The Mathematical Sophistication of Rashi and the Early Tosafists

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ואתה רואה מהכתובים האלה שהקב״ה ברא את עולמו במדידה ושיעור מתוכן ושקול. וחייב אדם להמשל לקונו בכל כחו ובכל מאודו וזהו שבחו של אדם לדעת כל החכמים.

———ר׳ אברהם בר חייא (בהקדמת המחבר לחיבור המשיחה והתשבורת)

Abstract

The mathematical sophistication of Rashi and the early Tosafists is investigated by quoting and analyzing several examples of Rashi and the Tosafists commentaries on talmudic excerpts of mathematical interest. In these examples, one may encounter fundamental mistakes, as well as some correct advanced mathematical ideas combined with clever cases as presented by the Tosafists. It seems reasonable to tentatively conclude that Rashi and the early Tosafists had no advanced education in geometry, although they were perfectly proficient in arithmetic. The more sophisticated geometrical material found is either sourced externally or is a result of the method of study instigated by Ri. This style of learning, distinct in its attention to detail and careful case construction, allows for realizations of geometric relationships. From a historical perspective, it can be argued these advanced ideas can be traced to Rabbi Abraham bar Hiyya, a Spanish mathematician c. 1100. It is notable that Christian Europe suffered from a lack of advanced mathematical study during the time of Rashi and the earlier Tosafists, and so the advanced ideas that entered into the Tosafists commentaries most likely derive from Arabic mathematics through Spanish influence.

I. Introduction

There has been much written about the scientific and mathematical world of the Jews throughout the ages. Studies into the mathematics of Spanish Jews such as Maimonides, Ibn Ezra, and Abraham bar Hiyya include those of Gandz [8], Freudenthal [7], Feldman [5], Levy [14]. Furthermore, studies of Babylonian mathematics relating to the Talmud discuss the relationship between the knowledge of Babylonian mathematics and the material appearing in the Talmud (such as some of Gandz's work [8]). Some works focus on general mathematical ideas within rabbinic works without a focus on any one given historical context (Garber and Tzaban [10]). However, to the best of the author's knowledge, there has not been a study focused on examining the mathematics of Rashi and the Tosafists, as seen in their commentaries to certain Talmudic *sugyot*. The goal of this work is to begin a conversation about the mathematical sophistication of Rashi and the early Tosafists by examining key examples in their writings, as related to arithmetic, algebraic and geometric approaches. Given that their exhibited knowledge and understanding of algebraic processes is fundamentally sound, we examine in more detail the geometric understanding. We focus on comments that show their sophisticated understandings of key geometrical relations as well as those that betray misapprehensions. We briefly look to their historical and geographical contexts and suggest a possible explanation which accounts for discrepancies in levels of mathematical sophistication across the writings we examine.

II. Rashi

When we examine Rashi's proficiency in math, we must differentiate between his knowledge of arithmetic and that of geometry. We will examine three cases of lengthy mathematical calculations Rashi does as well as smaller repeated entries in different forms across his commentary on the Talmud. We will see that there do not seem to be any major flaws in his arithmetic capabilities, even as there appear to be a number of geometric deficiencies in his commentary.

II.A Arithmetic

<u>Example II.A.1</u> ***Tzipori/Jerusalem***

The most common calculation Rashi explains throughout the Talmud is the change of measurements that took place in Jerusalem and then again in Tzipori (Sephoris). Rashi explains that there was an original system of measuring volume which the Jews used in the desert in the time of Moses (termed *midbariyot*), that was later changed in Jerusalem (see Rashi b. Kiddushin 46b s.v. hameshet revaim kemah). The new Jerusalem system of measurement added volume to each unit of measure. Six units from the system used in the desert was equivalent to five units in the Jerusalem system. That is, 1 *midbariyot* = $\frac{5}{6}$ *yerushalmiot*. The same conversion holds true between *yerushalmiyot* and *tziporiyot* after the system changed a second time.

Throughout the Talmud, when deemed necessary, Rashi explains this sequence of conversions. He describes the change as a sixth added onto the measurement. That is, the 6 units of *midbariyot* are equivalent to 5 units of *yerushalmiyot*. This appears in several sugyot, including:

רש"י מסכת חולין דף קלה עמוד ב כדי עריסותיכם

מסכת קידושין דף מו עמוד ב חמשת רבעים קמח

מסכת פסחים דף מח עמוד ב חמשת רבעים

מסכת יומא דף מד עמוד ב סאה מדברית

מסכת מנחות דף עו עמוד ב התודה

רש"י מסכת חולין דף קלה עמוד ב כדי עריסותיכם

מסכת עירובין דף פג עמוד ב כדי עיסותיכם

מסכת שבועות דף טו עמוד א שהן שש מדבריות

מסכת שבת דף טו עמוד א קב ומחצה

The first two of these describe the arithmetic process in great detail, and we transcribe and translate them below.

<u>רש"י מסכת קידושין דף מו עמוד ב</u>

חמשת רבעים קמח - חמשה לוגין צפורים חייבין בחלה שהן ו' ירושלמיות שהן שבע מדבריות וביצה וחומש ביצה דהיינו עריסותיכם כדי עיסת מדבר עומר לגולגולת והעומר עשירית האיפה והאיפה שלשה סאין והסאה ששה קבין והקב ארבע לוגין נמצא איפה ע"ב לוגין עשירית של ע' לוגין שבעה לוגין פשו להו שתי לוגין שהן שנים עשר ביצים עשירית שלהן ביצה וחומש ביצה הרי עומר שבעה לוגין וביצה וחומש ביצה הוסיפו בירושלים על המדות שתות והגדילו קבין ולוגין שתות מלבר דהוה חומשא מלגאו נתנו שש הראשונות בחמש פשו להו לוג מדברי וביצה וחומש והן לוג ירושלמי כיצד לוג מדברי ששה ביצים תנהו בחמש ביצים גדולים הרי לוג ירושלמי חסר ביצה גדולה עשה מביצה וחומש ביצה מדברית שבידך ביצה ירושלמית שחומש ביצה הוא שתות מלבר להוסיף על ביצת מדברית להשלים ביצה ירושלמית הרי שש לוגין ירושלמיות והם חמש ציפוריים שהציפוריים עודפות על ירושלמיות שתות

Five quarters of flour- five tzipori log, are required for the preparation of hala. This is six yerushalmi log, totaling seven midbari log, accompanied by an egg and a fifth of an egg. The reference to 'your kneading bowls' alludes to the quantity of dough used in the desert, signifying an omer per dough. An omer, in turn, represents a tenth of an ephah. The ephah comprises three *se'as, each se'a containing six kav, and a kav comprises four log. Therefore, an ephah is equivalent to 72 log. Extracting a tenth, we are left with 7 log. The remainder, 2 log, corresponds to 12 eggs. From this, a tenth is extracted, resulting in an egg and a fifth of an egg. Thus, the total is seven log, along with an egg and a fifth (midbari). In Jerusalem, an additional sixth was incorporated into the measurements, leading to an increment in a kav and a log by an external sixth or an internal fifth. This adjustment was implemented, rendering the original 6 log now reduced to 5 log. Consequently, there is a log midbari remaining, accompanied by an egg and a fifth of an egg, collectively constituting a log yerushalmi. The calculation involves converting a log midbari (equivalent to six eggs) into five large eggs. This makes a log yerushalmi short of a large egg. To remedy this, the egg and a fifth midbari are transformed into an egg yerushalmi. This process, utilizing an external sixth, allows for the addition onto an egg midbari, completing a yerushalmi egg. The outcome is 6 log yerushalmi, which is equivalent to 5 tzipori, as tzipori log surpass yerushalmi by a sixth.*

<u>רש"י מסכת פסחים דף מח עמוד ב</u>

חמשת רבעים - לוגין ציפורים, שהן שבעה ועוד מדבריים, שזה הוא שיעור העומר שהיתה עיסת מדבר שנאמר גבי חלה (במדבר טו) ראשית עריסותיכם - כדי עריסותיכם, דהיינו עיסת מדבר שהוא עומר לגולגולת, והוא עשירית האיפה, והאיפה שלש סאין, וסאה ששה קבין מדבריות, חשוב עישור שבהן, ותמצא שבעת רבעי קב וביצה וחומש ביצה, כיצד, חלק שלש סאין שהן שמונה עשרה קבין ועשה מהן רבעי קב, היינו לוגין - ותמצא בהן שבעים ושנים לוג, טול עישור של שבעים - הרי שבעה, והשנים הנשארים - שתים עשרה ביצים הן, עישור שלהן - ביצה וחומש ביצה, הרי שבעת רבעים ועוד, דהוא ביצה וחומש ביצה, ובירושלים הוסיפו על המדות שתות מלבר, שהוא חומש מלגו, שנתנו ששה מן הראשונים בתוך חמשה האחרונים - הרי ששת רבעים הראשונים נעשו חמשה, והרובע וביצה וחומש הנשארים נעשו רובע, שהרובע שש ביצים, עשה מהן חמש ביצים גדולות, והביצה וחומש ביצה נעשית ביצה גדולה שניתוסף עליה

חומשא מלגו, דהוא שתות מלבר - הרי שש ביצים גדולות, דהוא רובע קב גדול, הרי עומר המדבר ששה רבעים ירושלמיים, וסאה ציפורית עודפת על ירושלמית שתות מלבר, כדאמר בעירובין (פג, א), הרי עומר שהיה ששה רבעים ירושלמיים נכנס בחמש ציפוריים.

Five quarters - tzipori logs, totaling seven and some midbari, which is the measure of the omer that was the desert dough, as it is said about the challah (Numbers 15): "The first of your kneading bowls" - that is, the desert dough that is an omer for the round cakes. It is a tenth of an ephah, and the ephah is three se'as, and a se'a is six kav midbari. Calculate a tenth of this. You will find seven and a quarter kav and an egg and a fifth of an egg. How so? Divide three se'as, which are eighteen kav, and make from them a quarter kav, that is, a log. You will find seventy-two logs. Taking a tenth of seventy - you have seven logs, and the remaining two - these are twelve eggs. Their tenth is an egg and a fifth. Behold, seven and a quarter and some, which is an egg and a fifth. In Jerusalem, they added to the measures an external sixth which is an internal fifth. Six of them were given from the first ones within the last five - behold, the first six quarters became five, and the remaining quarter, an egg, and a fifth, became a quarter, which is six eggs. From them, five large eggs were made, and the egg and a fifth became a second large egg, to which an internal fifth was added which is an external sixth. Behold, six large eggs, which is a large quarter kav. Behold, the desert omer, is six quarters yerushalmi, and a se'a tzipori is an external sixth greater than a yerushalmi [se'a] as stated in Eruvin (83a): behold, the omer that was six Jerusalem quarters is in five tzipori.

In the Pesahim and Kidushin instances above, Rashi gives very detailed explanations that allow us to precisely understand his mathematical conversion process. In other cases, Rashi's explanation is extremely concise (e.g. Shevu'ot). In all cases, however, Rashi is very clear and provides a consistent and mathematically coherent and correct explanation.

Example II.A.2

The Talmud in Pesahim 109a states the measurements of a container which contains exactly a *revi'it,* a measure of volume. It gives the dimensions of a container containing forty *se'a*, another measure of volume, from which the dimensions for the *revi'it* can be obtained. Following the Talmud's lead, Rashi spells out the calculation in great detail.

רש"י מסכת פסחים דף קט עמוד ב

אמה על אמה ברום שלש אמות - מחזיקין ארבעים סאה, חלקהו תחלה כולו ברומו שלש אמות, יש בהן שמונה עשר טפחים, חלקם לששה עשר חלקים, ותמצא לכל חלק וחלק טפח וחצי אצבע, ומחזיק אחד מששה עשר בארבעים סאה כמה הן שני סאין וחצי סאה, טפח וחצי אצבע, כמה חומשי אצבעות הן - עשרים ושנים וחצי חומש, הטפח עשרים חומשים, חצי אצבע שני חומשין וחצי, חלוק עשרים ושנים חומשים וחצי לחמשה חלקים - תמצא בכל חלק ארבעה חומשין וחצי, חלוק שני סאין וחצי סאה לחמשה חלקים תמצא בכל חלק חצי סאה, נמצא דארבעה חומשין וחצי חומש מחזיק חצי סאה, ועוד ארבעה חומשין וחצי חומש מחזיק חצי סאה, [ועוד ארבעה חומשין וחצי חומש מחזיק חצי סאה], נמצא כל אמה על אמה ברום שלשה עשר חומשין וחצי חומש, מחזיק סאה וחצי סאה, חשוב סאה וחצי סאה ללוגין, תמצא שלשים וששה לוגין, כיצד, סאה וחצי סאה תשעה קבין, ששה קבים לסאה, והקב ארבעה לוגין, הרי שלשים וששה לוגין בכלי אמה על אמה ברום שלשה עשר חומשים וחצי חומש, חלוק הכלי באורך ורוחב לטפחים, תמצא האמה על אמה שלשים וששה טפחים בטפח על טפח, שהאמה ששה טפחים, וכשתחלקנו לאורכו ברצועות שהן רחבות טפח תמצא בו שש רצועות, כל אחת ואחת רחבה טפח וארכה ששה, חלוק כל רצועה לארכה לשש, תמצא בשש רצועות שלשים וששה טפחים, בטפח על טפח, ומחזיק שלשים וששה לוגין, נמצא כלי שהוא טפח על טפח ברום שלשה עשר חומשים וחצי חומש, מחזיק לוג, וטפח ארבע אצבעות, חלוק ארבע אצבעות על ארבע אצבעות שתי וערב

לארבעה, נמצא כל רביעית, אצבעים על אצבעים, נמצא כל אצבעים על אצבעים ברום שלושה עשר חומשים וחצי חומש, מחזיק רביעית הלוג, חשוב שלושה עשר חומשים וחצי, ותמצא בהן אצבעים וחצי אצבע וחומש אצבע.

The Talmud in Pesahim says that the dimensions of a vessel containing a *revi'it* of liquid are two finger widths by two finger widths by twelve and a half and a fifth (12.7) finger widths.[1] It brings a proof from the dimensions of a *mikva* which must contain a volume of forty *se'a*. The dimensions are one cubit by one cubit by 3 cubits. The commentators are left to their own devices to explain how the proof text supports the Gemara's claim. Rashi explains this by taking the dimensions of the *mikva* and scaling down first in height and then by area until he reaches the dimensions stated in the gemara for the *revi'it*. At the same time he scales down the volume step-by-step along with the dimensions to show that a container two finger widths by two finger widths by twelve and a half and a fifth holds a *revi'it*.

The conversion between measurements is

1 se'a = 6 kav = 24 log = 96 revi'it

1 ama = 6 tefahim = 24 etzba'ot

Rashi's method:

A mikva is 1x1x3 cubic *amot* and holds 40 *se'a*.

[1] Tosafot notes that Talmud Yerushalmi has another measurement for height. This leads to difficulties since we show that the measurements given in the Bavli are correct. Tosafot offers two possible suggestions for the discrepancy, both relying on differences in systems of measurements (*yerushalmiyot* vs. *midbariot*, and whether we are using a cylindrical container of a rectangular one), ending in a slight lack of accuracy in the Yerushalmi. The first answer we find is highly controversial (see Maharsha and Rashash, ad loc.) and is likely mathematically incorrect (depending on how one reads it). The second answer proffered is simple and not mathematically incorrect, and it is quoted in Tosafot ha-Rosh. We do not cover the first answer in the coming mistake section given sufficient other, clearer, examples. The reader is, however, encouraged to examine this case carefully, as it provides some insight into the arithmetic proficiency of the Tosafists, as well as their creative case building and solutions.

1. We divide the height by 16, what remains is 1 tefah and 1/2 *etzba*. Correspondingly divide the 40 se'a volume by 16 which yields 2.5 se'a.
2. We then convert the height to fifths of a finger width. That is 22.5 fifths of a finger width.
3. Then, we divide by 5 and multiply by 3 (multiply by 3/5ths), yielding 13.5 fifth *etzba'ot* (this is 2 *etzba'ot* + 1/2 + 1/5th *etzba*). Correspondingly, we multiply the volume by 3/5ths 2.5 *se'a* x 3/5 = 1.5 *se'a.*
4. Convert the volume to *log* and the base of the area in the mikva to smaller units.

   1.5 *se'a* = 9 *kav* = 36 *log*

   1x1 ama = 36 sq. tefahim. So 1x1 tefah with height 13.5 fifth etzba'ot is 1 log. Rashi describes in detail that 1x1 sq. ama makes 6 strips of 6 1x1 sq. ama totaling 36 1x1 sq. ama.
5. Apply the same process as in step 4 to divide this into four units. Thus, one revi'it is contained by 2x2 etzba'ot with height 13.5 fifth etzba'ot.

There is a small difference between the way Rashi and Rashbam conduct this process of scaling down. Rashbam follows an almost identical process, but in step 2, he reduces by ⅖ths instead of multiplying by ⅗ths.[2]

It is interesting to note that Rashi treats area and volume very differently. When it comes to volume, we see in steps 1-3 that Rashi cuts down the volume by height doing simple multiplication division. That is, when carrying out operations on volume Rashi treats volume as a unit of measurement one can operate on. However, once the correct height is reached and the process of reduction is limited to the length and width, we are left with doing operations on area

[2] We note that Rabbenu Tam is quoted by Tosafot here as having another method of conducting this calculation. However, it is not significant in its differences.

and Rashi does not do simple division. Instead, he breaks down the area into strips of 6 6x1 rectangles then the 6x1 into another 6 1x1 squares yielding 36 1x1 squares. Rashi does not simply divide by 36. This could be because Rashi wants to explain the final dimensions we reach, however, compare this with Rashi in Bava Batra 27a, which we will discuss in detail, from a different perspective, in Example 3. The Gemara in Bava Batra discusses the amount of land a tree draws nourishment from. To that end it focuses on the length of the roots and uses it to calculate the area of land a tree absorbs nutrients from. The Gemara starts off making a square around the tree and then transitions to a circle. That is, in the early stages of the *sugya*, the Talmud assumes that the shape of the land around the tree under discussion is a square. At a later point, the Talmud says that we are looking at a circle around the tree given that the roots spread out around the tree equally in each direction.

Rashi 26b s.v. *Kama* explains the Gemara stating the amount of land measured in a *beit se'a* describes how to measure the area of a *beit se'a* if it were all placed in a strip of land 1 ama wide. Throughout this sugya Rashi uses the same description of area. He keeps mentioning "by an ama". Why does Rashi feel the need to describe the area in this way instead of simply dividing the area as he does for volume? The most likely answer is that Rashi did not have the vocabulary to describe area in units but he had the language provided in the Talmud of *log* for volume.

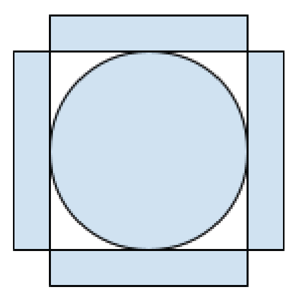
Example II.A.3

Rashi on that sugya in Bava Batra is another case where we see his arithmetic skill even when making (intentionally) incorrect calculations.

רש"י מסכת בבא בתרא דף כז עמוד א ד"ה וכמה מרובע יתר על העיגול רביע

דל רביע מאלף וכ"ד פשו להו תשס"ח אמות ברוחב אמה אכתי לא שוי שיעורי דהשתא פשו להו שיעורי דמתני' אדעולא פלגא דאמתא ונמצא אילן יונק י"ו אמה ומחצה לכל צד דהיינו ל"ג על ל"ג דשיעורא דעולא תשס"ח ושיעורא דמתני' תתל"ג ונפיש שיעורא דמתניתין אדעולא ס"ה אמות באמה רוחב חלוק הרצועה לרוחב הרי ק"ל אמה אורך ברוחב חצי אמה הקף מהם האילן ל"ב אמה לצפון וכנגדו לדרום נמצא מהצפון לדרום ל"ג שהרי הוספת עליו חצי אמה לכאן וחצי אמה לכאן ובידך נשארו ס"ו ברוחב חצי אמה תן חציין למזרח וחציין למערב נמצא מוקף ברבוע ולא דקדק הגמרא כל כך דנקט פלגא אמתא דהא עולא בעיגול קאמר ואילו בעיגול הייתי יכול להוסיף עליו שני שלישי אמה לכל צד אלא משום האי פורתא לא דק כולי האי ואית דגרסי פש ליה תרי תילתי דאמתא

At a later stage in the sugya, Rashi needs to add half an ama to the radius of a circle and calculate the new area. He takes the area of 768 square amot in the Gemara and adds strips of half ama width by 32 (or 33 length along the perpendicular side) around the circle. This yields 768+32/2*2+33/2*2 = 833 which is almost the correct area (the area stated in the Gemara is 833+⅓). However, one cannot place rectangular strips around a circle without creating gaps. He says the gemara is “lo dak” (inexact) because the Gemara is dealing with a circle not a square. Rashi seems to be implying that the method he just explained works well and adds the missing 65 square ama area when adding these strips onto a square (in which case adding rectangular strips onto a square would not create holes). However, the case of the Gemara is of a circle. He says if it were a circle then 2/3rds of an ama could be added onto the radius (and Rashi quotes girsaot like that). Rashi does not present the calculations for this. But this is actually correct: ⅔ of an ama added to the radius of a circle with radius 16 works out to be exactly 833+⅓ – using the assumption of the Talmud that pi=3.



In these calculations, Rashi explains the process clearly, although knowing that they are not the correct calculations for the case of the gemara (additionally, a square with length 32 amot would not have an area of 768 square amot so the case he describes does not conform to reality). Yet, Rashi explains them clearly and then very succinctly explains that the case is “inexact” and states the correct case.

II.B. Geometry

Example II.B.1.a: In the area of geometry we find that Rashi, at times, did not have a good understanding of fundamental relationships in triangles. For instance, in Eruvin 78, the gemara discusses when a wall splits an area in two, with respect to making an eruv hatzeirot.

תלמוד בבלי מסכת עירובין דף עח עמוד א

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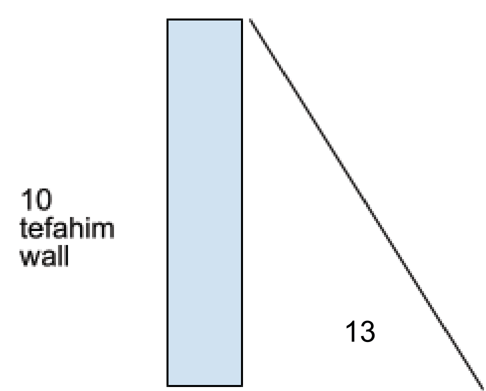
The gemara says that if there is a ladder leaning against the wall then the two courtyards are considered to be connected. There are a few opinions as to the height of the ladder relative to the wall necessary to connect the courtyards. The first opinion is that for a wall of height 10 tefahim the ladder must be 14 tefahim. Rashi explains that the ladder must be this tall so that the base of the ladder can be a distance of four tefahim away from the base of the wall while the top of the ladder is at the top of the wall. Rashi seems to be assuming that the length of the diagonal (the ladder) is the sum of the length of the sides (the height of the wall and the



distance of the base of the wall to the base of the ladder), that is 4+10=14. This is, of course, incorrect. We can easily use Pythagorean's theorem to calculate the correct length a ladder would need to be to reach the top of a ten tefahim high wall with a given distance from its base. Tosafot immediately points out, based on the rule of the Talmud that the diagonal of a square is 7/5ths the length of the side of the square, that if the base of the ladder is ten tefahim away from the base of the wall then the ladder of fourteen tefahim would still reach the top of a wall ten tefahim high. Tosafot says that therefore Rashi is simply not being accurate. In fact, however, Rashi is incorrect. Tosafot is either being respectful, or saying that Rashi was not concerned with the details of how far the base of the ladder was from the base of the wall.

Example II.B.1.b

We find another instance of Rashi's lack of understanding of the relationship between the sides of the triangles in Eruvin 5a. The Talmud discusses the minimum length of a mavoi. Abaye says it is 4 amot in length while Rav Yosef says it is 4 tefahim in length. Abaye brings a proof that it has to be more than 4 tefahim from the fact that the length of a mavoi must be greater than its width and one of the requirements for a mavoi is that hatzeirot have to open into it. The width of the door of a hatzeir must be at least 4 tefahim. Now there are two cases, either the door to a hatzeir is along the length of the mavoi or it is along the width, that is, in the back wall of the mavoi opposite the entrance. If the former is the case, then the length must be more than 4 tefahim. If the case is the latter, then the width of the mavoi is wider than 4 tefahim while the length is 4 tefahim. However, the Talmud states a rule that the length of a mavoi must be more than its width. Thus, there is no case where the length of a mavoi can be 4 tefahim. The Talmud says that Rav Yosef would reject this saying the door could be in a corner of the mavoi.

Rashi explains that one tefah of the door is along the width and the remaining three are along the length. Thus, the length of the mavoi is less than 4 tefahim while also being longer than the width. Rashi seems to say that the length of the door is then the sum of the sides of the triangle formed. As in the last example, Rashi assumes that the length of the diagonal is equal to the sum of the length of its sides when, in fact, they are related in a much more complicated way. Again, Rashi does not have a good understanding of the relationship between the length of the diagonal of a triangle and its sides.

Tosafot here points out that Rashi must be wrong (he says Rashi is just not exact in his example of length one and width three, the basic idea holds but would need slightly different numbers). He says the door would be the length of the hypotenuse of a 3x1 triangle which is less than four. Tosafot makes a similar comment on Rashi in 94b.

It is important to note that in these cases Tosafot said that Rashi was inexact in the examples he chose, but not that Rashi was fundamentally incorrect. We will see cases where Tosafot does say that Rashi is simply incorrect. Although Rashi's explanation of these talmudic passages remains correct, his comments betray a lack of understanding of geometric relationships.

A partial counterexample to this point is found in Sukkah 45a s.v. *ala ama*. There Rashi calculates the diagonal of a nine by three triangle. He says, correctly, that the length will be greater than nine and less than eleven tefahim. So, at the very least, Rashi here seems to know that the length of the hypotenuse is not the sum of the lengths of the sides.[3]

Example II.B.2.a:

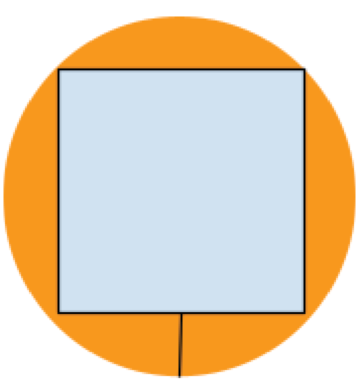
[3] One might suggest that this example in Sukkah indicates that at some point Rashi's mathematical understanding changed. It would be potentially interesting to investigate whether his mathematical understanding changed over time. However, to do this one would need a good sense of the order in which Rashi wrote his commentary on the various tractates. It would also require many more examples in Rashi of the sort we have presented to form a coherent theory of this.

Rashi also does not fully understand the relationship between squares and circles. In Eruvin 76 the gemara discusses cases of a window between two hatzeirot and the conditions that would allow for the two courtyards to combine and share an eruv. The mishna says that if the window is even partially below ten tefahim from the ground then the two courtyards can combine. The parameters for the window, four square tefahim, are given assuming a square window, R. Yochanan says that a circular window it would have to have circumference of twenty-four tefahim in order to contain a 4x4 square and must be two and some tefahim below the ten tefahim line in order for the inscribed square to be somewhat below the ten tefahim line.

<u>**רש"י מסכת עירובין דף עו עמוד א**</u>

צריך שיהא בהיקפו עשרים וארבעה - דבלאו הכי לא מצית למינקט בגווית חלון מרובע ארבעה על ארבעה, כל חלון עגול בתחתית אמצעיתו נמוך, ומאמצעיתו לכאן ולכאן הוה מגביה והולך, וצריך לזה שיהו שני טפחים ומשהו אורך מהקיפו בתוך עשרה מאמצעו לכאן טפח ומאמצעו לכאן טפח ועוד משהו משום דכי מרבעינן ליה מדלינן ליה מיניה שני טפחים מן ההיקף עגול שבין קרן לקרן לכל צד, דסתמינן להו, ומוקמינן לה אריבועתא, ונמצא אותו משהו הנשאר בסוף עשרה על פני רוחב החלון, כדאמר לקמן רבועא מגו עגולא פלגא בעית לדלויי, כלומר, חצי מדה הנותרת בריבוע ריבה העגול עליו, והיקף המרובע ששה עשר, נמצא העגול רבה עליו שמונה, הרי שני טפחים לכל צד.

Rashi erroneously explains this to mean two tefahim of the circumference of the circle must be below the ten tefahim line. He says that since the circumference of the circle is eight tefahim more than the perimeter of the square and it is split into four segments (between the points where there four corners of the square touches the circle) therefore, it would take two tefahim of circumference from one of those segments to reach the square. This is of course incorrect since each of the four segments Rashi describes would have one quarter of the total circumference of



2 tefahim

twenty-four tefahim, not the eight tefahim difference Rashi describes. Tosafot brings another explanation of the gemara but does not explicate what is wrong with Rashi's explanation.

Example II.B.2.b:

In Sukkah 8b, the Talmud presents the principle that a square inscribed in a circle is reduced by half[4]. Rashi explains that for a circle to contain a square of perimeter sixteen tefahim, the circle must have circumference twenty-four tefahim. Similarly, on 8a when explaining the Talmud's statement that a square is a quarter bigger than a circle inscribed in it, that is, the inscribed circle is 3/4ths the area of the square, Rashi says we see this by examining the relative perimeters. A circle of diameter four has a circumference 12, while a square of length four has perimeter 16. Thus, the square is an external quarter greater than the circle. Tosafot correctly objects to this line of explanation, stating that the ratio of areas is not the same as the ratios of perimeters because there isn't a linear relationship between perimeter and area. (Further discussion of the Tosafists argument is found below in Example III.B.3).

In the first case on 8a, Rashi is technically correct. That is, the relationship between the perimeter of a square with the perimeter of a circle inscribed in said square is 4:3 as it is for area. However, the case on 8b is not correct. The relationship between the perimeter of a circle and the perimeter of the square inscribed in it is not 3:2.[5] However, Rashi's incorrect explanation of the relationship between the perimeters of a circle and an inscribed square does not necessarily imply that he was not aware this was incorrect. It is important to remember that the gemara is about to reject this statement for its ignorance of reality. Rashi would have no problem suggesting an incorrect rationale for a statement that is deemed incorrect. On the other hand,

[4] This translation of the rule will be expounded upon in the Tosafot section in Example III.B.4 footnote 6.
[5] See the section on R. Abraham bar Hiyya for calculations and discussion of Tosafot on this point.

Tosafot has a way of salvaging the statement (as we shall discuss in detail in the section on R. Abraham bar Hiyya) and therefore wants to explain the statement in a way that is meaningful. However, this discussion is meaningful to us in that Rashi does not exhibit a level of geometric sophistication, while Tosafot does explain how this relationship is incorrect by employing examples built on calculations based on rules provided by the gemara.

<u>Example II.B.3</u> Eruvin 56b-57a

The Talmud in Eruvin 56 discusses how one should measure the *tehum* of a city. We look at the city as if it were a square and then make a larger square around it adding 2000 amot in each direction, the Gemara says. The Gemara has a statement about a city of Levites whose *tehum* is 2000 amot in each direction a thousand of which are set aside to be open fields. The statement says that the field portion is a quarter. The Gemara immediately asks that it is a half of the *tehum* not a quarter. It answers that it is not a quarter of the *tehum* but a quarter of the total area of the city with its *tehum*. The case is that the city is 2000 amot by 2000 amot so the area of the fields in each direction are a thousand by two thousand amot and in each corner there is an additional square of one thousand by one thousand amot of field. This yields 12 squares of one thousand by one thousand amot fields of the total 32 thousand by thousand square amot squares of *tehum* (eight on each side and four in each corner). This is more than a third of the total *tehum* not a quarter! Thus, the Gemara answers instead that the city itself is circular and the fields part of the *tehum* are a strip around the city. The field part of the *tehum* does not form a square with the circular city inside it. Instead it simply adds another thousand amot in radius of the city. This new calculation of the field will yield a field of size 9 thousand square ama which is a quarter of the total area of the city with its *tehum* (measured based on a squared city) of 36 thousand square ama.

Rashi calculates that the amount for the field is based on a circular city not square. He calculates this by saying that the ratio of field to city is 3/4ths. If the city were in a square of 2x2 thousand ama that would mean the surrounding fields have an area of 12 thousand square ama. Clearly, the field to city ratio is 3:1. Now, we downsize the city to a circle where a 16 thousand square ama area becomes a 12 thousand square ama circle. Then, since the field is ¾ of the total area it must take up 9 thousand square amot. Rashi first converts to a circle and then takes the ratio.

This was all for the case of a city of 2x2 thousand. The next case in the Gemara is a 1x1 thousand square ama city. There, he says the ratio of fields to the combined city and field is 8/9ths. He converts the 8 thousand square ama of field to a circle directly. Rashi does not convert the 9 square to a 6.75 circle then takes 8/9ths to get 6. Rashi chooses the easier method here. The order of operation is not mathematically significant but conceptually it may be easier to think in the first way Rashi does this. Doing the same type of calculation would be difficult in the second case so he does it the easier way. As part of his pedagogical method Rashi prioritizes converting the total area first and then taking the ratio. In a later stage, when those calculations would be confusing, Rashi chooses the more straightforward calculation after having already educated his audience in the earlier stage of the Gemara.

The reason the ratio between the fields and total area of the city with the fields remains the same whether we use a square city or a circular city is as follows. Let the length of the city be denoted by L. Let the additional area for the field be denoted by x. Assuming we treat the city as a square, the ratio is $\frac{2Lx+x^2}{L^2+2Lx+x^2}$. If we treat the city as a circle (treating pi as 3) then the ratio becomes $\frac{\frac{3}{4}2Lx+x^2}{\frac{3}{4}L^2+2Lx+x^2} = \frac{2Lx+x^2}{L^2+2Lx+x^2}$. Thus, the ratio does not change. Although this is a geometric case, the arithmetic argument that Rashi employs is correct and clearly explained.

II.C. Level of understanding: Rules

Often, Rashi will use certain mathematical principles but only accepts them as rules from *Chazal*, not as calculations that are explained. In Sukkah 8a, Rashi says that *Chazal* measured the diagonal of a square to be 7/5ths the length of the side of the square. In Eruvin 57a Rashi discusses the relationship between a square and the circle inscribed in it (see the ¾ rule mentioned above). He says that the square is a quarter larger than the circle both in perimeter as well as in area. For perimeter he quotes the gemara in Eruvin 14a that the circumference of a circle is three times its diameter based on the verse in 1 Kings 7: while for area he simply says that this is what *Chazal* measured the relationship to be.

When Rashi simply quotes a rule, or even a specific gemara with the rule, one can still say that Rashi independently knew or understood the mathematical (approximate) truth. However, in the instances where Rashi says כך שיערו רבנן, or an equivalent, it is hard to argue that Rashi is doing anything other than fully relying on the given rule from the gemara. This is particularly interesting given the cases in which Rashi produces incorrect calculations that seem to indicate a lack of mathematical rigor.

II.D. Summary

To summarize: From what we have seen Rashi was fully proficient in basic arithmetic, something not unexpected given that arithmetic was used for business. However, when it comes to geometry, we have seen many examples where Rashi does not appear to understand fundamental geometrical relationships. As described throughout, Tosafot corrects Rashi many times. We will now turn our attention towards the mathematical capabilities of the Tosafists.

III. Tosafot

III.A Historical Background & Overview

Before we begin our investigation into the mathematical understanding of the *Ba'alei ha-Tosafot* we must be careful with any evaluation of "their" mathematical understanding. There is no simple "they" which we can use. The text of Tosafot printed in our Talmud is a collection of various Tosafistic writings over the course of nearly two hundred years, varying from earlier sources such as *Tosafot Shanz* (the writings of R. Samson of Sens who was a student of Ri), to later Tosafists writing their comments based on those of earlier Tosafists, to the writings of the late Tosafists. In chapter 13, *Ha-Tosafot Shelanu* Urbach [15] goes through each tractate and discusses the authorship of Tosafot we have printed in that tractate. In general, the earlier the source, the clearer the writing tends to be, as later sources try to include as many questions and answers in the same context from a range of earlier sources. This often makes it more difficult to follow a full comment of *Tosafot* in terms of who offered which comments or analysis. We will be cognizant of the fact that whenever we read a comment of Tosafot, it may have emerged from different Tosafist study halls, ie., it does not necessarily come from a particular Tosafist. If a comment or calculation is quoted from an explicitly named Tosafist, we take note of that.

As will become clear, our main sources are from the earlier Tosafists Ri and Rash *mi-Shanz*. The main *Tosafot* sources used are those from Sukkah and Eruvin, along with those of Rash in his commentary to the Mshnah, as well as citations from Ri.

The first thing we have noticed about the math of Tosafot is all the places where Rashi's mistakes are corrected. We have discussed these already in the context of Rashi; however, those points should not be missed. Although not mathematically significant, the fact that there is an attention to this kind of detail is representative of the general style and method of learning among the Tosafists, which typically considered all facets, and examined every possible case in great detail. Other than the examples already mentioned above, there are a number of mathematical ideas of note we find in Tosafot. These include

- Ri's calculation of the area of a circle
- The relationship between the circumference of a circle and its area
- The inaccuracy of the Talmud's length of the hypotenuse of an isosceles right triangle, and the length of the hypotenuse of a scalene right triangle. (Or, as *Tosafot* refers to them: the diagonals of squares and rectangles.)

We will present these cases and the comments of the *Tosafot* below:

III.B. Circumference and Area of a Circle

Example III.B.1 **Area of a circle by strips**

In several places, *Tosafot* calculates the area of a circle. The issue is relevant in Sukkah to the discussion of the minimum required size of a Sukkah. The gemara has differing opinions as to the dimensions of a rectangular Sukkah. The Amora Rabbi Yohanan states the dimensions for a circular Sukkah. The gemara tries to understand the statement by comparing the dimensions of the circular Sukkah with those of the standard square Sukkah. *Tosafot* describes that the way one can easily calculate the area of a circle is by "unraveling" strips of the circumference and laying them on top of the previous one (this is an early example of the concept of the shell method of integration). This process, applied until the circle is reduced to a single point, yields a triangle. The length of the base of the triangle is the circumference of the original circle and the height is the radius. It is clear that the area of the circle is πr^2.[6] Tosafot assumes that $\pi = 3$, but otherwise his formula is correct. Tosafot in a few other places presents this method. Notably, in Eruvin 56b Tosafot quotes Ri as presenting this proof.

This simple and elegant proof is advanced far past the extent to which we would have expected the Tosafists to have figured this out for themselves, especially given what we have seen from Rashi. We find the basis of this method quoted from Ri in Bava Batra 27a. We refer the reader back to Example 2 in the arithmetic section of the discussion on Rashi. Tosafot presents another method of measuring the area of an additional 2/3rds of an ama to a circle of radius 16 quoted from Ri. Ri writes that he found a method which says to take a circle with radius 16+⅔ and unravel outer layers of the circle until the remaining circle has radius 16. Taking these unraveled layers and stacking them one on-top of the other yields a trapezoid with base length 100 amot (circumference of circle of diameter 33+1/3), while the length of the top is 96 amot (circumference of circle of diameter 32). We now cut off one end removing the length discrepancy and add the cut off section to the side (after rotating 180°) creating a rectangle of length 98 amot. This yields a rectangle of 98x2/3 square amot which is an area of the missing 65+⅓.

In that context, *Tosafot* presents multiple methods of calculating the area of a circle with radius 16+⅔. None of them involve a direct calculation using this unraveling method. Notice, Ri quotes this method for measuring the difference of area between circles not for a direct calculation of area. It is possible that Ri found this method and later extended the method (in

[6] R. Yair Hayyim Bachrach (d. 1702) in his responsa collection *Havot Yair* (Responsum 172, Jerusalem Edition, 1987), claims that this method of calculation is flawed as can be seen from the case of a square. If one sets a square with a corner as the base and peels away the layers of a square from the opposite corner then the triangle created will have base four times the length of the square and height half of the hypotenuse. The area will then be sqrt(2)*L^2. See Garber and Tzaban [9] for a rigorous proof of this method and Wilamowsky, Epstein, and Dickman [17] for an intuitive explanation of why the Havot Yair is incorrect.

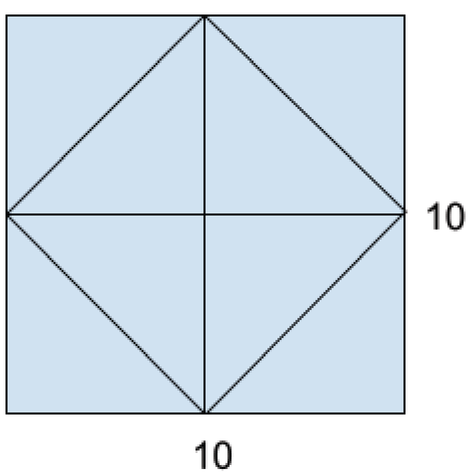
Eruvin 56b) continuing the unraveling process until the circle is reduced to a single point and a triangle is formed. This is not to say that Ri's source did not know how to extend this process, just that Ri did not know how to apply it at this time.

<u>Example III.B.2</u> Ratios of circles to squares

When *Tosafot* calculates the area of a circle, both in Eruvin 56b and Sukkah 8a, he does so to explain the statement in the gemara that a square is a quarter larger than a circle. That is, in the case of a square with a circle inscribed in it (i.e. the diameter of the circle is equal to the length of the square) the measure of the square is 4/3rds that of the circle. This rule was clearly stated in the Talmud, and we see above Rashi using it as well. However, *Tosafot* does not merely use this rule: in each place, the Tosafists state in slightly different ways that this ratio is true both for the perimeters as well as areas of the square and circle. If the length of the square is L then the perimeter is $4L$ while the circumference of the inscribed circle is πL. Assuming $\pi = 3$, the ratio holds. Similarly for area, the area of the square is L^2 while the area of the inscribed circle is $\pi \frac{L}{2}^2 = \frac{\pi}{4} L^2$. Again approximating $\pi = 3$ the ratio holds. That is, while the ratios between perimeters and areas do not follow a simple relationship such as the one stated (as we discuss in the next example), the ratio between the perimeter of a circle and the perimeter of a square, or the area of a circle and the area of a square, do have such a relationship.

<u>Example III.B.3</u> Relationship between perimeter and area

In Sukkah 8a, *Tosafot* is focused on dispelling the idea that one can use the ratio between the perimeters of squares and circles to demonstrate the ratio between the areas. He says that one cannot prove the ratio of areas based on the ratios of perimeters because there isn't a linear relationship between them. He shows this by comparing the area of a circle with diameter of four (with an area of twelve) with a 3x3 square (with an area of 9) even though they have the same perimeter. Tosafot calculates the area of a circle with diameter 4 by taking the circle inscribed in a square where the area of that square is 16 and applies the rule of the gemara for that scenario to get the area of the circle is 12 (the rule that the square is a quarter larger than the circle).[7] He brings another proof from a 5x1 rectangle which has perimeter 12 but area 5. Even within the world of rectangles one cannot say anything about the area based only on knowledge about the perimeter. Therefore, Tosafot concludes that the ratio between perimeters of a circle and a square is not sufficient to make any conclusions regarding the ratio of their areas.



Example III.B.4 Using $\sqrt{2}$

As mentioned, in Sukkah 8a (among other places) the Talmud states that the diagonal of a square is 7/5ths the length of that square. *Tosafo*t s.v. kol shows that the diagonal of a square is more than 7/5ths of the side. He creates a proof by contradiction. Make a 10x10 square, split this into 4 5x5 squares. Now, make a square from the inner half of each of the four squares where the diagonal of the 4 squares make up the sides of this square. The area of this square is clearly half of the big square (50) but the 7/5 ratio (making the length of the side of the inner square 7) would yield an area of 49. *Tosafot* Eruvin 57a s.v. *kol* states the same proof.

[7] It is interesting to note that there is a similar idea of linearity disproven by example in עירובין ע״ה ד״ה רבי with the ladder falling down if moved 10 away. It is different because there he just has to show it isn't a 1-1 correlation and here he has to show the ¾ ratio doesn't work which means he has to show it isn't linear. However, Tosafot does not present these differently. He presents them both as relations that one might think is 1-1 and shows by example that it is not.

In Sukkah 8b, Tosafot creates the same 10x10 square as before. Then he puts a circle in the outer square. Thus, the diagonal of the inner square is along the diameter of the circle and lies inside it. Comparing areas it is clear that the ratio of outer square to circle to inner square is 1:¾:½ based on the rules supplied by the Talmud and proved by Tosafot. Thus, the square in a circle is a half less than the circle.[8] He then says if the outer square is 7x7, instead of the 10x10 we have been working with, then the inner square will be 5x5 since the diagonal of the inner square is equal to the length of the outer square ($\frac{7}{7/5} = 5$), and the ratio of areas will then be 49:25 which is not 2:1 as we said it would be. He answers that this is similarly due to the error of the 7/5ths approximation of the diagonal. This is easy for us to confirm. The length of the side of the inner square is $\frac{7}{\sqrt{2}}$. The area of the inner square is then $\frac{49}{2}$ which is exactly what we expect.

<u>Example III.B.5</u> Diagonals

The Talmud in Eruvin 5a discusses how much of a damaged mavoi has to remain to still be considered a mavoi. Abaye says it is 4 amot and brings a proof that it has to be more than 4 tefahim from the fact that the length must be greater than the width and chatzeirot have to open into it. The door of a chatzeir must be at least 4 tefahim. Therefore, the only width of the mavoi must be at least 4 tefahim and the width must be more. The Talmud says that Rav Yosef would reject this source as a contradiction to his own opinion by explaining that the door could be in a corner and not along the width or length of the mavoi. Rashi explains that one tefah of the door

[8] Tosafot asks that the ratio between the circle and the inscribed square should be stated differently than it appears in the Talmud. The Talmud says the ratio is ½. However, Tosafot argues, it should be stated as ⅓ since the area of the square is ⅓ less than that of the circle. The language would then be parallel to the language of the ratio between a circle inscribed in a square and the square. There the Talmud says that the square is a ¼ larger than the circle. That is, the measurement is done from the perspective of the larger structure- the square. Therefore, the relation between the circle and the inscribed square should also be stated from the perspective of the larger structure- the circle- which is ⅓. To answer this he says that the relation stated is really between the square inscribed in the circle and the square in which the circle is inscribed. Relative to the outer square the inner square has ½ the area.

is along the width and the remaining three are along the length. Thus, the length is less than 4 while also being longer than the width. Tosafot brings rashi and says that this isn't exact because then the door would be the length of the diagonal with sides 3 and 1 (and not the sum 3+1=4 tefahim). Tosafot says it is clear to see that the length of the diagonal of a 3-1 doesn't reach the length of the diagonal of a 1-1.

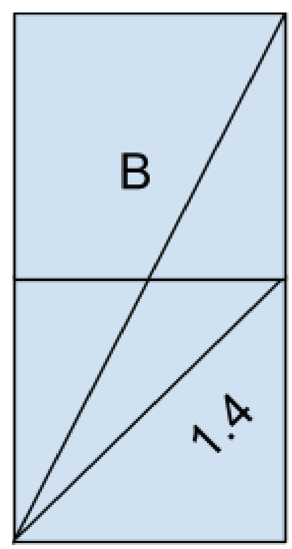
תוספות מסכת עירובין דף ה עמוד א ד״ה דפתח

פירש בקונט' טפח מן הפתח בדופן האמצעי ושלשה טפחים בדופן המשך וכן בצידו השני ולא דק דא"כ אין הפתח רחב אלא אלכסון של ג' טפחים על טפח והנה עינינו רואות שאין האלכסון של שלשה על אחד מגיע לאלכסון של טפח על טפח

It isn't immediately clear what Tosafot means by this.[9] However we find a similar idea in Tosafot in Shabbat 85b again correcting Rashi's inaccuracy

תוספות מסכת שבת דף פה עמוד ב באד״ה והא איכא

ומה שפירש"י והא איכא מקום קרנות ויכול להמשיך של זו לצפון ושל זו לדרום ואין צריך להמשיכו מכנגד אותו שבחברתה אלא טפח לא דק דא"כ אינם מרוחקין אלא כמו שעולה האלכסון של טפחיים על טפח ואינו עולה אפי' לשני טפחים ושני חומשים שאלכסון של טפח על טפחיים פחות מאלכסון של טפח על טפח



Tosafot says that the diagonal of a 2x1 square is not even 2.4 tefahim. He is clearly adding the diagonal of a 1x1 square, which is 1.4 according to the approximation of the Talmud we discussed above, and adding it to the additional 1 tefah length. Tosafot thus shows that as one dimension of a rectangle increases, the

[9]This line in Tosafot is so difficult that R. Levi ibn Habib (c. 1480 – c. 1545) dedicated an entire entry in his responsa collection *Maharlbah* (Responsum 13) to this. He goes so far as to say that ועל פי זה דברי התוס' כפי פשטן ודאי הם שקר גמור. The reader is encouraged to explore R. Levi ibn Habib's (clever and elegant) attempted explanation of Tosafot. However, we will present what we believe to be the correct explanation of Tosafot.

length of the new diagonal does not increase as fast as the increase of the dimension. Reading that back into Tosafot Eruvin 5a, he is saying that the diagonal of a 3x1 is even 3.4 (1.4+2), which is less than 4.

We must be careful placing an interpretation from one Tosafot into another. It must be noted, that our printed Tosafot in Eruvin is Tosafot Tuh based on *Tosafot Shanz*. In contrast, Tosafot in Shabbat is Tosafot Tuh, who frequently references Rash but incorporates a blend of sources, maintaining this approach until folio 122b, after which it predominantly quotes *Tosafot Shanz*. However, in this case we even find another similar example directly from R. Samson *mi-Shanz* in his commentary on kilayim

<u>ר"ש משאנץ כלאים פרק ה:ה</u>

ואלכסון של שלשים על עשרים אינו מעדיף על הריבוע אפילו כעודף האלכסון של כ' על כ' [ויושר י'][10] תדע דאם תקשור חוט שבקרן זוית ותוליכנו באלכסון לסוף כ' ואח"כ ביושר עד למטה י' אמות נמצא אורך החוט ל"ח אמות

So this does seem to be the correct interpretation of Tosafot in Eruvin.

III.C. Mistakes

Our exuberance over the apparent mathematical proficiency of the Ba'alei Tosafot must be tempered with the acknowledgment of their mistakes and misapprehensions. Although few, these mistakes are somewhat shocking to the mathematically modern reader.

<u>Example III.C.1</u>

[10] Although the words ‘ויושר י is an addition found in the Vilna printing and not in manuscripts, from the continuation of Rash that we quote, it is clear that this is what he meant. In fact, the language without the addition from the Vilna printing is more similar to Tosafot in Eruvin.

In Bava Batra 102a the Talmud discusses how to search the ground one is working on to ensure that there are no tombs in that area. Rabbi Shimon and the rabbis disagree on how one builds a family tomb. The rabbis say that one section of the tomb is 6x4 amot with 3 burial spots built into the walls along the lengths of 6 amot and two burial spots in the wall across the width of the tomb across from the entranceway. Rabbi Shimon says that one section of the tomb is 6x8 amot with 4 burial spots built into the walls along the lengths of 8 amot and 3 burial spots in the wall across the width of the tomb across from the entranceway and two additional burial spots one on either side of the entranceway. They further disagree on how many of these family sections there are in a tomb. The rabbis say that usually people build two of these sections one across from the other with a 6x6 central square from which one could enter the tomb sections. Rabbi Shimon says that there are four built around this central square. The Talmud has a statement that if one finds 3 graves he should search 20 amot out in each direction to make sure there is no other tomb section in the vicinity. The Talmud attempts to figure out whose opinion this statement subscribes to. According to the rabbis one would have to check 18 amot in each direction (6 amot from the length of both tomb sections and 6 amot from the central square). According to Rabbi Shimon one would have to check 22 amot (8 amot from the length of both tomb sections and 6 amot from the central square). Where does the opinion of 20 amot fit in? The Talmud answers that this is in accordance with the rabbis opinion however, one searches based on the length of the diagonal of one section and then the length of the central square and the length of the other section of the tomb (6 amot + 6 amot + diagonal of one section of the tomb).

רשב"ם מסכת בבא בתרא דף קב עמוד ב

ואלכסון של ד' אמות על ו' הרי ח' אמה כיצד ו' על ו' הוי ח' אמות וב' חומשין ה' על ה' הוי אלכסונן אמתים וה"ה לד' אמות על שש הוי אלכסון אמתים דמאי שנא ה' על ה' מארבע אמות על ו' אידי ואידי חד שיעורא הוא הלכך אלכסון שניהן שוה

Rashbam explains that the length of the diagonal of one section of the tomb is the diagonal of a 4x6 square according to the rabbis. The diagonal of a 6x6 is 8+⅖ based on the 7/5 rule. The diagonal of a 5x5 is two (that is, two amot in addition to the length of the side, totaling 7). The diagonal of a 6x4 is also two (more than the side) because it is the same in "measure" as the 5x5. Apparently, the length that Rashbam says to add 2 amot to for the length of the diagonal is the side of length 6, totaling 8 amot. As to why Rashbam does not think the 2 amot should be added to the side of length 4 totaling 4 amot, presumably this is because he is aware that the diagonal is longer than any given side of the rectangle. This "measure" that Rashbam refers to would seem to be the perimeter of the rectangle as both a 5x5 and a 4x6 have perimeter 20.

Tosafot quotes Rashbam and says that he must be incorrect. If we take a 4x4 amot square, the diagonal is 5+⅗. Now, we add a strip of 4x2 to one side creating a 4x6 rectangle. Adding the additional 2 amot length of the new strip to the diagonal of the 4x4 we have a length of 7+⅗ amot. Tosafot notes that this is longer than the length of the actual diagonal of a 4x6. Tosafot adds another point attempting to prove Rashbam incorrect. Tosafot claims the diagonal of a 5x5 must be more than that of a 6x4 because the area of a 5x5 is 25 which is more than the area of a 6x4 which is 24.

Rashbam is clearly wrong. One cannot calculate the diagonal by comparing the perimeters of rectangles.[11] However, we have not seen any impressive examples of Rashbam's

[11] Tosafot was not misled by the comparisons of perimeters. This is not surprising after we have seen in Sukkah 8a where Rash argued against using comparisons between perimeters to compare areas. Urbach (648-654) claims that

mathematical capabilities. It is very likely that, much like Rashi, Rashbam had a number of fundamental misunderstandings in the realm of geometry. After all, Rashbam grew up on the lap of his grandfather Rashi. The other Tosafists we discuss, Ri, Rash and their students, are at least one generation removed.

Tosafot's first comment is absolutely correct and a simple and familiar way of calculating an upper bound of the diagonal's length. We have seen this idea in Shabbat 85b, Eruvin 5a, and Rash on kilayim 5:5. The second comment, on the other hand, is not correct. It is clearly not true that a rectangle with a larger area has a longer diagonal than one with a smaller area. We see this clearly when we compare a 10x1 rectangle with a 5x5 square. Further, it is not true in this example of 5x5 and 6x4. The diagonal of a 5x5 is approximately 7.07 while the diagonal of a 6x4 is 7.2. The diagonal of the 6x4 is larger even though it has a smaller area.

Example III.C.2

It is tempting to attribute the previous mistakes to Tosafists less skilled in mathematics than what we saw from the Ri and Rash miShanz. However, we find in Rash miShanz's commentary on the *mishna* in *kilayim* 5:5 another glaring error.

The mishna discusses someone who plants a vegetable in a vineyard. Based on the law of kilayim some number of the vine plants around the vegetable, along with the vegetable, are considered kilayim and one is not allowed to benefit from them. The mishna describes depending on how far apart the vines are planted how many will be prohibited. The mishna says that if the vines are planted with a space of four or five amot between them, then a total of 45 vines will be

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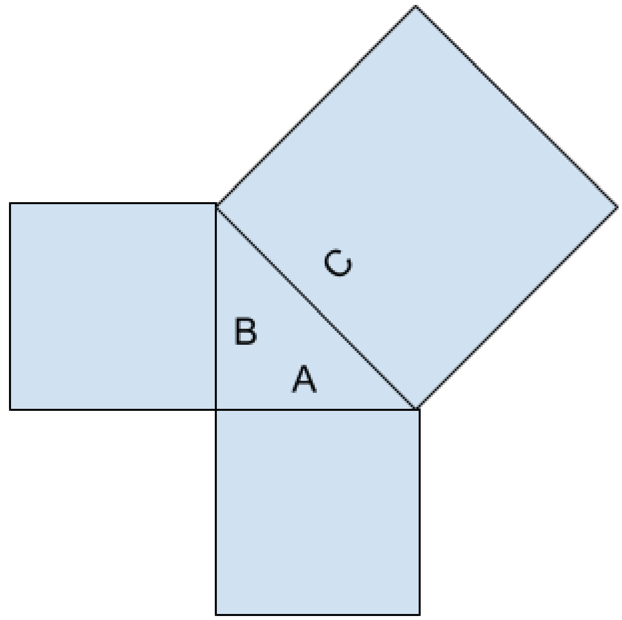
the Tosafot we have printed in Bava Batra (until folio 144b) are Tosafot Tuh whose sources were Rash *mi-Shanz*, other students of Ri, as well as Rivam.

kilayim. If they are planted at a distance of six or seven amot between them then all the vines within a 16 amot radius of the vegetable is kilayim.

Rash explains that when one has a vineyard of 7 by 7 vines each with a space of 4 or 5 amot between them, this creates either a square of 24x24 or of 30x30 amot (6 spaces between the 7 vines, 4 or 5 amot per space). If we create a circle around the center of this vineyard with radius 16 (as mentioned in the end of the mishna) then we have a circle with diameter 32 which covers all 49 vines in the vineyard except for the four, one in each corner. We see this, explains Rash, by calculating the diagonal of a 24x24 square. Following the 7/5ths rule the diagonal is 33.6 amot. Take half of this length for the distance from the center and it is 16.8 amot just outside the 16 amot radius we find in the end of the mishna. Similarly, says Rash, in the case where there are 5 amot between the vines, we remove the four vines in the corners. Now, the rows along each end have a length of 20 amot but all other rows are 30 amot long. If we look at the distance between the center of the vineyard and the furthest out vine (next to the recently removed corner) it is the length of the hypotenuse of a 15x10 right triangle, or as Rash puts it half the length of the diagonal of a 30x20. Rash, correctly, points out that this is less than the diagonal of a 20x20 amot (28 amot) with an additional 10 amot, making 38 amot. In fact it is just over 36.05 amot. However, here is where Rash gets into trouble. He says that it is not possible to calculate the diagonal of a rectangle with unequal width and length. It is clear from our mishna, Rash says, that the diagonal of a 30x20 is no more than 32 amot. In this way, we ensure that the outer vines are no more than 16 amot away from the center of the vineyard. Then our circle of radius 16 covers all the vines except the 4 vines in the corners.

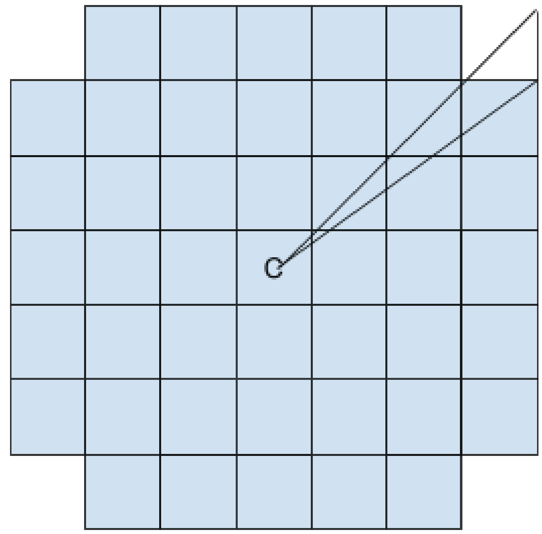
Rash continues to quote *"Hahmei haMidot"* who claim that one can, in fact, calculate the hypotenuse of any right triangle. He describes what we know to be Pythogorean's theorem. Rash describes that if we take a rectangle and make one square with side length equal to the length of the rectangle, and another square with side length equal to the width of the square, then the sum of the area of these two squares is equal to the area of a square with side length equal to the diagonal of the rectangle. $A^2 + B^2 = C^2$. Rash even presents a proof for an isosceles right triangle (the diagonal of a square) but rejects the rule for other rectangles as it would not fit with his explanation of the mishna in the latter case we explained as the area of a 32x32 square is less than the sum of the areas of a 20x20 and a 30x30 square.



The proof he presents for the case of a diagonal of a square is the same proof we find in Sukkah and Eruvin. That is, the area of the square made from the diagonal of one of the 4 small squares making a super-square is equal to the sum of the area of two of the squares.

However, Rash *mi-Shanz* is incorrect, the Pythagorean Theorem can and has been proven in a multitude of methods. Rash is clearly not familiar enough with practical geometry to know this as objective fact and is influenced by what he believes is the only explanation of the *mishna* in *kilayim*. The explanation of the *mishna* forces him to come to the conclusion that the diagonal of a 30x20 is no more than 32 which contradicts Pythagorean's theorem.

Rabbenu Asher b. Yehi'el (Rosh, d. c. 1325) in his commentary on this mishna cites Rash's claim that the distance between the center of the vineyard and the vine next to the corner is less than or equal to 16 amot. He shows that this cannot be correct as the distance between the center of the vineyard and the corner vine is half the diagonal of a 30x30 amot square which is 21 amot. Assume, by contradiction, that the distance between the center of the vineyard and the vine next to the corner is less than or equal to 16 amot. Then when we take the line connecting the center of the vineyard with the corner vine the length of that line (21 amot) is more than or equal to the sum of the line connecting the center of the vineyard with the vine next to the corner (<16 amot) and the line between the corner vine with its neighbor (given as 5 amot). Rabbenu Asher takes for granted the fact that the sum of two sides of a triangle are more than the third side. This is not surprising. Recall Tosafot's comments on Rashi in Eruvin 5a that the sum of the sides of a right triangle do not equal the hypotenuse.



<u>רא"ש מסכת כלאים פרק ה</u>

פירש רבינו שמשון שאין מגפן האמצעית עד הגפן שאצל הקרן יותר משש עשרה אמה הילכך מקדש מ"ה כאילו היו נטועות על ד' ד'. **ודבר זה נראה לעינים שאינו כן** שהרי מגפן אמצעית עד הגפן שבקרן יש ט"ו אמה וחצי אלכסון של ל' על ל' שהוא שש אמות שהם כ"א אמה, ואם אין מגפן אמצעית עד גפן שאצל הקרן אלא י"ו אמה נמצא כשתמתח חוט של כ"א אמה מגפן אמצעית עד גפן שאצל הקרן ואח"כ תמתח החוט מאותו הגפן יגיע עד גפן שבקרן שאין ביניהם אלא ה' אמות וכן נשאר מן החוט, הילכך ע"כ יש ביניהם יותר מי"ו אמה, **ולפי חשבון חכמי המדות** דכל מרובע שני קוים כמרובע אלכסונה, ואמתי הוא אפילו בריבוע שארכו יותר על רחבו כאשר אצייר לך יש מגפן האמצעית עד גפן שאצל הכרם י"ח אמות

Rabbenu Asher then quotes the same "*Hahmei haMidot*" mentioned by Rash and uses the Pythagorean Theorem to calculate the true distance between the center of the vineyard and the vine next to the corner.

Rabbenu Asher suggests the Rambam's explanation of the mishna in place of the Rash's. His explanation does not fall within the scope of our discussion; however, it should be noted that the Rambam had difficulty coming up with an alternative explanation of the mishna. Rash's explanation, were it mathematically feasible, would be the most obvious explanation[12].

<u>פירוש המשנה לרמב"ם מסכת כלאים פרק ה</u>

התבונן בהם היטיב, לפי שכשנשאל עליה רב מרבני התלמוד היתה תשובתו אחרי גמגום, ויש בדבר דקדוקין הרבה ואחרי כן יסייעו מן השמים ונצוה לפרשהו לכם

III.D Summary

We have seen sophisticated mathematics quoted from Ri and R. Samson of Sens. These include the proof for the area of a circle as well as the proof of the inaccuracy of the 7/5ths approximation of the square root of two. Many of Rashi's mathematical mistakes, or "inaccuracies", are pointed out and corrected by early Tosafists. These sources have been sourced in *Tosafot Shanz*, put together by Rash and his students. On the other hand, we have also seen the fundamental misunderstandings in geometry of Rashi, Rashbam, and even of Rash and other Tosafists in Bava Batra.

[12] However, Rash's explanation does not account for the change in form from the beginning of the mishna to the end. According to him, the entire mishna follows the rule that the kilayim extends 16 amot. Rambam has a different explanation for the first part of the mishna which then accounts for the shift in language in the mishna.

There appears to be a dissonance in the level of geometrical sophistication among the early Tosafists. What level of mathematical education could Ri have been privy to? More to the point: where did the advancement in mathematical understanding originate from, which allowed Ri and Rash to develop a deeper understanding of geometry than Rashi had? Further, how did it allow for such fundamental geometric mistakes? The explanation may lie in the history of the spread of mathematics in Christian Europe.

IV. Mathematical Culture in Christian Europe

It should be clear to the reader by this point that most of the exciting mathematical comments in Tosafot are either quoted from the Ri or are from the texts associated with Rash *mi-Shanz*, a student of the Ri. It is striking to see such a dramatic development in mathematical prowess from Rashi and Rashbam's time. Where did this development come from? Certainly not from Rashi!

When we examine the Tosafists' knowledge of non-Torah areas we must consider their intellectual milieu (as Kanarfogel says [2]). This may shed light on how they relate to certain material. In our case, it is necessary to understand what level of mathematical sophistication existed in Christian Europe in the early 12th century.

There is much discussion regarding the relationship between the Tosafists and the culture around them. Kanarfogel [3] argues that although there were interactions necessitated by business and potential influences that seeped through, the Tosafists were limited in their ability to learn from the christians around them by their ignorance of Latin and thus, could not have had any formal education from them. This is in stark contrast with Spanish Jewry who largely knew Arabic and thus could, and did, interact with their neighbors.

However, much of this discussion is a moot point for us. Katz [11] describes "The Mathematical Cultures of Medieval Europe" summarizing that

> *it was the Muslims who brought the Hindu-Arabic system to Europe from the East; the Jews learned it from them; and the Catholics eventually mastered it as well, learning both from translations from the Arabic and from the material Fibonacci brought back from his travels to Muslim lands. (Katz [12])*

Not only were the Christians not mathematically advanced in the time of the Ri and Rash but it was in part the Jews of that time, including Rabbi Abraham bar Hiyya whose translations opened up the world of mathematics to Christians by allowing Latin speaking Europe access to the works of Euclid. d'Alverny [1] points out that there was a translation of *Mathematica Alhandrei* from Arabic to Latin in the later tenth century however we do not find other scientific translations occurring in this period. In fact, there were warnings among Muslims in the eleventh century against selling scientific works to Jews and Christians who would translate them and pawn them off as their own works. While this indicates translations occurring already before the eleventh century, we do not find many from this period.

For this reason, even if there had been high levels of Jewish-Christian interaction at this time, the Jews of France in the 12th century could not have learned much mathematics from the Christians. Thus, when looking for sources of mathematical sophistication for the earlier Tosafists, we must look towards their neighbors from the east. It is only slightly later, once these mathematical works made their way into the Catholic education system, that we must start thinking about its effect on the Tosafists. However, by this time bar Hiyya had already written his original work in Hebrew for the French rabbinate.

But who was R. Abraham bar Hiyya?

V. R. Abraham bar Hiyya

Rabbi Abraham bar Hiyya (1065-1145) was a Jewish mathematician who lived in Barcelona and then later moved to Provence. Bar Hiyya translated many mathematical works from Arabic to Latin introducing Euclid's works to western Europe. He wrote works in mathematics, astronomy, astrology, philosophy, and the calendar. Our interest is focused on one specific work of his entitled the *Hibur haMeshiha v'hatishboret.* Unlike his other works in which he wrote in Latin and were intended for a wider audience, bar Hiyya wrote this work specifically for the rabbis of France in need of mathematical guidance and, thus, is written in Hebrew. In the introduction, Rabbi Abraham bar Hiyya describes the necessity of studying arithmetic and geometry in order to follow the commandments in the Torah. Proficiency in arithmetic is a prerequisite for calculating payments required by Torah such as the amount of money someone must pay to redeem his field before the Jubilee. Similarly, geometry is needed to correctly[13] split land among inheritors or to calculate distances for *tehumei Shabbat* (as we saw in the discussion of Shab. 56-7). Bar Hiyya declares his purpose for writing the *Hibur* is for the French rabbis without proper knowledge of geometry. Arithmetic was not needed, bar Hiyya says, because it is used in business so even the French know practical arithmetic. It is unclear how much influence this work had. Some claim that Ibn Tibbon who lived in Provence shortly after bar Hiyya did not see the *Hibur*. The language bar Hiyya used is thought to have influenced the book *meyasher aqov,* a Hebrew work of geometry by Alfonso of Valladolid. [6]

The *Hibur* consists of four sections: propositions and definitions, measurements of triangles and quadrilaterals and area of circle quadratics chord arcs, division of the above shapes, and three dimensional solids. At times bar Hiyya will include a proof for a given rule, however there are discrepancies between the various manuscripts: some will contain a proof while others omit it. One of such proofs is the onion method proof of the area of a circle we saw in Tosafot. Friedman and Garber suggest that it is possible bar Hiyya did not include the proof in the *Hibur* but that one of the scribes saw this proof in the text of Tosafot, or perhaps both saw it in a third source, and included it in his copy.

[13] Interestingly, Rabbi Abraham bar Hiyya explains that the mathematical inaccuracies found in the Talmud, such as equating pi with 3 and the square root of 2 with 7/5ths, only relate to laws pertaining to individual obligations towards God not to laws of interpersonal interactions.

As we have seen, the presence of the onion method in the writing of the Tosafists is not a later insertion. Rather, the Ri found the onion method either in a margin of a manuscript or in some other place[14]. Given that the Ri lived between 1115-1184, we can say that this method was present in France during or shortly after Rabbi Abraham bar Hiyya wrote the *Hibur*. While all the options presented by Friedman and Garber [6] remain possible, the timing of the proof makes it seem more likely that this proof was included in the *Hibur* and subsequently made its way to the Ri. However, given that the method as quoted by the Ri was not used for the total area of a circle but for a partial area, there is a possibility that the method was first used only for partial areas and the Ri (and potentially others) expanded it to the total area of a circle. Of course, this is a farfetched hypothesis.

As mentioned, R. Shimshon *mi-Shanz* in his commentary to Kilayim quotes geometric scholars claiming a relationship between the diagonal of a rectangle and its length and width or Pythagorean's theorem as we know and think of it. R. Abraham bar Hiyya states and proves the rule in his exercises following section two[15]. However, similar to Rash, bar Hiyya notes that it is easier to prove this relationship for a square than for a rectangle although he does so with a different method than Rash.

Additionally, Garber and Tzaban [10] claim that there is another example where Rabbi Abraham bar Hiyya's mathematics makes its way into Tosafists writings. The Talmud in Sukkah 8b and Eruvin 56 present a statement of the *Dayni d'Kesari* that the relationship between a square inscribed in a circle and the circle is "a half". This is interpreted to mean that the circle is a third larger than the inscribed square. That is, if one adds half the measure of the square to itself the resulting measure is the measure of the circle. The statement does not specify whether

[14] This fact is notably absent in Friedman and Garber's discussion.
[15] Found in exercises 1 and 50 as listed in Guttenberg.

this measure is of perimeter or area. The context of the Talmud dictates that it is referring to perimeter. If one examines the perimeter of the respective structures, purely using statements from the Talmud, this is clearly incorrect. Let L be the length of the inscribed square, then the perimeter has length 4L. The circle it is embedded in has diameter 7/5ths the length of the inscribed square (using the Talmud's approximation of the diagonal of a square). The circumference of the circle is then $\frac{21}{5}L$ which is just over 4L. If, however, we compare the areas of the circle and the inscribed square, this statement holds. The area of the square is L^2. The area of the circle is $3(\frac{7}{5}\frac{L}{2})^2 = \frac{147}{100}L^2$ which is approximately $\frac{3}{2}L$.

Tosafot in Sukkah 8b and Eruvin 76b resolve the issue by saying that the statement of the *Dayni d'Kesari* is referring to area rather than perimeter and the context of the Talmud using it for perimeter is an incorrect interpretation of the statement. Garber and Tzaban point out that Tosafot refer to this explanation as *yesh mefarshim* (in Eruvin). They therefore suggest that Tosafot are quoting this explanation from Rabbi Abraham bar Hiyya who, in his introduction to the Hibur, uses this statement in the context of measure of area. They defend the plausibility of this quotation by referencing Abraham bar Hiyya's stated intent for the Hibur, to educate French Rabbis, as well as the convenient timing of bar Hiyya's presence in Provence in the first half of the 12th century.

This connection, although possible, is tenuous. Firstly, Tosafot in Sukkah does not bring this interpretation of *Dayni d'Kesari* in the form of a *yesh mefarshim* and our text of Tosafot in Sukkah is *Tosafot Shanz* while the text in Eruvin is Tosafot Tuh based on *Tosafot Shanz*. Thus, this interpretation is not referred to as though it comes from an external source in the more authoritative text. This is not to say that the source could not have been bar Hiyya, only that the *language* of Tosafot does not indicate an external source.

Tzaban and Garber [10] point out that bar Hiyya's proof of the area of a circle is more mathematically rigorous, than Ri's method. In his definitions,[16] bar Hiyya defines a line as something with a length but no width. In this way, when he "unravels" the circle he can do so taking lines infinitesimally thin. Ri does not have a similar statement describing the lines. If one takes strips with a given length, a smooth triangle will not be formed. However, while Tzaban and Garber are correct that bar Hiyya's presentation is more mathematically rigorous, Ri still fundamentally understood the proof.

[16] Definition 4 in the first section of the Hibur

VI. Conclusion

While it may feel as though this thesis is at the crossroads between mathematics and the humanities, the author is keenly aware of the fact that all discussion and hypothesizing, lies, with the rest of the humanities, solely within the realm of argumentation without true proofs. Therefore, any and all “conclusions” reached here are entirely speculative in nature and the reader is invited to conduct their own analysis of the data and reach their own speculative conclusions. Furthermore, as the reader has surely noticed, we do not have many varied cases. Therefore, we are limited in our ability to properly analyze and understand the Tosafists' level of understanding in different areas within geometry. We are also limited in terms of which Tosafists we can discuss. As seen in our examples, the main Tosafists are Ri, Rash, and their students. With these caveats in mind we endeavor to come to some understanding of the Tosafists mathematical sophistication.

As we have shown, the main examples of mathematical sophistication that we have seen in the writings of the Tosafists come from external sources, which we have argued are from Rabbi Abraham bar Hiyya. However, this does not account for many cases where the Tosafists had either significant mathematical insights or were aware of mathematical truths which seemingly eluded Rashi. Such examples include the method by which Rash proved the pythagorean proof for an isosceles right triangle (the same one used in Sukkah to prove the diagonal of a square is more than 7/5ths the length of a side of the square), understanding that the relationship between the perimeter and area of a circle is not a linear one (or at least 1-1), and the length of a diagonal of a rectangle is not equal to the sum of the length and width of the rectangle.

For such examples, we suggest that the development of their mathematical knowledge from Rashi's fits with the general style of learning developed by Ri. Urbach [15] describes how the general method of study that we are familiar with in the Tosafists had its beginnings from the time of Rashi's son-in-laws but began to fully flourished in, and is forever associated with, the study house of Ri. That is, the style of exploring apparent contradictions between tractates and resolving them was the full focus and expertise of the Ri's study house. Urbach asserts this to be the place where "the Tosafot" were created (pg. 252).

This style, developed and promulgated by Ri (on the basis of what he learned from his uncle and major teacher, Rabbenu Tam), is dedicated to meticulous attention to detail. Oftentimes, to resolve contradictions, this style necessitates creating cases and claiming that this was the context of one of the entries in the Talmud, in this was resolving a contradiction. This habit of exploring/creating cases with great attention to detail cannot be limited to specific legal cases in the Talmud. Once practiced, this way of thinking extends to all areas of thought. We see this in a long comment of Tosafot in Bava Batra 101a s.v. *d'aved l'hu* where there is a back and forth between Ri and Rash pertaining to the exact dimensions of burial tombs.

We can hypothesize that some of the mathematical development seen between Rashi and Ri/Rash *mi-Shanz* is a result of newly developed and heightened attention to detail, including the examination of details resulting from their methods of study. Thus, Tosafot's understanding of the complex nature of the relationship between perimeters to areas of circles and squares is a natural result of a tendency to think of various cases. In this case, it would be to think of circles and squares with different lengths and constantly comparing them. The examples Tosafot provides in Sukkah 8a countering the claim that one can compare the ratio of perimeters of circles and squares with the areas of circles and squares would then come naturally.

However, even with this attention to detail, without a rigorous understanding of mathematics, mistakes, even fundamental ones, are to be expected. In this way, we can account for the varying examples in the Tosafists writings. We find some external influence (potentially from Rabbi Abraham bar Hiyya) providing some advanced mathematical ideas and proofs, their interactions with the mathematically destitute culture of Christian Europe of that time equipped the Tosafists with no rigorous mathematical education allowing for fundamental mistakes in geometry, and the attention to detail and curiosity to explore various cases stemming from their method of study allowed for noticing inaccuracies in Rashi's explanations.

That being said, this last hypothesis is the weakest of the author's conjectures. However, he finds comfort when we recall our motivation for this study was to begin a conversation about the mathematics of the Tosafists. We do not presume to have fully plumbed the depths of the sources we have found. Additionally, there are certainly many more works in the general history of mathematics that need to be reviewed while comparing and contrasting the mathematical works of the Tosafists. This may lead to a deeper and truer understanding of the mathematical sophistication of the Tosafitsts. Greater research into the writings of later Tosafists may open the doors to further understanding of the development of advanced mathematics among Jews in Christian Europe.

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