Exponentially fast dynamics in the Fock space of chaotic many-body systems

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We demonstrate analytically and numerically that in isolated quantum systems of many interacting particles, the number of states participating in the evolution after a quench increases exponentially in time, provided the eigenstates are delocalized in the energy shell. The rate of the exponential growth is defined by the width Γ of the local density of states (LDOS) and is associated with the Kolmogorov-Sinai entropy for systems with a well defined classical limit. In a finite system, the exponential growth eventually saturates due to the finite volume of the energy shell. We estimate the time scale for the saturation and show that it is much larger than $1/\Gamma$. Numerical data obtained for a two-body random interaction model of bosons and for a dynamical model of interacting spin-1/2 particles show excellent agreement with the analytical predictions.

Introduction.– After decades of intensive studies, the term "quantum chaos" [1–10] became widely disseminated and accepted in modern physics. Originally, it referred to quantum systems whose classical counterparts are chaotic. Paradigmatic examples are the kicked rotor model (KRM) [1, 2] and billiard models [3–5], both of which reveal quantum signatures of classical chaos [11, 12]. It was conjectured and numerically proved [4, 5] that quantum chaos might be quantified by specific properties of the fluctuations of energy spectra. In particular, it was found that in chaotic systems, the distribution of spacings between neighboring energy levels follows the Wigner-Dyson surmise, in contrast with the Poisson dependence that emerges in integrable systems.

Throughout the development of one-body quantum chaos, dynamics has played a crucial role. Numerical studies of the KRM [1, 2] discovered the unexpected existence of two time scales associated with the quantum-classical correspondence. It was confirmed that a complete correspondence between the quantum and classical behavior occurs only on a tiny time scale according to the Ehrenfest theorem. It was analytically shown in [13] that this time scale is given by $t_E \simeq \lambda^{-1} \ln(I/\hbar)$, where I represents a characteristic action and λ is the classical Lyapunov exponent. However, numerical data reported and discussed in [1, 2] demonstrated the existence of a much larger time scale on which the behavior of classical and quantum global observables are equivalent. This time scale was found to be $t_D \approx D/\hbar^2$, where D is the classical diffusion coefficient. After such time and in contrast with the classical case, quantum diffusion ceases. This phenomenon, called dynamical localization, was explained by the localization of the eigenstates in momentum space according to the relation $\ell \propto D$, where ℓ is the localization length [2, 14]. It was later argued that the dynamical localization found in the KRM can be also thought in terms of Anderson localization in pseudo-random potentials [15].

Contrary to one-body quantum chaos, in quantum many-body systems (MBS), level statistics is less informative than the structure of the eigenstates in a physically chosen basis [10, 16]. It is now understood, for example, that the relaxation of a quantum MBS to its thermal state requires the presence of chaotic eigenstates [8–10, 17]. The relaxation of a quantum MBS in the thermodynamic limit has been discussed before [18], but the time scale on which it occurs in finite systems is still an open question. To address this problem, we analyze the relaxation of observables of quantum MBS in the Fock space.

We consider the quench dynamics described by a Hamiltonian $H = H_0 + V$ in the region of parameters where the eigenstates are fully delocalized in the energy shell defined by the inter-particle interaction V [16, 19–22]. Specifically, we prepare the system in a single (unperturbed) eigenstate of H_0 and study how the state spreads in the unperturbed basis (Fock space) due to V. With the use of a semi-analytical approach, we show that the effective number of unperturbed states participating in the dynamics of quantum MBS increases exponentially in time.

We find that the exponential growth saturates at a time much larger than the characteristic time $1/\Gamma$ of the initial state decay, where Γ is the width of the local density of states (LDOS). We discuss the physical meaning of this novel time scale in connection with the quantum-classical correspondence for chaotic MBS and with the problem of thermalization in isolated quantum MBS. Our analytical estimates are fully confirmed by numerical data obtained for two different systems: a model of randomly interacting bosons and a one-dimensional (1D) system of spins 1/2 with deterministic couplings.

Models.– In both models, H_0 describes the non-interacting particles (or quasi-particles), while their interaction is contained in V. The first model represents

N identical bosons occupying M single-particle levels specified by random energies ϵ_s with a mean spacing $\langle \epsilon_s - \epsilon_{s-1} \rangle = 1$ setting the energy scale. The Hamiltonian reads

$$H = \sum \epsilon_s \, a_s^\dagger a_s + \sum V_{s_1 s_2 s_3 s_4} \, a_{s_1}^\dagger a_{s_2}^\dagger a_{s_3} a_{s_4}, \quad (1)$$

where a_s ($a_s^\dagger$) is the creation (annihilation) operator on level s, and the two-body matrix elements $V_{s_1 s_2 s_3 s_4}$ are random Gaussian entries with zero mean and variance v^2. The interaction conserves the number of bosons and connects many-body states that differ by the exchange of at most two particles. This two-body interaction (TBRI) model was introduced in [23, 24] and has been extensively studied for fermions [19, 25] and bosons [26]. The unperturbed many-body eigenstates $|k\,\rangle$ of $H_0 = \sum_k \mathcal{E}_k \,|k\,\rangle \langle k\,|$ are obtained by creating N bosons in M single-particles energy levels, so that $|k\rangle = a_{s_1}^\dagger ... a_{s_N}^\dagger |0\rangle$, where $1 \le s_1, .., s_N \le M$. The eigenstates $|\alpha\,\rangle$ of the Hamiltonian $H = \sum_\alpha E^\alpha \,|\alpha\,\rangle \langle \alpha\,|$ are represented in terms of the states $|k\rangle$ as $|\alpha\,\rangle = \sum_k C_k^\alpha \,|k\,\rangle$.

The other model studied has no random terms. It describes a dynamical system of interacting spins-1/2 on a 1D lattice of length L. Spin systems are intensively studied in experiments with nuclear magnetic resonance platforms, ion traps, and cold atoms. The Hamiltonians H_0 and V are given by

$$H_0 \; = \; \frac{J}{4} \sum_s \left(\sigma_s^x \sigma_{s+1}^x + \sigma_s^y \sigma_{s+1}^y + \Delta \sigma_s^z \sigma_{s+1}^z \right), \quad (2)$$

$$V \; = \; \lambda \frac{J}{4} \sum_s \left(\sigma_s^x \sigma_{s+2}^x + \sigma_s^y \sigma_{s+2}^y + \Delta \sigma_s^z \sigma_{s+2}^z \right), \quad (3)$$

where $\sigma_s^{x,y,z}$ are the Pauli matrices on site s. The coupling constant $J = 1$ sets the energy scale, Δ is the anisotropy parameter, and λ is the ratio between nearest-neighbor and next-nearest-neighbor couplings [27]. The Hamiltonian conserves the total spin in the z-direction, $\mathcal{S}^z = \sum_{s=1}^L \sigma_s^z/2$, which is here fixed to $\mathcal{S}^z = -1$. This choice of subspace implies that L is even and the number of up-spins (excitations) is given by $N = L/2 - 1$. When $V = 0$, the model is integrable, while as λ increases, it approaches the chaotic regime [16].

Basic relations.– We analyze the wave packet dynamics in the unperturbed basis $|k\,\rangle$ after switching on the interaction V. The system is initially prepared in a particular unperturbed state $|k_0\,\rangle$,

$$|\psi(0)\,\rangle = \sum_\alpha C_{k_0}^\alpha \,|\alpha\,\rangle \,. \quad (4)$$

The probability to find the evolved state in any basis state $|k\,\rangle$ at the time t is

$$P_k(t) = |\langle k|\psi(t)\rangle|^2 = \sum_{\alpha,\beta} C_{k_0}^{\alpha *} C_k^\alpha C_{k_0}^\beta C_k^{\beta *} e^{-i(E^\beta - E^\alpha)t}. \quad (5)$$

This probability can be written as the sum of a diagonal part, $P_k^d = \sum_\alpha |C_{k_0}^\alpha|^2 |C_k^\alpha|^2$, and a fluctuating time-dependent part, $P_k^f(t) = \sum_{\alpha \neq \beta} C_{k_0}^\alpha C_k^{\alpha *} C_{k_0}^\beta C_k^{\beta *} e^{-i(E^\beta - E^\alpha)t}$. After a long time and assuming a non-degenerate spectrum, P_k^f cancels out on average and only the diagonal part P_k^d survives.

With $P_k(t)$, we construct the quantity of our main interest, the number of principal components,

$$N_{pc}(t) = \left\{ \sum_k \left[P_k^d + P_k^f(t) \right]^2 \right\}^{-1}, \quad (6)$$

also known as participation ratio. It measures the effective number of unperturbed states $|k\,\rangle$ that composes the evolved wave packet. For weak interaction, $N_{pc}(t)$ oscillates in time. Our focus is, however, on strong values of V, where $N_{pc}(t)$ increases smoothly and eventually saturates to its infinite time average given by

$$\overline{N_{pc}^\infty} = \left[2 \sum_k (P_k^d)^2 - \sum_\alpha |C_{k_0}^\alpha|^4 \sum_k |C_k^\alpha|^4 \right]^{-1}. \quad (7)$$

This determines the total number of unperturbed many-body states inside the energy shell.

Dynamics in many-body space.– A distinctive property of the dynamics of a quantum MBS is that it cannot be described as either ballistic or diffusive in Fock space. A pictorial demonstration of how the initial state spreads in the Fock space is given in the Supplemental Material (SM) [28]. Specifically, on a small time scale only the basis states directly coupled to the initial state are excited. Their number is much smaller than the total number of basis states, due to the sparse structure of the Hamiltonian matrix. As time passes more basis states are populated inside the shell, until its ergodic filling. This takes place provided the perturbation V is sufficiently strong, so that the eigenstates of H are delocalized in the energy shell. The dynamics in the Fock space can be mathematically described as the evolution on a Cayley tree [25].

To describe the time dependence of $N_{pc}(t)$, we develop a cascade model to monitor the flow of probability to find the system in specific unperturbed states at different time steps. This is done by dividing the dynamical process in different time intervals associated with different sets of basis states (classes). At $t = 0$, only the $\mathcal{M}_0$ class is non empty: it has one element, which is the initial state $|k_0\,\rangle$. In the next time step, all states having a non-zero coupling with the initial basis state are populated. That is, the first class $\mathcal{M}_1$ contains the basis states $|k\rangle$ for which $\langle k_0|V|k\rangle \neq 0$. The second class $\mathcal{M}_2$ consists of those states which have non-zero matrix elements with all states from the first class. In the same manner, one can define all classes in the Fock space. For an infinite number of particles, there is an infinite hierarchy of equations describing the flow of probability from one class to the next one. However, in our case, the dimension $\mathcal{D}$ of the Fock space is finite and the number of states in the second class practically coincides with $\mathcal{D}$. For this reason, we restrict our consideration to two classes only. As shown below, this is indeed a good approximation.

Let us define the probability to find the system in class $\mathcal{M}_0$, as $W_0(t) \equiv P_{k_0}(t)$. This is the survival probability of the initial state. The probability for being in the class $\mathcal{M}_1$ is $W_1(t) \equiv \sum_{k\in\mathcal{M}_1} P_k(t)$. Neglecting the back flow to the initial state and assuming conservation of probability, $W_2(t) = 1 - W_0(t) - W_1(t)$. One thus obtains the following set of equations,

$$\begin{aligned} \frac{dW_0}{dt} &= -\Gamma(W_0 - \overline{W_0^\infty}), \\ \frac{dW_1}{dt} &= -\Gamma(W_1 - \overline{W_1^\infty}) + \Gamma(W_0 - \overline{W_0^\infty}), \end{aligned} \tag{8}$$

where the infinite time averages are $\overline{W_0^\infty} = \sum_\alpha |C_{k_0}^\alpha|^4$ and $\overline{W_1^\infty} = \sum_{k\in\mathcal{M}_1}\sum_\alpha |C_{k_0}^\alpha|^2 |C_k^\alpha|^2$. The decay rate Γ corresponds to the width of LDOS,

$$F_{k_0}(E) = \sum_\alpha |C_{k_0}^\alpha|^2 \delta(E - E^\alpha),$$

which is obtained by projecting the initial state $|k_0\rangle$ onto the energy eigenbasis. It was introduced in nuclear physics to describe the relaxation of excited heavy nuclei [29], where it is known as strength function.

The solution of Eq. (8) gives

$$\begin{aligned} W_0(t) &= e^{-\Gamma t}(1 - \overline{W_0^\infty}) + \overline{W_0^\infty}, \\ W_1(t) &= \Gamma t e^{-\Gamma t}(1 - \overline{W_0^\infty}) + \overline{W_1^\infty}(1 - e^{-\Gamma t}). \end{aligned} \tag{9}$$

With the expressions (9) one can derive the time dependence for $N_{pc}(t)$,

$$N_{pc}(t) \simeq \left[\sum_n W_n^2/\mathcal{N}_n\right]^{-1} \simeq \left[W_0^2 + W_1^2/\mathcal{N}_1\right]^{-1} \sim e^{2\Gamma t}, \tag{10}$$

where $\mathcal{N}_n$ is the number of states contained in the n-th class. This result shows that the number of basis states effectively participating in the evolution of the wave packet increases exponentially in time with the rate 2Γ. For a finite number of particles, this growth lasts until the saturation, given by Eq. (15).

Results for the TBRI model.– To verify the validity of our approach, we compare in Fig. 1(a) and (b) the numerical data for $W_0(t)$ and $W_1(t)$ with Eqs. (9). The chosen v is such that the eigenstates are strongly chaotic and extended in the energy shell [22]. The value of Γ used in the analytical expressions is obtained by fitting the numerical curve for $W_0(t)$. The agreement between numerical and analytical results is very good for the entire duration of the evolution, up to the saturation given by $\overline{W_0^\infty}$ and $\overline{W_1^\infty}$. These results confirm that the back flow can indeed be neglected and that one can take into account two classes only.

In Fig. 1(c), we show the evolution of the number of principal components N_{pc}. The numerical data (solid curve) corroborate the analytical prediction (dashed curve) from Eq. (10), namely the exponential behavior, $N_{pc}(t) \sim e^{2\Gamma t}$.

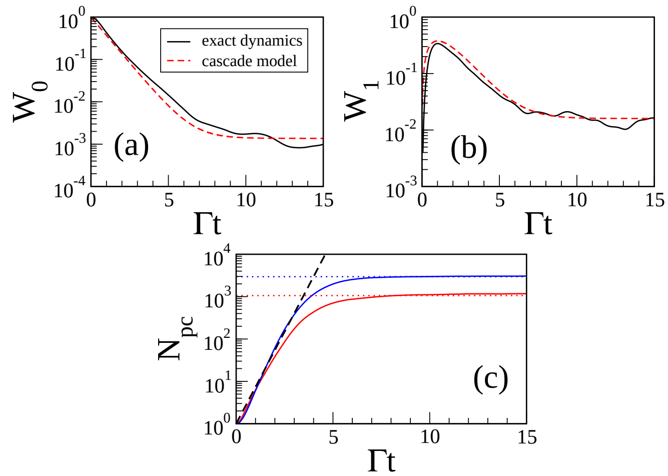

FIG. 1: TBRI model: Numerical data for $W_0(t)$ (a) and $W_1(t)$ (b) are shown by solid curves and compared with the analytical expressions (9) (dashed curves). The parameters are $N = 6$, $M = 11$, $v = 0.4$ (chaotic regime). In the initial state $|\psi(0)\rangle = (a_5^\dagger)^6 |0\rangle$ all particles initially occupy the 5-th single-particle level. The exponential rate $\Gamma = 2.8$ is obtained by fitting $W_0(t)$. In (c): Growth in time of N_{pc} for two initial conditions; from top to bottom: $|\psi(0)\rangle = (a_4^\dagger)^6 |0\rangle$ and $|\psi(0)\rangle = (a_5^\dagger)^6 |0\rangle$. The dashed line is $e^{2\Gamma t}$. Horizontal dotted lines are the analytical estimates given by Eq. (15). Average over 50 random realizations.

Our data manifest the existence of two time scales. The first one, $t_\Gamma \simeq 1/\Gamma$, corresponds to the characteristics decay time of $W_0(t)$, as shown in Eq. (9). The second, t_S, is the time scale for the saturation of the dynamics and can be estimated from $e^{2\Gamma t} \simeq \overline{N_{pc}^\infty}$, which gives

$$t_S \simeq \ln(\overline{N_{pc}^\infty})/2\Gamma. \tag{11}$$

Assuming a Gaussian shape for both the density of states and the LDOS, we show that the maximal value of $\overline{N_{pc}^\infty}$ is

$$N_{pc}^{max} = \eta\sqrt{1-\eta^2}\mathcal{D} \tag{12}$$

where $\eta = \Gamma/\sigma\sqrt{2}$ and σ is the width of the density of states (see details in SM [28]). For $M \sim 2N$ and for $M, N \gg 1$ one gets the estimate

$$t_S \sim N/\Gamma = N t_\Gamma. \tag{13}$$

This is the time scale for the thermalization in quantum MBS. As one can see from Eq. (13), when the number of particles is very large the two time scales are very different.

Results for the spin model– The analytical estimates obtained with the cascade approach are valid also for dynamical models. To show this, we study the evolution of the spin-1/2 system described by Eq. (3) in the limit of strong chaos ($\lambda = 1$) [21]. The analysis is analogous to the one developed with the TBRI model. We note, however, that H_0 is now initially written in the basis where each site has a spin pointing up or down in the

z-direction (site-basis). It is then diagonalized to obtain the mean-field basis. As a result, all matrix elements of the full Hamiltonian written in the mean-field basis become non-zero. Therefore, to properly determine the classes, we use the following procedure. In the first class we have all states m coupled to k_0 such that $|H_{k_0,m}| > \xi|H_{k_0,k_0} - H_{m,m}|$ with ξ being a threshold reasonably chosen. This procedure is repeated for higher classes.

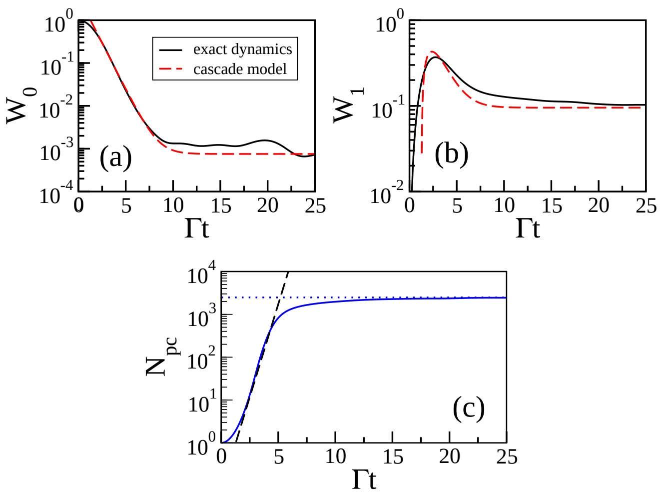

FIG. 2: Spin model: Numerical data (solid curves) for $W_0(t)$ and $W_1(t)$ compared with the analytical expressions (9) (dashed curves). In (c): Numerical data for the number of principal components $N_{pc}(t)$ (solid curve) and the infinite-time average in Eq. (15) (dotted line). The dashed line represents $e^{2\Gamma t}$. Parameters: $L = 16$, $\Delta = 0.48$, $\lambda = 1$, and $N = 7$ excitations. Average over 16 initial states with energy close to -0.5. Threshold for counting $\mathcal{N}_1$ is $\xi = 0.05$ and $\Gamma = 2.62$ is obtained by fitting $W_0(t)$.

Figure 2 compares the numerical results for $W_0(t)$, $W_1(t)$, and $N_{pc}(t)$ for the spin model with the analytical expressions in Eq. (9) and Eq. (10). The agreement is very good and the exponential increase in time of the number of principal components with rate 2Γ is confirmed. As for the TBRI model, we see that the back flow is not important and that two classes suffice to describe the dynamics. This validates our approach for realistic physical systems even in the absence of any random parameter. We recall that in the spin system, N appearing in the estimate (13) is the number of excitations.

Discussion.– We studied the dynamics of interacting quantum MBS whose eigenstates have a chaotic structure in the basis of non-interacting particles. We demonstrated that in the Fock space the relaxation is very different from either a diffusive or ballistic process. Instead, the wave packets spread exponentially fast in the unperturbed basis before reaching saturation, when all states of the energy shell get populated. Unexpectedly, we found that the time scale for saturation is much larger than the characteristic decay time of the survival probability. To describe the dynamical process, we developed a semi–analytical approach that allowed us to estimate the rate and the time scale of the relaxation, as well as the saturation value of the number of principal components in the wave packet. It is quite impressive that our simple phenomenological model with a single parameter – the width Γ of LDOS – reproduces so well the system dynamics at very different time scales.

The first analytical investigation of the properties of the LDOS was done by Wigner in his studies of banded random matrices [30]. In the context of quantum chaos, these matrices were employed in [31], where it was pointed out that the LDOS has a well defined classical limit. The classical LDOS is the projection of the unperturbed Hamiltonian onto the total Hamiltonian and can be obtained by solving classical equations of motion [32]. As shown in [31], the maximal width of the LDOS is given by the width of the energy shell. In the classical description, the energy shell corresponds to the phase-space volume obtained by the projection of the phase-space surface $H_0 = E_0$ onto the surface defined by the total Hamiltonian H. The dynamics of the classical packets created by H_0 is restricted to this shell [32, 33], which can be filled in time either partially or ergodically. In the quantum limit, these two alternatives correspond to either localized or delocalized wave packets.

Inspired by the above studies, our results for the exponential growth of N_{pc} can be treated in terms of the phase-space volume $\mathcal{V}_E$ occupied by the wave packet, $\mathcal{V}_E(t) \sim N_{pc}(t)/\rho(E)$, where $\rho(E)$ is the total density of states. We can write

$$\mathcal{V}_E(t) = \mathcal{V}_E(0)e^{2\Gamma t} \sim \mathcal{V}_E(0)e^{h_{KS}t}. \tag{14}$$

Here, we associate 2Γ with the Kolmogorov-Sinai entropy [34], h_{KS}, which gives the exponential growth rate of phase-space volumes for classically chaotic MBS [34]. Note that in many-body systems, h_{KS} is defined as the sum of *all* positive Lyapunov exponents and not only the largest one. The relation $h_{KS} = 2\Gamma$ allows one to establish a quantum-classical correspondence for MBS. Indeed, when the system admits a well defined classical limit in which there is strong chaos, the Kolmogorov-Sinai entropy is associated with the width of the classical LDOS.

We stress that Eq. (14) holds only up to the saturation time $t_S \sim N t_\Gamma$, which defines the time scale for the quantum-classical correspondence for the number of principal components N_{pc} participating in the dynamics. This time t_S is important for the problem of thermalization in isolated systems of interacting particles. It establishes the time scale for the complete relaxation of the system due to the ergodic filling of the whole energy shell. One sees that in the thermodynamic limit, $N \to \infty$, this time diverges (provided the width of the LDOS remains constant). This agrees with the quantum-classical correspondence principle, since statistical equilibrium in a classically chaotic MBS can be reached only in an infinite time.

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Supplemental Material: Exponentially fast dynamics in the Fock space of chaotic many-body systems

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I. DYNAMICS IN THE MANY-BODY SPACE

A main problem when studying the dynamics of systems with many interacting particles is that it cannot be described as either ballistic or diffusive in the many-body space. Instead, it is the initial unperturbed many-body state that spreads onto other unperturbed many-body states in a complicate way. A pictorial demonstration of how this happens is given in Fig. 3 , where we show $P_k(t) = |\langle k|\psi(t)\rangle|^2$ as a function of the unperturbed state $|k\rangle$ for different times for both the TBRI model (top panels (a)-(d)) and the the dynamical spin model (bottom panels (e)-(h)). The idea is similar to that of the Cayley tree. In a small time scale, only the basis vectors directly coupled to the initial state are excited [Fig. 3 (a) and (e)]. The number of these states is much smaller than the total number of basis vectors, which is a consequence of the sparse character of the Hamiltonian matrix. For longer times, as shown in Fig. 3 (b) (f) and Fig. 3 (c) (g), the participating states sparsely fill a large portion of the Fock space that is within the energy shell. As time passes, more basis states are populated inside the shell, until it gets ergodically filled [Fig. 3 (d) (h)]. This ergodic filling takes place provided the perturbation $\mathcal{V}$ is sufficiently strong, so that the eigenstates of H are delocalized in the energy shell.

II. ESTIMATE OF THE INFINITE TIME AVERAGE OF THE NUMBER OF PRINCIPAL COMPONENTS

The purpose of this section is to find an estimate of

$$[\overline{N_{pc}^{\infty}}]^{-1} = 2\sum_k (P_k^d)^2 - \sum_\alpha |C_{k_0}^\alpha|^4 \sum_k |C_k^\alpha|^4 \tag{15}$$

in terms of fundamental characteristics of the system, *i.e.* without the explicit diagonalization of the full Hamiltonian matrix.

First, we notice that the second term in the r.h.s of Eq. (15) is roughly $1/\mathcal{D}$ times smaller than the first one. This can be seen by taking uncorrelated components $C_k^\alpha \simeq (1/\sqrt{\mathcal{D}})e^{i\xi_{\alpha,k}}$, where $\xi_{\alpha,k}$ are random numbers. Thus

$$2\sum_k (P_k^d)^2 = 2\sum_{\alpha,\beta,k} |C_{k_0}^\alpha|^2 |C_k^\alpha|^2 |C_{k_0}^\beta|^2 |C_k^\beta|^2, \simeq \frac{\mathcal{D}^3}{\mathcal{D}^4} \simeq \frac{1}{\mathcal{D}} \tag{16}$$

while

$$\sum_{\alpha,k} |C_{k_0}^\alpha|^4 |C_k^\alpha|^4 \simeq \frac{\mathcal{D}^2}{\mathcal{D}^4} \simeq \frac{1}{\mathcal{D}^2} \tag{17}$$

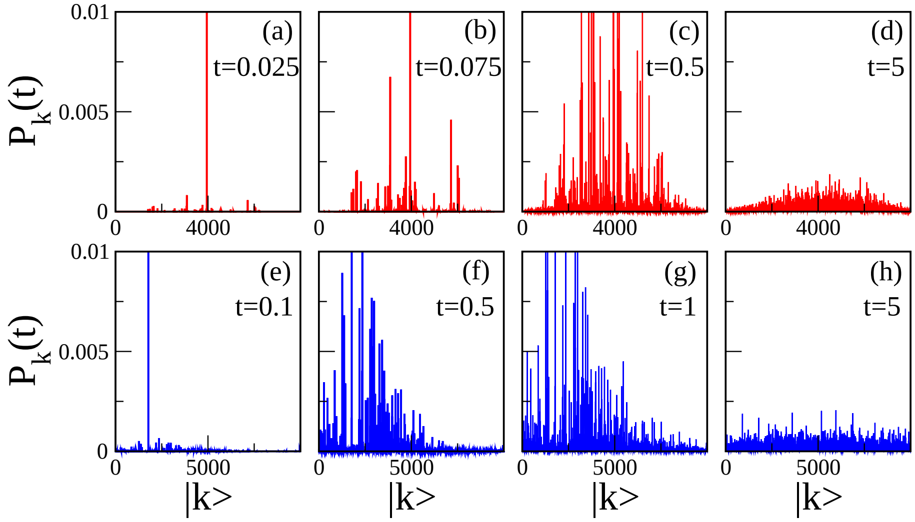

FIG. 3: Spread of the probability $P_k(t)$ onto the unperturbed basis states $|k\rangle$ for fixed times indicated in the panels for the TBRI model [(a)-(d)] and for the spin model [(e)-(h)]. For the TBRI model, $N = 6$, $M = 11$, $\mathcal{V} = 0.4$ and one realization of random potential. For the spin model, $L = 16$, 7 excitations, $\Delta = 0.48$, $\lambda = 1$, open boundaries. The energy of the initial basis state is chosen close to the middle of the spectrum.

We can then take the first term only,

$$\left[\,\overline{N_{pc}^{\infty}}\,\right]^{-1} \simeq 2 \sum_k (P_k^d)^2. \tag{18}$$

Let us now assume a Gaussian shape for the LDOS,

$$F_k(E) = \sum_\alpha |C_k^\alpha|^2 \delta(E - E^\alpha) \simeq \frac{1}{\Gamma\sqrt{2\pi}} \exp\left\{-\frac{(E - E_k^0)^2}{2\Gamma^2}\right\} \tag{19}$$

where Γ is the width of the LDOS and E_k^0 is the energy of the unperturbed state. We assume that Γ is independent of E_k^0. The LDOS is normalized, $\int dE F_k(E) = 1$. Let us also assume a Gaussian density of states, characterized by a width σ, such that

$$\rho(E) = \frac{\mathcal{D}}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{E^2}{2\sigma^2}\right\}, \tag{20}$$

where for simplicity we set the middle of the spectrum at the energy $E = 0$. The density of states is normalized to the dimension of the Fock space, $\int dE \rho(E) = \mathcal{D}$.

In the continuum, one has

$$P_k^d = \sum_\alpha |C_{k_0}^\alpha|^2 |C_k^\alpha|^2 \simeq \int dE\, \rho(E)^{-1} F_k(E) F_{k_0}(E) \equiv \mathcal{G}_{k_0}(E_k^0) \tag{21}$$

where the function

$$\mathcal{G}_{k_0}(E_k^0) = \frac{\sigma^2}{\Gamma\mathcal{D}\sqrt{2\sigma^2 - \Gamma^2}} \exp\left\{-\frac{(E_k^0)^2 + (E_{k_0}^0)^2}{2\Gamma^2} + \frac{(E_k^0 + E_{k_0}^0)^2}{2\Gamma^2(2\sigma^2 - \Gamma^2)}.\right\} \tag{22}$$

is defined only for $2\sigma^2 > \Gamma^2$. We can then approximate

$$[\,\overline{N_{pc}^{\infty}}\,]^{-1} \simeq 2 \sum_k (P_k^d)^2 \simeq 2 \int dE\, \rho_0(E) \mathcal{G}_{k_0}(E)^2. \tag{23}$$

Assuming a Gaussian shape also for the unperturbed density of states $\rho_0(E)$,

$$\rho_0(E) = \frac{\mathcal{D}}{\sigma_0\sqrt{2\pi}} \exp\left\{-\frac{E^2}{2\sigma_0^2}\right\}, \tag{24}$$

and taking into account that [22], $\sigma_0^2 = \sigma^2 - \Gamma^2$, Eq. (23) gives

$$\overline{N_{pc}^{\infty}} = \mathcal{D}\frac{\Gamma\sqrt{2\sigma^2 - \Gamma^2}}{2\sigma^2} e^{-E_{k_0}^2/\Gamma^2}. \tag{25}$$

One should note that a Gaussian shape for the LDOS, density of states of the full Hamiltonian, and density of the unperturbed states is a realistic assumption for the TBRI model [22] and spin model [16]. The maximal value of $\overline{N_{pc}^{\infty}}$ occurs in the middle of the energy spectrum, where $E_{k_0} = 0$. Defining $\eta = \frac{\Gamma}{\sqrt{2}\sigma}$, we have

$$\overline{N_{pc}^{max}} = \frac{\Gamma}{\sqrt{2}\sigma}\sqrt{1 - \left(\frac{\Gamma}{\sqrt{2}\sigma}\right)^2}\mathcal{D} = \eta\sqrt{1-\eta^2}\mathcal{D} \equiv \Xi(\eta)\mathcal{D}, \tag{26}$$

One sees that $\text{Max}_\eta\left[\Xi(\eta)\right] = 1/2$, so

$$\overline{N_{pc}^{max}} \leq \mathcal{D}/2. \tag{27}$$

III. TIME SCALES

As it is clear from the solution of the cascade model for the survival probability,

$$W_0(t) = e^{-\Gamma t}(1 - \overline{W_0^{\infty}}) + \overline{W_0^{\infty}},$$

the width Γ is strictly related to the time scale for the depletion of W_0, that is

$$t_\Gamma \simeq \frac{1}{\Gamma}.$$

This means that after the time t_Γ the probability to be in the 0^{th} class (i.e. the survival probability) is reduced by the factor $1/e$.

On the other hand, we have seen that global observables, such as the number of principal components N_{pc}, grow exponentially in time,

$$N_{pc}(t) \simeq e^{2\Gamma t},$$

up to the saturation point given by $\overline{N_{pc}^{\infty}}$.

It is quite natural to estimate the saturation time t_S as the time for which

$$e^{2\Gamma t_S} \simeq \overline{N_{pc}^{\infty}},$$

so that, using Eq. (26)

$$t_S \simeq \frac{1}{2\Gamma}\ln\left[\Xi(\eta)\mathcal{D}\right]. \tag{28}$$

In order to evaluate the dimension of the Fock space, we should distinguish between the TBRI and the spin model.

- The dimension of the Fock space for the TBRI model can be expressed in terms of the number of bosons N and the number of single particle energies M, as

  $$\mathcal{D} = \frac{(N+M-1)!}{N!(M-1)!}.$$

  In the limit of $N, M \gg 1$, using the Stirling approximation, one has

  $$\ln\mathcal{D} \approx N\ln\left(1 + \frac{M}{N}\right) + M\ln\left(1 + \frac{N}{M}\right). \tag{29}$$

  In the dilute limit, $M \simeq 2N$, and for $N, M \gg 1$, one finally gets the estimate for the saturation time,

  $$t_S = \frac{\ln\Xi(\eta)}{2\Gamma} + c_1\frac{N}{\Gamma}, \tag{30}$$

  where $c_1 = \ln(27/4)$ is a constant of order 1. Since the first term in the r.h.s term above is independent of the number of particles N, we have that in the thermodynamic limit and for a fixed ratio N/M ,

  $$t_S \approx N t_\Gamma.$$

- For the spin model with N excitations in L different sites one has

$$\mathcal{D} = \frac{L!}{N!(L-N)!}.$$

In the limit of $N, L \gg 1$, using the Stirling approximation, one has

$$\ln \mathcal{D} \approx N \ln\left(\frac{L}{N} - 1\right) - L \ln\left(1 - \frac{N}{L}\right). \tag{31}$$

For half filling, $L \simeq 2N$, and for $N, L \gg 1$, one finally gets the estimate for the saturation time,

$$t_S = \frac{\ln \Xi(\eta)}{2\Gamma} + c_2 \frac{N}{\Gamma}, \tag{32}$$

where $c_2 = \ln(4)$ is a constant of order 1. Since the first term in the r.h.s term above is independent of the number of excitations N, we have that in the thermodynamic limit and for a fixed ratio N/L ,

$$t_S \approx N t_\Gamma.$$

We notice the similarity between the many-body case and the one-body quantum chaotic systems. Both exhibit two different time scales. Here we have $t_S \gg t_\Gamma$. For the one-body case, we have the Ehrenfest time t_E and a much longer diffusive time t_D. t_E is related to the initial exponential spreading of the wave packets and is linked with the divergence of neighboring trajectories in the phase space of classically chaotic systems. Instead, t_D is associated with the quantum energy diffusion, in analogy with the classical unbounded diffusion.