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On the fractional Eulerian numbers and equivalence of maps with long term power-law memory (integral Volterra equations of the second kind) to Grünvald-Letnikov fractional difference (differential) equations

Mark Edelman

Department of Physics, Stern College at Yeshiva University, 245 Lexington Ave., New York, New York 10016, USA; Courant Institute of Mathematical Sciences, New York University, 251 Mercer St., New York, New York 10012, USA; and Department of Mathematics, BCC, CUNY, 2155 University Avenue, Bronx, New York 10453, USA

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In this paper, we consider a simple general form of a deterministic system with power-law memory whose state can be described by one variable and evolution by a generating function. A new value of the system's variable is a total (a convolution) of the generating functions of all previous values of the variable with weights, which are powers of the time passed. In discrete cases, these systems can be described by difference equations in which a fractional difference on the left hand side is equal to a total (also a convolution) of the generating functions of all previous values of the system's variable with the fractional Eulerian number weights on the right hand side. In the continuous limit, the considered systems can be described by the Grünvald-Letnikov fractional differential equations, which are equivalent to the Volterra integral equations of the second kind. New properties of the fractional Eulerian numbers and possible applications of the results are discussed. © *2015 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4922834]

Maps with memory have been investigated for many decades. The field of fractional difference equations is a few decades old. In this paper, we show that maps with power-law memory are equivalent to the Grünvald-Letnikov fractional difference equations. The fractional Eulerian numbers, introduced by Butzer and Hauss in 1993 in a paper which was cited only once in 1995 by Jean-Louis Nicolas, play the key role in the connection between maps with power-law memory and fractional difference equations. In the continuous limit, the relationship between maps with power-law memory and fractional difference equations leads to the equivalence of fractional differential equations and the Volterra integral equations of the second kind. Systems with power-law memory can be used to investigate chaos in continuous fractional systems of less than three dimensions.

I. INTRODUCTION

In paper,[1] we introduced α-families of maps (αFM), which correspond to a general form of fractional differential equations of systems experiencing periodic kicks

$$\frac{d^\alpha x}{dt^\alpha} + \tilde{G}_K(x(t-\Delta T)) \sum_{k=-\infty}^{\infty} \delta\left(\frac{t}{T} - (k+\varepsilon)\right) = 0, \quad (1)$$

where $\tilde{G}_K(x)$ is an arbitrary non-linear function, K is a parameter, $\varepsilon > \Delta > 0$, $\alpha \in \mathbb{R}$, $\alpha > 0$, in the limit $\varepsilon \to 0$, with the initial conditions corresponding to the type of the fractional derivative used. We investigated their general properties in Ref. 1 and in the following articles.[2–5] These maps are maps with power-law memory in which the new value of the variable x_{n+1} depends on all previous values x_k $(0 \le k \le n)$ of the same variable with weights proportional to the time passed $(n+1-k)$ to the power $(\alpha - 1)$. For example, in the case of the Caputo fractional derivatives, Eq. (1) leads to (for $T = 1$)

$$x_{n+1} = \sum_{k=0}^{N-1} \frac{x_0^{(k)}}{k!}(n+1)^k - \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \tilde{G}_K(x_k)(n-k+1)^{\alpha-1}, \quad (2)$$

where $x^{(k)}(t) = D_t^k x(t)$, $x_0^{(k)} = x^{(k)}(0)$, $0 \le N-1 < \alpha \le N$, $\alpha \in \mathbb{R}$, $N \in \mathbb{N}$.

Historically, the first maps with memory were considered as models for non-Markovian processes in general[6,7] and, with regards to thermodynamic theory of systems with memory,[8] as analogues of the integro-differential equations of non-equilibrium statistical physics[9–11] (see also a recent Stanislavsky's paper on maps with long-term memory[12]). The general form of the investigated maps was

$$x_{n+1} = \sum_{k=m}^{n} V(n,k) G(x_k), \quad (3)$$

where $V(n, k)$ characterizes memory effects. Maps Eq. (3) with $m = 0$ are called maps with long term memory. Maps in which the number of terms in the sum in Eq. (3) is bounded $(m = n - M + 1)$ are called maps with short term memory or M-step memory maps.

In this paper, we consider long term memory maps with power-law memory in the form

$$x_n = \sum_{k=0}^{n-1} (n-k)^{\alpha-1} G_K(x_k, h), \quad (4)$$

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where K is a parameter and h is a constant time step between t_n and t_{n+1}. These maps differ from the maps Eq. (2) by the sum of power functions depending on the initial conditions of Eq. (1). They coincide in the case of the zero initial conditions, $h=1$, and $G_K(x_k) = -\tilde{G}_K(x_k)/\Gamma(\alpha)$.

Interest in power-law memory maps is stimulated by the recent discovery of the large number of systems (mostly biological), not necessarily described by the fractional differential equations, with power-law memory. In the study of human memory, the accuracy on a memory tasks decays as a power law, $\sim t^{-\beta}$, with $0 < \beta < 1$.[13–17] In the study of human learning, the reduction in reaction times that comes with practice is a power function of the number of training trials.[18] Power-law adaptation has been used to describe the dynamics of biological systems in papers.[17,19–23] As it has been shown recently, even processing of external stimuli by individual neurons can be described by fractional differentiation.[24,25] Most of human organ tissues demonstrate viscoelastic properties.[26–37] This leads to their description by fractional differential equations with time fractional derivatives,[38–46] which implies the power-law memory. In most of the biological systems with the power-law behavior ($\sim t^{\beta}$), the power β is between -1 and 1, which leads to $0 < \alpha < 2$ in Eq. (4).

Biological systems are not the only natural systems with power-law memory. In the continuous case, these systems can be described by fractional differential equations and one may find many examples of such systems in the recent books on applications of fractional calculus.[39,47–59] In physics, for example, common and general examples of systems with power-law memory include: Hamiltonian systems, in which transport can be described by the fractional Fokker-Planck-Kolmogorov equation and memory is the result of stickiness of trajectories in time to the islands of regular motion;[49,60–62] dielectric materials, where electromagnetic fields are described by equations with time fractional derivatives due to the universal response—the power-law frequency dependence of the dielectric susceptibility in a wide range of frequencies;[51,63–65] materials with rheological properties and viscoelastic materials, in which non-integer order differential stress-strain relations give a minimal parameter set concise description of polymers and other viscoelastic materials with non-Debye relaxation and memory of strain history.[39,40,42–44] It is also interesting that the use of fractional calculus (power-law memory) in control (fractional order control) makes it possible to improve performance of traditional controllers.[53,55]

Another motivation for the present paper comes from the first results of the investigation of fractional (power-law memory, see, e.g., Eq. (2)[1–5,69–73]) and fractional difference (asymptotically power-law memory[3,4]) maps. It has been shown that fractional and fractional difference maps both demonstrate new type of attractors—cascade of bifurcations type trajectories (CBTT) (see Fig. 1) in which after a small number of iterations a trajectory converges to a period one trajectory (fixed point) which later bifurcates and becomes a $T=2$ sink and then follows the period doubling scenario typical for cascades of bifurcations in regular dynamics. The difference is that in regular dynamics a cascade of bifurcations is the result of a change in a non-linearity parameter and in CBTT a cascade of bifurcations occurs on a single attracting trajectory. CBTT were demonstrated in the examples of harmonic and quadratic maps with power-law (and falling factorial-law, which is asymptotically power-law) memory derived from differential equations with the Riemann-Liouville and Caputo fractional derivatives (and from Caputo fractional difference equations) with $\alpha \in (0,2)$. In regular continuous dynamical systems, the Poincaré-Bendixson theorem shows that chaos can only arise in systems with more than two dimensions. This is a consequence of the fact that phase space trajectories cannot intersect. Dependence of solutions of fractional differential equations on the whole history of the corresponding system's evolution makes intersection of trajectories possible (see Fig. 2) and one may consider a conjecture that chaos and CBTT are possible in fractional systems with less than two dimensions. One of the goals of the present paper is to investigate a possibility of preserving chaotic behavior during a transition from discrete to continuous fractional systems in less than two dimensions.

There is also a fundamental question of the origin of the Universe and a related question of the origin of the memory of living species. Were there seeds of memory present at the origin of the Universe? Were the fundamental laws of nature

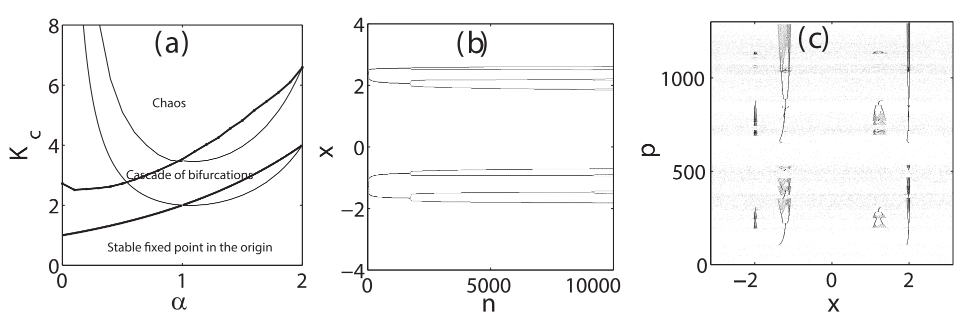

FIG. 1. Bifurcations and cascade of bifurcations type trajectories in fractional/(fractional difference) maps: (a) α-K diagrams for the Caputo fractional (thin lines) and fractional difference (bold lines) Standard Maps (see Ref. 3). Memory parameter α corresponds to the α in Eq. (4) and K is a non-linearity parameter, which in the case $\alpha=2$ coincides with the non-linearity parameter in the regular Standard Map.[74] Fixed point in the origin is stable below the lower curves and chaos exists above the upper curves. Period doubling cascades of bifurcations occur between the lower and upper curves; (b) a single trajectory (CBTT) for the Caputo fractional difference Standard Map with $\alpha=0.1$, $K=2.4$, and the initial condition $x_0=0.1$; (c) a single trajectory (intermittent CBTT) for the Riemann-Liouville fractional Standard Map with $\alpha=1.557$ and $K=4.21$.

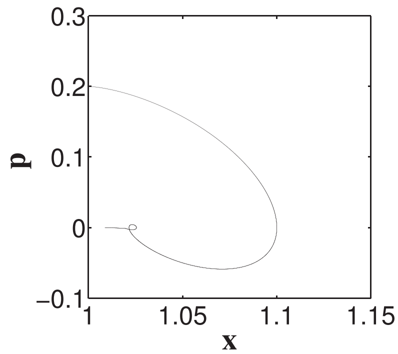

FIG. 2. A self intersecting phase space trajectory of the fractional Caputo Duffing equation ${}_0^C D_t^{1.5} x(t) = x(1-x^2)$, $t \in [0, 40]$ with the initial conditions $x(0) = 1$ and $dx/dt(0) = 0.2$. For the definition of the fractional Caputo derivative, see Eq. (65).

memoryless, or did they have some form of memory? One of the approaches is to assume that on the time and length scales smaller than Planck time and length the fundamental laws should have some memory and a feedback mechanism in order to manage its evolution. This is a purely philosophical question unless we show that the presence of memory may lead to a fundamentally different behavior of the Universe on the large scales and compare it with the observations. This is yet another motivation to investigate the very basic properties of systems with memory.

In what follows we prove the equivalence of the map Eq. (4) with the non-negative integer power-law memory ($\alpha = m > 0$) to the m-step memory map in Sec. II and prove a similar theorem for the maps with $\alpha \in \mathbb{R}$ in Sec. III. In Sec. IV, we consider behavior of the discrete maps with power-law memory and transition to the continuous limit as $h \to 0$; in this section, we also discuss some properties of the fractional Eulerian numbers. In Secs. V and VI, we summarize our results and discuss their possible applications.

II. MAPS WITH NON-NEGATIVE INTEGER POWER-LAW MEMORY

If we assume $\alpha = 1$, then the map Eq. (4) for $n > 0$ is equivalent to

$$x_1 = G_K(x_0, h), \quad x_n - x_{n-1} = G_K(x_{n-1}, h), \quad (n > 1) \tag{5}$$

and requires one initial condition x_0. Calculation of the second backward difference from Eq. (4) for x_n in the case $\alpha = 2$ for $n > 0$ yields

$$\begin{aligned} &x_1 = G_K(x_0, h), \quad x_2 = 2G_K(x_0, h) + G_K(x_1, h), \\ &x_n - 2x_{n-1} + x_{n-2} = G_K(x_{n-1}, h), \quad (n > 2) \end{aligned} \tag{6}$$

with the initial condition x_0. It is easy to see that for $\alpha = 3$ ($n > 3$) and $\alpha = 4$ ($n > 4$) calculating the third and the fourth backward differences for x_n we obtain correspondingly

$$\begin{aligned} &x_1 = G_K(x_0, h), \quad x_2 = 4G_K(x_0, h) + G_K(x_1, h), \\ &x_3 = 9G_K(x_0, h) + 4G_K(x_1, h) + G_K(x_2, h), \\ &x_n - 3x_{n-1} + 3x_{n-2} - x_{n-3} \\ &\quad = G_K(x_{n-1}, h) + G_K(x_{n-2}, h), (n > 3) \end{aligned} \tag{7}$$

and

$$\begin{aligned} &x_1 = G_K(x_0, h), \quad x_2 = 8G_K(x_0, h) + G_K(x_1, h), \\ &x_3 = 27G_K(x_0, h) + 8G_K(x_1, h) + G_K(x_2, h), \\ &x_4 = 64G_K(x_0, h) + 27G_K(x_1, h) + 8G_K(x_2, h) + G_K(x_3, h), \\ &x_n - 4x_{n-1} + 6x_{n-2} - 4x_{n-3} + x_{n-4} \\ &\quad = G_K(x_{n-1}, h) + 4G_K(x_{n-2}, h) + G_K(x_{n-3}, h), \quad (n > 4). \end{aligned} \tag{8}$$

Corresponding summations of Eqs. (5)–(8) with weights $(n-k)^{\alpha-1}$ yield Eq. (4).

Based on Eqs. (5)–(8), we may expect the following theorem:

Theorem 1. *Any long term memory map*

$$x_n = \sum_{k=0}^{n-1} (n-k)^{m-1} G_K(x_k, h), \quad (n > 0), \tag{9}$$

where $m \in \mathbb{N}$, is equivalent to the m-step memory map

$$\begin{aligned} &x_n = \sum_{k=0}^{n-1} (n-k)^{m-1} G_K(x_k, h), \quad (0 < n \le m), \\ &\sum_{k=0}^{m} (-1)^k \binom{m}{k} x_{n-k} = \delta_{m-1} G_K(x_{n-1}, h) + \sum_{k=0}^{m-2} A(m-1, k) \\ &\qquad \times G_K(x_{n-k-1}, h), (n > m). \end{aligned} \tag{10}$$

In Eq. (10), the alternating sum on the left hand side (LHS) is the mth backward difference for the x_n; δ_i is the Kronecker delta ($\delta_0 = 1$ and $\delta_{i \ne 0} = 0$); $A(n, k)$ are the Eulerian numbers

$$A(n, k) = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k+1-j)^n, \tag{11}$$

defined for $k, n \in \mathbb{N}_0$ ($\mathbb{N}_0 := \mathbb{N} \cup \{0\}$), which satisfy the recurrence formula

$$A(n, k) = (k+1)A(n-1, k) + (n-k)A(n-1, k-1). \tag{12}$$

Proof 1. To prove that Eq. (9) leads to Eq. (10), we modify the left side of Eq. (10) using Eq. (9)

$$\begin{aligned} &\sum_{k=0}^{m} (-1)^k \binom{m}{k} x_{n-k} \\ &= \sum_{k=0}^{m} (-1)^k \binom{m}{k} \sum_{i=0}^{n-k-1} (n-k-i)^{m-1} G_K(x_i, h) = S_1 + S_2, \end{aligned} \tag{13}$$

where S_1 and S_2 are the sums taken over the points in the upper triangular and the bottom rectangular areas in Fig. 3 correspondingly. After changing the order of summation in S_1, we have

$$S_1 = \sum_{i=n-m}^{n-1} G_K(x_i, h) \sum_{k=0}^{n-1-i} (-1)^k \binom{m}{k} (n-k-i)^{m-1}. \tag{14}$$

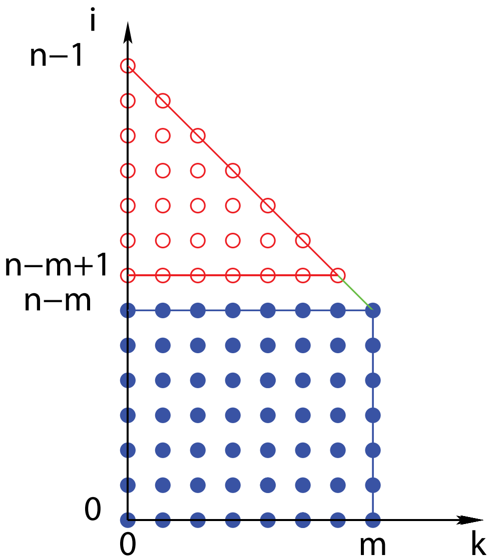

FIG. 3. The area of summation.

After introduction $j = n - i - 1$, we have

$$S_1 = \sum_{j=0}^{m-1} G_K(x_{n-j-1}, h) \sum_{k=0}^{j} (-1)^k \binom{m}{k} (j+1-k)^{m-1}$$
$$= \sum_{j=0}^{m-1} A(m-1, j) G_K(x_{n-j-1}, h)$$
$$= \delta_{m-1} G_K(x_{n-1}, h) + \sum_{k=0}^{m-2} A(m-1,k) G_K(x_{n-k-1}, h). \quad (15)$$

Here, we took into account that according to Eq. (21) below

$$A(m-1, m-1) = \delta_{m-1}. \quad (16)$$

For the second sum, we have

$$S_2 = \sum_{i=0}^{n-m-1} G_K(x_i, h) \sum_{k=0}^{m} (-1)^k \binom{m}{k} (n-k-i)^{m-1}$$
$$= \sum_{i=0}^{n-m-1} G_K(x_i, h) S_3(m, n-i), \quad (17)$$

where

$$S_3(m, j) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (j-k)^{m-1} \quad (18)$$

and $(m+1 \le j \le n)$.

Let us show that $S_3(m, j) = 0$

$$S_3(m,j) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (j-k)^{m-1}$$
$$= \sum_{k=0}^{m} (-1)^k \binom{m}{k} \sum_{i=0}^{m-1} (-1)^i k^i j^{m-1-i} \binom{m-1}{i}$$
$$= \sum_{i=0}^{m-1} (-1)^i j^{m-1-i} \binom{m-1}{i} S_4(m, i) = 0 \quad (19)$$

because

$$S_4(m, i) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} k^i = \begin{cases} 0, & \text{if } 0 \le i < m, \\ m!(-1)^m, & \text{if } i = m. \end{cases} \quad (20)$$

A simple proof of Eq. (20) by induction can be found in Ref. 66 and a very elegant and short proof using generating functions can be found on page 13 of Ref. 67.

For $m > 1$

$$A(m-1, m-1) = \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} (m-k)^{m-1}$$
$$= \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^{m-1}$$
$$= S_3(m, m) = 0. \quad (21)$$

This ends the first part of the proof.

Proof 2. Let us prove that if Eq. (9) is valid for $n - m \le k < n$ $(n > m)$ then, given Eq. (10), it is also valid for $k = n$. Equation (10) can be written as

$$x_n = \sum_{k=0}^{m-1} A(m-1, k) G_K(x_{n-k-1}, h) - \sum_{k=1}^{m} (-1)^k \binom{m}{k} x_{n-k}$$
$$= S_{1n} - S_{2n}. \quad (22)$$

Using the definition of $A(n, k)$, Eq. (11), in S_{1n} and substituting summation index k by $j = n - k - 1$, we have

$$S_{1n} = \sum_{j=n-m}^{n-1} G_K(x_j, h) \sum_{k=0}^{n-j-1} (-1)^k \binom{m}{k} (n-j-k)^{m-1}. \quad (23)$$

Using Eq. (9) and changing the order of summation in S_{2n}, we have

$$S_{2n} = \sum_{j=0}^{n-m-1} G_K(x_j, h) \sum_{k=1}^{m} (-1)^k \binom{m}{k} (n-j-k)^{m-1}$$
$$+ \sum_{j=n-m}^{n-2} G_K(x_j, h) \sum_{k=1}^{n-j-1} (-1)^k \binom{m}{k} (n-j-k)^{m-1}. \quad (24)$$

Now Eq. (22) can be written as

$$x_n = \sum_{j=n-m}^{n-1} (n-j)^{m-1} G_K(x_j, h)$$
$$- \sum_{j=0}^{n-m-1} G_K(x_j, h) \sum_{k=1}^{m} (-1)^k \binom{m}{k} (n-j-k)^{m-1}$$
$$= \sum_{j=0}^{n-1} (n-j)^{m-1} G_K(x_j, h)$$
$$- \sum_{j=0}^{n-m-1} G_K(x_j, h) \sum_{k=0}^{m} (-1)^k \binom{m}{k} [(n-j)-k]^{m-1}. \quad (25)$$

Using binomial formula and Eq. (20), it is easy to prove that the last sum is equal zero.

This ends the proof of Theorem 1.

III. MAPS WITH REAL POWER-LAW MEMORY

Let us consider the following total usually used to define the Grünvald-Letnikov fractional derivative (see Refs. 38 and 47):

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}x_{n-k} &= (-1)^n\binom{\alpha}{n}x_0+\sum_{k=0}^{n-1}(-1)^k\binom{\alpha}{k}\sum_{i=0}^{n-k-1}(n-k-i)^{\alpha-1}G_K(x_i,h)\\ &= (-1)^n\binom{\alpha}{n}x_0+\sum_{i=0}^{n-1}G_K(x_i,h)\sum_{k=0}^{n-i-1}(-1)^k\binom{\alpha}{k}(n-k-i)^{\alpha-1}\\ &= (-1)^n\binom{\alpha}{n}x_0+\sum_{i=0}^{n-1}G_K(x_i,h)A(\alpha-1,n-i-1), \end{aligned} \quad (26)$$

where α is a real number. Transformation from the first to the second line in Eq. (26) requires changing of the order of summations and can be seen on the same Fig. 3 if one assumes $m=n-1$. We used the standard definition (see Refs. 38 and 47)

$$\binom{\alpha}{n}=\frac{\alpha(\alpha-1)(\alpha-n+1)}{n!}=\frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha-n+1)} \quad (27)$$

and the definition of the Eulerian numbers with fractional order parameters introduced in Ref. 68

$$A(\alpha,k)=\sum_{j=0}^{k}(-1)^j\binom{\alpha+1}{j}(k+1-j)^{\alpha}. \quad (28)$$

Validity of Eq. (4) for $n=1$ follows from Eq. (26) with $n=1$. If we assume that Eq. (4) is true for $k\le n$, then from Eq. (26) written for $n+1$ follows:

$$\begin{aligned}x_{n+1} &= -\sum_{s=1}^{n}(-1)^s\binom{\alpha}{s}\sum_{k=0}^{n-s}(n-s-k+1)^{\alpha-1}G_K(x_k,h)\\ &\quad+\sum_{k=0}^{n}G_K(x_k,h)\sum_{s=0}^{n-k}(-1)^s\binom{\alpha}{s}(n-k-s+1)^{\alpha-1}\\ &= -\sum_{k=0}^{n-1}G_K(x_k,h)\sum_{s=1}^{n-k}(-1)^s\binom{\alpha}{s}(n-s-k+1)^{\alpha-1}\\ &\quad+\sum_{k=0}^{n}G_K(x_k,h)\sum_{s=0}^{n-k}(-1)^s\binom{\alpha}{s}(n-s-k+1)^{\alpha-1}\\ &= \sum_{k=0}^{n}(n-k+1)^{\alpha-1}G_K(x_k,h). \end{aligned} \quad (29)$$

Now we may formulate the following theorem:

Theorem 2. *Any long term memory map*

$$x_n=\sum_{k=0}^{n-1}(n-k)^{\alpha-1}G_K(x_k,h),\ \ (n>0), \quad (30)$$

where $\alpha\in\mathbb{R}$ and $n\in\mathbb{N}$, is equivalent to the map

$$\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}x_{n-k} = (-1)^n\binom{\alpha}{n}x_0+\sum_{k=0}^{n-1}G_K(x_{n-k-1},h)A(\alpha-1,k). \quad (31)$$

For $n=0$, Eq. (31) yields the identity $x_0=x_0$ and for $n=1$ it yields $x_1=G_K(x_0,h)$ (notice that $A(\alpha,0)=1$). In the case of a positive integer $\alpha=m$, Eq. (31) is equivalent to (in the case $n>m$) Eq. (10). This follows from equations

$$\binom{m}{k}=0 \text{ for } (k>m),\ \ A(m-1,k)=0 \text{ for } k>m-1 \quad (32)$$

and Eq. (16).

The property $A(m-1,k)=0$ for $k>m-1$ follows from Eq. (16) and repeated applications of the recurrence formula Eq. (12): diagonal elements $A(j,j)$ are equal to zero and each element $A(n,k)$ is a linear combination of the elements to the left $A(n,k-1)$ and below $A(n+1,k)$ with respect to this element.

IV. BEHAVIOR OF SYSTEMS WITH REAL POWER-LAW MEMORY

A. Discrete systems

For any finite h, systems with power-law memory are discrete systems. Their behavior for $\alpha>0$ was preliminarily investigated in papers.[1–5,69–73] In the most important for biological applications cases, $0<\alpha<2$, the investigation is more detailed and is done on the examples of the fractional Standard and Logistic maps. Maps with $m-1<\alpha\le m$, where $m\in\mathbb{N}$, are equivalent to m-dimensional maps. For integer values of $\alpha=m>1$, these maps are m-dimensional volume preserving maps with no (one-step) memory. It is easy to see that after the introduction

$$\begin{aligned} x_k^{(0)} &= x_k,\\ x_k^{(1)} &= x_k^{(0)}-x_{k-1}^{(0)},\\ &\ldots,\\ x_k^{(r)} &= x_k^{(r-1)}-x_{k-1}^{(r-1)},\\ &\ldots,\\ x_k^{(m-1)} &= x_k^{(m-2)}-x_{k-1}^{(m-2)}, \end{aligned} \quad (33)$$

where $k\ge m-1$, the map Eq. (10) can be written as

$$\begin{cases} x_n^{(m-1)}=x_{n-1}^{(m-1)}+\sum_{k=0}^{m-2}A(m-1,k)G_K\left(\sum_{i=0}^{k}(-1)^i\binom{k}{i}x_{n-1}^{(i)},h\right)\\ =x_{n-1}^{(m-1)}+F\left(x_{n-1}^{(0)},\ldots,x_{n-1}^{(m-2)}\right),\\ x_n^{(m-2)}=x_{n-1}^{(m-2)}+x_n^{(m-1)},\\ \ldots,\\ x_n^{(m-k)}=x_{n-1}^{(m-k)}+x_n^{(m-k+1)},\\ \ldots,\\ x_n^{(0)}=x_{n-1}^{(0)}+x_n^{(1)}. \end{cases} \quad (34)$$

The Jacobian matrix $(m \times m)$ of this transformation $J_{(x_{n+1}^{(0)}, x_{n+1}^{(1)}, \ldots, x_{n+1}^{(m-1)})}(x_n^{(0)}, x_n^{(1)}, \ldots, x_n^{(m-1)})$ is

$$\begin{vmatrix} 1+\frac{\partial F}{\partial x_n^{(0)}} & 1+\frac{\partial F}{\partial x_n^{(1)}} & 1+\frac{\partial F}{\partial x_n^{(2)}} & \ldots & 1+\frac{\partial F}{\partial x_n^{(m-2)}} & 1 \\ \frac{\partial F}{\partial x_n^{(0)}} & 1+\frac{\partial F}{\partial x_n^{(1)}} & 1+\frac{\partial F}{\partial x_n^{(2)}} & \ldots & 1+\frac{\partial F}{\partial x_n^{(m-2)}} & 1 \\ \frac{\partial F}{\partial x_n^{(0)}} & \frac{\partial F}{\partial x_n^{(1)}} & 1+\frac{\partial F}{\partial x_n^{(2)}} & \ldots & 1+\frac{\partial F}{\partial x_n^{(m-2)}} & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\partial F}{\partial x_n^{(0)}} & \frac{\partial F}{\partial x_n^{(1)}} & \frac{\partial F}{\partial x_n^{(2)}} & \ldots & 1+\frac{\partial F}{\partial x_n^{(m-2)}} & 1 \\ \frac{\partial F}{\partial x_n^{(0)}} & \frac{\partial F}{\partial x_n^{(1)}} & \frac{\partial F}{\partial x_n^{(2)}} & \ldots & \frac{\partial F}{\partial x_n^{(m-2)}} & 1 \end{vmatrix}.$$

The first column of this matrix can be written as the sum of the column with one in the first row and the remaining zeros and the column which is equal to $\partial F/\partial x_n^{(0)}$ times the last column. The determinant of the latter one is zero. It is easy to show recursively that determinant of the former one is equal to one and the map Eq. (34) indeed is volume preserving.

As it has been shown in paper,[1] the complexity of the behavior of discrete systems with positive power law memory increases with the increase in power. When the power is fractional, systems demonstrate the new types of behavior which include the new types of attractors and the non-uniqueness (dependence on the history) of solutions. The new types of attractors include cascade of bifurcations types trajectories (CBTT) and intermittent CBTT. As a result of the non-uniqueness, attractors may overlap and phase space trajectories intersect. Systems with $\alpha \le 0$ are not investigated.

B. Continuous systems

Let us assume, according to the general approach in the definition of the Grünvald-Letnikov fractional derivative, that

$$x = x(t),\ x_k = x(t_k),\ t_k = a + kh,\ nh = t - a \tag{35}$$

for $0 \le k \le n$. If one divides Eq. (10) by h^m in the case of positive integer values of α and considers a limit $h \to 0+$, then the left side of the resulting equation will give the mth derivative from $x(t)$ at the time t. If we assume

$$G_K(x,h) = \frac{1}{\Gamma(\alpha)} h^\alpha G_K(x), \tag{36}$$

where $G_K(x)$ is continuous, then $x(t) \in C^m$. The map Eq. (4) can be written as

$$x(t) = \frac{1}{\Gamma(\alpha)} h \sum_{k=0, nh=t-a}^{n-1} (t-t_k)^{\alpha-1} G_K(x(t_k)) \tag{37}$$

and in the limit $h \to 0$ Theorem 1 can be formulated as a well-known result.

Theorem 3. *The Volterra integral equation of the second kind*

$$x(t) = \frac{1}{\Gamma(m)} \int_a^t \frac{G_K(x(\tau))d\tau}{(t-\tau)^{1-m}},\quad (t > a), \tag{38}$$

where $m \in \mathbb{N}$ and $G_K(x) \in C^0$ on the range $D \in \mathbb{R}$ of the function $x(t)$ ($t \in [a,b]$), is equivalent on $[a,b]$ to the differential equation

$$\frac{d^m x(t)}{dt^m} = \frac{1}{\Gamma(m)} \sum_{k=0}^{m-1} A(m-1,k) G_K(x(t)) = G_K(x(t)), \tag{39}$$

where we used the classical result $\sum_{k=0}^{m-2} A(m-1,k) = \Gamma(m)$, with the zero initial conditions

$$c_k = \frac{d^k x(t)}{dt^k}(t=a) = 0,\quad k = 0,1,\ldots,m-1. \tag{40}$$

While discrete equations (9) and (10) have unique solutions for any function $G_K(x)$, the corresponding continuous equations (38) and (39) require the Lipschitz condition on $G_K(x)$ in D. Because this is not essential for this paper, in what follows, we always assume that the $G_K(x)$ satisfies the Lipschitz condition in D.

In the case $c_k \ne 0$, the well-known equivalence of the differential equation (39) to the Volterra integral equation of the second kind

$$x(t) = \sum_{k=0}^{m-1} \frac{c_k}{\Gamma(k+1)}(t-a)^k + \frac{1}{\Gamma(m)} \int_a^t \frac{G_K(x(\tau))d\tau}{(t-\tau)^{1-m}},\quad (t > a) \tag{41}$$

follows in the limit $h \to 0$ from the following generalization of Theorem 1.

Theorem 4. *Any long term memory map*

$$x_n = \sum_{k=0}^{m-1} \frac{c_k}{\Gamma(k+1)}(nh)^k + \frac{h^m}{\Gamma(m)} \sum_{k=0}^{n-1} (n-k)^{m-1} G_K(x_k),\quad (n > 0), \tag{42}$$

where $m \in \mathbb{N}$, is equivalent to the m-step memory map

$$\begin{aligned} x_n &= \sum_{k=0}^{m-1} \frac{c_k}{\Gamma(k+1)}(nh)^k + \frac{h^m}{\Gamma(m)} \sum_{k=0}^{n-1} (n-k)^{m-1} G_K(x_k),\quad (0 < n \le m), \\ \sum_{k=0}^{m} (-1)^k \binom{m}{k} x_{n-k} &= \frac{h^m}{\Gamma(m)} \sum_{k=0}^{m-1} A(m-1,k) G_K(x_{n-k-1}),\quad (n > m). \end{aligned} \tag{43}$$

Proof. The proof of this theorem is similar to the proof of Theorem 1.

(1) The first part of the proof uses the fact that for $n > m$ mth backward difference of the first sum in Eq. (42) is equal to zero

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \sum_{i=0}^{m-1} \frac{c_i}{\Gamma(i+1)} [(n-k)h]^i = \sum_{i=0}^{m-1} \frac{c_i h^i}{\Gamma(i+1)} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (n-k)^i. \tag{44}$$

After we apply the binomial formula to $(n-k)^i$ and use the identity Eq. (20), it is clear that the internal sum on the right hand side (RHS) is equal to zero.

(2) In the second part of the proof, an additional term on the RHS of Eq. (24) is

$$\sum_{k=1}^{m}(-1)^k\binom{m}{k}\sum_{i=0}^{m-1}\frac{c_i}{\Gamma(i+1)}[(n-k)h]^i = \sum_{i=0}^{m-1}\frac{c_ih^i}{\Gamma(i+1)}\sum_{k=1}^{m}(-1)^k\binom{m}{k}(n-k)^i = -\sum_{i=0}^{m-1}\frac{c_i}{\Gamma(i+1)}(nh)^i, \tag{45}$$

which completes the proof of Theorem 4.

(3) From Eq. (43) follows that $x(a)=x_0=c_0$ and for $0<n<m$

$$x^{(n)}(a) = \lim_{h\to 0}\frac{1}{h^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}x_{n-k} = \lim_{h\to 0}\frac{1}{h^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{i=0}^{m-1}\frac{c_ih^i}{\Gamma(i+1)}(n-k)^i = \lim_{h\to 0}\frac{1}{h^n}\sum_{i=0}^{m-1}\frac{c_ih^i}{\Gamma(i+1)}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^i = c_n. \tag{46}$$

In the last sum, all terms with $i<n$ are zeros because of Eq. (20); limit $h\to 0$ of all terms with $i>n$ is also zero; when $i=n$ the only term which gives non-zero sum over k in the binomial expansion of $(n-k)^n$ is $(-1)^nk^n$ and the corresponding sum is $n!$.

As we mentioned in Sec. I, a transition from discrete to continuous dynamical system in the case $m=2$ results in the disappearance of chaos, which, in general, should not be the case for systems with non-degenerate memory and for the case, which is important in applications, $0<\alpha<2$, we may expect that corresponding continuous systems will still have chaotic solutions.

Let us consider the limit $h\to 0$ for fractional $\alpha>0$ in Eq. (31) divided by h^α given in Eq. (35)

$$\lim_{\substack{n\to\infty\\ nh=t-a}} h^{-\alpha}\left\{\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}x_{n-k} = (-1)^n\binom{\alpha}{n}x_0+\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{n-1}h^\alpha G_K(x_{n-k-1})A(\alpha-1,k)\right\}. \tag{47}$$

The LHS of Eq. (47) coincides with the definition of the Grünvald-Letnikov fractional derivative

$$\lim_{\substack{n\to\infty\\ nh=t-a}} h^{-\alpha}\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}x_{n-k} = \lim_{\substack{h\to 0\\ nh=t-a}} h^{-\alpha}\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}x(t-kh) = {}_aD_t^\alpha x(t), \tag{48}$$

where $x(t)$ is assumed to be $\lceil\alpha\rceil$ times continuously differentiable on $[a,t]$. The first term on the RHS of Eq. (47) is equal to zero:

$$\lim_{\substack{n\to\infty\\ nh=t-a}} h^{-\alpha}(-1)^n\binom{\alpha}{n}x_0 = (-1)^nx_0(t-a)^{-\alpha}\lim_{n\to\infty}n^\alpha\binom{\alpha}{n} \tag{49}$$

and

$$\lim_{n\to\infty}\left|n^\alpha\binom{\alpha}{n}\right| = \lim_{n\to\infty}\left|\frac{n^\alpha\Gamma(\alpha+1)}{n!\Gamma(1-(n-\alpha))}\right| = \left|\frac{\Gamma(\alpha+1)\sin(\pi\alpha)}{\pi}\right|\lim_{n\to\infty}\frac{n^\alpha\Gamma(n-\alpha)}{n!} = \left|\frac{\Gamma(\alpha+1)\sin(\pi\alpha)}{\pi}\right|\lim_{n\to\infty}\frac{n^\alpha n^{-\alpha}(n-1)!}{n!} = \left|\frac{\Gamma(\alpha+1)\sin(\pi\alpha)}{\pi}\right|\lim_{n\to\infty}\frac{1}{n}=0. \tag{50}$$

Here, we used the well known properties of the Gamma-function: $\Gamma(1-z)\Gamma(z)=\pi/\sin(\pi z)$ and $\lim_{n\to\infty}\Gamma(n+\alpha)/[\Gamma(n)n^\alpha]=1$.

The evaluation of the last term in Eq. (47) will require some revision of the results obtained in Refs. 68 and 75:

(1) The last theorem (Theorem 9) proven in Ref. 68, which states that for any $\alpha>1$ and $k\in\mathbb{N}_0$

$$A(\alpha,k)=\Gamma(\alpha+1)\int_k^{k+1}p_\alpha(x)dx, \tag{51}$$

$$\sum_{k=0}^{\infty}A(\alpha,k)=\Gamma(\alpha+1), \tag{52}$$

where

$$p_\alpha(x) := \begin{cases} 0, & -\infty<x\le 0\\ \dfrac{1}{\Gamma(\alpha)}\sum_{0\le j<x}(-1)^j\binom{\alpha}{j}(x-j)^{\alpha-1}, & 0<x<\infty \end{cases} \tag{53}$$

is based on the results from Ref. 75 which are obtained for $\alpha>0$. The one line proof of Theorem 9 in Ref. 68 is nowhere violated for $0<\alpha\le 1$. Thus, we assume that Eqs. (51) and (52) are true for $\alpha>0$.

(2) According to the asymptotic formula for large k from the fifth page of Ref. 75 for $\alpha>0$, integer k, and $0<\Theta\le 1$

$$p_\alpha(k+\Theta)=O(k^{-\alpha-1})\Theta^{\alpha-1}+O(k^{-\alpha-1}+k^{\alpha-[\alpha]-2}). \tag{54}$$

Then

$$A(\alpha-1,k)=\sum_{j=0}^{k}(-1)^j\binom{\alpha}{j}(k+1-j)^{\alpha-1} = \Gamma(\alpha)p_\alpha(k+1)=O(k^{-\alpha-1}+k^{\alpha-[\alpha]-2}). \tag{55}$$

As a continuous function, $x(\tau)$ attains its maximum x_{max} and minimum x_{min} values on $[a,t]$ and is bounded ($|x|<M_1$).

Assuming that $G_K(x)$ is a continuous function on $[x_{min}, x_{max}]$, this function is also bounded ($|G_K(x)| < M_2$). This yields

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} |G_K(x_{n-k-1})A(\alpha-1,k)| \le \lim_{n\to\infty} \sum_{k=0}^{n-1} M_2 O(k^{-\alpha-1} + k^{\alpha-\lceil\alpha\rceil-2}) < \infty. \tag{56}$$

Now, for $\alpha > 0$, we may write

$$\lim_{\substack{n\to\infty \\ nh=t-a}} \sum_{k=0}^{n} G_K(x_{n-k})A(\alpha-1,k) = \lim_{\substack{n\to\infty \\ nh=t-a}} \sum_{k=0}^{N_1} G_K\left(x\left(t-\frac{k}{n}(t-a)\right)\right)A(\alpha-1,k) + \sum_{k=N_1+1}^{\infty} G_K(x_{n-k})A(\alpha-1,k), \tag{57}$$

where for an arbitrarily small $\varepsilon > 0$ there exists N such that for $\forall N_1 > N$ the following holds

$$\left|\sum_{k=N_1+1}^{\infty} G_K(x_{n-k})A(\alpha-1,k)\right| < \frac{\varepsilon}{2}. \tag{58}$$

In Eq. (57), by choosing $n > N_2 \gg N_1$, the argument of the function $x(\tau)$ in the first sum on the right can be made arbitrarily close to t so that due to the continuity of $x(\tau)$ and $G_K(x)$

$$\sum_{k=0}^{N_1}\left[G_K\left(x\left(t-\frac{k}{n}(t-a)\right)\right) - G_K(x(t))\right]A(\alpha-1,k) < \frac{\varepsilon}{2}. \tag{59}$$

Equations (56)–(59) yield

$$\lim_{\substack{n\to\infty \\ nh=t-a}} \sum_{k=0}^{n} G_K(x_{n-k})A(\alpha-1,k) = G_K(x(t)) \lim_{n\to\infty} \sum_{k=0}^{n} A(\alpha-1,k), \tag{60}$$

where the series on the right converges absolutely for $\alpha > 0$ according to Eq. (55). According to Eqs. (52) and (39) for $\alpha \ge 1$ the sum on the right is equal to $\Gamma(\alpha)$ and in the limit $h \to \infty$, we may formulate the following theorem.

Theorem 5. *For $\alpha \in \mathbb{R}$, $\alpha \ge 1$, the Volterra integral equation of the second kind*

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{G_K(x(\tau))d\tau}{(t-\tau)^{1-\alpha}}, \quad (t > a), \tag{61}$$

where $G_K(x(\tau))$ is a continuous on $x \in [x_{min}(\tau), x_{max}(\tau)]$, $\tau \in [a,t]$ function is equivalent to the fractional differential equation

$$ {}_aD_t^\alpha x(t) = G_K(x(t)), \tag{62}$$

where the derivative on the left is the Grünvald-Letnikov fractional derivative, with the zero initial conditions

$$c_k = \frac{d^k x(t)}{dt^k}(t=a) = 0, \quad k = 0,1,...,\lceil\alpha\rceil - 1. \tag{63}$$

The methods used in Refs. 68 and 75 do not allow us to prove Eq. (52) for $-1 < \alpha < 0$ but based on the convergence of the series in Eq. (60) we will formulate the following conjecture.

Conjecture 6 *Theorem 5 is valid for $0 < \alpha < 1$.*

Theorem 5 and Conjecture 6 are not new results. It is known (see Refs. 38, 47, and 48) that Riemann-Liouville and Caputo derivatives coincide in the case $c_k = d^k x(t)/dt(t=a) = 0$, $k = 0,1,...,\lceil\alpha\rceil$ and also that for $x(t) \in C^{\lceil\alpha\rceil}[a,T]$ and integrable $x^{\lceil\alpha\rceil+1}(t)$ in $[a,T]$ $(a<t<T)$ Riemann-Liouville and Grünvald-Letnikov fractional derivatives ${}_aD_t^\alpha x(t)$ coincide.

For $t>a$, the left-sided Riemann-Liouville fractional derivative is defined as

$${}_a^{RL}D_t^\alpha x(t) = D_t^n\, {}_aI_t^{n-\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{x(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}, \tag{64}$$

where $n-1 \le \alpha < n$, $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, $D_t^n = d^n/dt^n$, and ${}_0I_t^\alpha$ is a Riemann-Liouville fractional integral. In the definition of the left-sided Caputo fractional derivative, the order of integration and differentiation is switched

$${}_a^CD_t^\alpha x(t) = {}_aI_t^{n-\alpha} D_t^n x(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{D_\tau^n x(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}. \tag{65}$$

In Refs. 76 and 77, Kilbas and Marzan showed that fractional differential equation

$${}_a^CD_t^\alpha x(t) = G_K(t, x(t)), \quad 0 < \alpha, \; t \in [a,T] \tag{66}$$

with the initial conditions

$$\frac{d^k x(t)}{dt^k}(t=a) = c_k, \quad k = 0,1,...,\lceil\alpha\rceil - 1 \tag{67}$$

is equivalent to the Volterra integral equation of the second kind

$$x(t) = \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{c_k}{\Gamma(k+1)}(t-a)^k + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{G_K(\tau, x(\tau))d\tau}{(t-\tau)^{1-\alpha}}, \quad (t>a) \tag{68}$$

in the space $C^{\lceil\alpha\rceil-1}[a,T]$. A similar result for the equivalence of the equation with the Riemann-Liouville fractional derivative

$${}_a^{RL}D_t^\alpha x(t) = G_k(t, x(t)), \quad 0 < \alpha, \tag{69}$$

with the initial conditions

$$({}_a^{RL}D_t^{\alpha-k} x)(a+) = c_k, \;\; k = 1,2,...,\lceil\alpha\rceil \tag{70}$$

to the Volterra integral equation of the second kind

$$x(t)=\sum_{k=1}^{\lceil\alpha\rceil}\frac{c_k}{\Gamma(\alpha-k+1)}(t-a)^{\alpha-k}+\frac{1}{\Gamma(\alpha)}\int_a^t\frac{G_K(\tau,x(\tau))d\tau}{(t-\tau)^{1-\alpha}},\quad (t>a) \tag{71}$$

for $x(t)\in L(a,T)$ and $G(t,x(t))\in L(a,T)$ was proved by Kilbas *et al*. in Refs. 78 and 79.

On one hand, in the case of $x(t)\in C^{\lceil\alpha\rceil-1}[a,T]$ and the zero initial conditions, all above defined derivatives are equivalent and Eq. (62) is equivalent to Eq. (61). On the other hand, we saw that for $\alpha>0$ Eq. (61) is equivalent (see Eq. (60)) to

$${}_aD_t^\alpha x(t)=\frac{1}{\Gamma(\alpha)}G_K(x(t))\lim_{n\to\infty}\sum_{k=0}^{n}A(\alpha-1,k). \tag{72}$$

This proves Conjecture 6 and Eq. (52) for $\alpha>-1$.

We will end this section with the theorem which in the limit $h\to 0$ yields the equivalence of problem Eq. (69) and Eq. (70) to the problem Eq. (71) in the case $c_{\lceil\alpha\rceil}=0$, which corresponds to a finite value of $x(a)$.

Theorem 7. *Any long term memory map*

$$x_n=\sum_{k=1}^{\lceil\alpha\rceil-1}\frac{c_k}{\Gamma(\alpha-k+1)}(nh)^{\alpha-k}+\sum_{k=0}^{n-1}(n-k)^{\alpha-1}G_K(x_k,h), \tag{73}$$

where $\alpha\in\mathbb{R}$, *is equivalent to the map*

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}x_{n-k}-\sum_{i=1}^{\lceil\alpha\rceil-1}\frac{c_ih^{\alpha-i}}{\Gamma(\alpha-i+1)}\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k}A(\alpha-i,n-k-1)\\&=(-1)^n\binom{\alpha}{n}x_0+\sum_{k=0}^{n-1}G_K(x_{n-k-1},h)A(\alpha-1,k).\end{aligned} \tag{74}$$

Proof 1. The first part of the proof is the same as the proof of Theorem 2 plus the following result:

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}\sum_{i=1}^{\lceil\alpha\rceil-1}\frac{c_i}{\Gamma(\alpha-i+1)}[(n-k)h]^{\alpha-i}&=\sum_{i=1}^{\lceil\alpha\rceil-1}\frac{c_ih^{\alpha-i}}{\Gamma(\alpha-i+1)}\sum_{k=0}^{n-1}(-1)^k\binom{\alpha}{k}(n-k)^{\alpha-i}\\&=\sum_{i=1}^{\lceil\alpha\rceil-1}\frac{c_ih^{\alpha-i}}{\Gamma(\alpha-i+1)}\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k}A(\alpha-i,n-k-1).\end{aligned} \tag{75}$$

Here, we used the identity

$$\begin{aligned}\sum_{k=0}^{n-1}(-1)^k\binom{\alpha}{k}(n-k)^{\alpha-i}&=\sum_{k=0}^{n-1}(-1)^k\sum_{j=0}^{i-1}\binom{i-1}{j}\binom{\alpha-i+1}{k-j}(n-k)^{\alpha-i}=\sum_{j=0}^{i-1}\binom{i-1}{j}\sum_{k=j}^{n-1}(-1)^k\binom{\alpha-i+1}{k-j}(n-k)^{\alpha-i}\\&=\sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}\sum_{k=0}^{n-j-1}(-1)^k\binom{\alpha-i+1}{k}(n-k-j)^{\alpha-i}\\&=\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k}A(\alpha-i,n-k-1),\quad 0<i<\lceil\alpha\rceil.\end{aligned} \tag{76}$$

Proof 2. Equation (74) with $n=1$ yields Eq. (73). If we assume that Eq. (73) is true for $k\le n$, then we may write the equation for x_{n+1} as in Eq. (29) with two additional terms on the RHS

$$\begin{aligned}x_{n+1}&=\sum_{k=0}^{n}(n-k+1)^{\alpha-1}G_K(x_k,h)+\sum_{i=1}^{\lceil\alpha\rceil-1}\frac{c_ih^{\alpha-i}}{\Gamma(\alpha-i+1)}\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k}A(\alpha-i,n-k)-\sum_{k=1}^{n}(-1)^k\binom{\alpha}{k}\sum_{i=1}^{\lceil\alpha\rceil-1}\frac{c_ih^{\alpha-i}}{\Gamma(\alpha-i+1)}(n+1-k)^{\alpha-i}\\&=\sum_{k=0}^{n}(n-k+1)^{\alpha-1}G_K(x_k,h)+\sum_{i=1}^{\lceil\alpha\rceil-1}\frac{c_ih^{\alpha-i}}{\Gamma(\alpha-i+1)}\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k}A(\alpha-i,n-k)\\&-\sum_{i=1}^{\lceil\alpha\rceil-1}\frac{c_ih^{\alpha-i}}{\Gamma(\alpha-i+1)}\left[\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}(n+1-k)^{\alpha-i}-(n+1)^{\alpha-i}\right]\\&=\sum_{k=1}^{\lceil\alpha\rceil-1}\frac{c_k}{\Gamma(\alpha-k+1)}[(n+1)h]^{\alpha-k}+\sum_{k=0}^{n}(n-k+1)^{\alpha-1}G_K(x_k,h).\end{aligned} \tag{77}$$

Proof 3. From fractional calculus, it is known that the Grünvald-Letnikov fractional derivative of the power function $f(t) = (t-a)^\beta$ is

$$ {}_aD_t^\alpha (t-a)^\beta = \lim_{\substack{n\to\infty \\ nh=t-a}} h^{-\alpha} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} [(n-k)h]^\beta = \frac{\Gamma(\beta+1)}{\Gamma(-\alpha+\beta+1)} (t-a)^{\beta-\alpha}, \tag{78} $$

where $\alpha < 0$, $\beta > -1$ or $0 \le m \le \alpha < m+1$, $\beta > m$ (see Sec. 2.2.4 in Ref. 38). This yields for $\beta = \alpha - i$, $i \in \mathbb{Z}$, and $\beta, \alpha > 0$

$$ \lim_{\substack{n\to\infty \\ nh=t-a}} h^{-\alpha} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} [(n-k)h]^\beta = \begin{cases} \Gamma(\beta+1)(t-a)^{-i}/(-i)!, & i < 0; \\ \Gamma(\beta+1), & i = \alpha - \beta = 0; \\ 0, & i > 0. \end{cases} \tag{79} $$

For $k = 1, 2, ., \lceil\alpha\rceil - 1$, Eq. (73) leads to

$$ \begin{aligned} {}_aD_t^{\alpha-k} x(a+) &= \lim_{t\to a+} \lim_{\substack{n\to\infty \\ nh=t-a}} h^{k-\alpha} \sum_{j=0}^{n} (-1)^j \binom{\alpha-k}{j} x_{n-j}, = \lim_{t\to a+} \lim_{\substack{n\to\infty \\ nh=t-a}} h^{k-\alpha} \sum_{j=0}^{n} (-1)^j \binom{\alpha-k}{j} \sum_{i=1}^{\lceil\alpha\rceil-1} \frac{c_i}{\Gamma(\alpha-i+1)} [(n-j)h]^{\alpha-i} \\ &= \sum_{i=1}^{\lceil\alpha\rceil-1} \frac{c_i}{\Gamma(\alpha-i+1)} \lim_{t\to a+} \lim_{\substack{n\to\infty \\ nh=t-a}} h^{k-\alpha} \sum_{j=0}^{n} (-1)^j \binom{\alpha-k}{j} [(n-j)h]^{\alpha-i} \\ &= \sum_{i=1}^{\lceil\alpha\rceil-1} \frac{c_i}{\Gamma(\alpha-i+1)} \begin{cases} \lim_{t\to a+} \Gamma(\alpha-i+1)(t-a)^{k-i}/(k-i)!, & k > i; \\ \Gamma(\alpha-i+1), & i = k; \\ 0, & k < i \end{cases} = c_k. \end{aligned} \tag{80} $$

The direct calculation of the LHS of Eq. (79) with $m = -i \ge 0$ yields

$$ \begin{aligned} \lim_{\substack{n\to\infty \\ nh=t-a}} h^{-\alpha} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} [(n-k)h]^\beta &= \lim_{\substack{n\to\infty \\ nh=t-a}} h^{m} \sum_{k=0}^{n} (-1)^k \binom{\beta-m}{k} (n-k)^\beta \\ &= \lim_{\substack{n\to\infty \\ nh=t-a}} h^{m} \sum_{k=0}^{n} (-1)^k (n-k)^\beta \sum_{j_0=0}^{k} (-1)^{j_0} \binom{\beta-m+1}{k-j_0} \\ &= (t-a)^m \lim_{n\to\infty} n^{-m} \sum_{j_0=0}^{n-1} \sum_{k=0}^{n-j_0-1} (-1)^k \binom{\beta-m+1}{k} (n-j_0-k)^\beta \\ &= (t-a)^m \lim_{n\to\infty} n^{-m} \sum_{j_0=0}^{n-1} \sum_{k=0}^{j_0} (-1)^k \binom{\beta-m+1}{k} (j_0+1-k)^\beta \\ &= (t-a)^m \lim_{n\to\infty} n^{-m} \sum_{j_0=0}^{n-1} \sum_{j_1=0}^{j_0} \sum_{j_2=0}^{j_1} \ldots \sum_{k=0}^{j_m} (-1)^k \binom{\beta+1}{k} (j_m+1-k)^\beta \\ &= (t-a)^m \lim_{n\to\infty} n^{-m} \sum_{j_0=0}^{n-1} \sum_{j_1=0}^{j_0} \sum_{j_2=0}^{j_1} \ldots \sum_{j_m=0}^{j_{m-1}} A(\beta, j_m) = \frac{1}{m!} (t-a)^m \lim_{n\to\infty} \sum_{s=0}^{n-1} \frac{\Gamma(m+n-s)}{n^m \Gamma(n-s)} A(\beta, s) \\ &= \frac{1}{m!} (t-a)^m \lim_{n\to\infty} \sum_{s=0}^{n-1} D(m,n,s) A(\beta,s) = \frac{1}{m!} (t-a)^m \lim_{n\to\infty} S_n = \frac{1}{m!} \Gamma(\beta+1)(t-a)^m. \end{aligned} \tag{81} $$

The transition within the sixth line of this chain of transformations is based on Theorem 1 from Ref. 4, which states that for $\forall n \in \mathbb{N}$

$$ {}_a\Delta_t^{-n} f(t) = \frac{1}{(n-1)!} \sum_{s=a}^{t-n} (t-s-1)^{(n-1)} f(s) = \sum_{s^0=a}^{t-n} \sum_{s^1=a}^{s^0} \ldots \sum_{s^{n-1}=a}^{s^{n-2}} f(s^{n-1}), \tag{82} $$

where falling factorial function $t^{(\alpha)}$ is defined as

$$t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}. \tag{83}$$

For $m = 0$, the equality

$$\lim_{n\to\infty} \sum_{s=0}^{n-1} \frac{\Gamma(m+n-s)}{n^m \Gamma(n-s)} A(\beta, s) = \Gamma(\beta+1) \tag{84}$$

coincides with Eq. (52), which is true for $\beta > -1$. Series $\sum_{s=0}^{n-1} A(\beta, s)$ converges absolutely and $D(m, n, s)$, which is a product of m factors

$$D(m,n,s) = \left(1-\frac{s}{n}\right)\left(1-\frac{s-1}{n}\right)\ldots\left(1-\frac{s-m+1}{n}\right) < \left(1+\frac{m}{n}\right)^m, \tag{85}$$

is bounded. This means that S_n converges absolutely to some S. For $\forall \varepsilon > 0$, there exists N_1 such that for $\forall N \geq N_1$ simultaneously $|\sum_{s=N_1}^{N_2-1} D(m, N_2, s) A(\beta, s)| < \varepsilon/3$ and $|\Gamma(\beta+1) - \sum_{s=0}^{N_1-1} A(\beta, s)| < \varepsilon/3$. For $N_2 \gg N_1$ and $s \leq N_1$,

$$1 - m\frac{N_1}{N_2} < \left(1-\frac{N_1}{N_2}\right)^m < D(m, N_2, s) < \left(1+\frac{N_1}{N_2}\right)^m < 1 + \frac{m^2}{N_2} + o\left(\frac{m^2}{N_2}\right) \tag{86}$$

and

$$|D(m, N_2, s) - 1| < m\frac{N_1}{N_2}. \tag{87}$$

For $\forall N_2 > N_\varepsilon$, where

$$N_\varepsilon = \frac{3mN_1 \sum_{s=0}^{\infty} |A(\beta, s)|}{\varepsilon}, \tag{88}$$

we can write

$$|S_{N_2} - \Gamma(\beta+1)| = \left|\sum_{s=0}^{N_2-1} D(m, N_2, s) A(\beta, s) - \Gamma(\beta+1)\right| < \left|\sum_{s=N_1}^{N_2-1} D(m, N_2, s) A(\beta, s)\right| + \sum_{s=0}^{N_1-1} |D(m, N_2, s) - 1||A(\beta, s)| + \left|\sum_{s=0}^{N_1-1} A(\beta, s) - \Gamma(\beta+1)\right| < \varepsilon. \tag{89}$$

This means that $S = \Gamma(\beta+1)$.

If in Eq. (79) $i > 0$, then using Eq. (76), we may write

$$\lim_{\substack{n\to\infty \\ nh = t-a}} h^{-\alpha} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} [(n-k)h]^\beta = \lim_{\substack{n\to\infty \\ nh = t-a}} h^{-i} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} (n-k)^{\alpha-i} = (t-a)^{-i} \lim_{n\to\infty} n^i \sum_{k=0}^{i-1} (-1)^k \binom{i-1}{k} A(\alpha - i, n-k-1). \tag{90}$$

Comparing Eq. (90) with Eq. (79), we may formulate a new property of Eulerian numbers

$$\lim_{n\to\infty} n^i \sum_{k=0}^{i-1} (-1)^k \binom{i-1}{k} A(\alpha - i, n-k-1) = 0, \ \ (i > 0). \tag{91}$$

V. SUMMARY

Here, we summarize the main results obtained in this paper. We start with the fractional difference calculus. Theorem 2 can be formulated as the equivalence of maps with power-law memory (power $\alpha - 1$) generated by a function $G_K(x, h)$, where x is the map's variable, K is a parameter, and h is the map's step (constant time interval between two consecutive iterations), to fractional difference equations in which the Grünvald-Letnikov like fractional difference operator acting on the map's variable on the LHS is equal to the convolution of the values of the generating function from all previous steps k with the Eulerian numbers $A(\alpha - 1, k)$ on the RHS. In the case of an integer power-law memory, this theorem can be formulated as a simpler result (Theorem 1): any long term non-negative integer power-law memory (power $m - 1$) map is equivalent to a m-step memory map (the mth backward difference on the LHS is equal to the convolution of the generating functions from the $MAX(1, m-1)$ previous values of the map's variable with the Eulerian numbers $A(m-1, k)$ on the RHS). Maps with the long term positive integer ($m > 1$) power-law memory are equivalent to the m-dimensional volume preserving maps with no (one-step) memory.

In the continuous limit ($h \to 0$), Theorems 1 and 2 yield the well-known results of the equivalence of differential equations to the integral Volterra equations of the second kind in both integer and fractional cases. In the process of transition to the continuous limit, we were able to prove that the property of Eulerian numbers $\sum_{k=0}^{\infty} A(\alpha, k) = \Gamma(\alpha+1)$, Eq. (52), known for $\alpha > 1$, is true for $\alpha > -1$ and obtained a new property of Eulerian numbers [Eq. (91)].

VI. CONCLUSION

Phase space of discrete non-linear integer maps with power-law memory may demonstrate islands of stability and chaotic areas. These maps are well investigated for $m = 2$ but investigation of general properties of such maps for $m > 2$ is far from completion. Equation (5) yields the regular logistic map if we assume $G_K(x,h) = -G_K^L(x) = -x + Kx(1-x)$. Equation (34) with $G_K(x,h) = -G_K^{SM}(x) = -K\sin(x)$ yields the regular standard map. This is why we will call maps Eqs. (9), (10), (73), and (74) with $G_K(x,h) = -G_K^L(x)$ the logistic maps with memory or the fractional logistic maps and with $G_K(x,h) = -G_K^{SM}(x)$ the standard maps with memory or the fractional standard maps. Initial investigation of maps with long term fractional power-law memory in Refs. 1–5 and 69–73 has been done on the examples of the fractional logistic and standard maps with $0 < \alpha < 3$. New types of attractors (CBTT) were obtained for $0 < \alpha < 2$.

If we consider Eq. (74) with $G_K(x,h) = h^{\alpha}KG(x)$, then, up to the term depending on the initial conditions, solution of this fractional difference equation depends only on the product $h^{\alpha}K$. This type of systems includes fractional standard map ($G(x) = -\sin(x)$) and a system, which in the limit $h \to 0$ yields the fractional logistic differential equation ($G(x) = x(1-x)$). In the case $h = 1$ for $0 < \alpha < 2$, the fractional standard and logistic maps with $|K| \lesssim 1$ have only sinks (see Fig. 1(a)) (no chaos). We may conclude that for small h there will be no chaotic trajectories for $|K| \lesssim h^{-\alpha}$, which implies a possibility that in the limit $h \to 0$ the fractional logistic differential equation and the limit of the fractional standard map ($D^{\alpha}x(t)/Dt^{\alpha} = K\sin(x)$) will have no chaotic solutions for $0 < \alpha < 2$. This kind of reasoning may not work for all fractional systems. The stability of the $x = 1$ fixed point of the fractional logistic differential equation also follows from the elementary stability analysis (see, e.g., Ref. 80). In Ref. 81, on the basis of the analysis of two fractional order autonomous non-linear systems, authors conjectured that chaos may exist in autonomous non-linear systems with a total system's order of $2+\varepsilon$, where $0 < \varepsilon < 1$. Examples of fractional chaotic attractors in continuous systems of the order less than three can be found also in Ref. 82.

To the best of our knowledge, there is no proof that chaos cannot exist in fractional systems of the order less than two. To prove it or to find a counterexample is a challenging problem. Another challenging problem is to investigate if there are analogs of cascade of bifurcations type trajectories in continuous systems.

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