# Optimal Allocation of an Indivisible Good*

Ran Shao†
Yeshiva University

Lin Zhou‡
Shanghai Jiao Tong University

First Draft: October 2007
This Version: September 2016

**Abstract**

In this paper, we consider the problem of allocating an indivisible good efficiently between two agents with monetary transfers. We focus on allocation mechanisms that are dominant-strategy incentive compatible when agents' types are private information. Inefficiency of an allocation mechanism may come from two sources: misallocation of the indivisible good and an imbalanced budget. Unfortunately, as Green and Laffont (1979) demonstrate, no allocation mechanism can always overcome both kinds of inefficiency. We identify allocation mechanisms that maximize the expected total utilities of agents, and characterize optimal mechanisms for a large class of agents' type distributions. For strongly regular type distributions, we show that the optimal mechanisms must be budget-balanced: they are either fixed-price mechanisms or option mechanisms. The result may not hold for other type distributions. For certain type distributions, we show that optimal mechanisms are hybrids of Vickrey-Clarke-Groves mechanisms and budget-balanced mechanisms.

*The authors want to thank the the following scholars for their comments on the work reported in this paper: Hector Chade, Kim-Sau Chung, Jerry Green, Alejandro Manelli, Eric Maskin, David Miller, Benny Moldovanu, Stephen Morris, Ed Schlee, Michael Richter, four anonymous referees, and participants at the 2007 Far Eastern Meeting of the Econometric Society in Taipei, the 2008 Southwest Economic Theory Conference at UC Santa Barbara, the 2012 Spring Midwest Economic Theory Conference at Indiana University and the 2012 4th World Congress of the Game Theory Society in Istanbul. Lin Zhou's research is supported by the Ministry of Education of China through the Program for Changjiang Scholars and Innovative Research Team in University (IRT13030).

†Department of Economics, Yeshiva University, 215 Lexington Ave., New York, NY 10016. Email: rshao@yu.edu.

‡The corresponding author. Antai College of Economics and Management, Shanghai Jiao Tong University, The People's Republic of China. Email: zhoulin@sjtu.edu.cn

# 1 Introduction

In this paper, we consider the problem of allocating an indivisible good efficiently between two agents when agents' valuations of the good are private information. A typical example of such a problem is the bilateral bargaining problem, in which a seller and a buyer negotiate over whether and how to trade a particular good. Our focus is on dominant-strategy incentive compatible mechanisms. The research interest in this problem is derived from a fundamental dilemma of Green and Laffont (1979): When agents' valuations of the good are private information, it is impossible to always assign the good to the agent with the higher valuation without incurring any cost.

There are several methods that are commonly used in practice, including lotteries, seniority rankings, auctions. These methods either sometimes assign the good to the agent with the lower valuation or sometimes incur negative cash outflows from agents.

For scholars, two particular classes of methods have received more attention. The first class consists of all Vickrey-Clarke-Groves (VCG, henceforth) mechanisms (Vickrey, 1961; Clarke, 1971; Groves, 1973) that extend the conventional English auction scheme. The second class consists of all fixed-price mechanisms (Hagerty and Rogerson, 1987), in which the good is assigned to one agent (the seller) unless both agents are willing to trade the good at a predetermined price. VCG mechanisms always assign the good to the agent with the highest valuation, but they may incur outflow of money from agents (money burning). Fixed-price mechanisms do exactly the opposite.

Although extensive research has been conducted on VCG mechanisms and fixed-price mechanisms separately, they have never been scored against each other in any formal model, let alone in a model that allows for more-general mechanisms. Note that VCG and fixed-price mechanisms share two common features. First, they are dominant-strategy incentive compatible—i.e., it is always a dominant strategy for agents to reveal their types truthfully. Second, they are no-deficit—i.e., they have no need for money injection from outside to facilitate the agents. In this paper, we shall study all mechanisms that are dominant-strategy incentive compatible (DSIC) and no-deficit (ND). Our goal is to identify the optimal mechanisms among them.

To evaluate DSIC and ND mechanisms we assume a known Bayesian prior over the private types of the agents and look for mechanisms that perform well in expectation over types from this prior. Note that a corollary of the work by Green and Laffont (1979) is that there exists no mechanism that is always more efficient than others in every realization of agents' types. Our Bayesian objective is a standard one for mechanism design in environments where no mechanism is pointwise optimal.[1]

In Theorem 1, we present a characterization of optimal mechanisms when agents' type distributions are strongly regular.[2] An optimal mechanism is either a fixed-price mechanism or an option mechanism, depending on agents' type distributions. Hence, any optimal mechanism must be budget-balanced. Both fixed-price and option mechanisms are optimal if agents are identical ex ante. In an option mechanism, one agent is the temporary holder of the good, and the other agent is the recipient of a call option that allows him to purchase the good from the first agent at a predetermined price. The good changes hands whenever the option recipient wants to exercise his option. In comparison, under the fixed-price mechanism, the good changes hands only when both agents agree to the trade at a predetermined price. When agents' types are not strongly regular, the conclusions in Theorem 1 no longer hold. We study several such cases in Theorems 2 and 3 when agents are symmetric *ex ante*, obtaining characterizations of optimal mechanisms. Optimal mechanisms in these more general cases are not always budget-balanced, as they might be hybrids of VCG and budget-balanced mechanisms: An optimal mechanism may sometimes assign the good efficiently and sometimes impose budget-balance depending on the type profile.

We believe that our results make a significant contribution to the literature on mechanism design, as there are very few examples of closed-form optimal dominant-strategy incentive compatible mechanisms. Moreover, Theorem 1 highlights the importance of budget-balancedness for optimality with strongly regular type distributions. On the other hand, Theorems 2 and 3 demonstrate that the optimal mechanisms need not be either VCG mechanisms or budget-balanced mechanisms in other cases. They complement discoveries found by Miller (2011), Drexl and Kleiner (2015), and Schwartz and Wen (2012) through examples that either budget-balanced or VCG mechanisms can be outperformed by other mechanisms on average for different type distributions.

RELATED WORK. This paper considers dominant strategy incentive compatible and ex post no-deficit mechanisms to allocate a good between two agents to maximize the expected agents' utilities when the agents' types are drawn from a known distribution. Guo and Conitzer (2010) consider a generalization of our problem multiple goods and multiple agents and look for VCG mechanisms (which always choose the surplus maximizing allocation) that

[1]Previous works have considered the same setting but relaxing DSIC to Bayesian incentive compatibility or strengthening the Bayesian optimization criteria to a pointwise objective (but relaxing the optimality criteria to one of approximation). A comparison of these works to ours will be given in detail in the related work section.

[2]Our notion of strongly regular distribution requires that both the hazard rate and the reversed hazard rate are monotone. See Section 2 for details.

minimize the expectation of the outflow of money. This outflow of money can be reduced by redistributing the VCG payments among the agents (where the money not redistributed is burnt). Schwartz and Wen (2012) provide an example of a bilateral trade model in which the mechanism with money burning outperforms budget-balanced mechanisms for certain distributions. Miller (2011) shows that VCG mechanisms can never be optimal for a general class of agents' type distributions. Finally, in a contemporaneous paper, Drexl and Kleiner (2015) consider a variant of our problem, in which an additional ex post individual rationality (IR) condition is also imposed on mechanisms. Within this smaller set of mechanisms, they show that the optimal mechanisms are budget-balanced, a result similar to our Theorem 1. The advantage of their work is that their result is valid for all regular distributions[3], a more general class of distributions than ours. Nevertheless, when the IR condition is dropped, the optimal mechanisms are not necessarily budget-balanced for regular distributions as our Theorems 2 and 3 demonstrate. One must assume strong regularity in order to show that optimality implies budget-balancedness.

There is a line of research that considers a similar question but relaxes the DSIC requirement to Bayesian incentive compatibility. With this relaxation the mechanism of d'Aspremont and Gérard-Varet (1979) obtains the first-best welfare and, consequently, the no-deficit condition imposes no loss. There are two reasons to consider our mechanisms over these mechanisms. First, the proper working of Bayesian incentive compatible mechanisms is dependent on a strong common prior assumption.[4] Second, mechanisms with complicated transfers, like the AGV mechanism, tend not to be seen in practice.[5]

There is another line of research that strengthens the Bayesian optimization criterion to achieve guarantees for all types of the agents (i.e., pointwise) when there are many agents and units. Most of these papers focus on VCG mechanisms that aim to redistribute most of the agents' payments (and burn the remainder). Cavallo (2006) considers VCG mechanisms that minimize the outflow of money in the worst case. Guo and Conitzer (2009) consider VCG mechanisms that minimize the worst-case ratio of the outflow of money over the total Vickrey auction revenue. Moulin (2009) proposes another worst-case ratio measure and derives the optimal VCG mechanism. Non-VCG mechanisms were subsequently considered

[3]The hazard rates of the type distributions are monotone.

[4]Readers interested in the topic of dominant-strategy vs Bayesian incentive compatibility are referred to Chung and Ely, 2004; d'Aspremont and Gérard-Varet, 1979; Bergemann and Morris, 2005; and Jehiel et al., 2006.

[5]In recent work, Gershkov et al. (2013) derive an "equivalence" result between Bayesian and dominant-strategy incentive compatible mechanisms. For any Bayesian incentive compatible mechanism, they show that one can find another dominant-strategy incentive compatible mechanism that mimics the allocation of the Bayesian incentive compatible mechanism. However, their construction does not preserve the no-deficit condition, which is at the heart of our inquiry.

by Moulin (2009), Guo and Conitzer (2014) and de Clippel et al. (2014), who showed that they can outperform VCG mechanisms under the maxmin criterion. When the number of agents is large, the mechanisms from this literature are nearly optimal (in comparison to first-best); however, in the two-agent one-unit setting that we consider they tend to be trivial and provide only trivial guarantees.

# 2 Model and Main Results

We consider a model in which an indivisible private good is to be allocated between two agents. We refer to agent $i$'s valuation of the good, $\theta_i$, as his type, where $i = 1, 2$. We assume that each agent's type lies in a bounded positive interval, and without loss of generality, we normalize it as the unit interval $[0, 1]$. We also assume that agent $i$'s utility is quasi-linear in the monetary transfer—i.e., the agent's utility function is:

$$U_i(x_i, t_i; \theta_i) = \theta_i \, x_i + t_i \text{ for } i = 1, 2.$$

This says that agent $i$ obtains the good with probability $x_i$ and receives $t_i$.

An allocation mechanism, or simply a mechanism, $M = \{x_i, t_i\}_{i=1,2}$, consists of four real value functions $x_1(\theta_1, \theta_2)$, $x_2(\theta_1, \theta_2)$, $t_1(\theta_1, \theta_2)$, and $t_2(\theta_1, \theta_2)$. In this paper, we restrict our attention to deterministic mechanisms—i.e.,

$$x_i(\theta_1, \theta_2) \in \{0, 1\}, \text{ and } x_1(\theta_1, \theta_2) + x_2(\theta_1, \theta_2) = 1, \ \forall \theta_1, \theta_2.$$ [6]

A mechanism is *allocation efficient* if $x_i = 1$ whenever $\theta_i > \theta_j$, $i \neq j$. In words, the good is always given to the agent with the higher type.

Since agents' types are private information, they must be solicited. In order for a mechanism to work properly, it is important that agents are given an incentive to reveal their types truthfully. The strongest incentive property is the dominant-strategy incentive compatibility. It is required that it is always a dominant strategy for agents to reveal their true type—i.e.,

$$\begin{aligned} \theta_1 x_1(\theta_1, \theta_2) + t_1(\theta_1, \theta_2) &\geq \theta_1 x_1(\tilde{\theta}_1, \theta_2) + t_1(\tilde{\theta}_1, \theta_2), \ \forall \theta_1, \tilde{\theta}_1, \theta_2 \\ \theta_2 x_2(\theta_1, \theta_2) + t_2(\theta_1, \theta_2) &\geq \theta_2 x_1(\theta_1, \tilde{\theta}_2) + t_1(\theta_1, \tilde{\theta}_2), \ \forall \theta_1, \theta_2, \tilde{\theta}_2. \end{aligned} \tag{DIC}$$

The best known DIC mechanisms are Vickrey-Clarke-Groves mechanisms. In a VCG mechanism, allocation efficiency is always achieved through the clever choice of monetary

[6] Because of some technical difficulties, we do not allow the designer to withhold the good, so the good must be assigned to one of the agents. It is a natural condition in the bilateral trade model in which the seller originally owns the good.

transfers. However, VCG mechanisms are less specific regarding the monetary transfers to the agents. It is of less interest to us whether agents' utilities are inflated because of positive subsidies from outside. Hence, we will also impose the condition of no-deficit,

$$t_1(\theta_1, \theta_2) + t_2(\theta_1, \theta_2) \leq 0, \ \forall \theta_1, \theta_2. \tag{ND}$$

Moreover, if the equality in (ND) holds at all $(\theta_1, \theta_2)$, we say that the mechanism is *budget-balanced*. For a mechanism to achieve full efficiency, it must be both allocation efficient and budget-balanced. But Green and Laffont (1979) have already shown this cannot be true for any DIC mechanism.

In this paper, we use an average criterion to evaluate the efficiencies of various DIC mechanisms. We assign some probability distributions $F_1(\theta_1)$ and $F_2(\theta_2)$ to individual agents' types, and then we identify mechanisms that yield the highest total efficiency among all mechanisms that are (DIC) and (ND). For our full efficiency results, these distributions reflect agents' true type distributions. More generally, these distributions can reflect useful information available to the designer about agent's type distributions, that is, the designer's subjective beliefs. Here is our formal optimization problem.

For probability distributions $F_1$ and $F_2$, the average total utilities of both agents of mechanism $M$ are

$$TU(M) = \int_0^1 \int_0^1 (\theta_1 x_1(\theta_1, \theta_2) + \theta_2 x_2(\theta_1, \theta_2) + t_1(\theta_1, \theta_2) + t_2(\theta_1, \theta_2)) dF_1(\theta_1) \, dF_2(\theta_2).$$

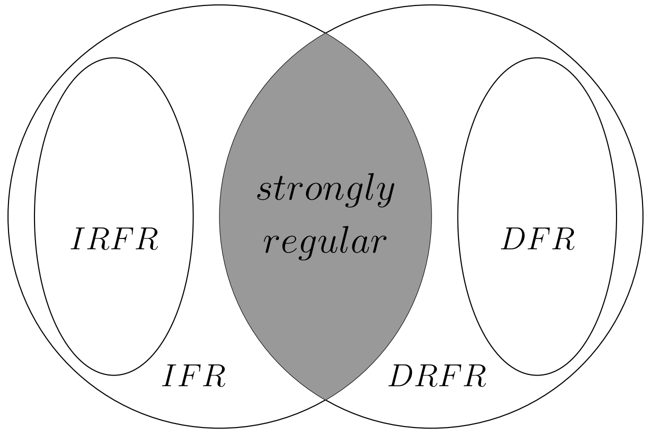
Denote the class of all feasible mechanisms that satisfy both (DIC) and (ND) by $\mathcal{M}$. Our task is to identify optimal mechanisms $M^* \in \mathcal{M}$ that yield the highest $TU$ value:

$$TU(M^*) = \max_{M \in \mathcal{M}} TU(M).$$

To find an optimal solution to the problem above, we must impose certain restrictions on $F_1$ and $F_2$. Borrowing from the existing literature, we consider the following conditions in this paper:

**IFR:** A distribution $F$ has an increasing failure rate if $f(\theta) / (1 - F(\theta))$ is increasing in $\theta$.
**DFR:** A distribution $F$ has a decreasing failure rate if $f(\theta) / (1 - F(\theta))$ is decreasing in $\theta$.
**IRFR:** A distribution function $F$ has an increasing reversed failure rate if $f(\theta) / F(\theta)$ is increasing in $\theta$.
**DRFR:** A distribution function $F$ has a decreasing reversed failure rate if $f(\theta) / F(\theta)$ is decreasing in $\theta$.

The most commonly used distributions satisfy one or two of the above conditions. While some conditions are incompatible, it can be shown that IRFR implies IFR, and DFR implies DRFR.




**Strong Regularity.** A distribution is *strongly regular* if both IFR and DRFR hold.

IFR is also known as the hazard rate condition, and DRFR means that $F$ is log-concave. The IFR and DRFR are commonly assumed in the mechanism design literature. The uniform distribution, truncated exponential distributions and truncated normal distributions are all strongly regular.[7]

Before we present our formal results, let us calculate $TU(M)$ for some well-known mechanisms. For simplicity, we carry out such calculations for the uniform distribution only.

First, the canonical pivotal mechanism (or the second-price auction mechanism) $M_{SP}$ has the total utilities $TU(M_{SP}) = \frac{1}{3}$, which is not very large. It is not even the best one among all VCG mechanisms. In a separate paper, we find the best VCG mechanism $M^*_{VCG}$ with $TU(M^*_{VCG}) = \frac{7}{12}$ (Shao and Zhou, 2008).

**Example 1.** Hagerty and Rogerson (1987) consider fixed-price mechanisms: Assuming that agent 1 is the seller and agent 2 is the buyer, a trade will take place at some fixed-price $p$ if and only if both the seller and the buyer agree. Formally, the fixed-price mechanism $M_{FP}$ with price $p$ is defined as follows (see Figure 1):

$$\begin{cases} x_1(\theta_1,\theta_2) = 0, \\ t_1(\theta_1,\theta_2) = p, \\ x_2(\theta_1,\theta_2) = 1, \\ t_2(\theta_1,\theta_2) = -p, \end{cases} \text{when } \theta_1 \le p \text{ and } \theta_2 \ge p; \text{ and } \begin{cases} x_1(\theta_1,\theta_2) = 1, \\ t_1(\theta_1,\theta_2) = 0, \\ x_2(\theta_1,\theta_2) = 0, \\ t_2(\theta_1,\theta_2) = 0, \end{cases} \text{otherwise.}$$

Among all fixed-price mechanisms, the mechanism with the price $p = \frac{1}{2}$ yields the highest

[7] A nice discussion of these conditions can be found in Bagnoli and Bergstrom (2005).

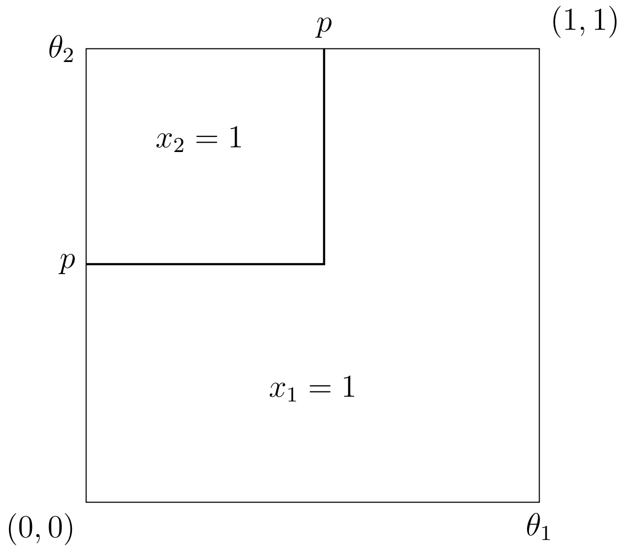



Figure 1: Allocation rule of a fixed-price mechanism

total utilities $TU(M_{FP}) = \frac{5}{8}$. (The same holds for the fixed-price mechanism in which agent 2 is the designated seller.)

**Example 2.** In this paper, we also consider another type of budget-balanced mechanisms, called option mechanisms, which are related to, but different from, fixed-price mechanisms. An option mechanism $M_O$ gives the good to agent 1 conditionally and, at the same time, issues a call option to agent 2 that allows him to buy the good from agent 1 at a fixed exercise price of $p$. Obviously, agent 2 will exercise the option if and only if $\theta_2 \geq p$. Formally, it is defined as follows (see Figure 2):

$$\begin{cases} x_1(\theta_1, \theta_2) = 0, \\ t_1(\theta_1, \theta_2) = p, \\ x_2(\theta_1 \theta_2) = 1, \\ t_2(\theta_1, \theta_2) = -p, \end{cases} \text{when } \theta_2 \geq p; \text{ and } \begin{cases} x_1(\theta_1, \theta_2) = 1, \\ t_1(\theta_1, \theta_2) = 0, \\ x_2(\theta_1, \theta_2) = 0, \\ t_2(\theta_1, \theta_2) = 0, \end{cases} \text{otherwise.}$$

Among all option mechanisms, the mechanism with the option price $p = \frac{1}{2}$ yields the highest total utilities $TU(M_O) = \frac{5}{8}$. (The same holds for the option mechanism in which agent 2 is the conditional owner of the good and agent 1 is awarded the option.)

There are two interesting observations from these examples. First, the best fixed-price mechanism and the best option mechanism yield the same level of total utilities. Assuming that agent 1 is the designated seller of the good, these two mechanisms differ only in the region $\theta_1 \geq \frac{1}{2}$ and $\theta_2 \geq \frac{1}{2}$, where both agents' types are greater than or equal to $\frac{1}{2}$. The fixed-price mechanism favors agent 1 by giving the good to agent 1 in the whole region,

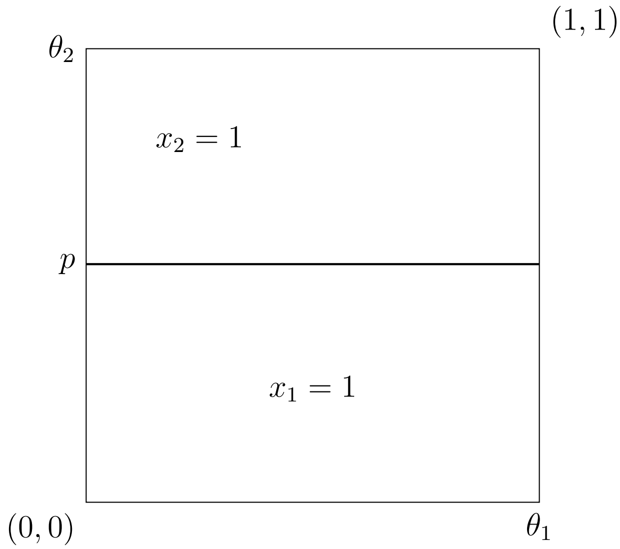



Figure 2: Allocation rule of an option mechanism

whereas the option mechanism favors agent 2. The total utilities are the same since agents' types are distributed symmetrically in these examples. Second, the numerical comparison also indicates that the budget-balanced mechanisms outperforms VCG mechanisms. In fact, our first result shows that both observations hold for more general distributions.

**Theorem 1.** When $F_1$ and $F_2$ are strongly regular, the optimal mechanisms are either fixed-price mechanisms or option mechanisms with optimally chosen prices $p^*$. Hence, optimal mechanisms must be budget-balanced. In addition, when $F_1 = F_2$, both fixed-price mechanisms and option mechanisms are optimal with the same $p^*$ equal to the mean of $F_1$.

Note that agents' distributions in the first part of Theorem 1 need not be identical. Whether option mechanisms or fixed-price mechanisms are optimal depends on probability distributions $F_1$ and $F_2$. By Theorem 1, it is sufficient to find the optimal mechanisms by restricting attention to fixed-price mechanisms and option mechanisms alone, which dramatically simplifies the actual optimization problem. We can even find closed forms of optimal mechanisms.

**Example 3.** All fixed-price mechanisms $M_{FP}$ are of the form given in Example 1. Since such mechanisms are budget-balanced, $t_1 + t_2 = 0$. When agent 1 is the designated seller,

$$\begin{aligned} TU\left(M_{FP}\right) = & \int_p^1 \int_0^p \theta_2 dF_1\left(\theta_1\right) dF_2\left(\theta_2\right) + \int_0^p \int_0^1 \theta_1 dF_1\left(\theta_1\right) dF_2\left(\theta_2\right) \\ & + \int_p^1 \int_p^1 \theta_1 dF_1\left(\theta_1\right) dF_2\left(\theta_2\right). \end{aligned}$$

Using the first order condition with respect to $p$, we can find the optimal price $p_1^*$. We denote such a mechanism as $M_{FP}(p_1^*)$. Similarly, we can solve for another candidate optimal mechanism $M_{FP}(p_2^*)$ when agent 2 is the designated seller.

**Example 4.** All option mechanisms $M_O$ are of the form given in Example 2. When agent 1 is the designated seller and agent 2 is given the option,

$$TU(M_O) = \int_0^p \int_0^1 \theta_1 dF_1(\theta_1)\, dF_2(\theta_2) + \int_p^1 \int_0^1 \theta_2 dF_1(\theta_1)\, dF_2(\theta_2).$$

Using the first order condition with respect to $p$, the optimal price $p_1^* = \mu_1$, in which $\mu_1$ is the mean of $F_1$. The mechanism is denoted as $M_O(p_1^*)$. Similarly, we can find the last candidate optimal option mechanism $M_O(p_2^*)$ when agent 2 is the designated seller with $p_2^* = \mu_2$.

By choosing from among the four candidate mechanisms those that yield the highest $TU$ value, we can identify the optimal mechanism(s). It is clear that if $F_1 = F_2$, all four mechanisms have the same total utilities and $p^* = \mu_1 = \mu_2$.

Theorem 1 highlights the importance of budget-balancedness for optimality. While both misallocation and money outflows are sources of inefficiency for a general mechanism, it is imperative for optimal mechanisms to eliminate money outflows completely. Consequently, whenever distributions are strongly regular, VCG mechanisms can never be optimal.

When $F_1$ and $F_2$ are not strongly regular, optimal mechanisms may no longer be budget-balanced. In the next two theorems, we obtain optimal mechanisms when probability distributions satisfy other conditions. It turns out that optimal mechanisms are neither VCG mechanisms nor budget-balanced mechanisms.

**Theorem 2.** Suppose that agents are ex ante identical, $F_1 = F_2 = F$, and IRFR holds for $F$; an optimal mechanism is

$$x_1(\theta_1, \theta_2) = \begin{cases} 1 & \text{if } \theta_1 > \theta_2 \text{ and } (\theta_1, \theta_2) \in [0,1] \times [0, c^*) \\ 0 & \text{otherwise} \end{cases}$$

$$t_i(\theta_1, \theta_2) = -\frac{c^*}{2} \text{ if } x_i(\theta_1, \theta_2) = 1$$

$$t_i(\theta_1, \theta_2) = \begin{cases} \theta_j - \frac{c^*}{2} & \text{for } \theta_j \in [0, c^*] \\ \frac{c^*}{2} & \text{for } \theta_j \in [c^*, 1] \end{cases} \quad i \neq j, \text{ if } x_i(\theta_1, \theta_2) = 0,$$

in which $c^* \in (0,1)$ is determined optimally. Another optimal mechanism is obtained by switching the roles of agents.

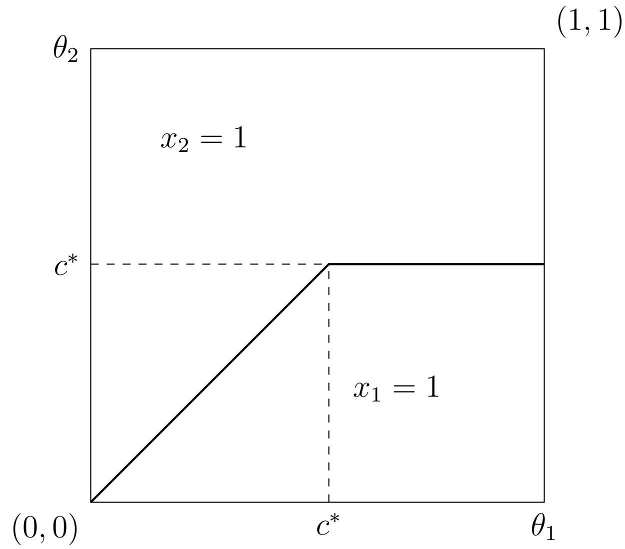



Figure 3: Allocation rule of the optimal mechanism when $F$ is IRFR

**Theorem 3.** Suppose that agents are ex ante identical, $F_1 = F_2 = F$, and DFR holds for $F$; an optimal mechanism is

$$x_1(\theta_1, \theta_2) = \begin{cases} 1 & \text{if } \theta_1 > \theta_2 \text{ and } (\theta_1, \theta_2) \in [d^*, 1] \times [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

$$t_i(\theta_1, \theta_2) = \begin{cases} -\frac{d^*}{2} & \text{for } \theta_j \in [0, d^*] \\ \frac{d^*}{2} - \theta_j & \text{for } \theta_j \in [d^*, 1] \end{cases} \quad i \neq j, \text{ if } x_i(\theta_1, \theta_2) = 1,$$

$$t_i(\theta_1, \theta_2) = \frac{d^*}{2} \text{ if } x_i(\theta_1, \theta_2) = 0,$$

in which $d^* \in (0, 1)$ is determined optimally. Another optimal mechanism is obtained by switching the roles of agents.

The allocation rules of the optimal mechanisms in Theorems 2 and 3 are illustrated in Figures 3 and 4. They are hybrids of VCG and budget-balanced mechanisms. However, at each type profile, either allocation efficiency or budget-balancedness is achieved. When IRFR holds, the good is allocated efficiently in all regions except where $\theta_1 \geq \theta_2$, and $\theta_2 \geq c^*$. In that region, we have $t_1(\theta_1, \theta_2) + t_2(\theta_1, \theta_2) = 0$. When DFR holds, the situation is similar.

Although the possibility that optimal mechanisms are neither VCG nor budget-balanced mechanisms has been illustrated by Schwartz and Wen (2012), Miller (2012), and Drexl and Kleiner (2012) through some numerical examples, Theorems 2 and 3 are the first general results that derive closed-form optimal mechanisms for irregular probability distributions.

Finally, let us discuss the individual rationality condition. We do not require that feasible

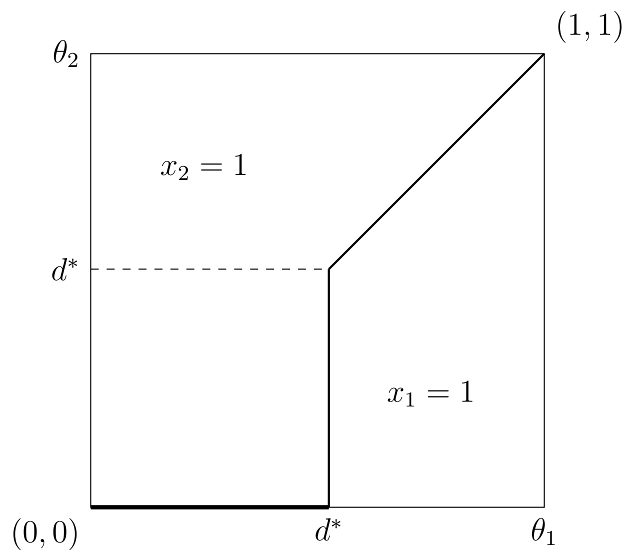



Figure 4: Allocation rule of the optimal mechanism when $F$ is DFR

mechanisms should satisfy (IR). Drexl and Kleiner (2015) study a variation of our model in which they also impose (IR) in addition to (DIC) and (ND). Hence, the class of feasible mechanisms is smaller in their paper than in ours. For the bilateral bargaining model, they show that any optimal mechanism for all mechanisms that satisfy (DIC), (ND), and (IR) must be budget-balanced. The imposition of (IR) allows them to reach their conclusion without any assumption on agents' type distributions. For our larger class of mechanisms, whether optimal mechanisms are budget-balanced depends crucially on agents' type distributions. When type distributions satisfy both IFR and DRFR, we find that the optimal mechanism for all mechanisms satisfying (DIC) and (ND) must be budget-balanced (Theorem 1). However, if type distributions do not satisfy DRFR, the conclusion no longer holds. Theorem 2 presents such a case in which type distributions are IRFR; the resulting optimal mechanism is no longer budget-balanced (even though IFR still holds, as IRFR implies IFR). Theorem 3 presents another case. Hence, with more information on underlying type distributions, we derive optimal mechanisms for a larger class of feasible mechanisms than those considered in Drexl and Kleiner (2015). For some distributions, the optimal mechanisms may be budget-balanced or even satisfy (IR), and for some distributions, they are not. In the latter case without (IR), our optimal mechanisms achieve higher efficiencies. When (IR) is imposed, no such distinction exists.

The general technical difficulty to further generalize our result to more than two agents, is the same as that is encountered in the *optimal transport* problem (a.k.a. "Monge-Kantorovich problem"),[8] which is well known in the mathematical programming literature. In fact, our

[8]The reader is referred to Villani (2008) for an introduction to this literature.

optimal mechanism problem is a specific variation of the optimal transport problem. As there is no general technique for solving this problem directly, we adopt an indirect approach. We first establish an upper bound for $TU$ values of all mechanisms that satisfy (DIC) and (ND). Then, we construct mechanisms, for which the $TU$ values can achieve this upper bound. These mechanisms, by construction, must be optimal. The details of our proofs are presented in the appendix.

# 3 Conclusion

The main contribution of this paper is the development of a general framework that can be used to evaluate the efficiency of dominant-strategy incentive compatible mechanisms. In earlier work, many authors focus their attention on VCG mechanisms whenever dominant-strategy incentive compatible mechanisms are concerned. While other authors study fixed-price mechanisms in the bilateral bargaining literature, they cannot relate their work to VCG mechanisms, as they usually impose budget-balancedness a priori. Since we soften the budget-balanced condition to the no-deficit condition, we allow for all sensible dominant-strategy incentive compatible mechanisms.

We have identified optimal mechanisms under alternative assumptions of the underlying probability distributions. While two sources might have contributed to the inefficiency of a dominant-strategy incentive compatible mechanism—misallocation of the good and money outflows necessary to induce truth-telling behavior—they hardly mingle with each other in any optimal mechanism. When probability distributions are strongly regular, optimal mechanisms are always budget-balanced. Although the results are not as striking in two other cases, it is still true that misallocation and money outflow do not co-exist at any profile for an optimal mechanism. Optimality entails budget-balancedness at all profiles (in the strongly regular case) or over a substantial region (in other cases). We conjecture that budget-balancedness still holds as long as the good is not efficiently allocated when probability distributions are non-degenerate.

The model becomes more complicated when non-deterministic mechanisms are also included. With the uniform distribution, we demonstrate that optimal mechanisms must be mixtures of fixed-price mechanisms and option mechanisms (Shao and Zhou, 2007). Although we believe this result should hold with identical strongly regular probability distribution, this remains an open question for further research.

# Appendix

Before we prove Theorems 1 to 3 separately, we first derive two common lemmas that will be used in all proofs. The first lemma is a detailed characterization of mechanisms that satisfy both (DIC) and (ND). The second lemma is an inequality that facilitates us in finding upper bounds of $TU$ values of optimal mechanisms. Throughout the appendix, whenever we see expressions involving both $i$ and $j$, it is always assumed that $i, j \in \{1, 2\}$ and $i \neq j$.

**Lemma A1. (i)** For any deterministic mechanism $M = \{x_i, t_i\}$, $M$ satisfies (DIC) if and only if both allocation rule $x_i(\theta_1, \theta_2)$ is (weakly) increasing in $\theta_i$, $i = 1, 2$, and transfers are given by

$$t_1(\theta_1, \theta_2) = -\theta_1 x_1(\theta_1, \theta_2) + \int_0^{\theta_1} x_1(\alpha, \theta_2) d\alpha + h_1(\theta_2), \tag{1}$$

$$t_2(\theta_1, \theta_2) = -\theta_2 x_2(\theta_1, \theta_2) + \int_0^{\theta_2} x_2(\theta_1, \beta) d\beta + h_2(\theta_1),$$

in which $h_i(\theta_j)$ is an arbitrary function of $\theta_j$.

**(ii)** Define

$$\phi_1(\theta_2) = \inf\{\alpha | x_1(\alpha, \theta_2) = 1\}, \text{ and}$$
$$\phi_2(\theta_1) = \inf\{\beta | x_2(\theta_1, \beta) = 1\}.$$

For any $M$ satisfying (DIC), $\phi_2(\theta_1)$ and $\phi_1(\theta_2)$ are increasing functions. Then, (ND) can be re-written as

$$h_1(\theta_2) + h_2(\theta_1) \leq \begin{cases} \phi_1(\theta_2) & \text{if } x_1(\theta_1, \theta_2) = 1 \\ \phi_2(\theta_1) & \text{if } x_2(\theta_1, \theta_2) = 1 \end{cases}, \text{ and equivalently} \tag{ND'}$$

$$h_1(\theta_2) + h_2(\theta_1) \leq \theta_1 x_1(\theta_1, \theta_2) + \theta_2 x_2(\theta_1, \theta_2) - \int_0^{\theta_1} x_1(\alpha, \theta_2) d\alpha - \int_0^{\theta_2} x_2(\theta_1, \beta) d\beta. \tag{ND''}$$

**Proof.(i)** This can be proved using the standard technique as in Myerson (1981). The first two terms of the right-hand side of (1) are the generalized pivotal-taxes. The third term $h_i(\theta_j)$ specifies rebates to agent $i$: $h_i(\theta_j)$ is the amount of money agent $i$ receives when agent $j$'s type is $\theta_j$. Since $h_i(\theta_j)$ is independent of agent $i$'s own type, $h_i$ does not affect $i$'s truth-telling behavior. Moreover, since $x_1 + x_2 = 1$, $x_i$ is decreasing in $\theta_j$.

(**ii**) Using (1), we can rewrite (ND) as (ND"). By definition of $\phi_i(\theta_j)$, we have

$$\theta_1 x_1(\theta_1,\theta_2) - \int_0^{\theta_1} x_1(\alpha,\theta_2)d\alpha = \begin{cases} \phi_1(\theta_2) & \text{if } x_1(\theta_1,\theta_2)=1 \\ 0 & \text{if } x_2(\theta_1,\theta_2)=1 \end{cases}, \text{ and}$$

$$\theta_2 x_2(\theta_1,\theta_2) - \int_0^{\theta_2} x_2(\theta_1,\beta)d\beta = \begin{cases} 0 & \text{if } x_1(\theta_1,\theta_2)=1 \\ \phi_2(\theta_1) & \text{if } x_2(\theta_1,\theta_2)=1 \end{cases}.$$

Since $x_i$ is decreasing in $\theta_j$, then function $\phi_i$ is increasing in $\theta_j$. And, (ND) can be rewritten as,

$$h_1(\theta_2) + h_2(\theta_1) \leq \begin{cases} \phi_1(\theta_2) & \text{if } x_1(\theta_1,\theta_2)=1 \\ \phi_2(\theta_1) & \text{if } x_2(\theta_1,\theta_2)=1 \end{cases}. \tag{ND'}$$

Note that the right-hand side of (ND') is always between zero and one. □

Geometrically, the type space $[0,1]\times[0,1]$ is divided into two regions $\{x_1(\theta_1,\theta_2)=1\}$ and $\{x_2(\theta_1,\theta_2)=1\}$. The union of the graphs of $\phi_1(\theta_2)$ and $\phi_2(\theta_1)$ forms the boundary between these two regions. (See Figure 5.)

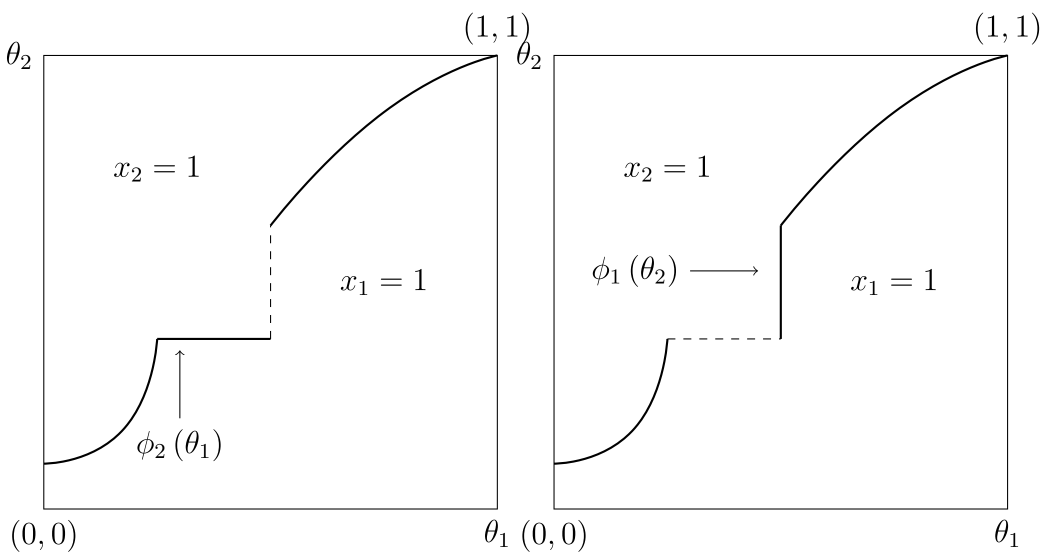



Figure 5: Boundary defined by the allocation rule

Given Lemma A1, we can reformulate the optimal mechanism design problem

$$\max_{M\in\mathcal{M}} \int_0^1 \int_0^1 (\theta_1 x_1(\theta_1,\theta_2) + \theta_2 x_2(\theta_1,\theta_2) + t_1(\theta_1,\theta_2) + t_2(\theta_1,\theta_2))dF_1(\theta_1)\,dF_2(\theta_2). \tag{P}$$

It now becomes

$$\max_{\{x_1,x_2,h_1,h_2\}} \int_0^1 \int_0^1 \left( \frac{1-F_1(\theta_1)}{f_1(\theta_1)} x_1(\theta_1,\theta_2) + \frac{1-F_2(\theta_2)}{f_2(\theta_2)} x_2(\theta_1,\theta_2) \right) dF_1(\theta_1)\, dF_2(\theta_2) \qquad \text{(P')}$$
$$+ \int_0^1 h_1(\theta_2) dF_2(\theta_2) + \int_0^1 h_2(\theta_1) dF_1(\theta_1)$$

s.t. (ND') and $x_i(\theta_1,\theta_2)$ is (weakly) increasing in $\theta_i$ for $i = 1,2$.

In this reformulation, it is clear that a major task is to estimate integrals of "rebate" functions $h_i$. Even though $h_i$ can be any functions from the incentive perspective, we need assume the integrability of $h_i$ so that (P') is well-posed.

**Lemma A2 (Ironing).** Assume that $A_1(\theta)$ and $A_2(\theta)$ are decreasing, defined on an arbitrary interval $\left[\underline{\theta}, \overline{\theta}\right] \subseteq [0,1]$. Consider any functions $x_1(\theta_1,\theta_2)$ and $x_2(\theta_1,\theta_2)$, where $x_i(\theta_1,\theta_2)$ is increasing in $\theta_i$ for $i=1,2$ and $x_1 + x_2 = 1$. Then,

$$\int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} \left(A_1(\theta_1) x_1(\theta_1,\theta_2) + A_2(\theta_2) x_2(\theta_1,\theta_2)\right) dF_1(\theta_1)\, dF_2(\theta_2)$$
$$\leq \max\left\{ \left(F_2(\overline{\theta}) - F_2(\underline{\theta})\right) \int_{\underline{\theta}}^{\overline{\theta}} A_1(\theta_1)\, dF_1(\theta_1), \left(F_1(\overline{\theta}) - F_1(\underline{\theta})\right) \int_{\underline{\theta}}^{\overline{\theta}} A_2(\theta_2)\, dF_2(\theta_2) \right\}.$$

The maximum can be achieved by letting either $x_1(\theta_1,\theta_2)$ or $x_2(\theta_1,\theta_2)$ be 1 for all $(\theta_1,\theta_2) \in \left[\underline{\theta}, \overline{\theta}\right]^2$. When $F_1(\theta) = F_2(\theta)$, the maximum can be achieved by any constant $x_1, x_2$ with $x_1 + x_2 = 1$.

**Proof.** Since $x_1(\theta_1,\theta_2)$ is increasing in $\theta_1$ and $A_1(\theta_1)$ is decreasing in $\theta_1$, fix $\theta_2$, by Chebyshev inequality[9],

$$\int_{\underline{\theta}}^{\overline{\theta}} A_1(\theta_1) x_1(\theta_1,\theta_2)\, dF_1(\theta_1) \leq \frac{1}{F_1(\overline{\theta}) - F_1(\underline{\theta})} \int_{\underline{\theta}}^{\overline{\theta}} A_1(\theta_1)\, dF_1(\theta_1) \int_{\underline{\theta}}^{\overline{\theta}} x_1(\theta_1,\theta_2)\, dF_1(\theta_1).$$

Similarly,

$$\int_{\underline{\theta}}^{\overline{\theta}} A_2(\theta_2) x_2(\theta_1,\theta_2)\, dF_2(\theta_2) \leq \frac{1}{F_2(\overline{\theta}) - F_2(\underline{\theta})} \int_{\underline{\theta}}^{\overline{\theta}} A_2(\theta_2)\, dF_2(\theta_2) \int_{\underline{\theta}}^{\overline{\theta}} x_2(\theta_1,\theta_2)\, dF_2(\theta_2).$$

[9]For increasing functions $m(x), n(y)$,

$$\frac{1}{F(\overline{\theta}) - F(\underline{\theta})} \int_{\underline{\theta}}^{\overline{\theta}} m(x)\, dF(x) \int_{\underline{\theta}}^{\overline{\theta}} n(y)\, dF(y) \leq \int_{\underline{\theta}}^{\overline{\theta}} m(x)\, n(x)\, dF(x).$$

Hence,

$$
\begin{aligned}
&\int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} \left(A_1(\theta_1)\, x_1(\theta_1,\theta_2) + A_2(\theta_2)\, x_2(\theta_1,\theta_2)\right) dF_1(\theta_1)\, dF_2(\theta_2) \\
&\leq \frac{1}{F_1(\overline{\theta}) - F_1(\underline{\theta})} \int_{\underline{\theta}}^{\overline{\theta}} A_1(\theta_1)\, dF_1(\theta_1) \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} x_1(\theta_1,\theta_2)\, dF_1(\theta_1)\, dF_2(\theta_2) \\
&+ \frac{1}{F_2(\overline{\theta}) - F_2(\underline{\theta})} \int_{\underline{\theta}}^{\overline{\theta}} A_2(\theta_2)\, dF_2(\theta_2) \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} x_2(\theta_1,\theta_2)\, dF_1(\theta_1)\, dF_2(\theta_2) \\
&\leq \max \left\{ \left(F_2(\overline{\theta}) - F_2(\underline{\theta})\right) \int_{\underline{\theta}}^{\overline{\theta}} A_1(\theta_1)\, dF_1(\theta_1),\, \left(F_1(\overline{\theta}) - F_1(\underline{\theta})\right) \int_{\underline{\theta}}^{\overline{\theta}} A_2(\theta_2)\, dF_2(\theta_2) \right\}.
\end{aligned}
$$

The last inequality is obtained by assigning $x_i = 1$ on $\left[\underline{\theta}, \overline{\theta}\right]^2$ for $i$ that is associated with the larger value of $\frac{1}{F_1(\overline{\theta}) - F_1(\underline{\theta})} \int_{\underline{\theta}}^{\overline{\theta}} A_1(\theta_1)\, dF_1(\theta_1)$ and $\frac{1}{F_2(\overline{\theta}) - F_2(\underline{\theta})} \int_{\underline{\theta}}^{\overline{\theta}} A_2(\theta_2)\, dF_2(\theta_2)$. $\square$

Similar results can also be found in Hartline and Lucier (2015).

**Proof of Theorem 1.** We already explained the basic idea of our proofs in Section 2. Conceptually, the proof consists of two steps. While the first step is to find an upper bound for $TU$ values of all mechanisms that satisfy (DIC) and (ND), the second step is to construct mechanisms that can achieve this bound. It is difficult to find a tight bound directly. Instead, we use a parameter $r \in [0,1]$ to index mechanisms that are potentially optimal, and we find an upper bound for $TU$ values of these $r$-indexed mechanisms for each $r \in [0,1]$. Then, an overall upper bound is obtained by optimizing over $r$.

The parameter $r$ is introduced as follows. For each mechanism $M = \{x_i, t_i\}$, we can make a constant transfer $\tau$ between agents—i.e., $t_1' = t_1 + \tau$ and $t_2' = t_2 - \tau$, with no change of incentives or total budget. Moreover, the $TU$ value remains the same. Hence, we consider only mechanisms that are normalized with $\max_{\theta_1} h_2(\theta_1) = 0$, and we use $r = \max_{\theta_2} h_1(\theta_2)$ as the parameter to index all potential optimal mechanisms. From (ND') in Lemma A1, $r \leq 1$. Also, any mechanism with $r < 0$ is never optimal, as a superior mechanism can be obtained by increasing the value of $h_1$ over all types (albeit with a different $r'$). Thus, we focus on $r \in [0,1]$ for the optimization exercise.

Let us demonstrate another property that any optimal mechanism must satisfy.

From (ND'), $\sup h_i(\theta_j)$ exist for both $i = 1, 2$, as both $h_i(\theta_j)$ are bounded from above. In the following proofs, we will assume that $\max h_i(\theta_j)$ exists for both $i = 1, 2$, in other words, $\sup h_i(\theta_j)$ can be achieved. (This assumption is made mainly for simplicity of exposition. The main argument is to show the existence of the upper bound of total utility of any given mechanisms. For the optimal mechanisms, $\sup h_i(\theta_j)$ always exists. If $\sup h_i(\theta_j)$ cannot be achieved, we can replace it with $\sup h_i(\theta_j) - \varepsilon$, which can be achieved. We then can carry out the subsequent proof with the same argument.) For any optimal mechanisms, the total efficiency subtracting, at most, $\varepsilon$ is bounded from above by the same upper bound. Hence, it is the upper bound for any mechanisms.For any mechanism $M$ with $\max_{\theta_1} h_2(\theta_1) = 0$ and $\max_{\theta_2} h_1(\theta_2) = r$, let

$$\theta_1' = \underset{\theta_1 \in [0,1]}{\operatorname{argmax}} h_2(\theta_1) \text{ and } \theta_2' \in \underset{\theta_2 \in [0,1]}{\operatorname{argmax}} h_1(\theta_2).$$

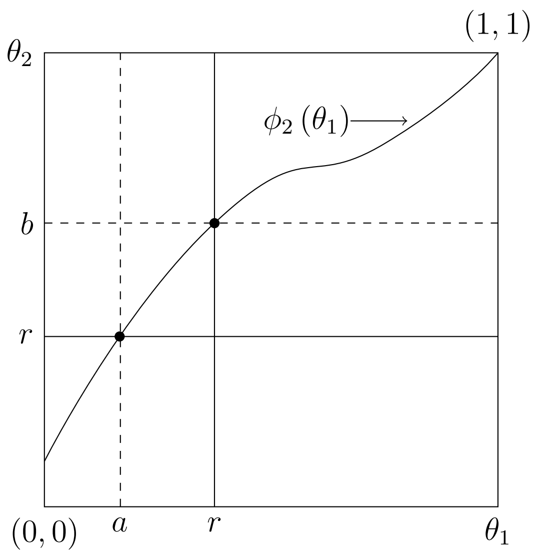



Figure 6: Definition of $a, b$

Then, $h_1(\theta_2') + h_2(\theta_1') = \max_{(\theta_1,\theta_2)} [h_1(\theta_2) + h_2(\theta_1)] = r$. We want to show that if $M$ is an optimal mechanism, then there exists some pair of maximizers $(\theta_1', \theta_2')$ with $\theta_1' \geq r$ and $\theta_2' \geq r$.

Without loss of generality, we assume that $(r, r)$ lies below the curve $\theta_2 = \phi_2(\theta_1)$ in the type space. (If $(r, r)$ lies above the curve $\theta_2 = \phi_2(\theta_1)$, then we may switch the identities of agents. In this case, $(r, r)$ lies below $\theta_1 = \phi_1(\theta_2)$ and the same proof applies.) Denote $a = \sup\{\alpha | \phi_2(\alpha) < r\}$ and $b = \sup\{\beta | \phi_1(\beta) < r\}$. Then, $a \leq r$ and $b \geq r$, as illustrated in Figure 6.

According to (ND'), $(\theta_1', \theta_2')$ cannot belong to $[0, 1] \times [0, r)$ or $[0, a) \times [r, 1]$ as $h_1(\theta_2) + h_2(\theta_1) \leq \phi_i(\theta_j) < r$.

If $(\theta_1', \theta_2') \in [a, r] \times [r, 1]$, then $\theta_2' > \phi_2(\theta_1')$. Otherwise, $h_2(\theta_1') + h_1(\theta_2') \leq \phi_1(\theta_2') < r$.

We now construct a new mechanism that yields a strictly larger $TU$ value than the original mechanism. We reset the allocation rule by letting $x_2'(\theta_1, \theta_2) = 1$ on $[0, r) \times [r, 1]$ (see Figure 7). With the modification, we have

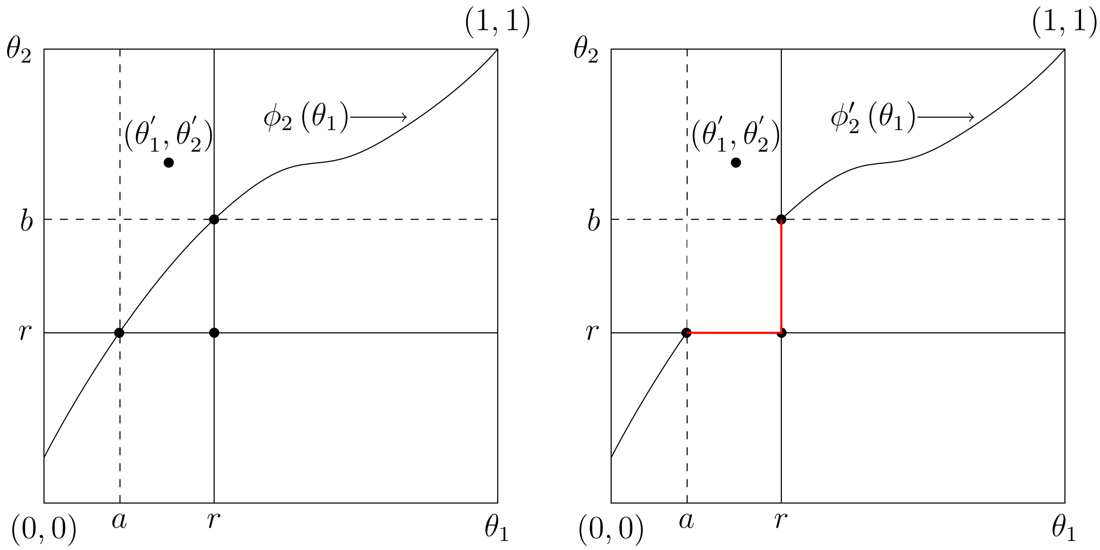



Figure 7: Local improvement

$$\phi_2'(\theta_1) = \begin{cases} r & \text{if } \theta_1 \in [a, r) \\ \phi_2(\theta_1) & \text{otherwise} \end{cases}$$

$$\phi_1'(\theta_2) = \begin{cases} r & \text{if } \theta_2 \in [r, b) \\ \phi_1(\theta_2) & \text{otherwise} \end{cases}$$

We also change the values of $h_2(\theta_1)$ to $h_2'(\theta_1) = 0$ for $\theta_1 \in [a, 1]$, and $h_1(\theta_2)$ to $h_1'(\theta_2) = r$ for $\theta_2 \in [r, 1]$. That is, we increase $h_2$ and $h_1$ to their maximum values on $[a, 1]$ and $[r, 1]$. Clearly, the new mechanism increases the allocation efficiency. However, does it always yield a higher aggregate transfer—$(t_1 + t_2)$? Since the value of $h_1 + h_2$ always increases, and

$$t_1 + t_2 = \begin{cases} h_1 + h_2 - \phi_1 & \text{if } x_1 = 1 \\ h_1 + h_2 - \phi_2 & \text{if } x_2 = 1 \end{cases},$$

we need to check where either $x_i$ changes or $x_i$ does not change, but $\phi_i$ increases. These happen only in the regions $[a, 1] \times [r, b]$, where the budget is balanced for the new mechanism. Hence, the new mechanism always yields higher aggregate transfers. It's not hard to verify that the new mechanism satisfies (ND'). Hence, it is, indeed, an improvement of the original mechanism. So we have shown: if some $r$-indexed mechanism is optimal, there must exist a pair of maximizers $(\theta_1', \theta_2')$ with $\theta_1' \geq r$ and $\theta_2' \geq r$.

Now, we can show that

$$h_2(\theta_1) \leq -\int_0^r x_2(\theta_1, \beta)\, d\beta \tag{2}$$

$$h_1(\theta_2) \leq r - \int_0^r x_1(\alpha, \theta_2)\, d\alpha \tag{3}$$

For any $\theta_1 \in [0, a]$, we pick $\theta_2 = \theta_2'$. Then, $x_1(\alpha, \theta_2') = 0$ for all $\alpha \in [0, a)$, and $x_2(\alpha, \beta) = 1$ on $[0, a] \times [r, \theta_2']$ (see Figure 8). (ND") implies

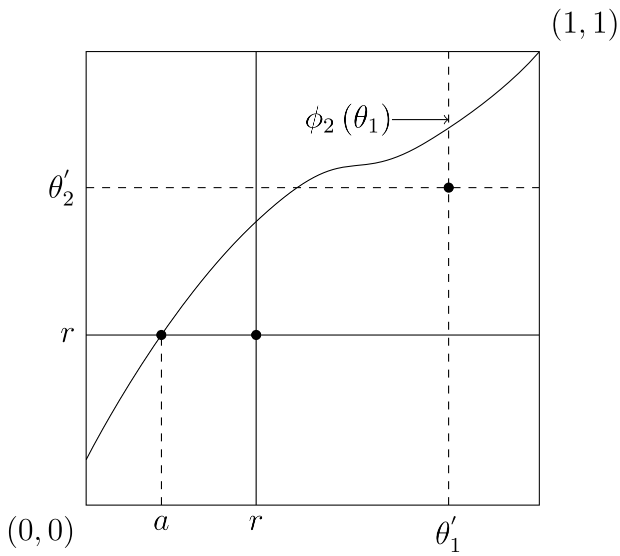



Figure 8: Illustration of $x_1(\alpha, \theta_2')$ with $\alpha \in [0, a)$, and $x_2(\alpha, \beta)$ on $[0, a] \times [r, \theta_2']$

$$\begin{aligned}
h_2(\theta_1) &\leq \theta_1 x_1(\theta_1, \theta_2') + \theta_2' x_2(\theta_1, \theta_2') - \int_0^{\theta_1} x_1(\alpha, \theta_2')\, d\alpha - \int_0^{\theta_2'} x_2(\theta_1, \beta)\, d\beta - h_1(\theta_2') \\
&= 0 + \theta_2' - 0 - (\theta_2' - r) - \int_0^r x_2(\theta_1, \beta)\, d\beta - r \\
&= -\int_0^r x_2(\theta_1, \beta)\, d\beta.
\end{aligned}$$

For any $\theta_1 \in (a, 1]$, $h_2(\theta_1) \leq 0$ by normalization, and $x_2(\theta_1, \beta) = 0$ for all $\beta \in [0, r]$. Therefore,

$$\begin{aligned}
h_2(\theta_1) &\leq 0 \\
&= -\int_0^r x_2(\theta_1, \beta)\, d\beta.
\end{aligned}$$

Similarly, we can prove

$$h_1(\theta_2) \leq r - \int_0^r x_1(\alpha, \theta_2)\, d\alpha.$$

We now return to the optimization problem (P')

$$\max_{\{x_1,x_2,h_1,h_2\}} \int_0^1 \int_0^1 \left( \frac{1-F_1(\theta_1)}{f_1(\theta_1)} x_1(\theta_1,\theta_2) + \frac{1-F_2(\theta_2)}{f_2(\theta_2)} x_2(\theta_1,\theta_2) \right) dF_1(\theta_1)\, dF_2(\theta_2) \qquad \text{(P')}$$
$$+ \int_0^1 h_1(\theta_2) dF_2(\theta_2) + \int_0^1 h_2(\theta_1) dF_1(\theta_1)$$

s.t. (ND') and $x_i(\theta_1, \theta_2)$ is (weakly) increasing in $\theta_i$ for $i = 1, 2$.

Using (2) and (3), we can estimate the mean value of agents' total utilities

$$\begin{aligned}
TU(M) \leq & \int_0^r \int_0^r \left( \frac{-F_1(\theta_1)}{f_1(\theta_1)} x_1(\theta_1,\theta_2) + \frac{-F_2(\theta_2)}{f_2(\theta_2)} x_2(\theta_1,\theta_2) \right) dF_1(\theta_1)\, dF_2(\theta_2) \\
& + \int_r^1 \int_r^1 \left( \frac{1-F_1(\theta_1)}{f_1(\theta_1)} x_1(\theta_1,\theta_2) + \frac{1-F_2(\theta_2)}{f_2(\theta_2)} x_2(\theta_1,\theta_2) \right) dF_1(\theta_1)\, dF_2(\theta_2) \\
& + \int_r^1 \int_0^r \left( \frac{-F_1(\theta_1)}{f_1(\theta_1)} x_1(\theta_1,\theta_2) + \frac{1-F_2(\theta_2)}{f_2(\theta_2)} x_2(\theta_1,\theta_2) \right) dF_1(\theta_1)\, dF_2(\theta_2) \\
& + \int_0^r \int_r^1 \left( \frac{1-F_1(\theta_1)}{f_1(\theta_1)} x_1(\theta_1,\theta_2) + \frac{-F_2(\theta_2)}{f_2(\theta_2)} x_2(\theta_1,\theta_2) \right) dF_1(\theta_1)\, dF_2(\theta_2) \\
& + r.
\end{aligned}$$

We can apply Lemma A2 to optimize the first two terms of the right-hand side,

$$\begin{aligned}
& \int_0^r \int_0^r \left( \frac{-F_1(\theta_1)}{f_1(\theta_1)} x_1(\theta_1,\theta_2) + \frac{-F_2(\theta_2)}{f_2(\theta_2)} x_2(\theta_1,\theta_2) \right) dF_1(\theta_1)\, dF_2(\theta_2) \\
& + \int_r^1 \int_r^1 \left( \frac{1-F_1(\theta_1)}{f_1(\theta_1)} x_1(\theta_1,\theta_2) + \frac{1-F_2(\theta_2)}{f_2(\theta_2)} x_2(\theta_1,\theta_2) \right) dF_1(\theta_1)\, dF_2(\theta_2) \\
& \leq \max\left\{ F_2(r) \int_0^r \frac{-F_1(\theta_1)}{f_1(\theta_1)} dF_1(\theta_1), F_1(r) \int_0^r \frac{-F_2(\theta_2)}{f_2(\theta_2)} dF_2(\theta_2) \right\} \\
& + \max\left\{ (1-F_2(r)) \int_r^1 \frac{1-F_1(\theta_1)}{f_1(\theta_1)} dF_1(\theta_1), (1-F_1(r)) \int_r^1 \frac{1-F_2(\theta_2)}{f_2(\theta_2)} dF_2(\theta_2) \right\}.
\end{aligned}$$

The third term is maximized by letting $x_2 = 1$, and the fourth term is maximized by letting

$x_1 = 1$—i.e.,

$$\begin{aligned}
&\int_r^1 \int_0^r \left( \frac{-F_1(\theta_1)}{f_1(\theta_1)} x_1(\theta_1, \theta_2) + \frac{1 - F_2(\theta_2)}{f_2(\theta_2)} x_2(\theta_1, \theta_2) \right) dF_1(\theta_1)\, dF_2(\theta_2) \\
&+ \int_r^1 \int_0^r \left( \frac{1 - F_1(\theta_1)}{f_1(\theta_1)} x_1(\theta_1, \theta_2) + \frac{-F_2(\theta_2)}{f_2(\theta_2)} x_2(\theta_1, \theta_2) \right) dF_1(\theta_1)\, dF_2(\theta_2) \\
&\leq F_1(r) \int_r^1 \frac{1 - F_2(\theta_2)}{f_2(\theta_2)} dF_2(\theta_2) + F_2(r) \int_r^1 \frac{1 - F_1(\theta_1)}{f_1(\theta_1)} dF_1(\theta_1).
\end{aligned}$$

Hence, the upper bound of $TU(M)$ for $r$-indexed potential optimal mechanisms is

$$\begin{aligned}
&\max \left\{ F_2(r) \int_0^r \frac{-F_1(\theta_1)}{f_1(\theta_1)} dF_1(\theta_1), F_1(r) \int_0^r \frac{-F_2(\theta_2)}{f_2(\theta_2)} dF_2(\theta_2) \right\} \\
&+ \max \left\{ (1 - F_2(r)) \int_r^1 \frac{1 - F_1(\theta_1)}{f_1(\theta_1)} dF_1(\theta_1), (1 - F_1(r)) \int_r^1 \frac{1 - F_2(\theta_2)}{f_2(\theta_2)} dF_2(\theta_2) \right\} \\
&+ F_1(r) \int_r^1 \frac{1 - F_2(\theta_2)}{f_2(\theta_2)} dF_2(\theta_2) + F_2(r) \int_r^1 \frac{1 - F_1(\theta_1)}{f_1(\theta_1)} dF_1(\theta_1) + r.
\end{aligned}$$

This upper bound is achieved by $x_1 = 1$ on $[r, 1] \times [0, r]$, $x_2 = 1$ on $[0, r] \times [r, 1]$, and either $x_1 = 1$ or $x_2 = 1$ on $[0, r]^2$ and $[r, 1]^2$. While values of $x_1$ and $x_2$ are definitive on the off-diagonal regions, there are four possible combinations for values of $x_1$ and $x_2$ on the two diagonal regions. In each case, we can find corresponding transfers that, together with $x_1$ and $x_2$, form a mechanism that satisfies both (DIC) and (ND).

Consider the case in which $x_1 = 1$ on both $[0, r]^2$ and $[r, 1]^2$ . If we let $t_1 = -t_2 = r$ when $x_2 = 1$, and $t_1 = t_2 = 0$ when $x_1 = 1$, we obtain the fixed-price mechanism with agent 1 being the seller (see, also, Example 1 with $p = r$). Consider another case, in which $x_1 = 1$ on $[0, r]^2$ and $x_2 = 1$ on $[r, 1]^2$. If we let $t_1 = -t_2 = r$ when $x_2 = 1$ and $t_1 = t_2 = 0$ when $x_1 = 1$, we obtain the option mechanism with agent 1 being the seller (see, also, Example 2 with $p = r$). The other two cases are similar.

When we choose one of the cases that achieves the upper bound, either a fixed-price mechanism or an option mechanism, we obtain the (conditional) optimal mechanism among all mechanisms that are indexed by $r$.

When we finally choose the best among all conditional optimal mechanisms, we have the overall optimal mechanism for (P').

A simple calculation shows that when $F_1 = F_2 = F$, for all index $r$, all four mechanisms yield the same total utilities, and the overall optimality is achieved at $r = \mu$, in which $\mu$ is the mean of $F$. □

**Proof of Theorem 2.** Although Theorem 2 is about deterministic mechanisms, for technical reasons, we work with some non-deterministic mechanisms with $x_i \in \left\{0, \frac{1}{2}, 1\right\}$. A mechanism $M = \{x_i, t_i\}$ is symmetric if $x_1(\theta_1, \theta_2) = x_2(\theta_2, \theta_1)$, and $t_1(\theta_1, \theta_2) = t_2(\theta_2, \theta_1)$.

Given any deterministic mechanism $M = \{x_i, t_i\}$, let

$$\begin{aligned} x_1'(\theta_1, \theta_2) &= x_2(\theta_2, \theta_1), \ x_2'(\theta_1, \theta_2) = x_1(\theta_2, \theta_1) \text{ and} \\ t_1'(\theta_1, \theta_2) &= t_2(\theta_2, \theta_1), \ t_2'(\theta_1, \theta_2) = t_1(\theta_2, \theta_1). \end{aligned}$$

The mechanism $M' = \{x_i', t_i'\}$ is also feasible and has the same $TU$ value as $M$. We construct a symmetric non-deterministic mechanism $\tilde{M} = \left\{\tilde{x}_i, \tilde{t}_i\right\}$:

$$\begin{aligned} \tilde{x}_i(\theta_1, \theta_2) &= \frac{1}{2}x_i(\theta_1, \theta_2) + \frac{1}{2}x_i'(\theta_1, \theta_2) \text{ and} \\ \tilde{t}_i(\theta_1, \theta_2) &= \frac{1}{2}t_i(\theta_1, \theta_2) + \frac{1}{2}t_i'(\theta_1, \theta_2) \text{ for } i = 1, 2. \end{aligned}$$

It is straightforward to see that $\tilde{M}$ is feasible, symmetric and $x_i \in \left\{0, \frac{1}{2}, 1\right\}$. Moreover, $TU(M) = TU\left(\tilde{M}\right)$ due to $F_1 = F_2 = F$. Hence, for any optimal mechanism for the original program, we can find another symmetric non-deterministic mechanism with $x_i \in \left\{0, \frac{1}{2}, 1\right\}$ that achieves the same $TU$ value. On the other hand, if we find the optimal mechanism for all symmetric mechanisms with $x_i \in \left\{0, \frac{1}{2}, 1\right\}$ and then find a deterministic mechanism that achieves the same $TU$ value, this deterministic mechanism must be optimal for the original problem.

The basic proof strategy is the same as above. We derive an upper bound of the objective in (P') first, and then construct a feasible deterministic mechanism, which achieves this upper bound.

Define a function $g(\theta)$ through the following equation:

$$F(g(\theta)) = 1 - F(\theta), \text{ or } g(\theta) = F^{-1}(1 - F(\theta)).$$

The function $g(\theta)$ is strictly decreasing with $g(0) = 1$, $g(1) = 0$. Let $G$ be the graph of $g(\theta)$, and it is symmetric—i.e., $g(\theta_1) = \theta_2$ if and only if $g(\theta_2) = \theta_1$. Also, $F\left(\hat{\theta}\right) = 1 - F\left(g^{-1}\left(\hat{\theta}\right)\right)$. Hence,

$$\int_0^1 h_1(g(\theta_1))\, dF(\theta_1) = \int_1^0 h_1(\theta_2)\, dF\left(g^{-1}(\theta_2)\right) = \int_0^1 h_1(\theta_2)\, dF(\theta_2).$$

Then we have

$$\int_0^1 h_1(\theta_2)\, dF(\theta_2) + \int_0^1 h_2(\theta_1)\, dF(\theta_1) = \int_0^1 \left(h_2(\theta_1) + h_1(g(\theta_1))\right) dF(\theta_1).$$

Integrating both sides of (ND”) with respect to $\theta_1$ along $G$, we have

$$\begin{aligned}
&\int_0^1 \left(h_2(\theta_1) + h_1(g(\theta_1))\right) dF(\theta_1) \\
&\leq \int_0^1 \int_0^{g(\theta_1)} \left(-\frac{1}{f(\theta_1)} x_1(\theta_1,\theta_2) - \frac{1}{f(\theta_2)} x_2(\theta_1,\theta_2)\right) dF(\theta_2)\, dF(\theta_1) \\
&+ \int_0^1 \left(\theta_1 x_1(\theta_1, g(\theta_1)) + g(\theta_1)\, x_2(\theta_1, g(\theta_1))\right) dF(\theta_1).
\end{aligned}$$

Now we derive an upper bound of (P’). For any mechanism $M$ with $x_i \in \left\{0, \frac{1}{2}, 1\right\}$,

$$\begin{aligned}
TU(M) &\leq \int_0^1 \int_0^{g(\theta_1)} \left(\frac{-F(\theta_1)}{f(\theta_1)} x_1(\theta_1,\theta_2) + \frac{-F(\theta_2)}{f(\theta_2)} x_2(\theta_1,\theta_2)\right) dF(\theta_2)\, dF(\theta_1) \qquad (4) \\
&+ \int_0^1 \int_{g(\theta_1)}^1 \left(\frac{1-F(\theta_1)}{f(\theta_1)} x_1(\theta_1,\theta_2) + \frac{1-F(\theta_2)}{f(\theta_2)} x_2(\theta_1,\theta_2)\right) dF(\theta_2)\, dF(\theta_1) \\
&+ \int_0^1 \left(\theta_1 x_1(\theta_1, g(\theta_1)) + g(\theta_1)\, x_2(\theta_1, g(\theta_1))\right) dF(\theta_1).
\end{aligned}$$

Since $x_1(\theta_1,\theta_2)$ is increasing in $\theta_1$ and decreasing in $\theta_2$, and $g(\theta)$ is decreasing in $\theta$, $x_1(\theta_1, g(\theta_1))$ is increasing in $\theta_1$. Due to $x_i \in \left\{0, \frac{1}{2}, 1\right\}$, the value of $x_1(\theta_1, g(\theta_1))$ must change from 0 to $\frac{1}{2}$ and then to 1 as $\theta_1$ changes from 0 to 1. Denote the value of $c = \inf\left\{\theta_1 | x_1(\theta_1, g(\theta_1)) = \frac{1}{2}\right\}$. We use $c$ as the parameter to index all mechanisms under consideration. By symmetry, $d = g(c) = \sup\left\{\theta_1 | x_1(\theta_1, g(\theta_1)) = \frac{1}{2}\right\}$. (See Figure 9). When $\left\{\theta_1 | x_1(\theta_1, g(\theta_1)) = \frac{1}{2}\right\}$ is empty, let $c = d$ such that $c = g(c)$. Without loss of generality, we assume $\left\{\theta_1 | x_1(\theta_1, g(\theta_1)) = \frac{1}{2}\right\}$ is non-empty in the following argument.

First, we claim that values of $x_1$ and $x_2$ along the curve $G$ are fixed. Consider $(\theta_1,\theta_2) \in [0,c) \times (d,1]$. By definition, $x_1(\theta_1,\theta_2) = 0$, $x_2(\theta_1,\theta_2) = 1$ for $(\theta_1,\theta_2)$ on the curve $\theta_2 = g(\theta_1)$. Applying monotonicity of $x_1$ to $(\theta_1,\theta_2)$ below the curve $G$ and applying the monotonicity of $x_2$ to $(\theta_1,\theta_2)$ above, we conclude that

$$x_1(\theta_1,\theta_2) = 0, x_2(\theta_1,\theta_2) = 1 \text{ for all } (\theta_1,\theta_2) \in [0,c) \times (d,1].$$

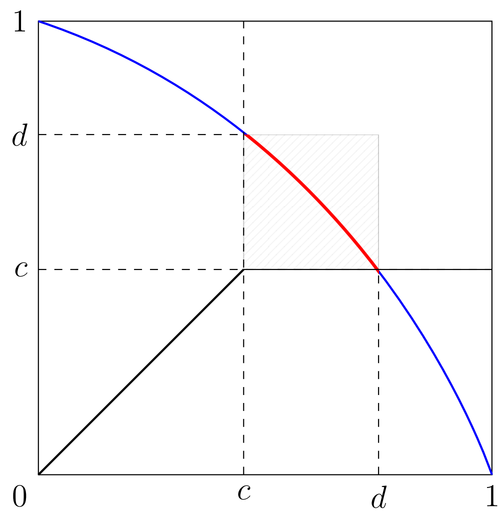


Figure 9: Estimation along off-diagonal curve $\theta_2 = g(\theta_1)$ when $F$ is IRFR

Similarly, we conclude that

$$x_1(\theta_1, \theta_2) = \frac{1}{2}, x_2(\theta_1, \theta_2) = \frac{1}{2} \text{ for all } (\theta_1, \theta_2) \in (c, d) \times (c, d), \text{ and}$$
$$x_1(\theta_1, \theta_2) = 1, x_2(\theta_1, \theta_2) = 0 \text{ for all } (\theta_1, \theta_2) \in (d, 1] \times [0, c).$$

As a consequence, the values of the integrals on the right-hand of (4) on these three regions are fixed: they are functions of $c$ only.

We estimate the upper bound in the remaining type space "region" by "region."

First, we consider region $[0, d] \times [0, c] \cup [0, c] \times [c, d]$, on which the integrals on the right-hand side of (4) are

$$\int_0^c \int_0^c \left( \frac{-F(\theta_1)}{f(\theta_1)} x_1(\theta_1, \theta_2) + \frac{-F(\theta_2)}{f(\theta_2)} x_2(\theta_1, \theta_2) \right) dF(\theta_2)\, dF(\theta_1)$$
$$+ \int_0^c \int_c^d \left( \frac{-F(\theta_1)}{f(\theta_1)} x_1(\theta_1, \theta_2) + \frac{-F(\theta_2)}{f(\theta_2)} x_2(\theta_1, \theta_2) \right) dF(\theta_2)\, dF(\theta_1)$$
$$+ \int_c^d \int_0^c \left( \frac{-F(\theta_1)}{f(\theta_1)} x_1(\theta_1, \theta_2) + \frac{-F(\theta_2)}{f(\theta_2)} x_2(\theta_1, \theta_2) \right) dF(\theta_2)\, dF(\theta_1).$$

Since IRFR means that $F/f$ is decreasing, the above is maximized by $x_i = 1$ for $\theta_i > \theta_j$.

Next, consider region $[d, 1] \times [c, d]$, on which the relevant integrals are

$$\int_c^d \int_d^1 \left( \frac{1 - F(\theta_1)}{f(\theta_1)} x_1(\theta_1, \theta_2) + \frac{1 - F(\theta_2)}{f(\theta_2)} x_2(\theta_1, \theta_2) \right) dF(\theta_1)\, dF(\theta_2).$$

Recall that $F/f$ decreasing (IRFR) implies that $(1 - F)/f$ is decreasing (IFR). Since $\theta_1$ is always greater than $\theta_2$ on $[d,1]\times[c,d]$, then $\frac{1-F(\theta_1)}{f(\theta_1)} \leq \frac{1-F(\theta_2)}{f(\theta_2)}$. Hence, the values of $x_1$ should be minimized. As $x_1 = \frac{1}{2}$ in the region to its left, the monotonicity of $x_1$ implies that $x_1 = \frac{1}{2}$ on it, as well. Hence, it is best to set $x_1 = \frac{1}{2}$ and $x_2 = \frac{1}{2}$ on this region.

Similarly, it is best to set $x_1 = \frac{1}{2}$ and $x_2 = \frac{1}{2}$ on region $[c,d]\times[d,1]$.

Finally, by Lemma A2, $x_1 = \frac{1}{2}$ and $x_2 = \frac{1}{2}$ also maximize the relevant integrals on region $[d,1]^2$.

Hence, for the given parameter $c$, we have found an upper bound $TU_c$ for $TU$ values of all symmetric non-deterministic mechanisms with $x_i \in \left\{0, \frac{1}{2}, 1\right\}$. This bound is given by the right-hand side of (4) with the allocation rule: $x_i = 1$ for $\theta_i > \theta_j$ when $(\theta_1,\theta_2) \in [0,1]^2 \setminus [c,1]^2$ and $x_i = \frac{1}{2}$ for all $(\theta_1,\theta_2) \in [c,1]^2$.

Can we find a mechanism that actually achieves $TU_c$? Note that this upper bound is derived by relaxing $\int_0^1 h_1(\theta_2)\, dF(\theta_2) + \int_0^1 h_2(\theta_1)\, dF(\theta_1)$ using (ND") along $G$. Hence, if we can find a feasible symmetric mechanism with the allocation rule given above, and (ND") is binding along $G$, this mechanism must achieve $TU_c$.

Let $M(c)$ be a deterministic mechanism defined by

$$x_1 = \begin{cases} 1 & \text{if } \theta_1 > \theta_2 \text{ and } (\theta_1,\theta_2) \in [0,1]\times[0,c) \\ 0 & \text{otherwise} \end{cases}$$

$$t_i(\theta_1,\theta_2) = -\frac{c}{2} \text{ if } x_i = 1 \text{ and}$$

$$t_i(\theta_1,\theta_2) = \begin{cases} \theta_j - \frac{c}{2} & \text{for } \theta_j \in [0,c] \\ \frac{c}{2} & \text{for } \theta_j \in [c,1] \end{cases} \quad \text{if } x_i = 0.$$

It is straightforward to verify that $M(c)$ is feasible. (ND") is binding along the curve $\theta_2 = g(\theta_1)$ since $t_1 + t_2 = 0$ on $(\theta_1,\theta_2) \in [0,1]^2 \setminus [0,c]^2$. By switching agents' identities in $M(c)$, one can derive another deterministic mechanism $\tilde{M}(c)$. The equally weighted combination of $M(c)$ and $\tilde{M}(c)$ is the desired symmetric mechanism.

To find the overall optimal mechanisms for the original problem, we need to first determine the parameter $c^*$ that maximizes $TU_c$. Then, both $M(c^*)$ and $\tilde{M}(c^*)$ described above are optimal for the original problem. Because of symmetry, $c \leq \theta^* < 1$ where $\theta^* = g(\theta^*)$. Apparently, $c^* = 0$ means that the good is always assigned to agent 2, which cannot be optimal. Hence, $c^*$ is strictly greater than 0 and less than 1. □

**Proof of Theorem 3.** The proof is similar to that of Theorem 2; we will focus on the non-deterministic symmetric mechanisms with $x_i \in \left\{0, \frac{1}{2}, 1\right\}$.

Inequality (4) in the proof of Theorem 2 still holds. Adapting the proof to the case when $F$ satisfies DFR, we can show that $TU_d$ where $d = g(c)$, the upper bound of all $c$-

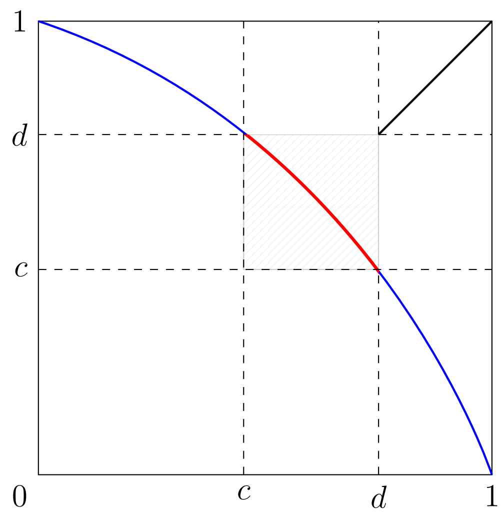



Figure 10: Estimation along off-diagonal curve $\theta_2 = g(\theta_1)$ when $F$ is DFR

indexed mechanisms, is achieved by the following allocation rule $x_i = 1$ for $\theta_i > \theta_j$ when $(\theta_1, \theta_2) \in [0,1]^2 \setminus [0,d]^2$ and $x_i = \frac{1}{2}$ for all $(\theta_1, \theta_2) \in [0,d]^2$.

The upper bound is achieved by the deterministic mechanism $M(d)$:

$$x_1 = \begin{cases} 1 & \text{if } \theta_1 > \theta_2 \text{ and } (\theta_1, \theta_2) \in [d,1] \times [0,1] \\ 0 & \text{otherwise} \end{cases}$$

$$t_i(\theta_1, \theta_2) = \begin{cases} -\frac{d}{2} & \text{for } \theta_j \in [0,d] \\ \frac{d}{2} - \theta_j & \text{for } \theta_j \in [d,1] \end{cases} \text{ if } x_i = 1,$$

$$t_i(\theta_1, \theta_2) = \frac{d}{2} \text{ if } x_i = 0.$$

By switching agents' identities of $M(d)$, one can derive another deterministic mechanism $\tilde{M}(d)$. Once we determine the optimal value of $d^*$ that maximizes $TU_d$, then, both $M(d^*)$ and $\tilde{M}(d^*)$ described above are optimal for the original problem. Moreover, due to symmetry, $d^* \geq \theta^* > 0$ where $\theta^* = g(\theta^*)$. If $d = 1$, it means that the good is always given to agent 2, which cannot be optimal. Hence, $1 > d^* > 0$. □

# References

[1] K. J. Arrow. The property rights doctrine and demand revelation under incomplete information. In M. Boskin, editor, "Economics and Human Welfare". Academic Press (1979).

[2] M. Bagnoli and T. Bergstrom. Log-concave probability and its applications. *Economic theory* **26**(2), 445–469 (aug 2005).

[3] D. Bergemann and S. Morris. Robust mechanism design. *Econometrica* **73**(6), 1771–1813 (2005).

[4] R. Cavallo. Optimal decision-making with minimal waste: Strategyproof redistribution of VCG payments. *Proceedings of the fifth international joint conference on Autonomous agents and multiagent systems* (2006).

[5] K. Chung and J. Ely. Foundations of dominant-strategy mechanisms. *The Review of Economic Studies* **74**(2), 447–476 (apr 2007).

[6] E. H. Clarke. Multipart Pricing of Public Goods. *Public Choice* **11**(1), 17–33 (sep 1971).

[7] C. D'Aspremont and L. Gérard-Varet. Incentives and incomplete information. *Journal of Public economics* **11**(1), 25–45 (feb 1979).

[8] G. de Clippel, V. Naroditskiy, M. Polukarov, A. Greenwald, and N. R. Jennings. Destroy to save. *Games and Economic Behavior* (nov 2012).

[9] M. Drexl and A. Kleiner. Optimal private good allocation: The case for a balanced budget. *Games and Economic Behavior* **94**, 169–181 (nov 2015).

[10] A. Gershkov, J. Goeree, A. Kushnir, B. Moldovanu, and X. Shi. On the equivalence of Bayesian and dominant strategy implementation. *Econometrica* **81**(1), 197–220 (2013).

[11] J. Green and J.-J. Laffont. "Incentives in Public Decision Making". North-Holland (1979).

[12] T. Groves. Incentives in teams. *Econometrica* **41**(4), 617–631 (jul 1973).

[13] M. Guo and V. Conitzer. Worst-case optimal redistribution of VCG payments in multi-unit auctions. *Games and Economic Behavior* **67**(1), 69–98 (sep 2009).

[14] M. Guo and V. Conitzer. Optimal-in-expectation redistribution mechanisms. *Artificial Intelligence* **174**(5-6), 363–381 (apr 2010).

[15] M. Guo and V. Conitzer. Better redistribution with inefficient allocation in multi-unit auctions. *Artificial Intelligence* **216**(0), 287–308 (2014).

[16] K. M. Hagerty and W. P. Rogerson. Robust Trading Mechanisms. *Journal of Economic Theory* **42**(1), 94–107 (jun 1987).

[17] J. D. Hartline and B. Lucier. Non-optimal mechanism design. *American Economic Review* **105**(10), 3102–3124 (2015).

[18] P. Jehiel, M. Meyer-ter Vehn, B. Moldovanu, and W. R. Zame. The limits of ex-post implementation. *Econometrica* **74**(3), 585–610 (may 2006).

[19] D. Miller. Robust collusion with private information. *The Review of Economic Studies* pp. 1–34 (sep 2011).

[20] H. Moulin. Almost budget-balanced VCG mechanisms to assign multiple objects. *Journal of Economic theory* **144**(1), 96–119 (jan 2009).

[21] R. B. Myerson. Optimal auction design. *Mathematics of Operations Research* **6**(1), 58–73 (feb 1981).

[22] J. Schwartz and Q. Wen. Robust trading mechanisms with budget surplus and partial trade. *Working Paper* (May) (2012).

[23] R. Shao and L. Zhou. Optimal Allocation of an Indivisible Good. *Working Paper* (2007).

[24] R. Shao and L. Zhou. Comparison of Groves Mechanisms and the Option Mechanisms. *Working Paper* pp. 1–8 (2008).

[25] R. Shao and L. Zhou. Voting and optimal provision of a public good. *Journal of Public Economics* **134**, 35–41 (feb 2016).

[26] W. Vickrey. Counterspeculation, auctions, and competitive sealed tenders. *The Journal of Finance* **16**(1), 8–37 (mar 1961).

[27] C. Villani. “Optimal transport: old and new”. Springer, 2009 ed. (2008).