that is, $f_1(y) = \frac{1}{\sqrt{t_1 - t_0}}\phi(\frac{y-a}{\sqrt{t_1-t_0}})$). Similarly, $F_{2|1}(z|y)$ stands for the conditional cumulative distribution function of $\mathfrak{L}_{t_1}^{a,t_0}$ under the condition $\mathfrak{B}_{t_1}^{a,t_0} = y$, again with σ replaced by 1. Note that $F_{2|1}(z|y)$ is the cumulative distribution function of $\mathfrak{L}_{t_1}^{a,t_0 \rightsquigarrow y,t_1}$. Let y be a real number and $y_n = \lfloor y\sqrt{n} \rfloor$.
The following two statements are well known for random walks (see for example [B89] and [R90]):

$$Prob\left(Y_n < y_n, \frac{Z_n}{\sqrt{n}} < z\right) \to Prob\left(\mathfrak{B}_{t_1}^{a,t_0} < y, \mathfrak{L}_{t_1}^{a,t_0} < z\right), \tag{4.12}$$

and

$$\frac{\sqrt{n}}{2} Prob\left(Y_n \in \{y_n, y_n + 1\}\right) \to f_1(y). \tag{4.13}$$

We want to prove that

$$\frac{\sqrt{n}}{2} Prob\left(Y_n \in \{y_n, y_n + 1\}, \frac{Z_n}{\sqrt{n}} < z\right) =: p_n(y, z) \to f_1(y)F_{2|1}(z|y). \tag{4.14}$$

Note that in (4.13) and (4.14) the division by 2 is needed because of the periodicity of the random walk (i.e. it can return to the origin only in even number of steps). Also notice that using (4.13), one easily sees that (4.14) is equivalent to the first statement of the Lemma for simple symmetric random walks. Further, we mention that (4.14) is proved in [T99] for the case $y = a$. The well known local limit theorem for random walks, and our previous computation yield that the k-th moment of $Z_n/\sqrt{n}$, under the condition $Y_n \in \{y_n, y_n + 1\}$, have the same asymptotics as $\mathbb{I}_n^k c_0^{-k} n^{-k/2}$, with b replaced by y, and $\sigma = 1$. Thus (4.11), the method of moments and (4.13) imply that the distribution of $Z_n/\sqrt{n}$ - under the condition $Y_n \in \{y_n, y_n + 1\}$ - weakly converges to a uniquely determined limit distribution. Whence, $p(y, z) := \lim_{n\to\infty} p_n(y, z)$ exists. Now suppose that there exist some y_0, z_0 such that $p(y_0, z_0) \neq f_1(y_0)F_{2|1}(z_0|y_0)$. For this fixed z_0, $f_1(y)F_{2|1}(z_0|y)$ is clearly continuous in y. Further, since the integral representation in (4.9) is continuous in b, the method of moments imply that the limit distribution, as $n \to \infty$, of $Z_n/\sqrt{n}$ - under the condition $Y_n \in \{y_n, y_n + 1\}$ -, continuously depends on y (with respect to the weak topology). Hence, $p(y, z_0)$ is also continuous in y. Thus one can find an interval I containing y_0 such that $\int_I p(y, z_0)dy \neq \int_I f_1(y)F_{2|1}(z_0|y)dy$, which is a contradiction to (4.12). So we have verified (4.14). But (4.14) together with (4.11) implies (4.10) and the first assertion of the Lemma. □

Now, we prove of the convergence of one dimensional distributions by a standard argument. That is, we need that for any open intervals A, B,

$$\mathbf{P}\left(\frac{S_{nt_0}^{(per)}}{\sqrt{n}} \in A, \frac{L_{nt_0}}{\sqrt{n}} \in B\right) \to Prob\left(\mathfrak{B}_{t_0} \in A, c_0\mathfrak{L}_{t_0} \in B\right). \tag{4.15}$$

The second statement of Lemma 4.6 implies the local version of (4.15) in the first coordinate, namely

$$\begin{aligned}
&\sqrt{n}\mathbf{P}\left(S^{(per)}_{\lfloor nt_0\rfloor}=\lfloor x\sqrt{n}\rfloor,\frac{L_{nt_0}}{\sqrt{n}}\in B\right) \qquad (4.16)\\
=\quad &\frac{1}{\sigma\sqrt{t_0}}\phi\left(\frac{x}{\sigma\sqrt{t_0}}\right)Prob\left(c_0\mathfrak{L}^{0,0\rightsquigarrow x,t_0}_{t_0}\in B\right)+o(1).
\end{aligned}$$

Now, define the real function φ_n, by setting $\varphi_n(x)$ to be equal to (4.16). Note that for fix n, φ_n is constant on the intervals $[k/\sqrt{n},(k+1)/\sqrt{n})$ for any integer k. We have for any x,

$$\varphi_n(x)\to\frac{1}{\sigma\sqrt{t_0}}\phi\left(\frac{x}{\sigma\sqrt{t_0}}\right)Prob\left(c_0\mathfrak{L}^{0,0\rightsquigarrow x,t_0}_{t_0}\in B\right)=:\varphi(x).$$

Thus, by Fatou's lemma,

$$\liminf_n\int_A\varphi_n(x)dx\geq\int_A\varphi(x)dx=Prob\left(\mathfrak{B}_{t_0}\in A,c_0\mathfrak{L}_{t_0}\in B\right). \qquad (4.17)$$

Analogously,

$$\liminf_n\int_{A^c}\varphi_n(x)dx\geq\int_{A^c}\varphi(x)dx=Prob\left(\mathfrak{B}_{t_0}\in A^c,c_0\mathfrak{L}_{t_0}\in B\right). \qquad (4.18)$$

As it was already mentioned in the beginning of the proof of Lemma 4.6, the rescaled local times converge to the appropriate limit. Thus,

$$\begin{aligned}
&\int_A\varphi_n(x)dx+\int_{A^c}\varphi_n(x)dx=\mathbf{P}\left(\frac{L_{nt_0}}{\sqrt{n}}\in B\right)\to\\
&Prob\left(c_0\mathfrak{L}_{t_0}\in B\right)=\int_A\varphi(x)dx+\int_{A^c}\varphi(x)dx. \qquad (4.19)
\end{aligned}$$

Now, using (4.17), (4.18) and (4.19), we conclude that the inequalities in (4.17) and (4.18) are, in fact, equalities and the lim inf can be replaced by lim. This, together with the observation that the difference of $\int_A\varphi_n(x)dx$ and the left hand side of (4.15) is bounded by a constant times $n^{-1/2}$, implies (4.15).
The convergence of any finite dimensional marginals can be proven analogously, as we proved the one dimensional ones. The only main difference is that one needs a multiple version of the statements of Lemma 4.6, but its proof is also analogous.

Now we turn to the proof of tightness. Proposition 4.4 implies that the first coordinate converges weakly to the desired limit (in $C[t_0,1]$ thus in $D[t_0,1]$ as well), hence is tight, too. We are going to establish the tightness of the local times. Then it will follow that

$$\left(\frac{S^{(per)}_{nt}}{\sqrt{n}},\frac{L_{nt}}{\sqrt{n}}\right)_{t\in[0,1]}$$

is tight, by definition.
Since the process L_{nt} is nondecreasing in t, tightness, in fact, can be deduced from the convergence of finite dimensional distributions. Namely, Theorem 15.2 in [B68] yields that we only have to verify the following two statements:

1. For each $\eta > 0$ there is a $d \in \mathbb{R}$ such that

$$\mathbf{P}\left(\frac{L_n}{\sqrt{n}} > d\right) < \eta,\ \ n \geq 1. \tag{4.20}$$

2. For each positive η and ε there is a δ, $0 < \delta < 1$ and an integer n_0 such that

$$\mathbf{P}\left(w_{\frac{L_{nt}}{\sqrt{n}}}(\delta) \geq \varepsilon\right) \leq \eta,\ \ n \geq n_0. \tag{4.21}$$

   Here,

$$w_\psi(\delta) = \inf_{\{t_i\}} \max_{0<i\leq r} \big(\lim_{\tau \nearrow t_i} \psi(\tau) - \psi(t_{i-1})\big),$$

   where the infimum is taken over finite sets $\{t_i\}$, for which $0 < t_1 < ... < t_r = 1$, $t_i - t_{i-1} > \delta$ for all i.

Since we have just verified that $L_n/\sqrt{n}$ converges weakly, (4.20) follows.
Again, the convergence of finite dimensional distributions implies that for fix $\eta > 0$ and $\varepsilon > 0$ one can find $\delta > 0$ and n_0 such that for all $n > n_0$, $0 \leq k_1 \leq \lfloor 1/\delta \rfloor$

$$\mathbf{P}\left(\frac{\#\{k : nk_1\delta < k < n(k_1+1)\delta, S_k^{(per)} \in \mathcal{J}\}}{\sqrt{n\delta}} > \frac{\varepsilon}{\sqrt{\delta}}\right) < \eta\delta,$$

Now the equidistant partition $\{t_i\}$ is enough to verify (4.21). Thus we have finished the proof of Proposition 4.5.

□

4.3 Proof of Theorem 4.2

Note that, though in its spirit our statement is very close to the results of [DSzV09], their proof cannot be applied here since the limiting process is not strong Markovian (see Remark 4.1) thus leaving no chance to apply the martingale method. Thus we need to argue in a more direct way, using Proposition 4.5. In Subsection 4.3.1 we prove the first statement of the theorem. That proof with trivial modifications is easily applicable to cases 2 and 3. The only non trivial modification is needed in case 4, which is treated in Subsection 4.3.2. Everywhere in this Section, we also use the notations introduced in Section 2.

4.3.1 Proof of case 1

In order to prove the statement, we need some technical lemmas.

Lemma 4.7. *Let E and F be any Polish spaces, X, X_n any random variables taking values in the space E such that $X_n \Rightarrow X$. Then for any continuous function $f : E \to F$ one has $(X_n, f(X_n)) \Rightarrow (X, f(X))$ in the product topology.*

Proof. Pick any $U \subset E \times F$ open set and define $V = \{x \in E : (x, f(x)) \in U\}$. If $x \in V$, then one can find an open product set $U_x = E_x \times F_x \subset U$ containing $(x, f(x))$. Since $f^{-1}(F_x)$ is open, $x \in E_x \cap f^{-1}(F_x)$ is also open. Now $x \in E_x \cap f^{-1}(F_x) \subset V$ implies that V is open, too. Thus

$$Prob((X_n, f(X_n)) \in U) = Prob(X_n \in V)$$

and the Portmanteau Theorem (see [B68], for instance) yield the statement. □

Next, we prove the following Lemma which is an extension of Proposition 4.5.

Lemma 4.8.

$$\left(\frac{S_{nt}^{(per)}}{\sqrt{n}}, \frac{L_{nt}}{\sqrt{n}}, \int_{\tau=t_0}^{t} \frac{1}{\sqrt{\tau}} d\frac{L_{n\tau}}{\sqrt{n}}\right)_{t\in[t_0,1]} \Rightarrow \left(\mathfrak{B}_t, c_0\mathfrak{L}_t, c_0 \int_{\tau=t_0}^{t} \frac{1}{\sqrt{\tau}} d\mathfrak{L}_\tau\right)_{t\in[t_0,1]},$$

where the left hand side is understood as a random variable with respect to the probability measure **P** *and* $\Rightarrow$ *stands for weak convergence in the Skorokhod space* $D_{\mathbb{R}^3}[t_0, 1]$.

Proof. Use Proposition 4.5 and Lemma 4.7 with the choice

$$\begin{aligned} E &= \{\psi = (\psi_1, \psi_2) \in D_{\mathbb{R}^2}[t_0, 1] : \psi_2 \text{ is non decreasing}\}, \\ F &= D[t_0, 1] \\ f((\psi_1, \psi_2)) &= \left(\int_{\tau=t_0}^{t} \frac{1}{\sqrt{\tau}} d\psi_2\right)_{t\in[t_0,1]} \end{aligned}$$

to infer Lemma 4.8. □

Note that we needed to restrict the processes of Proposition 4.5 to $D_{\mathbb{R}^2}[t_0, 1]$ in order to the above stochastic integrals be finite. (This technical difficulty can be avoided in the proof of case 2.) Finally, we will need Le Cam's famous inequality which was proven in [LC60].

Lemma 4.9. *Assume* Σ_m *is the sum of* m *independent, non-identically distributed Bernoulli random variables* ε_j; $1 \le j \le m$ *such that* $Prob(\varepsilon_j = 1) = p_j$. *Then*

$$\sum_{k=0}^{\infty} \left| Prob(\Sigma_m = k) - e^{-\lambda}\lambda^k/k! \right| \le 2\sum_{j=1}^{m} p_j^2,$$

where $\lambda = p_1 + \cdots + p_m$.

Now, we can proceed to the proof of case 1 of Theorem 4.2. First, we are going to prove a simplified version of the assertion, namely, the convergence of the measures $\mu_n^{\searrow}$ restricted to $C[t_0, 1]$. Then the statement of the first part of the Theorem will follow easily.
Note that one can think about our model as having two sources of randomness. The first one is the choice of x and the second is the choice of ξ's. In Section 2, we were only dealing with the first source, but now we are going to treat the second one, as well.
It would be more convenient to consider $S_n^{\searrow}$ as if the time instants of the reflections on the wall W_k

$(1 \le k \le n)$ were not computed. Since Proposition 4.5 and the scatterer configuration being symmetric to the y-axis imply that $|\{i \le n : \mathcal{F}_i^{\searrow} \dots \mathcal{F}_1^{\searrow} \in \mathcal{J}\}|$ is asymptotically of order $\sqrt{n}$, the diffusively scaled limits of $S_n^{\searrow}$ and of this "modified $S_n^{\searrow}$" (i.e. when we do not count the reflections on the wall) have the same limit. Thus it is sufficient to prove our statement for the "modified $S_n^{\searrow}$" - which will also be denoted by $S_n^{\searrow}$ in the sequel.
Note that the assumption of the periodic scatterer configuration being symmetric implies

$$|S_n^{\searrow}| = |S_n^{(per)}|. \tag{4.22}$$

Now for fix x, define $p(nt)$ as the probability, generated by the choice of ξ_n's, of the event that $S_{\lfloor nt \rfloor}^{\searrow}(x) S_{\lfloor nt \rfloor+1}^{\searrow}(x) < 0$, i.e. after step number $\lfloor nt \rfloor$, the particle crosses the hole. Lemma 4.8 implies that - by Skorokhod's representation theorem, cf. [B68] - there exists a probability space $(\Omega, \mathbb{Q})$ together with random variables $(\tilde{X}_n, \tilde{Y}_n, \tilde{Z}_n)$ having the same joint distribution as

$$\left(\left(\frac{S_{nt}^{(per)}}{\sqrt{n}} \right)_{t \in [t_0,1]}, \left(\frac{c}{c_1} \int_{\tau=t_0}^{t} \frac{1}{\sqrt{\tau}} d \frac{L_{n\tau}}{\sqrt{n}} \right)_{t \in [t_0,1]}, \left(\int_{\tau=t_0}^{t} p(n\tau) dL_{n\tau} \right)_{t \in [t_0,1]} \right)$$

with respect to $\mathbf{P}$, and also with random variables $(\tilde{X}, \tilde{Y})$ having the same joint distribution as

$$\left((\mathfrak{B}_t)_{t \in [t_0,1]}, \left(\frac{cc_0}{c_1} \int_{\tau=t_0}^{t} \frac{1}{\sqrt{\tau}} d\mathfrak{L}_\tau \right)_{t \in [t_0,1]} \right),$$

such that $(\tilde{X}_n, \tilde{Y}_n) \to (\tilde{X}, \tilde{Y})$ $\mathbb{Q}$-almost surely. Here, $c = \lim_n \alpha_n \sqrt{n}$ Now, for $\mathbb{Q}$-almost all $\omega \in \Omega$ we define the measures $\nu(\omega)$, $\nu_n(\omega)$, $\lambda_n(\omega)$ on $C[t_0, 1]$ in the following way. Consider the modulus of $\tilde{X}(\omega)$, i.e. $|\tilde{X}(\omega)| \in C[t_0, 1]$ (if $\tilde{X}(t_0)(\omega) > 0$; otherwise consider $-|\tilde{X}(\omega)|$), pick a Poisson point process - on some abstract probability space $(\Omega_\omega, \mathbb{Q}_\omega)$ - with intensity measure $d\tilde{Y}(\omega)$, and denote its point by $P_1 < P_2 <$ ... N.b. there are finitely many points. If it has m points, put $P_{m+1} = 1$. Now reflect the subgraph of $|\tilde{X}(\omega)|[P_{2i+1}, P_{2i+2}]$ to the origin for each i (if $\tilde{X}(t_0)(\omega) > 0$; otherwise reflect $-|\tilde{X}(\omega)|$). The distribution of the resulting random function - with respect to $\mathbb{Q}_\omega$ - generates a measure on $C[t_0, 1]$ which we denote by $\nu(\omega)$. The construction of $\nu_n(\omega)$ is similar, with two differences. The first is that one should replace $\tilde{X}$ and $\tilde{Y}$ by $\tilde{X}_n$ and $\tilde{Y}_n$ and the second is that instead of the Poisson point process, one introduces independent Bernoulli random variables for each discontinuity of the function $\tilde{Y}_n(\omega)$ with parameters being equal to the size of jump of $\tilde{Y}_n(\omega)$ at the corresponding discontinuity. Then denote by $P_1 < P_2 < \dots$ the positions, where the Bernoulli random variables are equal to 1. Finally, $\lambda_n(\omega)$ is defined the same way as $\nu_n(\omega)$ with $\tilde{Y}_n$ being replaced by $\tilde{Z}_n$.
Using Lemma 4.9, one can infer that for $\mathbb{Q}$-almost all ω, $\nu_n(\omega) \Rightarrow \nu(\omega)$ on $C[t_0, 1]$. Further, $\alpha_n \sqrt{n} \to c$ implies that for any fixed x and $\varepsilon > 0$, if $L_{\lfloor n\tau \rfloor - 1} < L_{\lfloor n\tau \rfloor}$, i.e. in $\lfloor n\tau \rfloor$ steps the particle arrives to $\mathcal{J}$, then $|p(\lfloor n\tau \rfloor) - c/(c_1 \sqrt{\lfloor n\tau \rfloor})| < \varepsilon/\sqrt{n}$ assuming that n is large enough. Whence, one can naturally couple the Bernoulli distributed random variables used by the definition of ν_n and λ_n in such a way that the resulting random functions in $C[t_0, 1]$ coincide on a subset of Ω_ω, whose $\mathbb{Q}_\omega$ measure tends to 1 as $n \to \infty$. Consequently, $\lambda_n(\omega) \Rightarrow \nu(\omega)$ on $C[t_0, 1]$ for $\mathbb{Q}$-almost all ω, too.

Define the measures ϱ and ϱ_n on $C[t_0, 1]$ by

$$\begin{aligned} \varrho(A) &= \int_\Omega \nu(\omega)(A) d\mathbb{Q}(\omega), \\ \varrho_n(A) &= \int_\Omega \lambda_n(\omega)(A) d\mathbb{Q}(\omega). \end{aligned}$$

Using that $\lambda_n(\omega) \Rightarrow \nu(\omega)$ for $\mathbb{Q}$-almost all ω, Fatou's lemma and the Portmanteau theorem, we obtain for any $A \subset C[t_0, 1]$ open set:

$$\begin{aligned} \liminf_n \varrho_n(A) &= \liminf_n \int_\Omega \lambda_n(\omega)(A) d\mathbb{Q}(\omega) \\ &\geq \int_\Omega \liminf_n \lambda_n(\omega)(A) d\mathbb{Q}(\omega) \geq \int_\Omega \nu(\omega)(A) d\mathbb{Q}(\omega) = \varrho(A). \end{aligned}$$

Whence - by the Portmanteau theorem, again - $\varrho_n \Rightarrow \varrho$ on $C[t_0, 1]$.
Observe that by construction, ϱ is the measure on $C[t_0, 1]$ generated by a QRBM$(cc_0/c_1, \sigma)$. Similarly, ϱ_n is the restriction of $\mathbf{W}_n^{\searrow}$ to $C[t_0, 1]$.
Now, one can easily prove the first part of the Theorem. Since the choice of t_0 was arbitrary, a limit theorem of any finite dimensional distributions is implied by the above computation. The tightness is also easy since the moduli of the random functions are tight. Thus we have finished the proof of the first part of the Theorem (in fact, with the constant $c_2 = c_0/c_1$).

4.3.2 Proof of case 4

As in the previous subsection, the tightness is trivial since the moduli of the random functions are tight. The convergence of one dimensional distributions follows from symmetry and Proposition 4.3. Here, we are only going to prove the convergence of two dimensional marginals since the convergence of any finite dimensional ones can be proven similarly.
The idea of the proof is that we know the convergence of $(|S^{\searrow}_{\lfloor nt_0 \rfloor}|/\sqrt{n}, |S^{\searrow}_{\lfloor nt_1 \rfloor}|/\sqrt{n})$ to the desired limit, thus we only need to care about the sign. For the latter, assume that $S^{\searrow}_{\lfloor nt_0 \rfloor}/\sqrt{n}$ is in a fixed positive interval, while $|S^{\searrow}_{\lfloor nt_1 \rfloor}|/\sqrt{n}$ is in another fixed positive interval. Using Proposition 4.5, and the results of the previous subsection, we can estimate the asymptotic probability of the local time being zero under the above condition. If the local time is zero, then the trajectory avoids the origin, hence $S^{\searrow}_{\lfloor nt_1 \rfloor} > 0$. If not, then the particle arrives at the origin eventually in $[nt_0, nt_1]$, and we need to verify that at time nt_1, it will end up in the positive half line with probability $1/2$. The heuristic reason for this is that once it is near the origin, since $\alpha_n \sqrt{n}$ is large, it will cross the holes many times, and thus forget that it came from the positive half-line. This argument will imply that the weak limit must be the two dimensional marginal of the BM.
Let us make the above argument precise. To do so, we will use the notations of the previous subsection. Especially, introduce the modification of $S_n^{\searrow}$ as in the previous subsection. Thus (4.22) still holds. Fix

$0 < t_0 < t_1 \leq 1$ and J_0, J_1 compact subintervals of $\mathbb{R}_+ \cup \{0\}$. Our aim is to prove that

$$\mathbf{P}\left(\frac{S^{\searrow}_{\lfloor nt_0 \rfloor}}{\sqrt{n}} \in J_0, \frac{S^{\searrow}_{\lfloor nt_1 \rfloor}}{\sqrt{n}} \in J_1\right) \to Prob\left(\mathfrak{B}_{t_0} \in J_0, \mathfrak{B}_{t_1} \in J_1\right), \tag{4.23}$$

and

$$\mathbf{P}\left(\frac{S^{\searrow}_{\lfloor nt_0 \rfloor}}{\sqrt{n}} \in J_0, \frac{S^{\searrow}_{\lfloor nt_1 \lfloor}}{\sqrt{n}} \in -J_1\right) \to Prob\left(\mathfrak{B}_{t_0} \in J_0, \mathfrak{B}_{t_1} \in -J_1\right), \tag{4.24}$$

as $n \to \infty$. Once we verify (4.23) and (4.24), by symmetry, they will also hold true for J_0 being a compact interval in $\mathbb{R}_- \cup \{0\}$, and hence the convergence of two dimensional marginals will follow. Define the probabilities

$$p_{J_0,J_1} = Prob\left(\forall s : t_0 < s < t_1 : \mathfrak{B}_s > 0 | \mathfrak{B}_{t_0} \in J_0, |\mathfrak{B}_{t_1}| \in J_1\right).$$

Now let A be the set of functions ψ in $C[0,1]$ for which $\psi(t_0) \in J_0$, $\forall s : t_0 < s < t_1 : \psi(s) > 0$, and $\psi(t_1) \in J_1$. Then the Wiener measure of ∂A is zero, thus Proposition 4.4 implies that

$$\mathbf{P}\left(\forall t_0 < s < t_1 : S^{(per)}_{ns} > 0 | \frac{S^{(per)}_{\lfloor nt_0 \rfloor}}{\sqrt{n}} \in J_0, \frac{|S^{(per)}_{\lfloor nt_1 \rfloor}|}{\sqrt{n}} \in J_1\right) \to p_{J_0,J_1}, \tag{4.25}$$

as $n \to \infty$. On the other hand, the strong Markov property of the BM obviously implies

$$Prob\left(\mathfrak{B}_{t_1} \in J_1 | \mathfrak{B}_{t_0} \in J_0, \exists s : t_0 < s < t_1 : \mathfrak{B}_s = 0, |\mathfrak{B}_{t_1}| \in J_1\right) = \frac{1}{2}. \tag{4.26}$$

Now, with the notation

$$\mathcal{A}_4(n) = \{x : \frac{S^{\searrow}_{\lfloor nt_0 \rfloor}(x)}{\sqrt{n}} \in J_0, \exists t_0 < s < t_1 : S^{(per)}_{ns}(x) = 0, \frac{|S^{\searrow}_{\lfloor nt_1 \rfloor}(x)|}{\sqrt{n}} \in J_1\},$$

we want to prove that

$$\mathbf{P}\left(\frac{S^{\searrow}_{\lfloor nt_1 \rfloor}}{\sqrt{n}} \in J_1 | \mathcal{A}_4(n)\right) \to \frac{1}{2}. \tag{4.27}$$

Note that combining Proposition 4.4, (4.22), (4.25), (4.26) and (4.27), one can deduce (4.23) and (4.24). Thus it remains to prove (4.27).

To do so, first observe that by Proposition 4.5, for every $\varepsilon > 0$ there exists $\delta > 0$ and N such that for all $n > N$,

$$\mathbf{P}\left(L_{[nt_0,nt_1]} > \delta\sqrt{n} | \mathcal{A}_4(n)\right) > 1 - \varepsilon. \tag{4.28}$$

Now consider the Markov transition matrices

$$A_p = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}$$

on the space $\{+,-\}$ and a time dependent Markov chain M_k such that $M_0 = +$ and the transition between k and $k+1$ is described by A_{p_k} with some numbers p_k. Now, for a fixed $x \in \mathcal{M}^{(per)}$ and n, define $D_k(x)$ as the k-th leftmost discontinuity of the function $s \to L_{\lfloor ns \rfloor}(x)$ on $s \in [t_0, t_1)$. With the choice $p_k(x) = \alpha_{nD_k(x)}/c_1$, $1 \leq k \leq L_{[\lfloor nt_0 \rfloor, \lfloor nt_1 \rfloor - 1]}(x)$ for each x, one easily sees that for n large enough, the probability in (4.27) is equal to

$$\frac{1}{\mathbf{P}(\mathcal{A}_4(n))} \int_{\mathcal{A}_4(n)} Prob(M_{L_{[\lfloor nt_0 \rfloor, \lfloor nt_1 \rfloor - 1]}(x)} = +) d\mathbf{P}(x). \tag{4.29}$$

If fact, this is only true for the case of $\mu_n^{\searrow}$, while in the case of $\mu_n^{\equiv}$, one needs to set $p_k(x) = \alpha_n/c_1$. On the other hand, elementary computations show that if one selects sequences $B(n) \to \infty, m(n) \to \infty$ and non-negative numbers $p_{k,n}, 1 \leq n, 1 \leq k \leq m(n)$, then with the transition matrices corresponding to $p_{1,n}, ..., p_{m(n),n}$,

$$\begin{aligned} & Prob(M_{m(n)} = +) \\ = & \begin{pmatrix} 1 & 0 \end{pmatrix} A_{p_{1,n}} \dots A_{p_{m(n),n}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ = & \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \prod_{k=1}^{m(n)} (1 - 2p_{k,n}) \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \\ = & 1/2 + o(1), \end{aligned}$$

as $n \to \infty$. Further, $o(1)$ converges to zero uniformly in m and p_k if $m(n) > \delta\sqrt{n}$ and $\min_{1 \leq k \leq m(n)}\{p_{k,n}\sqrt{n}\} > B(n)$. Now, choose $B(n) = \min_{N \geq \lfloor nt_0 \rfloor}\{\alpha_N \sqrt{N}/c_1\}$. This estimation together with (4.28) and (4.29) yields (4.27).

Chapter 5

A central limit theorem for time-dependent dynamical systems

5.1 Introduction

Time-dependent dynamical systems appear in various applications. Recently, [OSY09] could establish exponential loss of memory for expanding maps and, moreover, for one-dimensional piecewise expanding maps with slowly varying parameters. It also provided interesting motivations and examples for the problem. We also mention that the memory loss result of [OSY09] has been extended very recently to two dimensional Anosov diffeomorphisms in [S11]. For us - beside their work - an additional incentive was the following question: bound the correlation decay for a planar finite-horizon Lorentz process which is periodic apart form the 0-th cell; in it, the Lorentz particle encounters a particular scatterer of the 0-th cell moderately displaced at its each subsequent return to the 0-th cell. (Slightly similar is the situation in the Chernov-Dolgopyat model of Brownian Brownian motion, where - between subsequent collisions of the light particle with the heavy one - the heavy particle slightly moves away, cf. [ChD09b].)

The results of [OSY09] say that - for sequences of uniformly uniformly expanding maps - distances of images of a pair of different initial measures converge to 0 exponentially fast. In the same setup it is also natural to expect that probability laws of the Birkhoff-type partial sums of some given function - scaled, of course, by the square roots of their variances - are approximately Gaussian. The main theorem of this Chapter provides a positive answer though our conditions are surprisingly more restrictive than those of [OSY09]. Let us explain the difficulty and some related results.

In functional central limit theorems for functions of autonomous chaotic deterministic systems the zero-cohomology condition is - in quite a generality - known to be necessary and sufficient for the vanishing of the limiting variance (see [L96] for instance). For time-dependent systems, however, such a condition is only known for almost all versions of random dynamical systems (see [ALS09] and [CBR12]) and for other models the situation can be and definitely is completely different. In fact, for time-dependent systems, first [B95] had proved a Gaussian approximation theorem in quite a generality; its author, however, assumed that the variances of the Birkhoff-type partial sums tend to ∞ sufficiently fast; the paper, however, did not provide any example when this condition would hold. The more recent work [CR07] proves under some reasonable conditions a dichotomy: either the variances are bounded or the Gaussian approximation holds; the article also provides an example for the latter in the case when the time dependent maps are smaller and smaller perturbations of a given map. But still there is no general method for ascertaining whether the variance is bounded or not. Finally we note that [KK05] has interesting results for higher order cohomologies but its setup is different.

The present work is, in fact, the first one where non-random examples are also found, that are not small perturbations of a given map. The proof of our main theorem uses martingale approximation technique in the form introduced in [SV05] for treating additive functions of inhomogeneous Markov chains. The organization of this Chapter is simple: Section 5.2 contains our main theorem and provides examples when it is applicable. Section 5.2 is devoted to the proof of the theorem.

5.2 Results

Let A be a set of numbers and $(X, \mathcal{F}, \mu)$ a probability space. For each $a \in A$ define $T_a : X \to X$. Suppose that μ is invariant for all T_a's. Now consider a sequence of numbers from A, i.e. $\underline{a} : \mathbb{N} \to A$. Our aim is to prove some kind of central limit theorem for the sequence

$$f \circ T_{a_1}, f \circ T_{a_2} \circ T_{a_1}, ...$$

with some nice function $f : X \to \mathbb{R}$.

As usual,

$$\hat{T}_a g(x) = g(T_a x)$$

and $\hat{T}^*$ is the $L^2(\mu)$-adjoint of $\hat{T}$ (the so called Perron-Frobenius operator). Further, introduce the notation

$$\hat{T}_{[i..j]} = \begin{cases} \hat{T}_{a_i} \dots \hat{T}_{a_j} & \text{if } i \leq j \\ Id & \text{otherwise} \end{cases}$$

and for simplicity write $\hat{T}_{[j]} = \hat{T}_{[1..j]}$.

Similarly, define

$$\hat{T}^*_{[i..j]} = \begin{cases} \hat{T}^*_{a_j} \dots \hat{T}^*_{a_i} & \text{if } i \leq j \\ Id & \text{otherwise} \end{cases}$$

and $\hat{T}^*_{[j]} = \hat{T}^*_{[1..j]}$.

Further, define σ-algebras $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{F}_i = (T_{a_1})^{-1} \dots (T_{a_i})^{-1} \mathcal{F}_0$. We will need this sequence of σ-algebras to form a decreasing systems (cf. Assumption 2 of Theorem 5.1), restricting our approach to non-invertible maps. Let us assume that there is a Banach space $\mathcal{B}$ of $\mathcal{F}$-measurable functions on X such that $\|g\| := \|g\|_{\mathcal{B}} \geq \|g\|_\infty$ for all $g \in \mathcal{B}$.

Finally, for the fixed function f, introduce the notation

$$u_k = \sum_{i=1}^{k} \hat{T}^*_{[i+1..k]} f.$$

With the above notation, our aim is to prove a limit theorem for $S_n(x) = \sum_{k=1}^n \hat{T}_{[k]} f(x)$.

Theorem 5.1. *Assume that f, $\underline{a}$ and T_b, $b \in A$ satisfy the following assumptions.*

1. $\int f d\mu = 0$.

2. *T_b is onto but not invertible for all $b \in A$.*

3. *$f \in \mathcal{B}$ and there exist $K < \infty$ and $\tau < 1$ such that for all sequences $\underline{b}$ and for all k, $\|\hat{T}_{b_1}^* ... \hat{T}_{b_k}^* f\| < K\tau^k \|f\|$.*

4. (accumulated transversality) *Define χ_k as the L^2-angle between u_k and the subspace of $(T_{a_{k+1}})^{-1}\mathcal{F}_0$-measurable functions. Then*
$$\sum_{k=1}^{N} \min_{j \in \{k,k+1\}} (1 - \cos^2(\chi_j))$$
converges to ∞ as $N \to \infty$.

Then
$$Var(S_n) \to \infty$$
and
$$\frac{S_n(x)}{\sqrt{Var(S_n)}}$$
converges weakly to the standard normal distribution, where x is distributed according to μ.

Assumption 3 roughly tells that there is an eventual spectral gap of the operators $\hat{T}_{a_j}^*$ which is quite a natural assumption. Assumption 4 guarantees that there is no much cancellation in S_n, for instance f cannot be in the cohomology class of the zero function when $|A| = 1$.
Before proving the statement let us examine a special case.

Example 5.2. *Define $(X, \mathcal{F}, \mu) = (S^1,$ Borel, Leb$)$, $A = \{2, 3, \dots\}$, $T_a(x) = ax(\mod 1)$, $\mathcal{B} = C^1 = C^1(S^1)$,*
$$\|g\| := \sup_{x \in S^1} |g(x)| + \sup_{x \in S^1} |g'(x)|.$$
Fix a non constant function $f \in C^1$ satisfying $\int f dx = 0$. Then there exists some integer $L = L(f)$ such that with all sequences $\underline{a}$ for which
$$\#\{k : \min\{a_k, a_{k+1}, a_{k+2}\} > L\} = \infty$$
the assumptions of Theorem 5.1 are fulfilled.

Proof of Example 5.2. It is easy to see that for all $g \in C^1$ with zero mean, and for all $\underline{b} : \mathbb{N} \to A$,
$$\|\hat{T}_{b_1}^* g\| \le 2b_1^{-1} \|g\|$$
and similarly,
$$\|\hat{T}_{b_1}^* \dots \hat{T}_{b_k}^* g\| \le 2 \cdot 2^{-k} \|g\|. \tag{5.1}$$

Hence Assumption 3 is fulfilled.
In order to check Assumption 4, select $x, y \in S^1$, $\varepsilon > 0, \delta > 0$ such that

$$\min_{z\in[x,x+\varepsilon]} f(z) > \delta + \max_{z\in[y,y+\varepsilon]} f(z).$$

This can be done since f is not constant. Now choose

$$L > \max\{\frac{16\|f\|}{\delta}, \frac{2}{\varepsilon}\}.$$

Whence

$$\|\hat{T}_L^* f\| \leq \delta/8.$$

Thus if $a_k > L$, then

$$\|\sum_{i=1}^{k-1} \hat{T}^*_{[i+1..k]} f\| < 3\delta/8$$

is true independently of the choice of $a_1, \ldots a_{k-1}$. This yields

$$\min_{z\in[x,x+\varepsilon]} u_k(z) > \delta/4 + \max_{z\in[y,y+\varepsilon]} u_k(z).$$

Since $L > \frac{2}{\varepsilon}$, for all g which is $(T_{a_{k+1}})^{-1}\mathcal{F}_0$ measurable (with $a_{k+1} > L$), one can find $h : [0, \varepsilon/2) \to \mathbb{R}$ and $\varepsilon_1 \leq \varepsilon/2$ such that $g(y + \varepsilon_1 + z) = g(x + z) = h(z)$ for all $z \in [0, \varepsilon/2)$. Hence,

$$\begin{aligned}
&\|u_k - g\|_2^2 \\
\geq\ & \int_x^{x+\varepsilon/2} (u_k(z) - g(z))^2\, dz + \int_{y+\varepsilon_1}^{x+\varepsilon_1+\varepsilon/2} (u_k(z) - g(z))^2\, dz \\
=\ & \int_0^{\varepsilon/2} (u_k(x + z) - h(z))^2\, dz + \int_0^{\varepsilon/2} (u_k(y + \varepsilon_1 + z) - h(z))^2\, dz \\
\geq\ & \frac{1}{2}\int_0^{\varepsilon/2} (u_k(x + z) - u_k(y + \varepsilon_1 + z))^2\, dz \geq \frac{\delta^2\varepsilon}{64}
\end{aligned} \tag{5.2}$$

Since

$$\|u_k\|_2 < \|u_k\|$$

is bounded, (5.2) implies that $(1 - \cos^2(\chi_k))$ is uniformly bounded away from zero if $\min\{a_k, a_{k+1}\} > L$. Hence, Assumption 4 is fulfilled if there exist infinitely many indices k such that

$$\min\{a_k, a_{k+1}, a_{k+2}\} > L.$$

□

In Example 5.2, expanding maps with large derivative were needed in order to obtain the Gaussian approximation. Naturally arises the question that what happens in the case when one uses only finitely many dynamics, for instance, only T_2 and T_3 of Example 5.2. That is why we discuss the following example.

Example 5.3. *Define $X, \mathcal{F}, \mu, A, T_b, \mathcal{B}$ as in Example 5.2. If $\underline{a}$ is a sequence for which there is a $b \in A$ such that for all integer K, one can find a k for which*

$$a_k = a_{k+1} = ... = a_{k+K-1} = b,$$

and $f \in \mathcal{B}$, $\int f = 0$ is any function for which the equation $f = \hat{T}_b u - u$ has no solution u, then the assumptions of Theorem 5.1 are fulfilled.

Proof of Example 5.3. It is enough to verify Assumption 4. To do so, for $K \in \mathbb{Z}_+$ pick k such that

$$a_{k-K} = a_{k-K+1} = ... = a_{k+2} = b. \tag{5.3}$$

Then (5.1) implies that

$$\|u_j - \sum_{i=0}^{\infty} \left(\hat{T}_b^*\right)^i f\| < C2^{-K} \tag{5.4}$$

holds for $j = k, k+1$ with some C uniformly in K. Now, if $g := \sum_{i=0}^{\infty} \left(\hat{T}_b^*\right)^i f$ is not $(T_b)^{-1}\mathcal{F}_0$-measurable, then necessarily its L^2-angle with those functions is positive. Since (5.3) and (5.4) hold for infinitely many k's, $\min\{\chi_k, \chi_{k+1}\}$ exceeds a uniform positive number infinitely many times, inferring Assumption 4. On the other hand, if g is $(T_b)^{-1}\mathcal{F}_0$-measurable, then $g = \hat{T}_b \hat{T}_b^* g$ and $g - \hat{T}_b^* g = f$ imply that for $u = \hat{T}_b^* g$, $\hat{T}_b u - u = f$. □

Note, that in Example 5.3, $Var(S_n)$ can be arbitrarily small. Indeed, pick a C^1 function f, for which $f = \hat{T}_3 u - u$ has no solution u, but there is some v such that $f = \hat{T}_2 v - v$. Now, pick a sequence of integers $d_l, l \in \mathbb{N}$, $d_l \to \infty$ fast enough, and define

$$a_k = \begin{cases} 3 & \text{if } \exists l : d_l \le k < d_l + l \\ 2 & \text{otherwise.} \end{cases}$$

It is easy to see that (5.1) implies $\mathbb{E}(|\hat{T}_{[i]} f \cdot \hat{T}_{[j]} f|) \le 2^{|i-j|+1} \|f\|^2$ (formally it follows from (5.16)), which in turn yields that $Var(S_k)$ is bounded by some constant times k. Now, with the notation $l_n := \max\{l : d_l \le n\}$, write

$$\begin{aligned} Var(S_n) \quad \le \quad & 4Var(S_{d_{l_n-1}+l_n}) + 4Var(S_{d_{l_n}} - S_{d_{l_n-1}+l_n}) \\ & +4Var(S_{d_{l_n}+l_n} - S_{d_{l_n}}) + 4Var(S_n - S_{d_{l_n}+l_n}). \end{aligned}$$

On the other hand, $f = \hat{T}_2 v - v$ implies that $\hat{T}_2 f + ... + \hat{T}_2^m f$ is uniformly bounded in m. Thus the second and the last term in the above sum are bounded. Whence $Var(S_n)$ is smaller than some constant times d_{l_n-1}. Especially, if $d_l = 2^{2^{2^l}}$, then

$$\frac{Var(S_n)}{n^\alpha} \to 0$$

as $n \to 0$ for any α positive. Note that in this case the conditions of [B95] for the Gaussian approximation are not met.

We mention that the choice of T_a's in the above examples are very special (especially, they are commuting). In fact, we used the explicit form of them - the fact that a T_a-measurable function is $1/a$-periodic - only in Example 5.2. Indeed, Example 5.3, and the discussion after it, can be formulated with other dynamics that satisfy Assumptions 1-3. For instance, one could use T_2 and replace T_3 by the map

$$\tilde{T}_3(x) = \begin{cases} T_3(x) & \text{if } x \in [0, 2/3] \\ T_3(x) + a \pmod 1 & \text{otherwise} \end{cases}$$

with some constant $a \neq 0 \pmod 1$. The resulting maps are not commuting any more, but the Lebesgue measure is still invariant for both of them and they still satisfy Assumption 3 with $\mathcal{B}$ being the Banach space of functions of bounded variation. The latter statement is a consequence of [CR07]. For more examples of sets of maps that satisfy Assumption 3, we refer to [CR07].

5.3 Proof of Theorem 5.1

This section is devoted to the proof of Theorem 5.1.

As in [CR07], [L96] and [SV05], the proof is based on martingale approximation. That is, we are going to define a reverse martingale - just like in [CR07] and [L96] -, verify the conditions of some abstract martingale CLT, and prove that the difference between $S_n/\sqrt{Var(S_n)}$ and the reverse martingale is negligible.

First, observe that

$$\hat{T}^*_{[n]}\hat{T}_{[n]} = Id \tag{5.5}$$

and

$$\hat{T}_{[n]}\hat{T}^*_{[n]} \tag{5.6}$$

is the orthogonal projection onto the $\mathcal{F}_n$ measurable functions (for the proof of the latter, see [L96]). Now we proceed to the definition of our approximating reverse martingale, which is analogous to the one of [SV05]. To do so, first define $Z_0 = 0$ and

$$Z_k = \sum_{i=1}^k \mathbb{E}\left[\hat{T}_{[i]}f|\mathcal{F}_k\right] = \sum_{i=1}^k \hat{T}_{[k]}\hat{T}^*_{[k]}\hat{T}_{[i]}f = \sum_{i=1}^k \hat{T}_{[k]}\hat{T}^*_{[i+1..k]}f = \hat{T}_{[k]}u_k, \tag{5.7}$$

where we also used (5.5) and (5.6). Since

$$\begin{aligned} \hat{T}_{[i]}f &= Z_i - \mathbb{E}\left[Z_{i-1}|\mathcal{F}_i\right] && (5.8)\\ &= \left(Z_i - \mathbb{E}\left[Z_i|\mathcal{F}_{i+1}\right]\right) + \left(\mathbb{E}\left[Z_i|\mathcal{F}_{i+1}\right] - \mathbb{E}\left[Z_{i-1}|\mathcal{F}_i\right]\right), && (5.9)\end{aligned}$$

one obtains

$$S_n = \sum_{k=1}^{n-1}\left(Z_k - \mathbb{E}\left[Z_k|\mathcal{F}_{k+1}\right]\right) + Z_n.$$

Now, for fix n and $1 \le k \le n-1$, define

$$\xi^{(n)}_k = \frac{1}{\sqrt{Var(S_n)}}\left(Z_k - \mathbb{E}\left[Z_k|\mathcal{F}_{k+1}\right]\right).$$

Since $\mathbb{E}\left[\xi_k^{(n)}|\mathcal{F}_{k+1}\right]=0$, by definition, $\{\xi_k^{(n)}\}_{1\le k\le n-1}$ is a reverse martingale difference sequence for the σ-algebras $\mathcal{F}_1,\dots\mathcal{F}_n$ (which are indeed coarser and coarser due to Assumption 2). Thus, in particular

$$Var(S_n)=Var(Z_n)+\sum_{k=1}^{n-1}Var\left(Z_k-\mathbb{E}\left[Z_k|\mathcal{F}_{k+1}\right]\right). \tag{5.10}$$

Using our martingale approximation and the well known martingale CLT (see [SV05] for the specific form used here, or [HH80] for the proof and general discussion), it is enough to prove that the difference between the martingale approximant and $S_n/\sqrt{Var(S_n)}$ is negligible, and further, the following two conditions:

$$\max_{1\le i\le n}\|\xi_i^{(n)}\|_\infty\to 0 \tag{5.11}$$

and

$$\|\sum_{i=1}^{n}\mathbb{E}\left[\left(\xi_i^{(n)}\right)^2|\mathcal{F}_{i+1}\right]-1\|_2\to 0. \tag{5.12}$$

Heuristically, (5.11) means asymptotic negligibility of all components, while (5.12) is a law of large numbers for the conditional variances. To prove (5.11) and (5.12), we adopt the ideas of [SV05]. In order to verify (5.11), observe that by Assumption 3,

$$\begin{aligned}\|Z_k\|_\infty &\le \sum_{j=1}^{k}\|\hat{T}_{[k]}\hat{T}^*_{[j+1..k]}f\|_\infty\le\sum_{j=1}^{k}\|\hat{T}^*_{[j+1..k]}f\|_\infty\\ &\le \sum_{j=1}^{k}\|\hat{T}^*_{[j+1..k]}f\|\le\sum_{j=1}^{k}K\tau^{k-j}\|f\|\le C_f.\end{aligned} \tag{5.13}$$

Thus

$$\|\mathbb{E}\left[Z_k|\mathcal{F}_{k+1}\right]\|_\infty\le C_f \tag{5.14}$$

also holds. Now, we prove that the variance of S_n converges to infinity:

$$Var(S_n)=\mu(S_n^2)\to\infty \tag{5.15}$$

as $n\to\infty$. Since (5.13) implies that $Var(Z_n)$ is bounded, (5.10) can be written as

$$\begin{aligned}Var(S_n) &= O(1)+\sum_{k=1}^{n-1}\mathbb{E}(Z_k^2)+\mathbb{E}\left(\mathbb{E}[Z_k|\mathcal{F}_{k+1}]^2\right)-2\mathbb{E}\left(Z_k\mathbb{E}[Z_k|\mathcal{F}_{k+1}]\right)\\ &= O(1)+\sum_{k=1}^{n-1}\mathbb{E}(Z_k^2)-\mathbb{E}\left(\mathbb{E}[Z_k|\mathcal{F}_{k+1}]^2\right)\\ &= O(1)+\sum_{k=1}^{n-1}\|u_k\|_2^2-\|u_k\|_2^2\cos^2\chi_k.\end{aligned}$$

Here, we used (5.7), and the fact that $\hat{T}_{[k]}$ is $L^2(\mu)$-isometry. Now, since

$$Var(f)=Var(\hat{T}_{[i]}f)\le 2Var(Z_i)+2Var(\mathbb{E}[Z_{i-1}|\mathcal{F}_i])\le 2\|u_i\|_2^2+2\|u_{i-1}\|_2^2,$$

one obtains

$$Var(S_n) \geq O(1) + \frac{1}{4} Var(f) \sum_{k=1}^{n-1} \min_{j \in \{k,k+1\}} \left(1 - \cos^2 \chi_j\right),$$

which converges to infinity as $n \to \infty$ by Assumption 4. Thus we have verified (5.15).

Now, (5.13), (5.14) and (5.15) together imply (5.11) and that the difference between the martingale and $S_n/\sqrt{Var(S_n)}$ is negligible, i.e.

$$\frac{\|Z_n\|_\infty}{\sqrt{Var(S_n)}} \to 0,$$

as $n \to \infty$.

To verify (5.12), first observe that for $i > j$

$$\begin{aligned} \|\mathbb{E}\left[\hat{T}_{[j]} f | \mathcal{F}_i\right] \|_\infty &= \|\hat{T}_{[i]} \hat{T}^*_{[i]} \hat{T}_{[j]} f\|_\infty = \|\hat{T}_{[i]} \hat{T}^*_{[j+1..i]} f\|_\infty = \|\hat{T}^*_{[j+1..i]} f\|_\infty \\ &\leq K\tau^{i-j} \|f\|. \end{aligned} \tag{5.16}$$

Then one can prove the assertion obtained from Lemma 4.4 in [SV05] by replacing $v_l^{(n)}$ with

$$\mathbb{E}\left[\left(\xi_{n-l}^{(n)}\right)^2 |\mathcal{F}_{n-l+1}\right]$$

the same way as it was done in [SV05], which yields (5.12).

Chapter 6

Tail asymptotics of free path lengths for the periodic Lorentz process. On Dettmann's geometric conjectures.

6.1 Introduction

In planar periodic Lorentz processes, the limiting distribution of the rescaled displacement is Gaussian and that of the rescaled orbit is a Wiener process. The scaling, however, is either the diffusive $\sqrt{n}$ or the slightly super-diffusive $\sqrt{n \log n}$ depending on whether the billiard has finite or infinite horizon. In the first case the limiting covariance is given by the Green-Kubo formula (cf. [BS81], [BSCh91]), which - though explicit - nevertheless does not permit precise calculations (the formula contains an infinite sum of time correlations of the free flight vector). In the infinite horizon case, however, - as it was conjectured by [B92] and established by [SzV07], [ChD09a] - the stronger $\sqrt{n \log n}$ scaling suppresses time correlations and the limiting covariance has a simple form expressed by geometric parameters of the billiard in question. For multidimensional Sinai billiards - under the complexity hypothesis! - exponential decay of correlations is known in the finite horizon case, only (cf. [BT08]). Then the central limit theorem with the diffusive scaling is a consequence and the limiting covariance is again given by the Green-Kubo formula. Physicists are always emphatically interested in expressions that are easy to calculate and check. Dettmann, [D12], motivated by a problem of [Sz08], was assuming that the aforementioned 2D infinite horizon case picture is also valid for multidimensional dispersing billiards and made a guess as to how the limiting covariance looks like. The difficulty is that, in this case, the structure of the horizons, i. e. orbits which never meet any scatterer, is much more complicated than in the planar case.

In fact, Dettmann formulates three conjectures for $\mathbf{Z}^d$-periodic Lorentz processes. The first two make claims for the tail asymptotic of the free path length. Roughly speaking the first one is related to the generic cases whereas the second one to certain degenerate cases. (In both cases a Wiener limit is expected with diffusive or super-diffusive scaling.) These conjectures are of purely geometric nature and the main goals of our work is to establish them. We do this in a wider generality: 1) for semi-dispersing billiards, 2) possibly with corner points, 3) and permitting arbitrary lattices $\mathcal{L}$ of finite covolume rather than only $\mathbf{Z}^d$. By accepting the dynamical hypothesis that the multidimensional picture is analogous to the 2D one (i. e. I. there is an exponential decay of cross correlations, and II. whether there is super-diffusive or diffusive behavior only depends on the tail asymptotic of the free path length), the first conjecture, among others, implies that - similarly to the planar case - the super-diffusivity covariance has a simple form that can be calculated from the geometry of the billiard. The second conjecture supports the hypothesis that, indeed, degenerate billiards, without open configuration sets of collision-free orbits, always have diffusive behavior. These questions are related to Pólya's visibility problem (1918) (cf. [P18],[K08]), to theories of Bourgain-Golse-Wennberg (1998-) (cf. [BGW98]) and of Marklof-Strömbergsson (2010-) (cf. [MS10]). The results also provide the asymptotic covariance of the periodic Lorentz process assuming it has a limit in the super-diffusive scaling, a fact if $d = 2$ and the horizon is infinite. Dettmann's third conjecture supports the previously mentioned dynamical hypothesis since it is about exponential decay of correlations being the subject of future progress of the theory.

The Chapter is structured as follows. In Section 6.2, we provide the definitions and formulate Dettmann's conjectures together with our results. In Section 6.3, we prove some finiteness lemmas and introduce an important tool which is the fattening of the configuration space (or shrinking of the scatterers, in other words). The key lemma of our proofs is the so-called Proportionality lemma, which we discuss in Section 6.4. Section 6.5 and Section 6.6 are devoted to the proofs of Dettmann's first and second conjectures, respectively. In Section 6.7, we present interesting examples where the super-diffusive limiting covariance matrix can be calculated: one of them is the first multidimensional semi-dispersing billiard whose ergodicity got ever proved: a three-dimensional toric billiard with two cylindrical scatterers (cf. [KSSz89]). The second one is the model of two hard balls on $\mathbb{T}^d : d \geq 3$. Finally, we make some concluding remarks in Section 6.8.

6.2 Setup and main results

6.2.1 $\mathcal{L}$-periodicity and the dynamics

Definition 6. *For a finite dimensional real, Euclidean vector space V we call a* lattice *a discrete additive subgroup $\mathcal{L} \subset V$, from which a basis of the vector space can be chosen.*

The discreteness in the above definition is equivalent to saying that every compact set of the vector space contains only finitely many elements of the lattice.

Definition 7. *A linear subspace is called a* lattice subspace *if it can be generated by lattice vectors.*

Periodicity We consider an infinite configuration space $\tilde{Q} \subset \mathbb{R}^d$ and a lattice $\mathcal{L}$ defining the periodicity of the Lorentz gas. Namely, the configuration space is invariant under translations in $\mathcal{L}$. We will also consider the compact configuration space $Q = \tilde{Q}/\mathcal{L}$.

Scatterers The complement of the compact configuration space consists of finitely many open, convex sets $\mathbb{R}^d/\mathcal{L} \setminus Q = \bigcup_{i=1}^n \mathcal{O}_i$ (called scatterers, or obstacles). Equivalently $\mathbb{R}^d \setminus \tilde{Q} = \bigcup_{i=1}^n \bigcup_{l \in \mathcal{L}} (\mathcal{O}_i + l)$. We assume that the boundary of each O_i is a $\mathcal{C}^3$-smooth hypersurface.

Notice that we do not require the scatterers to be disjoint, nota bene different scatterer configurations can lead to identical configuration spaces, if the differences are covered by other scatterers. Points in the boundary intersections $q \in \partial\mathcal{O}_i \cap \partial\mathcal{O}_j$ are called corner points.

Curvature upper bound We also require that, at any point of the boundary ∂Q, the curvature operator K is uniformly bounded from above: there exists a universal constant $\kappa_{\max}$, such that for every tangent vector v of the hypersurface ∂Q, the inequality $0 \leq K(v,v) \leq \kappa_{\max}\|v\|^2$ holds.

Dynamics and phase space The continuous time dynamics Φ_t acts on the phase space $\tilde{M} = \tilde{Q} \times \mathbb{R}^d / \sim$ where $\sim$ is the identification of pre-collisional and post-collisional velocities on $\partial\tilde{Q}$, which are mirror images with respect to the tangent hyperplane of the boundary at that point. We also write $\Phi_{[t_1,t_2]}x$ for the set $\{\Phi_s x | t_1 \leq s \leq t_2\}$. For later definitions and statements if we write $x = (q, v)$ with $q \in \partial\tilde{Q}$, then v is chosen as the post collisional one. At corner points there are more than one such hyperplanes, and mirroring generally does not commute, so the dynamics is either not defined, or has multiple values. (Since the speed $|v|$ is invariant under the dynamics, in the literature one usually takes the phase space $\tilde{M} = \tilde{Q} \times S^{d-1}/\sim$ but for our purpose it is more suitable to consider $\tilde{M}$ as introduced above.)

The action is free flight $\Phi_t(q, v) = (q + tv, v)$ as long as $q + vt \notin \partial\tilde{Q}$. On the boundary the velocity is reset to the post collisional one, and free flight follows with that vector. Moreover, even if it hits a scatterer and if the collision is tangent (sometimes called grazing), the dynamics is still free flight, since the velocity is in the mirroring plane, so it does not change. The dynamics is invariant under $\mathcal{L}$-translations, so the compact phase space of the flow is $M = Q \times S^{d-1}/\sim$. For simplicity, we will use the same notation Φ_t for the flow on the compact phase space as well. The Lorentz dynamics has natural invariant measures, the Liouville-ones: $d\mu = \text{const.}dqdv$ on $\tilde{M}$. The const. $= 1$ measure is called Lebesgue. Similarly the invariant probability measure for the billiard dynamics on M is $d\mu = c_\mu dqdv$ with $c_\mu = (\operatorname{vol} Q \operatorname{vol} S^{d-1})^{-1}$. We will also use the notation $\lambda_{d'}$ for the Lebesgue measure in dimension $d' \leq d$.

Billiard and Lorentz process

Definition 8. *Under the aforementioned conditions, the dynamics $\Phi_t (t \in \mathbb{R})$ on the phase space M is called a* semi-dispersing billiard *and that on the phase space $\tilde{M}$ a (semi-dispersing)* Lorentz process. *If the scatterers are strictly convex, then the billiard is called a* dispersing *one or a* Sinai-billiard.

In this Chapter we will consider a fixed semi-dispersing billiard (or the corresponding Lorentz process) satisfying the aforementioned conditions.

The free flight function For $x = (q, v)$

$$\tau(x) = \inf\{t > 0 \mid q + tv \in \cup_i \mathcal{O}_i\}$$

as usual, the infimum of the empty set is ∞. This definition is slightly different from the usual definition. In fact, at points where the first collision is tangential, the new definition gives a larger value. The advantage of this change is seen by the semi-continuity Claim 6.10. It is obviously invariant under $\mathcal{L}$ translations, so we will not distinguish whether the function is defined on the compact or on the non-compact phase space.

Our main focus will be on the tail distribution of the free path length: $F(t) = \mu(\tau > t)$, i. e. of the probability of surviving without collision for time t.

6.2.2 Horizons

Definition 9. *For a configuration point $q \in \tilde{Q}$ a* free subspace V *is a maximal (for containment) linear subspace of $\mathbb{R}^d$, such that $q + V \subset \tilde{Q}$. This latter is equivalent to requiring $\tau(q + v, w) = \infty$ for all $v, w \in V$.*

Claim 6.1. *Any free subspace V is a lattice subspace.*

Proof. If we have a vector $v \in V$, then by invariance $q + tv + l \in \tilde{Q}$ for all $t \in \mathbb{R}$ and $l \in \mathcal{L}$. If this vector v is not parallel to a lattice vector, then the set $tv + l$ is dense in some lattice subspace V', concluding $q + V' \subset \tilde{Q}$, so $V' \subset V$ by maximality. □

Now, we proceed to the definition of the *horizons*. Pick a phase point (q, v) such that $\tau(q, v) = \infty$ and choose a free subspace V at q which contains v (note that there might be more such free subspaces). Project orthogonally all the scatterers of the periodic Lorentz process to $q + V^\perp$ (here, $V^\perp$ is the linear subspace perpendicular to V), and denote by $\tilde{Q}_{q,V^\perp}$ the complement of the projection of all the scatterers. Obviously, $q \in \tilde{Q}_{q,V^\perp}$, and V is a free subspace for q. For all $q' \in \tilde{Q}_{q,V^\perp}$, either V is a free subsapce for q', or there exists some infinite line $e \subset \tilde{Q}_{q,V^\perp}$ containing q'. Indeed, by the definition of $\tilde{Q}_{q,V^\perp}$, $\tau(q', v) = \infty$ for any $v \in V$, so the only reason for V not being a free subspace for q' is the possible lack of maximality (see Figure 6.1). Now, the maximal connected subset $\tilde{B}_H$ of $\tilde{Q}_{q,V^\perp}$ containing q and such points q' for which V is a free subspace, is called the basis of the horizon. The horizon itself is $\tilde{H} = \tilde{B}_H \times V \subset \tilde{Q}$. Thus a horizon is a subset of the infinite configuration space. Note that by construction, for every $q \in \tilde{H}$, V is a free subspace for q. Further, a horizon determines its basis $\tilde{B}_H$ uniquely (up to translations in V) and its free subspace, which will be sometimes referred to as V_H. Accordingly, the dimension of a horizon is $d_H = \dim V_H$.
We also define the horizons as subspaces of the compact configuration space Q. Given a horizon $\tilde{H} \in \tilde{Q}$, the corresponding horizon $H \in Q$ consists of the image of all points $q \in \tilde{Q}$ under the natural projection

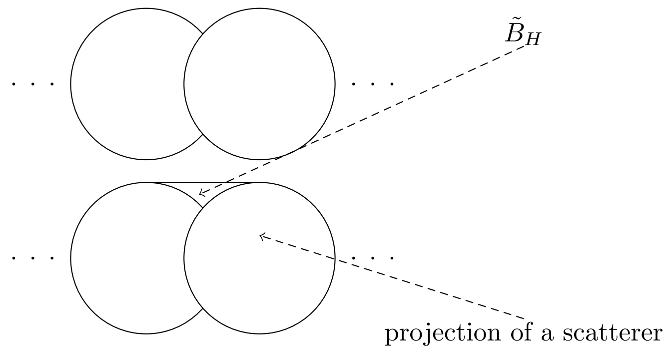

Figure 6.1: Basis of a horizon with $V_H = (\lambda, \lambda, 0)$ in $\mathbb{Z}^3$, and spherical scatterers of radius $1/2 - \epsilon$. Figure in $V^\perp$.

$\mathbb{R}^d \to \mathbb{R}^d/\mathcal{L}$. Observe that for two disjoint horizons $\tilde{H}_1$ $\tilde{H}_2$ in the infinite configuration space, both intersecting with a parallelepiped defining $\mathcal{L}$, the corresponding horizons H_1 and H_2 in the compact configuration space can coincide. Finally, we write $\tilde{\mathcal{H}} = \tilde{H} \times V \subset \tilde{M}$ and $\mathcal{H} = H \times V \subset M$ for the phase space analogues. If we talk about the base B_H of a horizon H, then we think of $\tilde{B}_H$ for an arbitrary $q \in H$ as represented in Q.

The conjectures and the results will use the probability of remaining within a horizon H that is

$$F_H(t) = \mu\left(\{(q, v) \in M \mid q + sv \in H, \ \ \forall s \in [0, t]\}\right)$$

a quantity easier to calculate exactly. (cf. Equ. (26) of [D12] or (6.1)).

Definition 10. *([D12])*

- *A* maximal horizon *is one of the highest dimension for the given billiard (or Lorentz process).*
- *A* principal horizon *is one of the highest dimension possible, which is $d-1$ if there are scatterers.*
- *A horizon H is* incipient *if its basis B_H has ($d - d_H$ dimensional) measure zero.*

Denote the set of maximal non-incipient horizons by $\mathbb{H}$. It can be empty if all maximal horizons are incipient, or there are no horizons at all.

We conclude this point with a simple lemma.

Lemma 6.2. *The boundary of the basis of a horizon consists of $\mathcal{C}^3$, concave pieces except for principal horizons when it consists of two endpoints of an interval.*

6.2.3 Dettmann's conjectures, [D12]

Conjecture 6.3. *Consider an $\mathcal{L}$-periodic Lorentz process with at least one non-incipient maximal horizon. Then, as $t \to \infty$ we have*

$$F(t) \sim \sum_{H \in \mathbb{H}} F_H(t).$$

Conjecture 6.4. *Consider an $\mathcal{L}$-periodic Lorentz process with incipient (but no actual) principal horizon. Then, as $t \to \infty$, we have*

$$F(t) \asymp \begin{cases} t^{-2}, & d < 6 \\ t^{-2} \log t, & d = 6 \\ t^{-\alpha_d} \qquad (1 < \alpha_d < 2), & d > 6 \end{cases}$$

These two conjectures are of purely geometric nature, whereas the following one concerns the dynamics, too.

Conjecture 6.5. *Consider an $\mathcal{L}$-periodic Lorentz process and let $f, g : M \to \mathbb{R}$ denote zero-mean (wrt the invariant measure μ) Hölder functions. Then, as $t \to \infty$, we have*

$$\int_{\{x \in M | \tau(x) < t\}} (f)(g \circ \Phi_t) d\mu = o(F(t)).$$

6.2.4 Main results

Now we can formulate the main results of this Chapter.

Theorem 6.6. *Consider an $\mathcal{L}$-periodic semi-dispersing Lorentz process (possibly with corner points). Assume it has at least one non-incipient maximal horizon. Then, as $t \to \infty$ we have*

$$F(t) \sim \sum_{H \in \mathbb{H}} F_H(t).$$

Theorem 6.7. *Consider an $\mathcal{L}$-periodic semi-dispersing Lorentz process (possibly with corner points). Assume it has at least one incipient (but no actual) principal horizon. Then, as $t \to \infty$, we have*

$$F(t) = \begin{cases} O(t^{-2}), & 3 \le d \le 5 \\ O(t^{-2} \log t), & d = 6 \\ O\left(t^{\frac{2+d}{2-d}}\right), & d > 6. \end{cases}$$

Further, if we also assume that the curvature is bounded away from 0 (from below) uniformly at every point of ∂Q (dispersing case), then

$$F(t) \asymp \begin{cases} t^{-2}, & 3 \le d \le 5 \\ t^{-2} \log t, & d = 6 \\ t^{\frac{2+d}{2-d}}, & d > 6. \end{cases}$$

Remark. According to the dynamical theory of semi-dispersing billiards super-diffusive behavior can only arise if the asymptotics of $F(t)$ is non-integrable. Therefore Theorems 6.7,6.6 and (6.1) suggest that, in the absence of principal, non-incipient horizon, no super-diffusive behavior is possible (cf. Section 6.7). Moreover, in the case of super-diffusivity the scaling is $\sqrt{t \log t}$ - again by (6.1).

6.3 The method of fattening, finiteness and stability lemmas

6.3.1 Lattice geometry

Lemma 6.8. *For any $K > 0$ the number of lattice subspaces V, such that* $\operatorname{vol}(V/V \cap \mathcal{L}) < K$ *is finite.*

Proof. Consider the Grassmannian algebra $\Lambda(\mathbb{R}^d)$, and construct a lattice $\Lambda(\mathcal{L})$ as follows. Choose a base a_i of the lattice $\mathcal{L}$, by definition this is a base in the linear sense for $\mathbb{R}^d$. All wedge products of a_i forms a base for the Grassmannian. Consider it as a base for a lattice. Observe, that this construction does not depend on the choice of the basis, only on the lattice $\mathcal{L}$. Since $\Lambda(\mathcal{L})$ is a lattice there are only finitely many points with length not greater than K.

If we have a lattice subspace V, then choose a base in it b_j. The length of $\wedge_j b_j \in \Lambda(\mathcal{L})$ is exactly the covolume of $V \cap \mathcal{L}$ inside V. Therefore all the less than K covolume sublattices are in the above defined finite set, and these are distinct elements, consequently there are only finitely many of those. □

By this lemma the minimal covolume of k dimensional sublattices exists, and we will denote it by ℓ_k. For example ℓ_1 is the minimal length of nonzero lattice vectors, $\ell_d = \operatorname{vol}\left(\mathbb{R}^d/\mathcal{L}\right)$, and $\ell_0 = 1$ as usual for empty products.

Claim 6.9. *If we have a lattice subspace V, and we take its orthogonal complement $V^\perp$, and we project $\mathcal{L}$ orthogonally onto $V^\perp$ to get $\mathcal{L}_V^\perp$, then we have*

$$\operatorname{vol}\left(\mathbb{R}^d/\mathcal{L}\right) = \operatorname{vol}(V/V \cap \mathcal{L}) \operatorname{vol}\left(V^\perp/\mathcal{L}_V^\perp\right).$$

Proof. Take a basis $\{a_i\}_{i=1}^{\dim(V)}$ for $\mathcal{L} \cap V$, and extend this to a basis $\{a_i\}_{i=1}^{d}$ of $\mathcal{L}$. Then $|\det(a_i)| = \operatorname{vol}\left(\mathbb{R}^d/\mathcal{L}\right)$. The determinant does not change if we project the last $d - \dim(V)$ vectors orthogonally to the orthocomplement of the first $\dim(V)$ vectors. The projections give rise to a basis of $\mathcal{L}_V^\perp$, and by orthogonality $|\det(a_i)| = |\wedge_{i=1}^{\dim(V)} a_i|| \wedge_{i=\dim(V)+1}^{d} a_i^\perp|$, which is the claim. □

Now we can provide the asymptotic form of $F_H(t)$. Indeed, in our notations, Equ. (26) of [D12] reads as

$$F_H(t) \sim \frac{\operatorname{vol} S_{d_H-1} \int_{B_H} \int_{B_H} \Delta_{B_H}^{\text{vis}}(q,q')dqdq'}{(1-\mathcal{P}) \operatorname{vol} S_{d-1} \operatorname{vol}\left(V^\perp/\mathcal{L}_V^\perp\right)} \frac{1}{t^{d-d_H}} =: C_H \frac{1}{t^{d-d_H}} \tag{6.1}$$

where $\mathcal{P} = 1 - \frac{\operatorname{vol} Q}{\operatorname{vol}(\mathbb{R}^d/\mathcal{L})}$ is the volume fraction covered by scatterers and $\Delta_{B_H}^{\text{vis}}(q,q')$ is the visibility function providing the number of possible connecting intervals $\overline{q,q'}$, lying in B_H, of the points q, q' (toric geometry!). Note that the value of the integral is invariant under V-shifts of B_H and is finite since $\Delta_{B_H}^{\text{vis}}(q,q')$ is bounded. So as to verify the latter, assume by contradiction that for each $n > 0$ one finds q_n, q_n' such that $\Delta_{B_H}^{\text{vis}}(q_n,q_n') > n$. Since the sets $\Delta_n = \{(q,q') | \Delta_{B_H}^{\text{vis}}(q,q') > n\}$ are closed subsets of each other, they have a nonempty intersection containing some (q_∞, q_∞') with $\Delta_{B_H}^{\text{vis}}(q_\infty,q_\infty') = \infty$. Thus an infinite line is part of B_H, which contradicts to its definition.

Remark. In the much interesting case of a *principal horizon* H, B_H is an interval and the previous formula becomes simpler:

$$F_H(t) \sim \frac{2 \operatorname{vol} S_{d_H-1} |B_H|^2}{(1-\mathcal{P}) \operatorname{vol} S_{d-1} \operatorname{vol}\left(V^\perp/\mathcal{L}_V^\perp\right)} \frac{1}{t^{d-d_H}} \tag{6.2}$$

6.3.2 Fattening, and its properties

The curvature upper bound implies in particular that, at any point of the boundary, a tangent sphere of radius $\kappa_{\max}^{-1}$ is contained completely in the scatterer. This allows us to define the shrinking of the scatterers, or equivalently the fattening of the configuration space by $0 \le \delta < \kappa_{\max}^{-1}$ as a parallel domain (which is typically not homothetic to the original one). Indeed, define $\mathcal{O}_i^\delta$ as the centers of all balls of radius δ, which are contained in $\mathcal{O}_i$:

$$\mathcal{O}_i^\delta = \{q \in \mathcal{O}_i \mid \mathrm{dist}(q, \partial\mathcal{O}_i) > \delta\}.$$

This leads to new configuration spaces $\tilde{Q}^\delta = \mathbb{R}^d \setminus \cup_i \cup_{l\in\mathcal{L}} (\mathcal{O}_i^\delta + l)$, and $Q^\delta = \tilde{Q}^\delta/\mathcal{L}$, which satisfy all the above assumptions, with $(\kappa_{\max}^{-1} - \delta)^{-1}$ as a curvature upper bound.

The definition can be extended to negative values of δ. Also note our previous comment that different scatterer configurations can lead to the same configuration space. Since fattening is defined from scatterers, the same configuration space can have different fattenings for the same δ. The semigroup property of this operation holds $Q^0 = Q$, and $\left(Q^\delta\right)^{\delta'} = Q^{\delta+\delta'}$ as long as δ, δ' and $\delta + \delta'$ are all smaller than $\kappa_{\max}^{-1}$. (By the latter restrictions this is not exactly a semigroup.)

It is then natural to denote the corresponding dynamics by Φ_t^δ. We denote by τ^δ the free flight function on the fattened space.

Claim 6.10. *τ^δ as a function on $(-\infty, \kappa_{\max}^{-1}) \times \tilde{M}$ is upper semi-continuous (to be abbreviated as USC) in all of its variables (δ, x).*

Proof. This only requires a proof at points $x = (q, v)$, where $\tau^\delta(x) < \infty$. By the definition of τ, for a small $\epsilon > 0$ we have $q + (\tau^\delta(x) + \epsilon)v \in \mathcal{O}_i^\delta$ for some i. Since both the free flight dynamics and the fattening are continuous, we have for nearby points x', and nearby parameters δ', that $q' + (\tau^\delta(x) + \epsilon)v' \in \mathcal{O}_i^{\delta'}$. The dynamics of a nearby point x' may differ from the free flight dynamics only if it had a jab (non tangent) collision 'before', but then $\tau^{\delta'}(x')$ is even smaller than $\tau^\delta(x) + \epsilon$. ('before' permits equality as well thus the argument is also valid for simultaneous collisions at corner points.) □

Monotonicity Of course, the fattening of the configuration space makes free flights longer. We will use it not only for the above defined parallel domain, but for a larger set of inclusion relations, too.

Denote by $\mathcal{V}^\varepsilon(q)$ (or by $\mathcal{V}(q)$) the set of free subspaces at $q \in \tilde{Q}^\varepsilon$ (or at $q \in \tilde{Q}$, respectively).

Claim 6.11. *For $q \in Q$, $\mathcal{V}^\varepsilon(q)$ is an increasing function of $\varepsilon \in [0, \kappa_{\max}^{-1})$ in the sense that $\forall 0 < \varepsilon < \varepsilon'$ and $\forall V \in \mathcal{V}^\varepsilon(q)$ $\exists V^* \in \mathcal{V}^{\varepsilon'}(q)$ such that $V \subset V^*$.*

Moreover, in the case of the previously defined fattening, the equality $\tau(x) = \tau^\varepsilon(x)$ holds with $\varepsilon > 0$ if and only if $\tau(x) = \infty$.

Also, if for $(q, v) \in M$ $\tau^\varepsilon(q, v) = \infty$ for every $\varepsilon > 0$, then $\tau(q, v) = \infty$, too.

Proof. If $Q \subset Q'$, then for any $x \in M$ we have $x \in M'$, too. Then we can consider both free flights τ and τ' and we have $\tau(x) \le \tau'(x)$. □

6.3.3 Local stability and finiteness of free subspaces

For $q \in \tilde{Q}$ and $\delta > 0$ denote by $B(q, \delta)$ the $\delta-$neighborhood of q. Unless specified otherwise, it will be considered as a neighborhood in $\tilde{Q}$. (We use the same notation analogously for $q \in Q$.)

Lemma 6.12 (**Local stability**). *For any $q \in \tilde{Q}$ there exists $\xi > 0$ such that, for every $q' \in B(q, \xi) \cap \tilde{Q}$ and any free subspace $V^\xi(q')$ for Φ^ξ at q', there is a free subspace $V(q)$ for Φ at q such that $V^\xi(q') \subset V(q)$.*

In other words, the set of free subspaces as a function of the base point $q \in Q$ and of $\varepsilon > 0$ is upper semi-continuous at $q \in Q, \varepsilon = 0$ in the sense that for any $q_n \to q$ and $\varepsilon_n \searrow 0$ and for any $V \in \lim_{n\to\infty} \cup_{k \geq n} \mathcal{V}^{\varepsilon_k}(q_k)$ there is a $V^* \in \mathcal{V}(q)$ such that $V \subset V^*$.

Proof. We prove the claim in its second form. Assume the contrary. Then there exists a velocity v_∞ and sequences $q_n \to q$, $v_n \to v_\infty$ and $\varepsilon_n \to \varepsilon$ such that $\tau^\varepsilon(q, v_\infty) < \infty$ and $\tau^{\varepsilon_n}(q_n, v_n) = \infty$. This contradicts Lemma 6.10. □

Lemma 6.13. *For any configuration point $q \in \tilde{Q}$ the set of free subspaces is finite.*

Proof. The proof is inductive by codimension $d - \dim(V)$. If $\dim V = d$, then there are no scatterers at all and $\mathbb{R}^d$ is the only free subspace. Assume we have proven the statement for dimensions larger than $d'(< d)$.

The induction step is indirect. We are going to show that, if the number of d' dimensional free subspaces is infinite, then for every positive ϵ there exists a free subspace of higher dimension in $\tilde{Q}^\epsilon$. We will apply the inductive condition to $\tilde{Q}^\epsilon$ ($\epsilon > 0$ sufficiently small) to derive a contradiction.

For any given $\delta > 0$ there are only finitely many d' dimensional lattice subspaces, for which the lattice translates are δ-separated. By the indirect condition we have a free subspace, for which the lattice translates are δ-dense in a higher dimensional subspace. This higher dimensional subspace is therefore free in $\tilde{Q}^\epsilon$ (as long as $\epsilon < (1/7)\delta^2\kappa_{\max}$), but is not free in $\tilde{Q}$ (free subspaces can not contain each other by maximality).

By the inductive condition the number of higher (i. e. $> d'$) dimensional free subspaces is finite. For each ϵ we create a vector $\vec{n}^\epsilon$ such that the first coordinate is the number of d dimensional free subspaces for q in $\tilde{Q}^\epsilon$, the second is the number of $d-1$ dimensional free subspaces for q in $\tilde{Q}^\epsilon$, and so on, the last coordinate is the number of $d'+1$ dimensional free subspaces for q in $\tilde{Q}^\epsilon$. We consider the lexicographical ordering on these vectors, so the biggest is $(1, 0, \ldots, 0)$, and $(0, 2, 3, 0, 1) > (0, 2, 2, 23, 11)$. The set of possible vectors is not finite, but well ordered.

We claim that $\vec{n}^\epsilon$ does not increase as ϵ decreases, and that $\vec{n}^\epsilon$ is right continuous in ϵ. For the first claim, observe that new free subspaces can only appear, if they were covered by higher dimensional free subspaces for higher ϵ values. So the first changing coordinate is decreasing. For the second claim, observe that $\tilde{Q} = \cap_{\epsilon > 0} \tilde{Q}^\epsilon$, so if a free subspace is present for all small enough $\epsilon > 0$, then it is also present for $\epsilon = 0$. Therefore

$$\lim_{\epsilon \searrow 0} \vec{n}^\epsilon = \min_{\epsilon > 0} \vec{n}^\epsilon = \vec{n}^0$$

the first equality follows from monotonicity and well-orderedness, the second from right continuity.

This is a contradiction with the previously proven statement: $\vec{n}^\epsilon > \vec{n}^0$ for all $\epsilon > 0$.

□

Lemma 6.14. *There are finitely many maximal horizons.*

Proof. (see also Lemma 1 in [D12] and Lemma A.2.2. of [Sz94]) For every $q \in Q$ pick a stability neighborhood using Lemma 6.12. Since Q is compact, one can choose a finite cover of Q by such neighborhoods. This yields that there are only finitely many maximal dimensional free subspaces. It remains to prove that for such a free subspace V, there are only finitely many corresponding horizons. For this, project the scatterer configuration to $V^{\perp}$. Note that there is no higher dimensional free subspace than V, thus the complement of the images of the scatterers is the union of the bases of horizons with free subspace V. Since the complement of finitely many convex sets has finitely many connected components, the statement follows. □

6.4 The Proportionality lemma

The next lemma states that any long enough free flight has a fixed proportion of its time spent in a horizon without leaving it. The technical formulation is a little bit different, and formally we will use the statement below, where instead of a horizon we use the vicinity of a free subspace.

Lemma 6.15. *For every $\epsilon > 0$ there exist $T > 0$ and $c \in (0,1)$, such that for any $x \in M$ if $\infty > \tau(x) > T$ then there exist $\tau(x) > s > t > 0$ with $s - t > c\tau(x)$ and a free subspace $p + V \subset \tilde{Q}$ such that the configuration component of $\Phi_u(x)$ is ϵ close to $p + V$ in $\tilde{Q}$ for every $s > u > t$.*

Proof. The proof is indirect. We are going to suppose that there exists an $\epsilon > 0$ such that for all $T > 0$ and $c \in (0,1)$ there exists $x \in M$ with $\infty > \tau(x) > T$ such that for any free subspace $p + V \subset \tilde{Q}$ and for any time segment $\tau(x) > s > t > 0$ if the configuration component of $\Phi_u(x)$ is ϵ close to $p + V$ for $s > u > t$, then $\tau(x) > (s - t)/c$.

Choice of constants Choose $T_n \to \infty$, and $c_n \to 0$, and choose $(q_n, v_n) = x_n \in M$ according to the indirect assumption. By compactness of M we have an accumulation point $x_\infty = (q_\infty, v_\infty)$. Apply lemma 6.12 to get ξ as the stability fattening factor for q_∞. We have an ϵ from the indirect statement. Choose η, such that $\frac{3}{2}\eta < \epsilon$, and $2^d\eta < \xi$. For $1 \leq k \leq d$ let us define

$$r_k = \frac{\ell_{k-1}}{\ell_d} \left(\frac{\eta}{2}\right)^{d-k} D^{d-k}, \tag{6.3}$$

where D^j is the j-dimensional volume of the j-dimensional unit ball, and $D^0 = 1$. Choose n such that $|q_n - q_\infty| < \eta/2$ and

$$T_n > \frac{1}{r_1}, \tag{6.4}$$

$$c_n < 2^{-d}\eta \min_{1 \leq k \leq d} r_k. \tag{6.5}$$

Inductive assumptions We are going to prove the following statements in an inductive fashion for $1 \le k \le d$.

- We have linearly independent lattice vectors $\{l_i\}_{i=1}^k$, all from a free subspace for q_∞ in $\tilde{Q}$.
- We have $0 < t_k < \tau(x_n)$, such that the parallelepiped $q_n + \sum_{i=1}^k \lambda_i l_i,\ \lambda_i \in [0,1]$ is contained in the $(2^k-1)\eta$ radius tubular neighborhood of the trajectory segment $\Phi_{[0,t_k]}x_n$ (the ρ tubular neighborhood of a line segment $[a,b]$ is the set of such points in $\mathbb{R}^d$ which are ρ-close to the line line segment $[a,b]$, and whose orthogonal projection to the line defined by a and b lies between a and b).
- Denote by $v_n^{\perp k}$ the component of v_n which is orthogonal to $\mathrm{span}\{l_i\}_{i=1}^{k-1}$ (this gives v_n for $k=1$). We require that:
$$t_k < \sum_{i=1}^{k} 2^{k-i} \frac{1}{|v_n^{\perp i}|\, r_i} \tag{6.6}$$

The last statement is purely technical.

Start of induction By condition (6.4) the tubular $\eta/2$ neighborhood of the free flight trajectory of x_n has a bigger volume than $\mathrm{vol}\left(\mathbb{R}^d/\mathcal{L}\right)$, therefore it has a self intersection in $Q = \tilde{Q}/\mathcal{L}$. This means that, in this tubular neighborhood, there are two points q', and $q'+l_1$ which are lattice translates with $0 \ne l_1 \in \mathcal{L}$. Moreover $|q'-q_n| < \eta/2$ and $|q'+l_1-(q_n+t_1v_n)| < \eta/2$ and $0 < t_1 < \tau(x_n)$. Consequently in the fattened space $\tilde{Q}^{\eta/2}$ the line segment $q', q'+l_1$ is collision free and periodic, hence $\tau^{\eta/2}(q', l_1) = \infty$. Applying the stability lemma we conclude that l_1 is part of a free subspace for q_∞ in $\tilde{Q}$. We also note that the line segment q_n, q_n+l_1 is in the tubular η neighborhood of the trajectory segment $\Phi_{[0,t_1]}x_n$.

Note that we only used $\tau(x_n) > 1/r_1$ about the length of the free flight, so actually $t_1 < 1/r_1$, which gives equation (6.6) for $k=1$.

Inductive step Suppose we have all the inductive statements for $k-1$. For simplicity we denote the lattice subspace $V = \mathrm{span}\{l_i\}_{i=1}^{k-1}$, and its orthocomplement $V^\perp$. Consider the orthogonal projection of the free flight $\Phi^\perp_{[0;\tau(x_n)]}(x_n)$. Since $|q_n - q_\infty| < \eta/2$ and $\epsilon > 3\eta/2$ the projection of the free flight lies in at least η length (and equivalently for at least $\eta/|v_n^{\perp k}|$ time) in the ϵ neighborhood of $q_\infty^\perp$, meaning that the non projected free flight spends the same $\eta/|v_n^{\perp k}|$ time in the ϵ neighborhood of the free subspace containing V. By the indirect condition, the complete length of the projection $|v_n^{\perp k}|\tau(x_n)$ is at least $\eta/c_n > 1/r_k$, therefore
$$\tau(x_n) > \frac{\eta}{|v_n^{\perp k}| c_n} \tag{6.7}$$

By the definition of r_k we have that the ($(d-k+1)$-dimensional) volume of the tubular $\eta/2$ neighborhood of the projected free flight trajectory, is bigger than ℓ_d/ℓ_{k-1}, so in particular bigger than $\mathrm{vol}\left(\mathbb{R}^d/\mathcal{L}\right)/\mathrm{vol}\left(V/\mathcal{L}\cap V\right)$, which is by Claim 6.9 the covolume of the projected lattice $\mathcal{L}_V^\perp$. Therefore this neighborhood contains a pair of points $q'^\perp$, and $q'^\perp + l_k^\perp$, with $0 \ne l_k^\perp \in \mathcal{L}_V^\perp$. The latter means that there is $l_k \in \mathcal{L} \setminus V$, such that $l_k^\perp$ is its projection. We can choose q' such that $|q'-q_n| < \eta/2$ and
$$|q' + l_k - (q_n + tv_n) - v| < \eta/2, \tag{6.8}$$

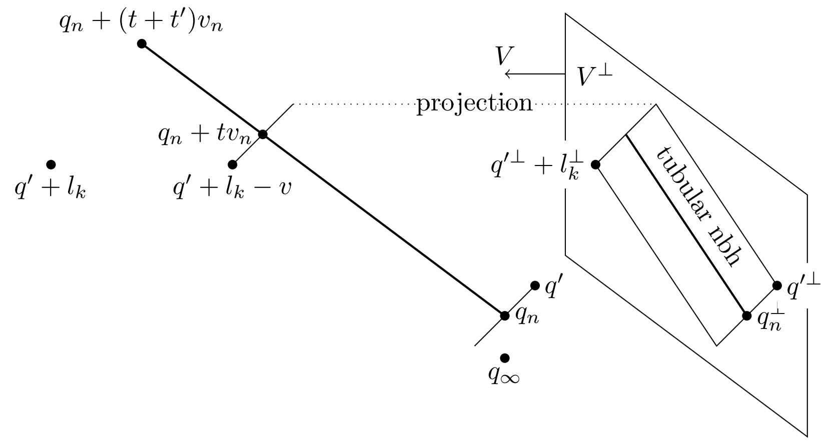

Figure 6.2: Constellation of vectors in the inductive step in the proof of lemma 6.15

for some $\tau(x_n) > t > 0$, and some $v \in V$. We can suppose, that v is in the parallelepiped $\sum_{i=1}^{k-1} \lambda_i l_i$, since the lattice component can be added to l_k, it does not change the property that $l_k \in \mathcal{L} \setminus V$. The inductive condition gives that v is in the tubular $(2^{k-1}-1)\eta$ neighborhood of the trajectory segment $\Phi_{[0,t_{k-1}]} x_n$, we have from equation 6.8 that

$$|q' + l_k - (q_n + (t+t')v_n)| < \left(2^{k-1} - \frac{1}{2}\right)\eta, \tag{6.9}$$

where $0 < t' < t_{k-1}$. The positivity of t' comes from the sign of v in equation 6.8 and the fact that all l_i has positive scalar product with v_n by construction. It follows, that the line segment $q_n, q_n + l_k$ is in the tubular $2^{k-1}\eta$ neighborhood of $\Phi_{[0,t+t']} x_n$, and therefore the parallelepiped $q_n + \sum_{i=1}^{k} \lambda_i l_i$ is in the $\left(2^k - 1\right)\eta$ neighborhood of $\Phi_{[0,t+t'+t_{k-1}]} x_n$.

We declare $t_k = t + t' + t_{k-1}$, and note that in the construction of t we have only used $\tau(x_n) > 1/r_k \left|v_n^{\perp k}\right|$ about the length of the free flight, so actually $t < 1/r_k \left|v_n^{\perp k}\right|$. Using $t' < t_{k-1}$, and equation 6.6 from the inductive condition for $k-1$ we get

$$t_k < \frac{1}{r_k \, |v_n^{\perp k}|} + 2\sum_{i=1}^{k-1} 2^{k-1-i} \frac{1}{|v_n^{\perp i}| \, r_i} = \sum_{i=1}^{k} 2^{k-i} \frac{1}{|v_n^{\perp i}| \, r_i},$$

which is equation 6.6 for k. To show $t_k < \tau(x_n)$, observe that $\left|v_n^{\perp i}\right|$ is decreasing with i, hence

$$t_k < \sum_{i=1}^{k} 2^{k-i} \frac{1}{|v_n^{\perp i}| \, r_i} < \sum_{i=1}^{k} 2^{k-i} \frac{1}{|v_n^{\perp k}| \min r_i} < 2^k \frac{1}{|v_n^{\perp k}| \min r_i} \le \frac{\eta}{|v_n^{\perp k}| \, c_n}.$$

The last inequality follows from equation 6.5. The last expression in the row, and hence t_k is smaller than $\tau(x_n)$ by equation 6.7.

In the fattened space $\tilde{Q}^{(2^k-1)\eta}$ we have a k dimensional lattice parallelepiped, and (by $\mathcal{L}$ periodicity) the generated lattice subspace free of scatterers. By the choice of η we can apply the stability lemma to conclude that $\{l_i\}_{i=1}^{k}$ are from a free subspace for q_∞ in $\tilde{Q}$.

Contradiction The last ($k = d$) claim in the above induction states the existence of a d dimensional free subspace, which means that there are no scatterers. Even in that case the indirect condition states that the trajectory leaves this free subspace, which is the whole configuration space. $\square$

6.5 Proof of Theorem 6.6

Here, we prove the generalization of Dettmann's first conjecture (i.e. Theorem 6.6).

6.5.1 Lower estimate

First, we prove the lower estimate, namely

$$\limsup_{t\to\infty} \sum_{H\in\mathbb{H}} F_H(t)/F(t) \leq 1. \tag{6.10}$$

Since $\cup_H \{(q,v) \in M \mid q + sv \in H,\ \forall s \in [0,t]\} \subset \{(q,v) \in M \mid \tau(q,v) > t\}$, (6.1) implies that (6.10) follows, whenever

$$\mu\left(\{(q,v) \in M \mid \exists H_1 \neq H_2 \in \mathbb{H},\ \forall s \in [0,t],\ \ q + sv \in H_1 \cap H_2\}\right) = o\left(t^{d_H - d}\right)$$

is established. Since there are finitely many maximal horizons, it suffices to prove that for every pair $(H_1, H_2) \in \mathbb{H}^2$,

$$F_{H_1,H_2}(t) = \mu\left(\{(q,v) \in M \mid q + sv \in H_1 \cap H_2,\ \forall s \in [0,t]\}\right) = o\left(t^{d_H - d}\right).$$

Now assume that for fix (H_1, H_2) and for every $n > 1$, one can find $x_n \in M$ such that the trajectory segment $\Phi_{[0,n]}x_n$ lies entirely in $H_1 \cap H_2$ (if not, then obviously $F_{H_1,H_2}(t) = 0$ for t large enough). Since maximal horizons are closed, there is an accumulation point $x_\infty = (q_\infty, v_\infty)$ with $\Phi_{[0,\infty]}x_\infty \in H_1 \cap H_2$. Thus the set $V_{H_1,H_2} = V_{H_1} \cap V_{H_2}$ is a non-empty subspace of V_{H_1}. Obviously it is strictly smaller than V_{H_1}, otherwise H_1 and H_2 would coincide. Now project the scatterer configuration to $V^{\perp}_{H_1,H_2}$. In this projection, the intersection of the images of $\tilde{H}_1$ and $\tilde{H}_2$ does not contain any subspace (indeed, if it contained a line, that could be added to V_{H_1,H_2}). Then the same argument used to prove (6.1) provides $F_{H_1,H_2}(t) = O\left(t^{d_H - 1 - d}\right)$.

6.5.2 Upper estimate

The estimate will work as an induction by dimension. If $d = 1$ the claim is trivial, the $d = 2$ case was proved in [SzV07].

The idea of the present proof is briefly the following. The measure of points for which the trajectory up to time t is spent in a horizon of dimension d' is of order $t^{d'-d}$. In order to prove the upper bound, one needs to overcome two difficulties. First, there are trajectories which travel from one horizon to another - this problem is solved by the Proportional lemma. The second problem is that although there are finitely many maximal horizons, but there are infinitely many lower dimensional "attached" horizons,

thus the above naive estimation cannot be summed up. To solve this problem, we slightly extend the maximal horizons in the estimation - this way, they swallow all, but finitely many attached horizons, while their leading constant (C_H) do not change a lot.
Formally, in the general d dimensional case, we prove the following statement. For every $\delta > 0$ there exists a $T < \infty$ such that for every $t > T$,

$$F(t) \leq (1+\delta) \sum_{H \in \mathbb{H}} C_H t^{d_{\max}-d}, \tag{6.11}$$

where $d_{\max}$ is the dimension of the maximal horizons. To prove this, let us introduce the fattened version of the maximal horizons. Since $\cap_{\varepsilon>0} \tilde{Q}^\varepsilon = \tilde{Q}$, for ε small enough, the maximal horizons of the fattened configuration space $\tilde{Q}^\varepsilon$ are in one to one correspondence with those of $\tilde{Q}$, and are slightly thicker then those. Thus one can choose $\varepsilon > 0$ such that

$$\sum_{H \in \mathbb{H}} C_{H^{3\varepsilon}} < (1+\delta/4) \sum_{H \in \mathbb{H}} C_H,$$

where $H^{3\varepsilon}$ is the fattened version of the horizon H - which can also be written $B_H^{3\varepsilon} \times V_H$ - and $C_{H^{3\varepsilon}}$ is the corresponding constant defined in (6.1). Note that the 3ε neighborhood of H (B_H, resp.) is a proper subset of $H^{3\varepsilon}$ ($B_H^{3\varepsilon}$, resp.). Fix this ε for the rest of the proof.

Estimator environments Now, for any fixed $\varepsilon > 0$, we construct a finite net of environments, called estimator environments, which will be used by the estimate. In fact, this finiteness will have an essential role in our arguments so despite of its simplicity we formulate the statement in a lemma.

Lemma 6.16. *Given $\varepsilon > 0$, one can find a finite set of points $q_1, \dots, q_{\bar{i}}$ with $\mathcal{V}(q_i) = \{V_j(q_i) : 1 \leq j \leq j(q_i)\}$ such that for arbitrary $q \in Q$ and any free subspace $V \in \mathcal{V}(q)$, there are some q_i and $j : 1 \leq j \leq j(q_i)$ such that q is in the ε neighborhood of q_i and $V \subset V_j(q_i)$.*

Consequently, the 2ε neighborhood of $q_i + V_{i,j}$ contains the ε neighborhood of $q + V$.

Proof. Using Lemma 6.12, for every point $q \in Q$ pick a stability neighborhood $U(q)$ of radius $\xi(q) < \varepsilon$. By the compactness of Q, fix a finite subcover $\cup_{i=1}^{\bar{i}} U(q_i)$ of Q from these environments and remember that by Lemma 6.13 each $\mathcal{V}(q_i) = \{V_j(q_i) : 1 \leq j \leq j(q_i)\}$ is finite. Then by the definition of stability neighborhoods, we have that for arbitrary $q \in Q$ and any free subspace $V \in \mathcal{V}(q)$, there are some q_i and $j : 1 \leq j \leq j(q_i)$ such that q is in the ε neighborhood of q_i and $V \subset V_j(q_i)$. □

Remark Those $q_i + V_{i,j}$'s with $\dim V_{i,j} = d_{\max}$ are necessarily subsets of some maximal horizons. Since $H^{2\varepsilon}$ contains the 2ε neighborhood of H, the 2ε neighborhoods of these $q_i + V_{i,j}$'s are covered by the $H^{2\varepsilon}$'s. Thus we call the sets $H^{2\varepsilon}$ for $H \in \mathbb{H}$, and the 2ε neighborhoods of the remaining $q_i + V_{i,j}$'s ($i \in I, j \in J_i$) *estimator environments.* Remind that $\dim V_{i,j} < d_{\max}$ for all $i \in I, j \in J_i$, and that the ε neighborhood of any affine free subspace is covered by some estimator environment - thus the Proportionality lemma asserts that the c portion of a long enough free flight is spent in an estimator environment.

Proof of (6.11) In this paragraph, we finish the proof of Theorem 6.6. First, with the already fixed ε, use the Proportionality lemma to obtain some c and T. From now on, we always assume $t > T$. For the estimation of $\mu(\tau > t)$, we distinguish three cases.

Case 1 Such points $x = (q, v) \in M$ with $\tau(x) > t$, for which the time interval $[s_1, s_2]$ with $0 < s_1 < s_1 + c\tau(x) < s_2 < \tau(x)$ guaranteed by the Proportionality lemma is spent in the 2ε neighborhood of $q_i + V_{i,j}$ for some $i \in I, j \in J_i$.
Since there is a line segment of length at least $ct/2$ spent in the neighborhood of $q_i + V_{i,j}$, the angle of v and $V_{i,j}$ is necessarily smaller than $2/(ct)$. As it was also used by the proof of (6.1), the $d-1$ dimensional Lebesgue measure on S^{d-1} of such velocity vectors v is asymptotically

$$\left(\frac{2}{ct}\right)^{\dim V_{i,j} - d}.$$

Since $\dim V_{i,j} < d_{\max}$ and there are finitely many estimator environments, for t large enough the μ-measure of points of Case 1 are smaller than

$$\delta/4 \sum_{H \in \mathbb{H}} C_H t^{d_{\max} - d}.$$

Case 2 (Main term) Such points $x \in M$ with $\tau(x) > t$, where the configuration component of $\Phi_{[0,t]}x$ is a subset of $H^{3\varepsilon}$ for some $H \in \mathbb{H}$.
The same argument used to prove (6.1) implies that the μ-measure of such points is asymptotically not larger than

$$\sum_{H \in \mathbb{H}} C_{H^{3\varepsilon}} t^{d_{\max} - d},$$

thus for t large enough, is smaller than

$$(1 + \delta/2) \sum_{H \in \mathbb{H}} C_H t^{d_{\max} - d}.$$

Case 3 Such points $x \in M$ with $\tau(x) > t$ not treated in Case 2, for which the time interval $[s_1, s_2]$ with $0 < s_1 < s_1 + c\tau(x) < s_2 < \tau(x)$ guaranteed by the Proportionality lemma is spent in $H^{2\varepsilon}$ for some $H \in \mathbb{H}$. It is worth noting that one difficulty of this case comes from the fact that it covers an infinite number of lower dimensional "attached" horizons.
Note that $\Pi_Q \Phi_{[0,\tau(x)]}x$ for such an x has a part of length at least $c\tau(x)/2$ in the region $H^{3\varepsilon} \setminus H^{2\varepsilon}$ and also crosses this region in the sense that intersects with both $H^{2\varepsilon}$ and the complement of $H^{3\varepsilon}$. Thus there are some s_3, s_5 with $0 < s_3 < s_3 + c\tau(x)/2 < s_5 < \tau(x)$ such that $\Pi_Q \Phi_{s_3}x$ is in $\partial B_{H^{2\varepsilon}} \times V_H$ and $\Pi_Q \Phi_{s_5}x$ is in $\partial B_{H^{3\varepsilon}} \times V_H$ (or $\Pi_Q \Phi_{s_3}x$ is in $\partial B_{H^{3\varepsilon}} \times V_H$ and $\Pi_Q \Phi_{s_5}x$ is in $\partial B_{H^{2\varepsilon}} \times V_H$, which case can be treated analogously). As a starting idea, one can think about this trajectory segment as a long free flight in a $d_{\max}$ dimensional billiard, which guarantees that the Lebesgue measure of points of Case 3 are not large. More precisely, write

$$\Phi_{s_3}x = (q^{\perp} + q^{\parallel}, v^{\perp} + v^{\parallel}),$$

where $q^\perp$ and $v^\perp$ are in $V_H^\perp$, while $q^\|$ and $v^\|$ are in V_H. Note that $q \in Q$ by definition, but the components $q^\perp, q^\|$ are in $\mathbb{R}^d$. The projection of the trajectory segment $\Pi_{\tilde{Q}}\Phi_{[s_3,s_5]}x$ to $V_H^\perp$, prescribed by $q^\perp$ and $v^\perp$, is going to be used to construct the billiard table of dimension $d_{\max}$, while $q^\|$ and $v^\|$ are going to define the trajectory in this lower dimensional billiard table. There is a point $z \in \partial B_H$ such that in the intersection point of $z + V_H$ and ∂Q the d dimensional sphere of radius $\kappa_{\max}^{-1}$ touching the appropriate scatterer from inside has a center, the projection of which to $V_H^\perp$ is collinear with z and $q^\perp$. (See Figure 6.3.) Let us denote this sphere by S. Now, consider the affine subspace $q^\perp + V_H$. By definition, there exists a point $q^\perp + p$ in this affine subspace such that the d dimensional ball of radius 2ε and center $q^\perp + p$ is contained completely in S and hence in a scatterer.
Now let us define a $d_{\max}$ dimensional billiard configuration space: the periodicity is $\mathcal{L} \cap V_H$, there is one spherical scatterer of radius ε and the center of this spherical scatterer is p (when $\mathbb{R}^{d_{\max}}$ is identified with V_H). Denote its configuration space by $\tilde{Q}_{d_{\max}}$. Note that the intersection of $\tilde{Q}$ and $q^\perp + V_H$ is contained in $\tilde{Q}_{d_{\max}}$ (again, with $\mathbb{R}^{d_{\max}}$ being identified with V_H). Further, we claim that with the notation

$$s_4 = s_5 \wedge \min\{s > 0 : \mathrm{dist}(q^\perp, q^\perp + (s - s_3)v^\perp) > \varepsilon\},$$

for every $s_3 < s < s_4$, the intersection of $\tilde{Q}$ and $q^\perp + (s-s_3)v^\perp + V_H$ is also contained in $\tilde{Q}_{d_{\max}}$. Indeed, since $\mathrm{dist}(q^\perp, q^\perp + (s - s_3)v^\perp) < \varepsilon$, the d dimensional ball of radius ε and center $q^\perp + (s - s_3)v^\perp + p$ is contained in the ball of radius 2ε and center $q^\perp + p$. The latter statement is in general not true for $s = s_5$, since $q^\perp + (s_5 - s_3)v^\perp$ can be outside of the projection of S (see Figure 6.3), that is why we needed to introduce s_4.
Now, we can easily map a long free flight in this $d_{\max}$ dimensional billiard to our trajectory segment $\Phi_{[s_3,s_5]}(x)$. Namely, let us choose the free flight of the phase point $(q^\|, v^\|)$ in $Q_{d_{\max}}$. Due to the construction, this free flight is longer than $(s_4 - s_3)/2$. We claim that this is longer then a universal constant (in the sense that does not depend on x but may depend on ε and also on H since there are finitely many of them) times t, i.e.

Lemma 6.17. *There is a constant $c'(\varepsilon)$, such that $s_3 < s_3 + 2c'(\varepsilon)\tau(x) < s_4 \le s_5$.*

Proof. It is enough to prove that there exists some $c''(\varepsilon)$ such that $s_3 + c''(\varepsilon)(s_5 - s_3) < s_4$. Since $|(s_4 - s_3)v^\perp| > \varepsilon$, it is enough to give an upper bound for $|(s_5 - s_3)v^\perp|$. Thus we need that the function

$$\Delta(y, z) = \max\{r | \exists l \in \mathcal{L} : \overline{y, z + l} \subset B_{H^{3\varepsilon}} \text{and}\, \mathrm{dist}(y, z + l) = r\} \tag{6.12}$$

on $B_{H^{3\varepsilon}} \times B_{H^{3\varepsilon}}$ is bounded (then ε divided by this bound is an appropriate choice for $c''(\varepsilon)$).
In order to see that (6.12) is bounded, first we prove that the set $\{\Delta(y, z) \ge n\}$ is closed for any integer n. Choose any convergent sequence $(y_i, z_i) \to (y_\infty, z_\infty)$ from the above set. There are corresponding $l_i \in \mathcal{L}$ vectors by the definition of $\Delta(y, z)$. Then the set $\{l_i : i \ge 1\}$ cannot be infinite, since if it was, then one could choose a convergent subsequence of $l_i/|l_i|$ and the line with this direction containing y_∞ would be a subset of $B_{H^{3\varepsilon}}$ which is a contradiction. Thus the set $\{l_i : i \ge 1\}$ is finite. Hence one can choose a subset $(y_{i_k}, z_{i_k}) \to (y_\infty, z_\infty)$ with $l_{i_k} = l$, yielding $\Delta(y_\infty, z_\infty) \ge n$. Whence $\{\Delta(y, z) \ge n\}$ is closed. Now assume by contradiction that $\Delta(y, z)$ is not bounded, thus the sets $\{\Delta(y, z) \ge n\}$ for $n \ge 1$

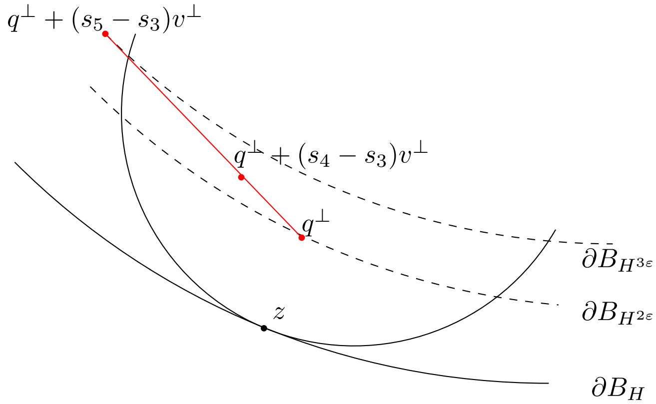

Figure 6.3: Construction of the $d_{\max}$ dimensional billiard table - figure in $V_H^\perp$

are closed subsets of each other. Thus there is a pair (y, z) such that $\Delta(y, z) = \infty$. Just like before, one can easily deduce the existence of an infinite line in $B_{H^{3\varepsilon}}$ through y which is a contradiction. Thus we have proved that (6.12) is bounded and thus verified the existence of an appropriate $c'(\varepsilon)$. □

Now, we finish the proof of Case 3. Since at least ct time of the free flight is spent in $H^{2\varepsilon}$, the angle of v and V_H is smaller then C_1/t with some C_1. Thus the points $x = (q, v)$ of Case 3 are elements of the set $\mathbb{R}^d/\mathcal{L} \times V(t)$, where

$$V(t) = \{v \in S^{d-1} : \angle(v, V_H) < C_1/t\}.$$

As before, $\lambda_{d-1}(V(t)) < C_2 t^{d_{\max}-d}$. Every point $(q, v) \in \mathbb{R}^d/\mathcal{L} \times V(t)$ can uniquely be written in the form

$$(q, v) = (q_0^\perp + q_0^\parallel, v^\perp + v^\parallel)$$

with $q_0^\perp, v^\perp \in V_H^\perp$, $q_0^\parallel, v^\parallel \in V_H$. The conditional measure of $\lambda_d \times \lambda_{d-1}$ on $\mathbb{R}^d/\mathcal{L} \times V(t)$ to such points where $q_0^\perp, v^\perp$ are fixed, is also Lebesgue on the possible set of pairs $(q_0^\parallel, v^\parallel)$. Note that since $|v^\perp|$ is small, the set of possible $v^\parallel$'s is a $d_{\max}$ dimensional sphere of radius close to one. But the set of possible $q_0^\parallel$'s depends on $q_0^\perp$, since $V_H^\perp$ is not necessarily generated by lattice vectors. Thus write

$$\mathfrak{q}(q) = \{\bar{q} \in \mathbb{R}^d/\mathcal{L} : \bar{q}_0^\perp = q_0^\perp\}.$$

One can easily prove that there exists some $\eta > 0$ such that

$$\lambda_d\{q : \lambda_{d_{\max}}(\mathfrak{q}(q)) < \eta\} < \frac{C_H \delta}{8 c_\mu C_2}.$$

Thus

$$\mu(\{q : \lambda_{d_{\max}}(\mathfrak{q}(q)) < \eta\} \times V(t)) < \frac{\delta}{8} C_H t^{d_{\max}-d}.$$

Now we can assume that

$$\lambda_{d_{\max}}(\mathfrak{q}(q)) > \eta. \tag{6.13}$$

Since the c portion of the line segment $\Pi_{V_H^\perp}\Pi_Q\Phi_{[0,\tau(x)]}x$ is spent in $H^{2\varepsilon}$, once $q_0^\perp, v^\perp$ are fixed, the number of possible $q^\perp$'s (that is, the projection of $\Pi_Q\Phi_{s_3}x$ to $V_H^\perp$) is bounded. This, the inductive hypothesis (used on the billiard table $\tilde{Q}_{d_{\max}}$), Lemma 6.17 and (6.13) imply that once $q_0^\perp, v^\perp$ are fixed, the $\lambda_{d_{\max}} \times \lambda_{d_{\max}-1}$ measure of such coordinates $(q_0^\parallel, v^\parallel)$ with which the free flight is longer than t is bounded by some universal constant times t^{-1}. Consequently, for t large enough, the μ measure of points in Case 3 are smaller than

$$\frac{\delta}{4}\sum_{H\in\mathbb{H}} C_H t^{d_{\max}-d}.$$

6.6 Proof of Theorem 6.7

6.6.1 Lorentz process with small scatterers

First, we recall the following result of Bourgain, Golse and Wennberg (see [BGW98] and [GW00]). Consider a billiard table with periodicity $\mathbb{Z}^D$ ($D \geq 2$) and one spherical scatterer of radius $r < 1/2$. Define $\mu_{\mathbb{Z},r}$ and $\tau_{\mathbb{Z},r}$ for this billiard table as before. Then there exist $c'(D)$ and $C'(D)$ such that

$$\frac{c'(D)}{tr^{D-1}} \leq \mu_{\mathbb{Z},r}(\tau_{\mathbb{Z},r} > t) \leq \frac{C'(D)}{tr^{D-1}} \tag{6.14}$$

is true whenever

$$t > r^{1-D}. \tag{6.15}$$

In the case $t \approx r^{D-1}$, the so-called Boltzmann-Grad limit, much more is known than (6.14), see [MS10]. In order to prove Theorem 6.7, we need a slightly extended version of the above estimation.
Let $\mathcal{L}'$ be any D-dimensional lattice and let $q_1, \dots q_{n'} \in \mathbb{R}^D/\mathcal{L}'$. Consider the billiard table with periodicity $\mathcal{L}'$ and finitely many disjoint spherical scatterers of radius r centered at $q_1, \dots q_{n'}$. Let Q', M', μ' and τ' be defined accordingly.

Lemma 6.18. *There exist $c'(\mathcal{L}')$ and $C'(\mathcal{L}')$ such that*

$$\frac{c'(\mathcal{L}')}{tr^{D-1}} \leq \mu'(\tau' > t) \leq \frac{C'(\mathcal{L}')}{tr^{D-1}} \tag{6.16}$$

is true whenever

$$t > r^{1-D}. \tag{6.17}$$

Remark Obviously, Lemma 6.19 also implies that for any fixed $\eta > 0$, (6.16) is true if $t > \eta r^{1-D}$, with some $c'(\mathcal{L}')$ and $C'(\mathcal{L}')$ depending also on η. Thus, whenever we refer to (6.17), it may be true only with some η, but in order to make the exposition simpler, we do not keep track of the η's.

Proof. First, we prove the upper estimate. Pick a basis $\{a_i\}_{i=1}^D$ of the lattice $\mathcal{L}'$ and denote by A the matrix whose i-th column is a_i. Also write σ_i for the i-th smallest singular value of A^{-1}. Further, identify $\mathbb{R}^D/\mathbb{Z}^D$ with the unit cube and $\mathbb{R}^D/\mathcal{L}'$ with the parallelepiped $(a_i)_{i=1}^D$. Without loss of generality, we

may assume that one of the spherical scatterers is centered at the origin (i.e. $q_1 = 0$).
Now assume that for some $x' = (q', v') \in M'$, $\tau'(x') > t$. Then for the point

$$\phi(x') := x_{\mathbb{Z}} = (A^{-1}q', \frac{A^{-1}v'}{\|A^{-1}v'\|}),$$

we have

$$\tau_{\mathbb{Z}, r\sigma_1}(x_{\mathbb{Z}}) > t\sigma_1.$$

Indeed, the image under A^{-1} of the sphere of radius r centered at the origin contains the sphere of radius $r\sigma_1$ (the images of the possible other scatterers are simply omitted). The Lebesgue measure on Q' is transformed by ϕ to $\det(A^{-1})$ times the Lebesgue measure in $\mathbb{R}^D/\mathbb{Z}^D$ minus an ellipse centered at the origin, which is dominated by the Lebesgue measure on $\mathbb{R}^D/\mathbb{Z}^D \setminus B(0, r\sigma_1)$. The image of the Lebesgue measure on S^{D-1} by ϕ is $\frac{1}{\|A^{-1}v'\|}dv'$.
Thus, using (6.14), one can prove the second part of (6.16) with

$$C'(\mathcal{L}') = \det(A^{-1})\sigma_n \sigma_1^{-D-1} C'(D)$$

at least, for $t > \sigma_1^{-D} r^{D-1}$, but consequently for $t > r^{D-1}$ too, possibly with a different $C'(\mathcal{L}')$.
Now, we prove the lower estimate. Observe that it is enough to prove the statement for the special case $\mathcal{L}' = \mathbb{Z}^D$. Indeed, once $c(\mathbb{Z}^D)$ is found, one can prove the existence of $c(\mathcal{L}')$ for any $\mathcal{L}'$ the same way as in the upper estimation.
Thus the statement we going to prove is indeed a slight modification of the first part of (6.14): the difference is that we have n' spherical scatterers of radius r centered at arbitrary points $q_1, \dots q_{n'}$, instead of just one scatterer. We claim that an obvious modification of the proof of Golse and Wennberg applies here. Indeed, if q is an integer vector with $g.c.d.(q) = 1$ and one projects the scatterer configuration to the line with direction q, then observes a gap of length at least $(1/|q| - 2n'r)/n'$ among the images of the scatterers, assuming of course that $r < (2n'|q|)^{-1}$. Hence there is a principal horizon perpendicular to q (or "sandwich layer") whose middle third has width

$$a_{q,r} = \frac{1}{3}\left(\frac{1}{|q|} - 2n'r\right)\frac{1}{n'}.$$

Considering only those q's for which $|q| < q_{\max} = (4n'r)^{-1}$, the density of the middle third layers is larger than $(12n')^{-1}$ (instead of 1/6, see page 1158 in [GW00] for more details). With these modifications, the proof of [GW00] yields the statement.

□

6.6.2 Upper estimate

We assume that there is one principal incipient horizon, if there were more, an analogous proof would apply. As in Subsection 6.5.2, let us fix an ε, define the estimator environments - one of them is the 2ε neighbourhood of the principal incipient horizon ($H^{2\varepsilon}$), the others have dimension at most $d-2$. The proportionality lemma implies that the c portion of a long enough flight is spent in one of the estimator environments. The μ-measure of such points for which this is not $H^{2\varepsilon}$ is $O(t^{-2})$ as in Case 1 of Subsection 6.5.2.

The essence of the proof is the following statement:

Lemma 6.19. *For a fixed ε small enough,*

$$\begin{aligned}\lambda_d \times \lambda_{d-1} \quad & (\{x=(q,v)|q\in H^{2\varepsilon}, \tau(x)>s, \Pi_Q\Phi_{[0,s]}x\subset H^{2\varepsilon}\}) \\ = \quad & \begin{cases} O(s^{-2}), & 3\le d\le 5 \\ O(s^{-2}\log s), & d=6 \\ O\left(s^{\frac{2+d}{2-d}}\right), & d>6. \end{cases}\end{aligned}$$

Proof. Denote by V the $d-1$ dimensional hyperplane defining the incipient horizon. Without loss of generality, we may assume that the origin is in this horizon, that is $H=V$. Since V is a lattice subspace, one can choose a lattice vector $v_d\in\mathcal{L}\setminus V$ such that $V\cap\mathcal{L}$ and v_d generate $\mathcal{L}$. Since $\mathbb{R}/\mathcal{L}$ can be identified with a parallelepiped generated by $v_1,\dots,v_d$ with $v_1,\dots v_{d-1}\in V$, for every $q\in Q\cap H^{2\varepsilon}$, there is a unique decomposition

$$q = q_V + q_W$$

with $q_V\in V$, $q_W \parallel v_d$ and $|q_W|<2\varepsilon\cot\alpha$, where α is the angle of V and v_d. We also write

$$v = v^{\parallel} + v^{\perp},$$

where $v\in S^{d-1}, v^{\parallel}\in V, v^{\perp}\in V^{\perp}$.

The idea of the proof is reminiscent to that of Case 3 in Subsection 6.5.2. If there is a long flight in $H^{2\varepsilon}$, then v is close to V. Thus we can think of this trajectory as a long free flight in a $d-1$ dimensional billiard. Note that here, the $d-1$ dimensional scatterer size can be arbitrary small, since the trajectory is close to V. Thus a delicate analysis of this scatterer size, and the upper estimation of (6.16) are needed.

Let us chop the set of possible q_W's and $v^{\perp}$'s into the following pieces:

$$\begin{aligned} V_i &= \{v^{\perp}\in V^{\perp}||v^{\perp}|\in[2^{-i},2^{-i+1})\} \qquad i>\log s-\log 2\varepsilon \\ Q_j &= \{av_d||a|\in[2^{-j}\cot\alpha, 2^{-j+1}\cot\alpha)\} \quad j>-\log 2\varepsilon. \end{aligned}$$

Accordingly, we write

$$H_j = H^{2^{-j+1}}\setminus H^{2^{-j}}.$$

Here, and also in the sequel, log always stands for $\log_2$.

Now assume that $v^{\perp}\in V_i$ and $q_W\in Q_j$ for some fix i,j. We want to estimate the $\lambda_{d-1}\times\lambda_{d-2}$ measure of parameters q_V, $v^{\parallel}$ with which (q,v) is an element of the set

$$Q_{long} = \{x=(q,v)|q\in H^{2\varepsilon}, \tau(x)>s, \Pi_Q\Phi_{[0,s]}x\subset H^{2\varepsilon}\}.$$

We can assume that the projection of q_W and $v^{\perp}$ are oppositely oriented. If they are not, a simpler version of the forthcoming proof is applicable.

From now, we distinguish four cases.

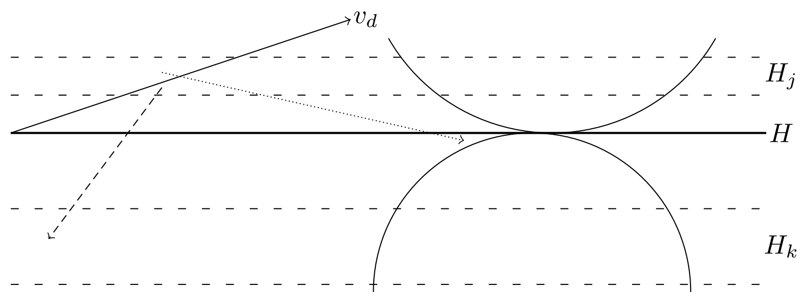

Figure 6.4: $H^{2\varepsilon}$ - a d dimensional picture. Densely dotted trajectory: $j < i - \log s$. Densely dashed trajectory: $j \geq i - \log s$.

- **Case a** $i < \frac{d}{d-2} \log s$ and $j \leq i - \log s$.
  In this case, there is a line segment of $\Pi_Q \Phi_{[0,s]} x$ of length at least $s/5$ spent in the strip H_{j+1}. Note that for every $q \in H_{j+1}$, the intersection of Q and $q + V$ is a billiard configuration of dimension $d-1$. Further, this billiard configuration is contained in a larger one, where there is only one spherical scatterer of radius approximately $\sqrt{\kappa_{\max}^{-1} 2^{-j}}$. Indeed, there is at least one d dimensional scatterer touching V from the appropriate side. If one takes the d dimensional ball of radius $\kappa_{\max}^{-1}$ touching V in this point and considers the intersection of the ball and a close enough affine hyperplane, obtains a $d-1$ dimensional ball of the desired radius (which is roughly the square root of the distance of the hyperplanes). As in Case 3 of Subsection 6.5.2, by projecting the previously obtained trajectory segment of length $s/5$ to the "lower boundary of H_{j+1}" (i.e. $\partial H^{2^{-j}}$) we obtain a free flight of length at least $s/6$ (if ε is small enough) in a $d-1$ dimensional billiard table with periodicity $\mathcal{L} \cap V$ and one spherical scatterer of radius $\sqrt{\kappa_{\max}^{-1} 2^{-j}}$. Note that this mapping to the lower dimensional billiard is simpler then that of Subsection 6.5.2, since $V^{\perp}$ is one dimensional, thus the billiard configuration space in $q + V$ is increasing as q moves from $\partial H^{2\varepsilon}$ to V (the issue of moving scatterers is simply absent). Observe that $i < \frac{d}{d-2} \log s$ and $j \leq i - \log s$ imply $j \leq \frac{2}{d-2} \log s$ which yields that (6.17) is satisfied by $t = s$, $r = \kappa_{\max}^{-1/2} 2^{-j/2}$ and $D = d-1$. Thus the second part of (6.16) implies that whenever $v^{\perp} \in V_i$ and $q_W \in Q_j$ are fixed, the $\lambda_{d-1} \times \lambda_{d-2}$ measure of parameters q_V, $v^{\|}$ with which $(q, v) \in Q_{long}$ is $O(s^{-1} 2^{j(d-2)/2})$.

- **Case b** $\frac{d}{d-2} \log s \leq i < \frac{d+2}{d-2} \log s$ and $j \leq \frac{2}{d-2} \log s$.
  The same estimation as in Case a yields that the $\lambda_{d-1} \times \lambda_{d-2}$ measure of parameters q_V, $v^{\|}$ with which $(q, v) \in Q_{long}$ is $O(s^{-1} 2^{j(d-2)/2})$.

- **Case c** $i < \frac{d}{d-2} \log s$ and $j > i - \log s$.
  Note that distance of $\Pi_V^{\perp} \Pi_Q x$ and $\Pi_V^{\perp} \Pi_Q \Phi_s x$ (here, $\Pi_V^{\perp}$ is the orthogonal projection to $V^{\perp}$) is at least $s 2^{-i}$, which is larger than 2^{-j}. Hence there is a k such that a line segment of $\Pi_Q \Phi_{[0,s]} x$ of length at least $s/8$ is spent in H_k and 2^{-k} is larger than $s 2^{-i}/4$. Now using the same estimation as in Case a in the strip H_k, one obtains that the $\lambda_{d-1} \times \lambda_{d-2}$ measure of parameters q_V, $v^{\|}$ with which $(q, v) \in Q_{long}$ is $O(s^{-1-\frac{d-2}{2}} 2^{i(d-2)/2})$.

- **Case d** $\frac{d}{d-2}\log s \le i < \frac{d+2}{d-2}\log s$ and $j > \frac{2}{d-2}\log s$, or $i \ge \frac{d+2}{d-2}\log s$.
  In this case, we simply estimate the measure of the appropriate parameters q_V, $v^{\|}$ by a constant.

Note that $\lambda_1(V_i) \sim 2^{-i}$ and $\lambda_1(Q_j) = 2^{-j}$. Taking into account this fact and the estimations of Cases a-d, one obtains that $\mu(Q_{long})$ is bounded from above by some constant times the following expression:

$$\sum_{i=\log s-\log 2\varepsilon}^{\frac{d}{d-2}\log s}\left[\left(\sum_{j=-\log 2\varepsilon}^{i-\log s} 2^{-i}2^{-j}s^{-1}2^{j(d-2)/2}\right) + 2^{-i}s2^{-i}s^{-1-\frac{d-2}{2}}2^{i(d-2)/2}\right]$$
$$+ \sum_{i=\frac{d}{d-2}\log s}^{\frac{d+2}{d-2}\log s}\left[\left(\sum_{j=-\log 2\varepsilon}^{\frac{2}{d-2}\log s} 2^{-i}2^{-j}s^{-1}2^{j(d-2)/2}\right) + 2^{-i}s^{\frac{2}{2-d}}\right] + s^{\frac{2+d}{2-d}}.$$

An elementary computation shows that this is the same order of magnitude as stated in the lemma.

□

In order to finish the proof of the upper estimate, we need to bound the measure of points $x = (q, v)$ for which $\tau(x) > t$ and the proportionality lemma gives the estimator environment $H^{2\varepsilon}$. Observe that in this case, the angle of v and V is necessarily smaller than $2\varepsilon/t$. The Lebesgue measure of point for which $q \in H^{2\varepsilon}$ is bounded by the desired order of magnitude due to Lemma 6.19. Thus assume that $q \notin H^{2\varepsilon}$. For every such point $x = (q, v)$, there is a point $\phi(x) = x_b = (q_b, v)$, which is the initial point of the free flight segment in $H^{2\varepsilon}$ (i.e. $\exists s < (1-c)\tau(x) : \Phi_s(x) = (q_b, v)$, $q_b \in \partial H^{2\varepsilon}$, $\Pi_Q\Phi_{[s,s+c\tau(x)]}x \subset H^{2\varepsilon}$). The proportionality lemma also implies that for any such x_b,

$$\lambda_1(\phi^{-1}(x_b)) < \frac{1}{c}\max\{s : s < \tau(x_b), \Pi_Q\Phi_{[0,s]}x_b \subset H^{2\varepsilon}\}.$$

Thus, also using Lemma 6.19 (with $s = ct/2$), the integral

$$\int_{\partial H^{2\varepsilon}\times\{v:\angle(v,V)<2\varepsilon/t\}} \lambda_1(\phi^{-1}(x_b))d\lambda_{d-1}(q_b) \times \lambda_{d-1}(v)$$

can be bounded by the desired order of magnitude which yields the upper estimate of Theorem 6.7.

6.6.3 Lower estimate

Now, we prove the second part of Theorem 6.7, which is a lower estimate in the dispersing case.
In dimension $d \le 5$, the statement is straightforward, since obviously there are horizons of codimension 2 "attached" to the incipient horizon (indeed, a hyperplane parallel to the incipient horizon and close to it, intersects the scatterers in tiny convex bodies - approximate ellipsoids - which depend continuously on the distance of the hyperplanes). Then the same argument used to prove (6.1) provides a subset of the phase space of measure $O(t^{-2})$ consisting of points having free flight longer than t.
In dimension $d \ge 6$, we use a simplified version of the proof of Lemma 6.19. The main observation is that due to the lower bound on the curvature, the scatterers touch the incipient horizon in finitely many points (in $q_1, \dots q_{n'}$, say). Further, the intersection of the scatterers and a hyperplane parallel to the incipient horizon at distance h from it, is contained in n' spheres of radius $\sqrt{\kappa_{\min}^{-1}h}$ centered at $q_1, \dots q_{n'}$.

Thus in Cases a-d of Lemma 6.19, by such a choice of i and j, where $s2^{-i} \approx 2^{-j}$, using the first part of (6.16) instead of the second, one easily obtains a lower bound of the same order of magnitude. In fact, for $d > 6$, only one pair of indices (i, j) is enough. Namely, choose

$$i = \lceil \frac{d}{d-2} \log s \rceil$$

and $j = \lceil i - \log s \rceil$. With this choice and the notation $r = \sqrt{\kappa_{\min}^{-1} s 2^{-i}}$, $s = t$, (6.17) is fulfilled, hence the Lebesgue measure of points $x = (q, v)$ with $v^\perp \in V_i$ and $q_W \in Q_j$ having free flight longer than s is at least some constant times $2^{-i}2^{-j}$, thus another constant times $s^{\frac{2+d}{2-d}}$.
In dimension $d = 6$, one needs to consider all indices i with $\log s - \log 2\varepsilon < i < 3/2 \log s - \log \kappa_{\min}$ and for a fix i, the index $j = i - \log s$. Similarly to the case $d > 6$, the lower estimation of order $s^{-2} \log s$ follows.

6.7 Examples

Equ. (35) of [D12] provides the form of the limiting covariances for the super-diffusive limit of dispersing Lorentz processes assuming his Conjectures 1 and 3 hold. His derivation of Equ. (35) from the conjectures can be extended to the semi-dispersing case thus our Theorem 6.6 can be used. His Conjecture 3 is of dynamical nature and for clarity we briefly summarize what is known and what we expect in general. For brevity - beside [D12] - we rely here on the works [Y98, BT08] where, for instance, the complexity hypothesis is also used and the precise forms of exponential decay of correlations (EDC) and of the central limit theorem (CLT) are given.

- [BT08] For multidimensional ($d > 2$) dispersing billiards with finite horizon satisfying the complexity hypothesis, EDC and CLT hold and the diffusivity covariance is given by Green-Kubo;
- **Conjecture A** (Dynamical) For multidimensional ($d > 2$) semi-dispersing billiards without a principal horizon and satisfying the complexity hypothesis, EDC and CLT hold and the diffusivity covariance is given by Green-Kubo;
- **Conjecture B** (Dynamical) For multidimensional ($d > 2$) semi-dispersing billiards with at least one principal horizon and satisfying the complexity hypothesis, EDC and the super-diffusive limit statement with scaling $\sqrt{n \log n}$ or $\sqrt{t \log t}$ hold. (cf. [SzV07, ChD09a] for $d = 2$).

Example 1: Two hard balls of radii $0 < r < 1/4$ on $\mathbb{T}^d$. Under the complexity hypothesis it follows from [BT08] and from Theorem 6.6 that the super-diffusive limiting covariance of the system is

$$\mathcal{D}^2 = \frac{1}{1 - (2r)^d |B_d|} \frac{|B_{d-1}|}{|S_{d-1}|} (\frac{1}{2} - 2r)^2$$

Here B_d is the d-dimensional unit ball, and S_{d-1} is its surface (cf. Equ. (37) of [D12]).

Example 2: Cylindrical billiard on $\mathbb{T}^3$. (We note that this was the first semi-dispersing billiard whose ergodicity had been established (cf. [KSSz89]).) We assume that on $\mathbb{T}^3$ we are given two non-intersecting cylindrical scatterers C_1 and C_2 - for simplicity - of equal radii $0 < r < 1/4$. Suppose that the generator of C_i is parallel to the coordinate direction e_i, $i = 1, 2$ and the distances between the two cylinders - in the coordinate direction 3 - are z and w. In this case we have two principal horizons of widths z and w parallel to the coordinate plane (e_1, e_2) and super-diffusion is expected in the directions e_1, e_2 whereas regular one in the direction of the axis e_3.

$$\mathcal{D}_{11} = \mathcal{D}_{22} = \frac{1}{4(1 - 2r^2\pi)}(z^2 + w^2)$$

$$\mathcal{D}_{33} = 0$$

Of course, - if in the direction of the axis e_3 - we apply diffusive scaling, then the limiting covariance should again be given by the Green-Kubo formula.

6.8 Concluding remarks

1. In order to prove the above Conjecture B, a first step could be determining the limiting joint distribution of τ and the forthcoming free flight (i.e. $\tau \circ \Phi_{\tau+}$, where $\Phi_{\tau+}$ means that the velocity is the post-collisional one), when τ is large (see also Conjecture 3 in [D12] and [B92], [SzV07] in the planar case). Thus we formulate another conjecture.

   - **Conjecture C** (Geometric) In a d dimensional dispersing billiard with at least one principal, non-incipient horizon, if τ is large, then $\tau \circ \Phi_{\tau+}$ is typically of order $\tau^{1/d}$.

   Now we explain why we expect Conjecture C to be true. Note that if $\tau(x)$ is larger than some large t, then $x = (q, v)$ - with probability close to one - is such that q is in a principal horizon H, and the angle of v and V_H is roughly $1/t$. Further, the component of v in V_H is uniformly distributed. After some time, the free flight from x reaches the boundary $\partial B_H \times V_H$ of the horizon. Now we claim that the remaining time until the collision is typically $t^{\frac{d-2}{d}}$, or in other words, the distance of $\Pi_{V_H^\perp}\Pi_Q\Phi_{\tau(x)}x$ and B_H is roughly $t^{-2/d}$. Indeed, in the hyperplane $q_h + V_H$ at distance h from $B_H + V_H$, there are $d-1$ dimensional scatterers (approximate ellipsoids of bounded eccentricity due to the dispersing assumption) of diameter $\sqrt{h}$. Thus (6.16) yields that in this hyperplane, a $\lambda_{d-1} \times \lambda_{d-2}$-typical phase point does not collide until time ht if and only if $ht << h^{\frac{2-d}{2}}$. Now a similar argument used to prove of Lemma 6.19, implies that typically the distance of $\Pi_{V_H^\perp}\Pi_Q\Phi_{\tau(x)}x$ and B_H is roughly $h = t^{-2/d}$. Denote the post collisional velocity by v'. We expect that the angle of v' and V_H is typically of order $t^{-1/d}$ which would provide Conjecture C. Nevertheless, this angle can be smaller.

2. [Sz08] also raised the problem of the limiting behavior of a quasi-periodic Lorentz process, for instance that of the Penrose-Lorentz one. As [W12] points out the tail distribution of the free path

length is exponential in random Lorentz processes with non-intersecting scatterers whereas - as we have seen - it is algebraic in the presence of horizons. The simulations of the author suggest that for a 1-dimensional quasi-periodic paradigm of the Lorentz process, this tail behavior is not exponential. On the other hand, [KS12] stresses that for the random non-intersecting Lorentz process one has normal diffusion and observes computationally three different regions for a 2-dimensional quasi-periodic Lorentz process showing super-diffusion, diffusion and subdiffusion.

Chapter 7

Lorentz process with infinite horizon and the martingale method

7.1 Introduction

As it was mentioned in Chapter 1, the super-diffusively rescaled trajectory in planar periodic Lorentz processes with infinite horizon converges to the Brownian motion. This was first conjectured by [B92], later proven in [SzV07] (in a slightly weaker form, namely the convergence of finite dimensional distributions), and recently in [ChD09a]. The two proofs are essentially different: [SzV07] uses Young towers, while [ChD09a] combines the standard pair technique with Bernstein's big block- small block technique from Probability theory.

In case of finite horizon, local perturbation does not spoil the Brownian limit, as it was proven in [DSzV09]. The proof of this result is based on the standard pair technique, but the probabilistic ingredient is the martingale method of Stroock and Varadhan [SV06]. Thus there is some hope that the martingale method might be useful in attacking the corresponding problem in infinite horizon (i.e. Conjecture 1.1), too. This is the motivation of our present work.

Here, we identify the possible limit points of the super-diffusively rescaled trajectory in planar periodic Lorentz processes (which is, of course, Wiener process, solely) with the use of the martingale method. This is almost the same, as giving a third proof for the Brownian limit in periodic Lorentz process with infinite horizon - what missing, is checking that the weak limit indeed exists, i.e. proving tightness. We also stress the fact that our proof is strongly based on the one of [DSzV09] and also uses similar cut-offs (although not the same) as, and other technical results from [ChD09a].

In Section 7.2 we formulate our statement and provide the basic definitions and lemmas for its proof, while in Section 7.3, we present the actual proof.

7.2 Preliminaries

Here, we summarize very briefly the most important notions and notations from Sinai billiards needed in the present work. For a much ampler description, consult [CM06]. Define $\mathcal{D} = \mathbb{R}^2 \setminus \cup_{i=1}^{\infty} B_i$, where $B_1, \ldots, B_k$ are disjoint strictly convex domains inside the unit torus, whose boundaries are C^3-smooth and whose curvatures are bounded from below. $B_{k+1}, B_{k+2}, ...$ are the translational copies of $B_1, \ldots, B_k$ with translations in $\mathbb{Z}^2$. The billiard flow is the dynamics of a point particle in $\mathcal{D}$, which consists of free flight inside $\mathcal{D}$ and specular reflection on $\partial\mathcal{D}$. It is common to take the Poincaré section on the boundaries of the scatterer (billiard ball map). The phase space of the billiard ball map is

$$\mathcal{M} = \{x = (q, v) \in \partial\mathcal{D} \times S^1, \langle v, n \rangle \geq 0\},$$

where n is the normal vector of ∂D at the point q pointing inside $\mathcal{D}$, and the map itself is denoted by $\mathcal{F} : \mathcal{M} \to \mathcal{M}$. The natural invariant measure on $\mathcal{M}$, which we denote by μ, is the projection of the Lebesgue measure on the phase space of the billiard flow. In fact, $d\mu = \mathit{const} \cos\phi dr d\phi$, where r is the arc length parameter on $\partial\mathcal{D}$ and $\phi \in [-\pi/2, \pi/2]$ is the angle of v and n. We will write $q(x)$ for the the projection of the point x to its first coordinate (that is $q(x) \in \partial\mathcal{D}$). The free flight vector is $\Delta_0(x) = q(\mathcal{F}(x)) - q(x)$. We will also write $q_k(x) = q(\mathcal{F}^k(x))$ and $\Delta_k(x) = \Delta_0(\mathcal{F}^k(x)) = q_{k+1}(x) - q_k(x)$. Analogously, one can define the Sinai billiard on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Then one needs to introduce $\mathcal{D}_0 = \mathbb{T}^2 \setminus \cup_{i=1}^{k} B_i$, and define $\mathcal{M}_0$, $\mathcal{F}_0$ and μ_0 as before. Now μ is the periodic extension of μ_0. Since μ is infinite and μ_0 is finite, we choose the constant in the definition of μ such that μ_0 is a probability measure. The functions q_k and Δ_k can also be defined on $\mathcal{M}_0$ as the restriction of the corresponding functions on $\mathcal{M}$.

Hyperbolicity and ergodicity of $\mathcal{F}_0$ (nice properties) were proven by Sinai [S70]. An unpleasant property of the billiard map is the presence of singularities (corresponding to grazing collisions). An elegant and flexible approach to overcome this problem and prove statistical properties is the standard pair method developed by Chernov and Dolgopyat [ChD09b]. What follows, is an informal description of this method. For almost every $x \in \mathcal{M}_0$, stable and unstable manifolds through x exist. There is a factor of stretching in the unstable direction, which is bounded from below by some $\Lambda > 1$. Nevertheless, these factors are not bounded from above (the closer is x to the grazing collisions, $\{\cos\phi = 0\}$, the stronger is the expansion), which makes difficult to control the distortion of unstable manifolds. That is why it is common to introduce the following additional (secondary) singularities

$$S_{\pm k} = \{(r, \phi) : \phi = \pm\pi/2 \mp k^{-2}\}$$

for k larger than some k_0, yielding bounded distortion of an unstable manifold disjoint to all singularities. An unstable curve is some curve $W \subset \mathcal{M}$ such that at every point $x \in W$, the tangent space $T_x W$ is in the unstable cone (slightly weaker property than the unstable manifold). Further, W is homogeneous, if does not intersect any singularity. A pair $l = (W, \rho)$ is called a standard pair, if W is a homogeneous unstable curve and ρ is a regular probability density supported on W. In order to define the desired regularity of the density ρ, we need one more notion. For two points x, y on W, we write $s_{\pm}(x, y)$ for the smallest integer n such that $\mathcal{F}_0^{\pm n}(x)$ and $\mathcal{F}_0^{\pm n}(y)$ are separated by some singularity curve. A function f

is called dynamically Hölder continuous, if there exists some $\theta_f < 1$ such that for any x and y lying on some unstable (stable, resp.) curve W, the following inequality holds

$$|f(x) - f(y)| < K_f \theta_f^{s_\pm(x,y)}.$$

Now, the regularity property required for the density ρ is that $\log \rho$ should be dynamically Hölder continuous. For a standard pair $l = (W, \rho)$, we write $\mathbb{E}_l$ for the integral with respect to ρ, $\mathbb{P}_l(A) = \rho(A)$ and $length(l) = length(W)$. Once we have a standard pair, its image under the map $\mathcal{F}_0$ is a bunch of unstable curves and some measures living on them.

A nice property of standard pairs is that this image is in fact a weighted sum of standard pairs. That is why we call weighted sums of standard pairs standard families. Formally, a standard family is a set $\mathcal{G} = \{(W_a, \nu_a)\}, a \in \mathfrak{A}$ of standard pairs and a probability measure $\lambda_\mathcal{G}$ on the index set $\mathfrak{A}$. This family defines a probability measure on $\mathcal{M}_0$ by

$$\mu_\mathcal{G}(B) = \int \nu_a(B \cap W_a) d\lambda_\mathcal{G}(a).$$

Every $x \in W_a$ (for some $a \in \mathfrak{A}$), chops W_a into two pieces. The length of the shorter one is denoted by $r_\mathcal{G}$. Now the $\mathcal{Z}$-function of $\mathcal{G}$ is defined by

$$\mathcal{Z}_\mathcal{G} = \sup_{\varepsilon > 0} \frac{\mu_\mathcal{G}(r_\mathcal{G} < \varepsilon)}{\varepsilon}.$$

Note that if $\mathcal{G}$ consists of one standard pair, then $\mathcal{Z}_\mathcal{G} = 2/|W|$. In any case, we assume $\mathcal{Z}_\mathcal{G} < \infty$.

A most important property of the billiard map is that while the unstable curves are expanded due to hyperbolicity, they are also cut by the singularities of $\mathcal{F}_0$; and in some sense, the expansion prevails fragmentation. Formally, the following Growth lemma holds true:

Lemma 7.1 ([DSzV09] Prop 1.)**.** *Let $l = (W, \rho)$ be some standard pair. Then*

$$\mathbb{E}_l(A \circ \mathcal{F}_0^n) = \sum_a c_{a,n} \mathbb{E}_{l_{an}}(A),$$

where $c_{a,n} > 0$, $\sum_a c_{a,n} = 1$; $l_{an} = (W_{an}, \rho_{an})$ are standard pairs such that $\cup_a W_{an} = \mathcal{F}_0^n W$ and ρ_{an} is the push-forward of ρ by $\mathcal{F}_0^n$ up to a multiplicative factor. Finally, there are universal constants $\varkappa, C_1$ (depending only on $\mathcal{D}$), such that if $n > \varkappa |\log length(W)|$, then

$$\sum_{length(l_{an}) < \varepsilon} c_{a,n} < C_1 \varepsilon.$$

Another way of stating basically the same lemma is that there are universal constants $\theta < 1, C_2, C_3$ (depending only on $\mathcal{D}$) such that for a standard family $\mathcal{G} = \{(W_a, \nu_a)\}, a \in \mathfrak{A}$, and $\mathcal{G}_n = \mathcal{F}_0^n(\mathcal{G})$, one has

$$\mathcal{Z}_{\mathcal{G}_n} < C_2 \theta^n \mathcal{Z}_\mathcal{G} + C_3.$$

If we fix some large constant C_p and call a standard family proper if its $\mathcal{Z}$ function is smaller than C_p, then briefly one can say that the image of $\mathcal{G}$ becomes proper in $\log \mathcal{Z}_\mathcal{G}$ steps.

The essence of the standard pair technique is that the measures carried on two proper standard families can be coupled together exponentially fast. Then one of the two standard families is chosen to be μ_0 itself (it can be proven that there exists $\mathcal{G}$ such that $\mu_{\mathcal{G}} = \mu_0$). As a result, one obtains the following Equidistribution statement.

Lemma 7.2 ([Ch06b] Theorem 4)**.** *Let $\mathcal{G}$ be a proper standard family. For any dynamically Hölder continuous f there exists some $\theta_f < 1$ such that for any $n \geq 0$,*

$$|\int_{\mathcal{M}_0} f \circ \mathcal{F}_0^n d\mu_{\mathcal{G}} - \int_{\mathcal{M}_0} f d\mu_0| \leq B_f \theta_f^n.$$

We want to identify the possible limit points of the rescaled trajectory of the particle in case of infinite horizon. We assume that there are at least two non-parallel infinite corridors. The observable, we are interested in, is Δ_0. One difficulty is that Δ_0 is not dynamically Hölder continuous if the horizon is infinite. To overcome this problem, we introduce a cut-off of the free flight vector, thus we obtain a dynamically Hölder continuous version. That is, we define $\hat{\Delta}_j$ to be equal to Δ_j once the distance of the scatterers hit by the particle at time j and $j+1$ is less than R, otherwise let $\hat{\Delta}_j$ be zero (we will set $R = \sqrt{N} \log^\beta N$). Note that this is slightly different than $\Delta_j 1_{\{\Delta_j < R\}}$. With this notation, write

$$\hat{q}_j = q_0 + \sum_{i=0}^{j-1} \hat{\Delta}_i.$$

The following lemma is proven in its form in [ChD09a] (note also that the first half of part (a) is a purely geometric statement, (b) is a consequence of the Equidistribution, (c) was essentially proven in [SzV07]).

Proposition 7.3. *Let l be a standard pair.*

(a) $\mathcal{M}_0$ is divided by the singularity curves of $\mathcal{F}_0$ to cells D_m, such that $\Delta_0 \sim Cm$ on D_m. Then for any $n > 0$,

$$\mu_0 \left[D_{m_1} \cap \mathcal{F}^{-n}(D_{m_2}) \right] < min\{C m_2^{-3}, C m_1^{-9/4} m_2^{-2}\}.$$

(b) For any $i \geq 0$,

$$\mathbb{E}_l \left(\hat{\Delta}_i \right) = \mu_0 \left(\hat{\Delta}_i \right) + O\left(\theta^i R\right) = O\left(\theta^i R\right),$$

and analogously, for $k > 1$ and $i_1 \leq i_2 \leq \cdots \leq i_k$,

$$\mathbb{E}_l \left(\hat{\Delta}_{i_1} \otimes \cdots \otimes \hat{\Delta}_{i_k} \right) = \mu_0 \left(\hat{\Delta}_{i_1} \otimes \cdots \otimes \hat{\Delta}_{i_k} \right) + O\left(\theta^{i_1} R^k\right),$$

and

$$\mathbb{E}_l \left(\|\hat{\Delta}_{i_1}\| \ldots \|\hat{\Delta}_{i_k}\| \right) = \mu_0 \left(\|\hat{\Delta}_{i_1}\| \ldots \|\hat{\Delta}_{i_k}\| \right) + O\left(\theta^{i_1} R^k\right).$$

(c)

$$\mu_0 \left(\hat{\Delta}_j \otimes \hat{\Delta}_j \right) = 2\sigma^2 \log R + O(1),$$

where σ^2 is a non degenerate 2×2-matrix, explicitly given in [SzV07].

(d)

$$\mu_0\left(\|\hat{\Delta}_j \otimes \hat{\Delta}_k\|\right) < C\theta^{|j-k|},$$

whenever $j \neq k$.

Now, we proceed to our main statement. Let $\mathbf{W}_N(x)$ be element of $C([0,1],\mathbb{R}^2)$ with $\mathbf{W}_N(x)(j/N) = q_j(x)/\sqrt{N\log N}$ and linearly interpolated between j/N and $(j+1)/N$. When x is chosen according to some standard pair l, $\mathbf{W}_N(x)$ generates a measure $\mathbf{W}_{N,l}$ on $C[0,1]$. In this Chapter, we are going to prove the following statement.

Theorem 7.4. *Suppose that for some fixed l there is a sequence of integers N_k such that $\mathbf{W}_{N_k,l}$ is weakly convergent. Then its limit is the Wiener measure with covariance matrix σ^2.*

Remark 7.5. *The statement of Theorem 7.4 is also true for l being replaced by some proper standard family $\mathcal{G}$. Consequently, for the invariant measure, too.*

7.3 Proof by martingale method

Here, we are going to prove Theorem 7.4 by the martingale method of Stroock and Varadhan. That is, we prove the following statement: For any $\mathbf{W}$ limit point of the super-diffusively scaled billiard trajectory,

$$\phi(\mathbf{W}(t)) - \phi(\mathbf{W}(0)) - \frac{1}{2}\int_0^t \sum_{a,b\in\{1,2\}} D^2_{ab}\phi(\mathbf{W}(s))\sigma^2_{ab}ds$$

is a martingale, where σ^2_{ab} is an element of the super-diffusive covariance matrix, D^2_{ab} stands for partial derivatives of second order, and $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function with compact support. In order to see this, it suffices to prove that for any smooth functions $\psi_1, ...\psi_m$ and for any $0 < s_1 < s_2 < \cdots < s_m < t_1 < t_2$,

$$\mathbb{E}_l\left(\left[\phi(\mathbf{W}(t_2)) - \phi(\mathbf{W}(t_1)) - \frac{1}{2}\int_{t_1}^{t_2} \sum_{a,b\in\{1,2\}} D^2_{ab}\phi(\mathbf{W}(s))\sigma^2_{ab}ds\right]\prod_{j=1}^{k}\psi_j(\mathbf{W}(s_j))\right) = 0. \tag{7.1}$$

We will prove the following simplified version of (7.1) (it will be clear, how its proof provides also the more general statement (7.1)):

$$\mathbb{E}_l\left(\phi(\hat{\mathbf{W}}(t)) - \phi(\hat{\mathbf{W}}(0)) - \frac{1}{2}\int_0^1 \sum_{a,b\in\{1,2\}} D^2_{ab}\phi(\hat{\mathbf{W}}(s))\sigma^2_{ab}ds\right) = 0, \tag{7.2}$$

(where $\hat{\mathbf{W}}$ is a limiting point of the super diffusively scaled variant of the process $\hat{q}_j$ and l is a standard pair) and

$$\max_{m\leq N} \frac{q_m - \sum_{j=0}^{m-1}\hat{\Delta}_j}{\sqrt{N\log N}} \Rightarrow 0, \tag{7.3}$$

where the weak convergence is with respect to the measure generated by l.

For this fixed $l = (W,\rho)$ and any $x \in W$, $n \geq 0$, define $r_n(x)$ the following way. The image of W under

$\mathcal{F}_0^n$ is cut into several homogeneous unstable curves $W_1, W_2, \ldots$. There is an i such that $\mathcal{F}_0^n x \in W_i$. Now, W_i is cut by $\mathcal{F}_0^n x$ into two pieces, the length of the shorter one is denoted by $r_n(x)$. Observe that the growth lemma implies the existence of some C depending on l such that for every $\varepsilon > 0$ and every $n \in \mathbb{Z}_+$,

$$\mathbb{P}_l(r_n(x) < \varepsilon) < C\varepsilon.$$

Important remark: from now on, every appearance of C might mean a different constant. Nevertheless, each C depends only on the length of l (and of course on $\mathcal{D}$). Similarly, O has some involved constant which only depend on $\mathcal{D}$ and the length of l.
Since the free flight of length m is attained by point on $\mathcal{M}$ which belong to some homogeneity strip of width m^{-2}, one obtains

$$\mathbb{P}_l(\hat{\Delta}_n(x) > \sqrt{N} \log^\beta N) < \mathbb{P}_l(r_n(x) < N^{-1} \log^{-2\beta} N) < CN^{-1} \log^{-2\beta} N.$$

Thus

$$\mathbb{P}_l(\exists 1 \leq n \leq N : \hat{\Delta}_n \neq \Delta_n) = O(\log^{-2\beta} N) \tag{7.4}$$

which implies (7.3).

The rest of this Chapter is devoted to the proof of (7.2). Let $\alpha > 0$ be small, $m_p = pN^\alpha$, and fix some $p \geq 2$ integer. For a smooth ϕ with compact support:

$$\phi\left(\frac{\hat{q}_{m_{p+1}}}{\sqrt{N \log N}}\right) - \phi\left(\frac{\hat{q}_{m_p}}{\sqrt{N \log N}}\right) = \sum_{j=m_p}^{m_{p+1}-1} \phi\left(\frac{\hat{q}_{j+1}}{\sqrt{N \log N}}\right) - \phi\left(\frac{\hat{q}_j}{\sqrt{N \log N}}\right) =$$

$$\begin{aligned}
= \quad & \sum_{j=m_p}^{m_{p+1}-1} \frac{1}{\sqrt{N \log N}} \left\langle D\phi\left(\frac{\hat{q}_j}{\sqrt{N \log N}}\right), \hat{\Delta}_j \right\rangle \\
+ \quad & \frac{1}{2} \sum_{j=m_p}^{m_{p+1}-1} \frac{1}{N \log N} \left\langle D^2\phi\left(\frac{\hat{q}_j}{\sqrt{N \log N}}\right) \hat{\Delta}_j, \hat{\Delta}_j \right\rangle + O\left(\frac{\sum_{j=m_p}^{m_{p+1}-1} \|\hat{\Delta}_j\|^3}{(N \log N)^{3/2}}\right) \\
=: \quad & S_1^1 + S_2^1 + S_3^1
\end{aligned}$$

Now, using Proposition 7.3 b, we have

$$\mathbb{E}_l\left(\|\hat{\Delta}_j\|^3\right) < C \sum_{k=1}^{\sqrt{N} \log^\beta N} \frac{1}{k^3} k^3 + O\left(\theta^j N^{3/2} \log^{3/2} N\right) = O\left(N^{1/2} \log^\beta N\right), \tag{7.5}$$

(here, we also used that due to $p \geq 2$, we have $\theta^j << N$ for any $m_p < j$). Thus we conclude

$$\mathbb{E}_l(S_3^1) = O\left(N^\alpha N^{1/2} \log^\beta N N^{-3/2} \log^{-3/2} N\right) = O\left(N^{\alpha-1} \log^{\beta-3/2} N\right).$$

Further, for $m_p \leq j < m_{p+1}$,

$$\begin{aligned}
& D\phi\left(\frac{\hat{q}_j}{\sqrt{N \log N}}\right) = D\phi\left(\frac{\hat{q}_{m_{p-1}}}{\sqrt{N \log N}}\right) + \frac{1}{\sqrt{N \log N}} \sum_{k=m_{p-1}}^{j-1} D^2\phi\left(\frac{\hat{q}_{m_{p-1}}}{\sqrt{N \log N}}\right) \hat{\Delta}_k \\
& +O\left(\frac{1}{N \log N} D^3\phi\left(\frac{\hat{q}_{m_{p-1}}}{\sqrt{N \log N}}\right) (\hat{q}_j - \hat{q}_{m_{p-1}})^{\otimes 2}\right) =: S_1^2 + S_2^2 + S_3^2,
\end{aligned}$$

where S_3^2 is the error term in the Taylor expansion. Now, we want to substitute $S_1^2+S_2^2+S_3^2$ to S_1^1. The substitution of $S_1^2+S_2^2$ will be computed, while the one of S_3^2 is an error term. To estimate the latter, observe that both coordinates of S_3^2 are bounded by

$$C\frac{1}{N\log N}\left(\sum_{k=m_{p-1}}^{j-1}\|\hat{\Delta}_k\|\right)^2.$$

Thus, when substituting S_3^2 to S_1^1, one obtains a term whose modulus has l-expectation not larger than some constant times

$$\frac{1}{N^{3/2}\log^{3/2}N}\mathbb{E}_l\Big[\sum_{j=m_p}^{m_{p+1}-1}\sum_{m_{p-1}\le k_1\le k_2\le j-1}\|\hat{\Delta}_{k_1}\|\|\hat{\Delta}_{k_2}\|\|\hat{\Delta}_j\|\Big]. \tag{7.6}$$

To estimate (7.6), we introduce a second cut-off: let $\hat{\hat{\Delta}}_j$ to be equal to Δ_j once the distance of the scatterers hit by the particle at time j and $j+1$ is less than A_N, otherwise let $\hat{\hat{\Delta}}_j$ be zero (we will set $A_N=N^{9\alpha}$).

Now, we compute the contribution of the Δ's for fixed $k_1\le k_2<j$ with (the indices n_1,n_2,n_3 stand for the realization of $\hat{\Delta}_{k_1}$, $\hat{\Delta}_{k_2}$, $\hat{\Delta}_j$, respectively):

$$\begin{aligned}
&\mathbb{E}_l\left[\|\hat{\Delta}_{k_1}-\hat{\hat{\Delta}}_{k_1}\|\|\hat{\Delta}_{k_2}\|\|\hat{\Delta}_j\|\right]<\mu_0\left[\|\hat{\Delta}_{k_1}-\hat{\hat{\Delta}}_{k_1}\|\|\hat{\Delta}_{k_2}\|\|\hat{\Delta}_j\|\right]+O(\theta^{k_1}N^{3/2}\log^{3\beta}N)\\
<\ & C\sqrt{N}\log^\beta N\sum_{n_1=A_N}^{\sqrt{N}\log^\beta N}\sum_{n_3=1}^{\sqrt{N}\log^\beta N}n_1n_3n_1^{-9/4}n_3^{-2}\\
<\ & \sqrt{N}\log^\beta NA_N^{-1/4}\log N.
\end{aligned}\tag{7.7}$$

In the first inequality, we used the Equidistribution (note that $\hat{\Delta}_0-\hat{\hat{\Delta}}_0$ is also dynamically Hölder continuous), while in the second one, we used Proposition 7.3 (a) and the fact that $p\ge 2$. One can similarly compute that for $k_1\le k_2<j$,

$$\begin{aligned}
&\mathbb{E}_l\left[\|\hat{\hat{\Delta}}_{k_1}\|\|\hat{\Delta}_{k_2}\|\|\hat{\Delta}_j\|\right]<A_N\mathbb{E}_l\left[\|\hat{\Delta}_{k_2}\|\|\hat{\Delta}_j\|\right]+O(A_N\theta^{k_2}N\log^{2\beta}N)\\
<\ & CA_N\sum_{n_2=1}^{\sqrt{N}\log^\beta N}\sum_{n_3=1}^{\sqrt{N}\log^\beta N}n_2n_3n_2^{-9/4}n_3^{-2}<CA_N\log N.
\end{aligned}$$

Combining the above estimations we conclude that (7.6) is bounded by

$$\begin{aligned}
&CN^{-3/2}\log^{-3/2}N\Big(N^{3\alpha}\cdot\sqrt{N}\log^{\beta+1}NA_N^{-1/4}+N^{3\alpha}\cdot A_N\log N\Big)\\
=\ & o(N^{\alpha-1}),
\end{aligned}$$

if we choose $A_N=N^{9\alpha}$, and α is small. At the last step, we want to substitute

$$D^2\phi\left(\frac{\hat{q}_j}{\sqrt{N\log N}}\right) \tag{7.8}$$

in S_2^1 with

$$D^2\phi\left(\frac{\hat{q}_{m_{p-1}}}{\sqrt{N\log N}}\right). \tag{7.9}$$

It is easy to see that the difference between S_2^1 and the formula obtained from S_2^1 with (7.8) being replaced by (7.9) is in

$$O\left(\frac{1}{(N\log N)^{3/2}}\sum_{m_p\le j\le m_{p+1}-1}\ \sum_{m_{p-1}\le k<j}\|\hat{\Delta}_k\|\|\hat{\Delta}_j\|^2\right) \tag{7.10}$$

As before, we have

$$\mathbb{E}_l\left[\|\hat{\hat{\Delta}}_k\|\|\hat{\Delta}_j\|^2\right]\le CA_N\sum_{n_2=1}^{\sqrt{N}\log^\beta N} n_2^{-3}n_2^2+O\left(A_N\theta^jN\log^{2\beta}N\right)<CA_N\log N. \tag{7.11}$$

This, and (7.7) imply that the expectation of (7.10) with respect to l is bounded by

$$CN^{-3/2}\log^{-3/2}N\cdot N^{2\alpha}\left(\sqrt{N}\log^{\beta+1}NA_N^{-1/4}+A_N\log N\right)=o(N^{\alpha-1}).$$

Hence, for $p\ge 2$ we obtain

$$\begin{aligned}
&\phi\left(\frac{\hat{q}_{m_{p+1}}}{\sqrt{N\log N}}\right)-\phi\left(\frac{\hat{q}_{m_p}}{\sqrt{N\log N}}\right)\\
=&\sum_{j=m_p}^{m_{p+1}-1}\frac{1}{\sqrt{N\log N}}\left\langle D\phi\left(\frac{\hat{q}_{m_{p-1}}}{\sqrt{N\log N}}\right),\hat{\Delta}_j\right\rangle\\
+&\frac{1}{N\log N}\left[\frac{1}{2}\sum_{j=m_p}^{m_{p+1}-1}\left\langle D^2\phi\left(\frac{\hat{q}_{m_{p-1}}}{\sqrt{N\log N}}\right)\hat{\Delta}_j,\hat{\Delta}_j\right\rangle\right.\\
+&\left.\sum_{m_p\le j<m_{p+1},m_{p-1}\le k<j}\left\langle D^2\phi\left(\frac{\hat{q}_{m_{p-1}}}{\sqrt{N\log N}}\right)\hat{\Delta}_k,\hat{\Delta}_j\right\rangle\right]+h_p,
\end{aligned}\tag{7.12}$$

where $\mathbb{E}_l(h_p)=o\left(N^{\alpha-1}\right)$.
With the notation

$$f_p(x)=\phi\left(\frac{\hat{q}_{m_{p+1}}(x)}{\sqrt{N\log N}}\right)-\phi\left(\frac{\hat{q}_{m_p}(x)}{\sqrt{N\log N}}\right),$$

for any $x\in\mathcal{M}$,

$$\phi\left(\frac{\hat{q}_N(x)}{\sqrt{N\log N}}\right)-\phi\left(\frac{\hat{q}_0(x)}{\sqrt{N\log N}}\right)=\sum_{p=0}^{N^{1-\alpha}}f_p(x)$$

Thus, in order to verify (7.2), we need to prove

$$\mathbb{E}_l\sum_{p=0}^{N^{1-\alpha}}\left[f_p(x)-N^{\alpha-1}\frac{1}{2}\sum_{a,b\in\{1,2\}}D^2_{ab}\phi\left(\frac{\hat{q}_{m_p}(x)}{\sqrt{N\log N}}\right)\sigma^2_{ab}\right]=o(1). \tag{7.13}$$

First, we verify that $\mathbb{E}_l(f_0+f_1)=o(1)$. Note that Proposition 7.3 (b) implies

$$\mathbb{E}_l\|\hat{q}_{m_2}-\hat{q}_0\|<\sum_{k=0}^{m_2}\mu_0(\|\hat{\Delta}_k\|)+O(\theta^k\sqrt{N}\log^\beta N)=O(\sqrt{N}\log^\beta N).$$

Since $\beta<1/2$, the Markov inequality and the fact that ϕ has compact support and is in C^1, implies $\mathbb{E}_l(f_0+f_1)=o(1)$.

For $p \geq 2$, as in [DSzV09], we want to use the Markov decomposition at time $\tau_p = (m_{p-1} + m_p)/2$. Now, we will also need some Markov decomposition at time $\tilde{\tau}_p = (3m_{p-1} + m_p)/4$. However, observe that since $\hat{q}_{m_p}$ is not necessarily equal to q_{m_p}, $f_p(x)$ and

$$f'_p(x) = \phi\left(\frac{\hat{q}_{\frac{3}{2}N^\alpha}(\mathcal{F}^{m_{p-1}+N^\alpha/2}x)}{\sqrt{N\log N}}\right) - \phi\left(\frac{\hat{q}_{\frac{1}{2}N^\alpha}(\mathcal{F}^{m_{p-1}+N^\alpha/2}x)}{\sqrt{N\log N}}\right),$$

which is easier to deal with using the Markov decomposition, are not equal in general. That is why we need some more computation. Observe that the decomposition (7.12) can also be written in terms of f'_p instead of f_p. The difference is that $\hat{q}_{m_{p-1}}$ should be replaced by

$$\hat{q}'_{m_{p-1}} = q_{(m_{p-1}+m_p)/2} - \sum_{j=1}^{N^\alpha/2} \hat{\Delta}_j,$$

and h_p should be replaced by some h'_p. Observe that our previous computation also yields $\mathbb{E}_l(h'_p) = o(N^{\alpha-1})$.

Now, we claim that

$$\lim_{N\to\infty} \sum_{p=2}^{N^{1-\alpha}} \mathbb{E}_l\left(f_p - f'_p\right) = 0. \tag{7.14}$$

To prove (7.14), observe that with the notation

$$\mathcal{L} = \{x \in \mathcal{M} : \exists j \leq N : \Delta_j(x) \neq \hat{\Delta}_j(x)\}$$

for the set of points having long flight, f_p coincides with f'_p on $\mathcal{M} \setminus \mathcal{L}$, which has l-measure at least $1 - C\log^{-2\beta} N$ by (7.4). Since $\sum_p f_p$ is bounded,

$$\int_{\mathcal{L}} \sum_p f_p dl = o(1).$$

Thus in order to prove (7.14), is is enough to establish

$$\int_{\mathcal{L}} \sum_p f'_p dl = o(1). \tag{7.15}$$

This statement is not obvious, since $\sum_p f'_p$ is not bounded. However, with the notation

$$L(x) = \#\{p < N^{1-\alpha} : \exists j \in [m_p, m_{p+1}], \Delta_j(x) \neq \hat{\Delta}_j(x)\},$$

we have for any $x \in \mathcal{L}$,

$$\sum_p f'_p(x) < 2(L(x)+1)\|\phi\|.$$

Thus, it is enough to prove

$$\int_{\mathcal{L}} L(x) dl = o(1). \tag{7.16}$$

We will prove that

$$\int_{\mathcal{L}} L_{even}(x)dl = o(1), \tag{7.17}$$

where

$$L_{even}(x) = \#\{p < N^{1-\alpha}, p \text{ is even and } \exists j \in [m_p, m_{p+1}]\Delta_j(x) \neq \hat{\Delta}_j(x)\}.$$

This, together with a very same computation for the odd p's implies (7.16). To prove (7.17), first observe that

$$\mathbb{P}_l(L_{even}(x) = 1) < C \log^{-2\beta} N.$$

We claim that analogously,

$$\mathbb{P}_l(L_{even}(x) = k) < C^k \log^{-2\beta k} N \tag{7.18}$$

for every k positive integer, with a uniform C. This implies (7.17), since $\sum_{k>1} kC^k \log^{-2\beta k} N = o(1)$. Now pick any $1 \leq i < i + N^\alpha \leq j \leq N$. We prove that

$$\mathbb{P}_l(\Delta_i \neq \hat{\Delta}_i, \Delta_j \neq \hat{\Delta}_j) < C^2 N^{-2} \log^{-4\beta} N, \tag{7.19}$$

which obviously implies (7.18) for $k = 2$ (for larger k's, the proof goes the same way). We have

$$\mathbb{P}_l(\Delta_i \neq \hat{\Delta}_i, \Delta_j \neq \hat{\Delta}_j) = \sum_a c_a \mathbb{P}_{l_a}(\Delta_{j-i} \neq \hat{\Delta}_{j-i}), \tag{7.20}$$

where $\{l_a\}_a$ is the collection of standard pairs in the image of l under $\mathcal{F}^{i+1}$, for which for any point x in γ_a, $\Delta_0(\mathcal{F}^{-1}x) \neq \hat{\Delta}_0(\mathcal{F}^{-1}x)$. We already know that $\sum_a c_a < CN^{-1} \log^{-2\beta} N$. Let $S_1 + S_2$ be the sum in (7.20), where S_1 corresponds to a's, for which $length(l_a) < N^{-3}$. Then the growth lemma implies $S_1 < CN^{-3}$. For a's, where $length(l_a) > N^{-3}$, the image of l_a becomes proper in $C \log N$ steps, thus (the proof of (7.4)) implies $S_2 < \sum_a c_a CN^{-1} \log^{-2\beta} N$ with a uniform C. Thus we have verified (7.19), and finished the proof of (7.14).

As it was already mentioned, we will also need a Markov decomposition at time $(3m_{p-1} + m_p)/4$. Thus, we still need to slightly adjust f_p, that is to define

$$\hat{q}''_{m_{p-1}} = q_{(3m_{p-1}+m_p)/4} - \sum_{j=1}^{N^\alpha/4} \hat{\Delta}_j,$$

and

$$\begin{aligned} f''_p(x) \quad := \quad & \sum_{j=m_p}^{m_{p+1}-1} \frac{1}{\sqrt{N \log N}} \left\langle D\phi\left(\frac{\hat{q}'_{m_{p-1}}}{\sqrt{N \log N}}\right), \hat{\Delta}_j \right\rangle \\ + \quad & \frac{1}{N \log N} \Big[\frac{1}{2} \sum_{j=m_p}^{m_{p+1}-1} \left\langle D^2\phi\left(\frac{\hat{q}'_{m_{p-1}}}{\sqrt{N \log N}}\right) \hat{\Delta}_j, \hat{\Delta}_j \right\rangle \\ + \quad & \sum_{m_p \leq j < m_{p+1}, m_{p-1} \leq k \leq m_{p-1} + \frac{3}{8}N^\alpha} \left\langle D^2\phi\left(\frac{\hat{q}'_{m_{p-1}}}{\sqrt{N \log N}}\right) \hat{\Delta}_k, \hat{\Delta}_j \right\rangle \\ + \quad & \sum_{m_p \leq j < m_{p+1}, m_{p-1} + \frac{3}{8}N^\alpha < k < j} \left\langle D^2\phi\left(\frac{\hat{q}''_{m_{p-1}}}{\sqrt{N \log N}}\right) \hat{\Delta}_k, \hat{\Delta}_j \right\rangle \Big] + h'_p. \end{aligned} \tag{7.21}$$

Next, we prove that the contribution of this adjustment asymptotically vanishes, i.e.

$$\mathbb{E}_l(|f'_p - f''_p|) = o\left(N^{\alpha-1}\right). \tag{7.22}$$

In order to prove (7.22), write

$$\mathcal{L}_p = \{x \in \mathcal{M} : \exists j \in [m_{p-1} + \frac{1}{4}N^\alpha, m_{p-1} + \frac{1}{2}N^\alpha] : \Delta_j(x) \neq \hat{\Delta}_j(x)\}$$

for the set of phase points, where $f'_p \neq f''_p$. Further, observe that for any $x \in \mathcal{L}_p$, the only difference between $f'_p(x)$ and $f''_p(x)$ is that in the fourth line of (7.21) $\hat{q}''_{m_{p-1}}$ is replaced by $\hat{q}'_{m_{p-1}}$ in the case of $f'_p(x)$. Thus, for any $x \in \mathcal{L}_p$,

$$|f'_p - f''_p| < C\frac{1}{N \log N} N^\alpha \sqrt{N} \log^\beta N \sum_{m_p \leq j < m_{p+1}} \|\hat{\Delta}_j\|.$$

Consequently, the Markov decomposition at time $m_{p-1} + N^\alpha/2$ implies

$$\int_{\mathcal{L}_p} |f'_p - f''_p| dl < CN^{-1+\alpha+1/2} \sum_a c_a \sum_{m_p \leq j < m_{p+1}} \mathbb{E}_{l_a}(\|\hat{\Delta}_j\|), \tag{7.23}$$

where $\{l_a\}_a$ is the collection of standard pairs in the image of l under $\mathcal{F}^{m_{p-1}+N^\alpha/2}$, for which for any point x in γ_a, there is a $j \in [0, N^\alpha/4]$, such that $\hat{\Delta}_0(\mathcal{F}^{-j}x) \neq \Delta_0(\mathcal{F}^{-j}x)$. Note that $\sum_a c_a = l(\mathcal{L}_p)$. If we denote by $S_1 + S_2$ the sum in (7.23), where S_1 corresponds to a's with $length(l_a) < N^{-2}$, then using the obvious estimation $\mathbb{E}_{l_a}(\|\hat{\Delta}_j\|) < \sqrt{N} \log^\beta N$ and the growth lemma, we obtain $S_1 < CN^{2\alpha-2} \log^\beta N$. Since in $C \log N$ steps the standard pairs of the sum S_2 develop to proper families, Proposition 7.3 (b) and (7.4) imply $S_2 < N^{-1+\alpha+1/2} l(\mathcal{L}_p) N^\alpha < N^{-1+\alpha+1/2+\alpha-1+\alpha}$. (7.22) follows.

Combining (7.14) and (7.22), (7.13) is equivalent to the statement

$$\mathbb{E}_l\left(f''_p\right) = N^{\alpha-1}\frac{1}{2}\sum_{a,b\in\{1,2\}} \mathbb{E}_l\left(D^2_{ab}\phi\left(\frac{\hat{q}_{m_p}(x)}{\sqrt{N \log N}}\right)\right)\sigma^2_{ab}(1+o(1)). \tag{7.24}$$

Since $\mathbb{E}_l(h'_p)$ is in $o(N^{\alpha-1})$, in order to prove (7.24), it suffices to verify that $T_1 + T_2 + T_3 + T_4$ is equal to the right hand side of (7.24), where T_i is the l-expected value of line i in formula (7.21) - except for T_4, where we omit h'_p. The proof of this is similar to the one in [DSzV09].

So as to estimate T_1, T_2 and T_3, we use Markov decomposition at time $(m_{p-1} + m_p)/2$. Since in any case, the first three lines of (7.21) are bounded by $CN^{2\alpha}$, the standard pairs that are shorter than N^{-2} contribute to $T_1 + T_2 + T_3$ with a term which is bounded by $CN^{-2+2\alpha}$. If we denote by T'_1, T'_2 and T'_3 the contribution of the standard pairs that are longer than N^{-2}, then we have

$$T'_1 = O\left(\frac{1}{\sqrt{N \log N}} N^\alpha \theta^{\frac{1}{2}N^\alpha} \sqrt{N} \log^\beta N\right). \tag{7.25}$$

Indeed, the value of $D\phi\left(\frac{\hat{q}'_{m_{p-1}}}{\sqrt{N \log N}}\right)$ on some points which will form a standard pair at time $(m_{p-1}+m_p)/2$ is some constant with error $O(\theta^{\frac{1}{2}N^\alpha})$, thus Proposition 7.3 (b) implies (7.25).

Now, using Proposition 7.3 (c) one can analogously compute that

$$T_2' = N^{\alpha-1}\frac{1}{2}\sum_{a,b\in\{1,2\}} \mathbb{E}_l\left(D^2_{ab}\phi\left(\frac{\hat{q}_{m_p}(x)}{\sqrt{N\log N}}\right)\right)\sigma^2_{ab}(1+o(1)). \tag{7.26}$$

Similarly to the estimation of T_1', we can bound T_3'. Note that $D^2\phi\left(\frac{\hat{q}'_{m_{p-1}}}{\sqrt{N\log N}}\right)\hat{\Delta}_k$ on some points which will form a standard pair at time $(m_{p-1}+m_p)/2$ is some constant with error $O(\theta^{\frac{1}{2}N^\alpha})$, but now, this constant is only bounded by $\sqrt{N}\log^\beta N$. Thus again, Proposition 7.3 (b) yields

$$T_3' = O\left(\frac{1}{N\log N}N^{2\alpha}\sqrt{N}\log^\beta N\theta^{\frac{1}{8}N^\alpha}\sqrt{N}\log^\beta N\right). \tag{7.27}$$

Finally, we use Markov decomposition at time $(3m_p+m_{p+1})/4$ to estimate T_4. As before, since the last line of (7.21) is bounded by $CN^{2\alpha}$, the standard pairs that are shorter than N^{-2} can be neglected. Using the same argument as in the proof of (7.25), and now also Proposition 7.3 (d), we conclude that the contribution of the longer standard pairs is in

$$O\left(\frac{1}{N\log N}\sum_{m_p\le j<m_{p+1},\, m_{p-1}+\frac{3}{8}N^\alpha<k<j}\left(\theta^{j-k}+\theta^{\frac{1}{8}N^\alpha}N\log^{2\beta}N\right)\right).$$

This, together with our previous estimations, yields (7.24). We have finished the proof of (7.2).

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