

# Tail asymptotics of free path lengths for the periodic Lorentz process. On Dettmann's geometric conjectures.

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## Abstract

In the simplest case, consider a  $\mathbb{Z}^d$ -periodic ( $d \geq 3$ ) arrangement of balls of radii  $< 1/2$ , and select a random direction and point (outside the balls). According to Dettmann's first conjecture, the probability that the so determined free flight (until the first hitting of a ball) is larger than  $t \gg 1$  is  $\sim \frac{C}{t}$ , where  $C$  is explicitly given by the geometry of the model. In its simplest form, Dettmann's second conjecture is related to the previous case with tangent balls (of radii  $1/2$ ). The conjectures are established in a more general setup: for  $\mathcal{L}$ -periodic configuration of - possibly intersecting - convex bodies with  $\mathcal{L}$  being a non-degenerate lattice. These questions are related to Pólya's visibility problem (1918), to theories of Bourgain-Golse-Wennberg (1998-) and of Marklof-Strömbergsson (2010-). The results also provide the asymptotic covariance of the periodic Lorentz process assuming it has a limit in the super-diffusive scaling, a fact if  $d = 2$  and the horizon is infinite.

## 1 Introduction

The subject of our paper is the verification of the first two, purely geometric, conjectures of Dettmann, circumscribed in the abstract (the final, third one is of dynamical feature). A substantial motivation for the conjectures - and for us, too - came from the dynamical theory of Brownian motion, more concretely from that of the periodic Lorentz process. Therefore the introduction will consist of two parts. In the first one we restrict ourselves to the geometric problems, whereas in the second one we treat the motivation coming from the dynamical theory of the periodic Lorentz process. (We also hope that our forecast for the final picture of the limit theorems of the dynamical theory will speed up filling out the missing details.)

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## 1.1 The geometric conjectures

In the simplest case, consider a  $\mathbb{Z}^d$ -periodic ( $d \geq 3$ ) arrangement of balls of radii  $r < 1/2$ , and select a random direction and point (outside the balls). According to Dettmann's first conjecture,

1. the probability that the so determined free flight  $\tau_r$  (until the first hitting of a ball) is larger than  $t \gg 1$  is  $\sim \frac{C_r}{t}$ , and
2. the constant  $C_r$  is explicitly given by the geometry of the model.

[BGW98, GW00] provided upper and lower bounds for the aforementioned tail probability, whereas [MS10] gave precise description of the rescaled tail distribution of  $r^{d-1}\tau_r$  when  $r \rightarrow 0$ . This latter is the well-known Boltzmann-Grad limit of statistical physics when the average length of the free path tends to a constant. (Both teams observed the surprising phenomenon that the Boltzmann-Grad limit of the Lorentz-process, i. e. of the billiard dynamics is not the classical linear Boltzmann equation.) The answer to Pólya's 1918 visibility problem (cf. [K08]) is a simple consequence of the results above.

In its simplest form, Dettmann's second conjecture is related to the case with tangent balls (of radii  $1/2$ ). In this case, the expected asymptotics is  $\asymp t^{-2}$  if  $d < 6$  (with a logarithmic correction if  $d = 6$ ) and is  $\asymp t^{-\alpha_d}$  if  $d > 6$  where  $1 < \alpha_d < 2$ .

The main goal of our work is the proof of these two conjectures. In fact, we establish them in a much more general setup for any  $\mathcal{L}$ -periodic configuration of - possibly intersecting - convex bodies with  $\mathcal{L}$  being a non-degenerate lattice. We also emphasize that here  $r$  is fixed and does not tend to zero. Moreover, in case of the second conjecture we also provide the exact values of the exponents  $\alpha_d$ . We note that the essential mathematical difficulties of both proofs are already present in the aforementioned simplest cases.

## 1.2 Motivation: The dynamical problem

In 1905, Hendrik Lorentz [L05] introduced Lorentz gas as a model of motion of classical electrons in a metal. The (periodic) *Lorentz process* is the dynamics of just one electron in a crystal. It is the  $\mathbb{Z}^d$ -extension of a *toric Sinai billiard* (i. e. of one with strictly convex smooth scatterers on the  $d$ -torus,  $d \geq 2$ ). Unfortunately stochastic properties of Sinai-billiards - and more generally of semi-dispersing ones - have been established in the planar case, only. Nevertheless, if the dynamical theory of these billiards will prove those properties as expected, then our results will also 1.) forecast when exactly one has super-diffusive scaling rather than diffusive one and, moreover, 2.) provide the asymptotic covariance under the super-diffusive scaling.

Let us explain the previous ideas in more detail. It is known that for the *planar Lorentz process* the limiting distribution of the rescaled displacement is Gaussian and that of the rescaled orbit is a Wiener process. The scaling, however, is either the diffusive  $\sqrt{n}$  or the slightly super-diffusive  $\sqrt{n \log n}$  depending on whether the billiard has finite or infinite *horizon*, resp. (we say that

the horizon is finite if the free flight time is finite). In the first case the limiting covariance is given by the Green-Kubo formula (cf. [BS81], [BSCh91]), which - though explicit - nevertheless does not permit precise calculations (the formula contains an infinite sum of time correlations of the free flight vector). In the infinite horizon case, however, - as it was conjectured by [B92] and established by [SzV07], [ChD09] - the stronger  $\sqrt{n \log n}$  scaling suppresses time correlations and the limiting covariance has a simple form expressed by geometric parameters of the billiard in question.

For multidimensional Sinai billiards - even under the complexity hypothesis, expected to hold typically - exponential decay of correlations is known in the finite horizon case, only (cf. [BT08]). Then the central limit theorem with the diffusive scaling is a consequence and the limiting covariance is again given by the Green-Kubo formula. Physicists are always emphatically interested in expressions that are easy to calculate and check. Dettmann, [D12], motivated by a problem of [Sz08] and the most precious - computationally supported - observations of [S08], was assuming that the aforementioned 2D infinite horizon case picture is also valid for multidimensional dispersing billiards and made a guess as to how the limiting covariance looks like. The difficulty is that, in this case, the structure of the horizons, i. e. orbits which never meet any scatterer, is much more complicated than in the planar case.

In fact, Dettmann formulates three conjectures for  $\mathbb{Z}^d$ -periodic Lorentz processes. The first two make claims for the tail asymptotic of the free path length. Roughly speaking the first one is related to the generic cases whereas the second one to certain degenerate cases. (In both cases a Wiener limit is expected with diffusive or super-diffusive scaling.) These conjectures are of purely geometric nature and the main goal of our work is to establish them. We do this in a wider generality: 1) for semi-dispersing billiards, 2) possibly with corner points, 3) and permitting arbitrary lattices  $\mathcal{L}$  of finite covolume rather than only  $\mathbb{Z}^d$ . By accepting the dynamical hypothesis that the multidimensional picture is analogous to the 2D one (i. e. **A.** there is an exponential decay of cross correlations, and **B.** whether there is super-diffusive or diffusive behavior only depends on the tail asymptotic of the free path length), the first conjecture, among others, implies that - similarly to the planar case - the super-diffusivity covariance has a simple form that can be calculated from the geometry of the billiard. The second conjecture supports the hypothesis that, indeed, degenerate billiards, i. e. those without an open configuration subset of collision-free subspaces of maximal dimension  $d-1$ , always have diffusive behavior. It is worth mentioning that our Theorem 2 also provides exact values of the exponents  $\alpha_d$  in cases  $d > 6$  where Dettmann only guessed  $1 < \alpha_d < 2$ . Dettmann's third conjecture, also a dynamical one, supports the previously mentioned dynamical hypothesis since it is about sufficiently strong decay of correlations being the subject of future progress of the theory.

The paper is structured as follows. In Section 2, we provide the definitions and formulate Dettmann's conjectures together with our results. In Section 3, we prove some finiteness lemmas and introduce an important tool which is the fattening of the configuration space (or shrinking of the scatterers, in other

words). A key lemma to our proof of Theorem 1 is the so-called Proportionality lemma, which we discuss in Section 4. Sections 5 and 6 are devoted to the proofs of Dettmann’s first and second conjectures, respectively (it is worth noting that their methods are completely different). In Section 7, we present instructive examples where the super-diffusive limiting covariance matrix can be calculated: one of them is the first multidimensional semi-dispersing billiard whose ergodicity got ever proved: a three-dimensional toric billiard with two cylindrical scatterers (cf. [KSSz89]). The second one is the model of two hard balls on  $\mathbb{T}^d : d \geq 3$ . Finally, we make some concluding remarks in Section 8. In particular we also describe briefly the relation of our setup to that of Bourgain-Golse-Wennberg, [BGW98] and of Marklof-Strömbergsson, [MS10].

## 2 Setup and main results

### 2.1 $\mathcal{L}$ -periodicity and the dynamics

**Periodicity** We consider an infinite configuration space  $\tilde{Q} \subset \mathbb{R}^d$  and a lattice (i. e. a discrete additive subgroup  $\mathcal{L} \subset \mathbb{R}^d$  of finite covolume) defining the periodicity of the Lorentz gas. Assuming that the configuration space is invariant under translations in  $\mathcal{L}$  one can also consider the compact configuration space  $Q = \tilde{Q}/\mathcal{L}$ . For latter reference we recall that a linear subspace is called a *lattice subspace* if it can be generated by lattice vectors.

**Scatterers** The complement of the compact configuration space consists of finitely many open, convex sets  $\mathbb{R}^d/\mathcal{L} \setminus Q = \bigcup_{i=1}^n \mathcal{O}_i$  (called scatterers, or obstacles). Equivalently  $\mathbb{R}^d \setminus \tilde{Q} = \bigcup_{i=1}^n \bigcup_{l \in \mathcal{L}} (\mathcal{O}_i + l)$ . We assume that the boundary of each  $\mathcal{O}_i$  is a  $C^3$ -smooth hypersurface.

Notice that we do not require the scatterers to be disjoint, nota bene different scatterer configurations can lead to identical configuration spaces, if the differences are covered by other scatterers. Points in the boundary intersections  $q \in \partial\mathcal{O}_i \cap \partial\mathcal{O}_j$  are called corner points.

**Curvature upper bound** It is required that, at any point of the boundary  $\partial Q$ , the curvature operator  $K$  is uniformly bounded from above: there exists a universal constant  $\kappa_{\max}$ , such that for every tangent vector  $v$  of the hypersurface  $\partial Q$ , the inequality  $0 \leq K(v, v) \leq \kappa_{\max} \|v\|^2$  holds.

**Dynamics and phase space** The continuous time dynamics  $\Phi_t$  acts on the phase space  $\tilde{M} = \tilde{Q} \times \mathbb{R}^d / \sim$ , where  $\sim$  is the identification of pre-collisional and post-collisional velocities on  $\partial\tilde{Q}$ , which are mirror images with respect to the tangent hyperplane of the boundary at that point. We also write  $\Phi_{[t_1, t_2]}x$  for the set  $\{\Phi_s x | t_1 \leq s \leq t_2\}$ . For later definitions and statements if we write  $x = (q, v)$  with  $q \in \partial\tilde{Q}$ , then  $v$  is chosen as the post collisional one. At corner points there are more than one such hyperplanes, and mirroring generally does not commute, so the dynamics is either not defined, or has multiple values. (Since the speed

$|v|$  is invariant under the dynamics, in the literature one usually takes the phase space  $\tilde{M} = \tilde{Q} \times S^{d-1} / \sim$ . It will not lead to a contradiction that for us often it will be more convenient to consider  $\tilde{M}$  as introduced above.)

The action is free flight  $\Phi_t(q, v) = (q + tv, v)$  as long as  $q + vt \notin \partial\tilde{Q}$ . On the boundary the velocity is reset to the post collisional one, and free flight follows with that vector. Moreover, even if the orbit hits a scatterer and the collision is tangent (sometimes called grazing), the dynamics is still free flight since in this case the velocity does not change. The dynamics is invariant under  $\mathcal{L}$ -translations, so the compact phase space of the flow is  $M = Q \times S^{d-1} / \sim$ . For simplicity, we will use the same notation  $\Phi_t$  for the flow on the compact phase space as well. The Lorentz dynamics has natural invariant measures, the Liouville-ones:  $d\mu = \text{const} \cdot dqdv$  on  $\tilde{M}$ . The  $\text{const} = 1$  measure is called Lebesgue. Similarly the invariant probability measure for the billiard dynamics on  $M$  is  $d\mu = c_\mu dqdv$  with  $c_\mu = (\text{vol } Q \text{ vol } S^{d-1})^{-1}$ . We will also use the notation  $\lambda_{d'}$  for the Lebesgue measure in dimension  $d' \leq d$ .

## Billiard and Lorentz process

**Definition 1.** *Under the aforementioned conditions, the dynamics  $\Phi_t(t \in \mathbb{R})$  on the phase space  $M$  is called a semi-dispersing billiard and that on the phase space  $\tilde{M}$  a (semi-dispersing) Lorentz process. If the scatterers are strictly convex, then the billiard is called a dispersing one or a Sinai-billiard.*

In this paper we will consider a fixed semi-dispersing billiard (or the corresponding Lorentz process) satisfying the aforementioned conditions.

**The free flight function** For  $x = (q, v)$

$$\tau(x) = \inf\{t > 0 \mid q + tv \in \cup_i \mathcal{O}_i\}$$

as usual, the infimum of the empty set is  $\infty$ . This definition is slightly different from the usual definition. In fact, at points where the first collision is tangential, the new definition gives a larger value. The advantage of this change is seen by the semi-continuity Claim 5. It is obviously invariant under  $\mathcal{L}$  translations, so we will not distinguish whether the function is defined on the compact or on the non-compact phase space.

Our main focus will be on the tail distribution of the free path length:

$$F(t) = \mu(\tau > t) \tag{1}$$

i. e. of the probability of surviving without collision for time  $t$ .

## 2.2 Horizons

**Definition 2.** *For a configuration point  $q \in \tilde{Q}$  a free subspace  $V$  is a maximal (for containment) linear subspace of  $\mathbb{R}^d$ , such that  $q + V \subset \tilde{Q}$ . This latter is equivalent to requiring  $\tau(q + v, w) = \infty$  for all  $v, w \in V$ . (Sometimes we also call the affine subspace  $q + V$  a free subspace.)*

**Claim 1.** *Any free subspace  $V$  is a lattice subspace.*

*Proof.* If we have a vector  $v \in V$ , then by invariance  $q + tv + l \in \tilde{Q}$  for all  $t \in \mathbb{R}$  and  $l \in \mathcal{L}$ . If this vector  $v$  is not parallel to a lattice vector, then the set  $tv + l$  is dense in some lattice subspace  $V'$ , concluding  $q + V' \subset \tilde{Q}$ , so  $V' \subset V$  by maximality.  $\square$

**Definition 3.** *If  $\tau(q, v) = \infty$ , consider a free subspace  $V \ni v$  for  $q$ , and the subspace  $V^\perp$  orthogonal to it. A maximal connected subset  $\tilde{B}_H \subset \tilde{Q} \cap (q + V^\perp)$  containing points  $q'$ , for all of which  $V$  is a free subspace is called a basis for the horizon  $\tilde{H} = \tilde{B}_H \times V \subset \tilde{Q}$ . The dimension  $d_{\tilde{H}}$  of the horizon is the dimension of the free subspace  $V$ .*

**Remark** Of course, for one and the same horizon the set of possible bases is invariant under  $V$ -shifts. If we talk about the basis  $\tilde{B}_H$  of a horizon  $\tilde{H}$ , then we think of  $\tilde{B}_H$  as represented in  $\tilde{Q}$  for an arbitrary  $q \in \tilde{H}$ .

**Definition 4.** *By taking  $Q = \tilde{Q}/\mathcal{L}$  one obtains the horizon  $H = \tilde{H}/\mathcal{L}$  in the compact configuration space. Its basis  $B_H$  is  $\tilde{B}_H$  as represented in  $Q$ , and its dimension  $d_H$  coincides with  $d_{\tilde{H}}$ .*

**Remark** At factorization, for a  $q \in Q$ ,  $q + V^\perp$  can contain several copies of the basis  $B_H$ .

**Remark** We will use the same names, though different notations, for the phase space counterpart  $\tilde{\mathcal{H}} = \tilde{H} \times V \subset \tilde{M}$  of a horizon  $\tilde{H}$  and for the corresponding sets  $H, \mathcal{H}$  in the compact spaces. Also, to denote the relation between the horizon and its velocity component, we will sometimes use the notation  $V_H$ .

A substantial observation of Dettmann is that  $F(t)$  (see (1)) can asymptotically be expressed as a sum for a finite number of horizons  $H$  of times a free flight spends in  $H$ . Concretely, he introduced the probability of remaining within a horizon  $H$  for time  $t$ , that is

$$F_H(t) = \mu(\{(q, v) \in M \mid q + sv \in H, \forall s \in [0, t]\})$$

a quantity that can be calculated exactly. (cf. Equ. (26) of [D12] or (2) to be given later).

**Definition 5.** ([D12])

- A maximal horizon is one of the highest dimension for the given billiard (or Lorentz process).
- A principal horizon is one of the highest dimension possible, which is  $d - 1$  if there are scatterers.
- A horizon  $H$  is incipient if its basis  $B_H$  has  $(d - d_H)$  dimensional measure zero.

Denote the set of maximal non-incipient horizons by  $\mathbb{H}$ . It can be empty if all maximal horizons are incipient, or there are no horizons at all.

We conclude this point with a simple lemma.

**Lemma 2.** *The boundary of the basis of a horizon consists of  $C^3$ , concave pieces except for principal horizons when it consists of two endpoints of an interval (they may coincide).*

### 2.3 Dettmann's conjectures, [D12]

**Conjecture 1.** *Consider an  $\mathcal{L}$ -periodic Lorentz process with at least one non-incipient maximal horizon. Then, as  $t \rightarrow \infty$  we have*

$$F(t) \sim \sum_{H \in \mathbb{H}} F_H(t).$$

**Conjecture 2.** *Consider an  $\mathcal{L}$ -periodic Lorentz process with incipient (but no actual) principal horizon. Then, as  $t \rightarrow \infty$ , we have*

$$F(t) \asymp \begin{cases} t^{-2}, & d < 6 \\ t^{-2} \log t, & d = 6 \\ t^{-\alpha_d} \quad (1 < \alpha_d < 2), & d > 6 \end{cases}$$

These two conjectures are of purely geometric nature, whereas the following one concerns the dynamics, too.

**Conjecture 3.** *Consider an  $\mathcal{L}$ -periodic Lorentz process and let  $f, g : M \rightarrow \mathbb{R}$  denote zero-mean (wrt the invariant measure  $\mu$ ) Hölder functions. Then, as  $t \rightarrow \infty$ , we have*

$$\int_{\{x \in M | \tau(x) < t\}} (f)(g \circ \Phi_t) d\mu = o(F(t)).$$

### 2.4 Main results

Now we can formulate the main results of our paper.

**Theorem 1.** *Consider an  $\mathcal{L}$ -periodic semi-dispersing Lorentz process (possibly with corner points). Assume it has at least one non-incipient maximal horizon. Then, as  $t \rightarrow \infty$  we have*

$$F(t) \sim \sum_{H \in \mathbb{H}} F_H(t).$$

**Theorem 2.** *Consider an  $\mathcal{L}$ -periodic semi-dispersing Lorentz process (possibly with corner points). Assume it has at least one incipient (but no actual) principal horizon. Then, as  $t \rightarrow \infty$ , we have*

$$F(t) = \begin{cases} O(t^{-2}), & 3 \leq d \leq 5 \\ O(t^{-2} \log t), & d = 6 \\ O\left(t^{\frac{2+d}{2-d}}\right), & d > 6. \end{cases}$$

Further, if we also assume that the curvature is bounded away from 0 (from below) uniformly at every point of  $\partial Q$  (dispersing case), then

$$F(t) \asymp \begin{cases} t^{-2}, & 3 \leq d \leq 5 \\ t^{-2} \log t, & d = 6 \\ t^{\frac{2+d}{2-d}}, & d > 6. \end{cases}$$

**Remark.** According to the dynamical theory of semi-dispersing billiards super-diffusive behavior can only arise if the asymptotics of  $F(t)$  is non-integrable. Therefore Theorems 1, 2 and (2) suggest that, in the absence of principal, non-incipient horizon, no super-diffusive behavior is possible (cf. Section 7). Moreover, in the case of super-diffusivity the scaling is  $\sqrt{t \log t}$  - again by (2).

### 3 The method of fattening, finiteness and stability lemmas

#### 3.1 Lattice geometry

The following statement is well-known, in fact, quantitative results are also known, see for instance [Sch68].

**Lemma 3.** *For any  $K > 0$  the number of lattice subspaces  $V$ , such that  $\text{vol}(V/V \cap \mathcal{L}) < K$  is finite.*

By this lemma the minimal covolume of  $k$  dimensional sublattices exists, and we will denote it by  $\ell_k$ . For example  $\ell_1$  is the minimal length of nonzero lattice vectors,  $\ell_d = \text{vol}(\mathbb{R}^d/\mathcal{L})$ , and  $\ell_0 = 1$  as usual for empty products.

**Lemma 4.** *If we have a lattice subspace  $V$ , and we take its orthogonal complement  $V^\perp$ , and we project  $\mathcal{L}$  orthogonally onto  $V^\perp$  to get  $\mathcal{L}_V^\perp$ , then we have*

$$\text{vol}(\mathbb{R}^d/\mathcal{L}) = \text{vol}(V/V \cap \mathcal{L}) \text{vol}(V^\perp/\mathcal{L}_V^\perp).$$

*Proof.* Take a basis  $\{a_i\}_{i=1}^{\dim(V)}$  for  $\mathcal{L} \cap V$ , and extend this to a basis  $\{a_i\}_{i=1}^d$  of  $\mathcal{L}$ . Then  $|\det(a_i)| = \text{vol}(\mathbb{R}^d/\mathcal{L})$ . The determinant does not change if we project the last  $d - \dim(V)$  vectors orthogonally to the orthocomplement of the first  $\dim(V)$  vectors. The projections give rise to a basis of  $\mathcal{L}_V^\perp$ , and by orthogonality  $|\det(a_i)| = |\wedge_{i=1}^{\dim(V)} a_i| |\wedge_{i=\dim(V)+1}^d a_i^\perp|$ , which is the claim.  $\square$

Now we can provide the asymptotic form of  $F_H(t)$ . Indeed, in our notations, Equ. (26) of [D12] reads as

$$F_H(t) \sim \frac{\text{vol} S_{d_H-1} \int_{B_H} \int_{B_H} \Delta_{B_H}^{\text{vis}}(q, q') dq dq'}{(1 - \mathcal{P}) \text{vol} S_{d-1} \text{vol}(V^\perp/\mathcal{L}_V^\perp)} \frac{1}{t^{d-d_H}} =: C_H \frac{1}{t^{d-d_H}} \quad (2)$$



where  $\mathcal{P} = 1 - \frac{\text{vol } Q}{\text{vol}(\mathbb{R}^d/\mathcal{L})}$  is the volume fraction covered by scatterers and  $\Delta_{B_H}^{\text{vis}}(q, q')$  is the visibility function providing the number of possible connecting intervals  $\overline{q, q'}$ , lying in  $B_H$ , of the points  $q, q'$  (toric geometry!). Note that the value of the integral is invariant under  $V$ -shifts of  $B_H$  and is finite since  $\Delta_{B_H}^{\text{vis}}(q, q')$  is bounded. So as to verify the latter, assume by contradiction that for each  $n > 0$  one finds  $q_n, q'_n$  such that  $\Delta_{B_H}^{\text{vis}}(q_n, q'_n) > n$ . Since the sets  $\Delta_n = \{(q, q') \mid \Delta_{B_H}^{\text{vis}}(q, q') > n\}$  are closed subsets of each other, they have a nonempty intersection containing some  $(q_\infty, q'_\infty)$  with  $\Delta_{B_H}^{\text{vis}}(q_\infty, q'_\infty) = \infty$ . Thus an infinite line is part of  $B_H$ , which contradicts to its definition.

**Remark.** In the much interesting case of a *principal horizon*  $H$ ,  $B_H$  is an interval and the previous formula becomes simpler:

$$F_H(t) \sim \frac{2 \text{vol } S_{d-2} |B_H|^2}{(1 - \mathcal{P}) \text{vol } S_{d-1} \text{vol}(V^\perp/\mathcal{L}_V^\perp)} \frac{1}{t} \quad (3)$$

### 3.2 Fattening, and its properties

The curvature upper bound implies in particular that, at any point of the boundary, a tangent sphere of radius  $\kappa_{\max}^{-1}$  is contained completely in the scatterer. This allows us to define the shrinking of the scatterers, or equivalently the fattening of the configuration space by  $0 \leq \delta < \kappa_{\max}^{-1}$  as a parallel domain (which is typically not homothetic to the original one). Indeed, define  $\mathcal{O}_i^\delta$  as the centers of all balls of radius  $\delta$ , which are contained in  $\mathcal{O}_i$ :

$$\mathcal{O}_i^\delta = \{q \in \mathcal{O}_i \mid \text{dist}(q, \partial\mathcal{O}_i) > \delta\}.$$

This leads to new configuration spaces  $\tilde{Q}^\delta = \mathbb{R}^d \setminus \cup_i \cup_{l \in \mathcal{L}} (\mathcal{O}_i^\delta + l)$ , and  $Q^\delta = \tilde{Q}^\delta/\mathcal{L}$ , which satisfy all the above assumptions, with  $(\kappa_{\max}^{-1} - \delta)^{-1}$  as a curvature upper bound.

**Upper semi-continuity** The definition can be extended to negative values of  $\delta$ . Also note our previous comment that different scatterer configurations can lead to the same configuration space. Since fattening is defined from scatterers, the same configuration space can have different fattenings for the same  $\delta$ . The semigroup property of this operation holds  $Q^0 = Q$ , and  $(Q^\delta)^{\delta'} = Q^{\delta+\delta'}$  as long as  $\delta, \delta'$  and  $\delta + \delta'$  are all smaller than  $\kappa_{\max}^{-1}$ . (By the latter restrictions this is not exactly a semigroup.)

It is then natural to denote the corresponding dynamics by  $\Phi_t^\delta$ . We denote by  $\tau^\delta$  the free flight function on the fattened space.

**Lemma 5** (Upper semi-continuity).  $\tau^\delta$  as a function on  $(-\infty, \kappa_{\max}^{-1}) \times \tilde{M}$  is upper semi-continuous (to be abbreviated as USC) in all of its variables  $(\delta, x)$ .

Moreover, in the case of the previously defined fattening, the equality  $\tau(x) = \tau^\varepsilon(x)$  holds with  $\varepsilon > 0$  if and only if  $\tau(x) = \infty$ .

Also, if for  $(q, v) \in M$   $\tau^\varepsilon(q, v) = \infty$  for every  $\varepsilon > 0$ , then  $\tau(q, v) = \infty$ , too.

*Proof.* This only requires a proof at points  $x = (q, v)$ , where  $\tau^\delta(x) < \infty$ . By the definition of  $\tau$ , for a small  $\epsilon > 0$  we have  $q + (\tau^\delta(x) + \epsilon)v \in \mathcal{O}_i^\delta$  for some  $i$ . Since both the free flight dynamics and the fattening are continuous, we have that, for nearby points  $x'$ , and nearby parameters  $\delta'$ ,  $\exists \epsilon > 0$  such that  $q' + (\tau^{\delta'}(x) + \epsilon)v' \in \mathcal{O}_i^{\delta'}$ . The dynamics of a nearby point  $x'$  may differ from the free flight dynamics only  $S^{[0, \tau^{\delta'}(x)]}(x)$  had a jab (non tangent) collision 'before', but then  $\tau^{\delta'}(x')$  is even smaller than  $\tau^\delta(x) + \epsilon$ . ('before' permits equality as well thus the argument is also valid for simultaneous collisions at corner points.)  $\square$

**Monotonicity** Of course, the fattening of the configuration space makes free flights longer. We will use it not only for the above defined parallel domain, but for a larger set of inclusion relations, too.

Denote by  $\mathcal{V}^\epsilon(q)$  (or by  $\mathcal{V}(q)$ ) the set of free subspaces at  $q \in \tilde{Q}^\epsilon$  (or at  $q \in \tilde{Q}$ , respectively).

**Lemma 6 (Monotonicity).** *For  $q \in Q$ ,  $\mathcal{V}^\epsilon(q)$  is an increasing function of  $\epsilon \in [0, \kappa_{\max}^{-1})$  in the sense that  $\forall 0 < \epsilon < \epsilon'$  and  $\forall V \in \mathcal{V}^\epsilon(q) \exists V^* \in \mathcal{V}^{\epsilon'}(q)$  such that  $V \subset V^*$ .*

*Proof.* If  $Q \subset Q'$ , then for any  $x \in M$  we have  $x \in M'$ , too. Then we can consider both free flights  $\tau$  and  $\tau'$  and we have  $\tau(x) \leq \tau'(x)$ .  $\square$

**Local stability** For  $q \in \tilde{Q}$  and  $\delta > 0$  denote by  $B(q, \delta)$  the  $\delta$ -neighborhood of  $q$ . Unless specified otherwise, it will be considered as a neighborhood in  $\tilde{Q}$ . (We use the same notation analogously for  $q \in Q$ .)

**Lemma 7 (Local stability).** *For any  $q \in \tilde{Q}$  there exists  $\xi > 0$  such that, for every  $q' \in B(q, \xi) \cap \tilde{Q}$  and any free subspace  $V^\xi(q')$  for  $\Phi^\xi$  at  $q'$ , there is a free subspace  $V(q)$  for  $\Phi$  at  $q$  such that  $V^\xi(q') \subset V(q)$ .*

*In other words, the set of free subspaces as a function of the base point  $q \in Q$  and of  $\epsilon > 0$  is upper semi-continuous at  $q \in Q, \epsilon = 0$  in the sense that for any  $q_n \rightarrow q$  and  $\epsilon_n \searrow 0$  and for any  $V \in \lim_{n \rightarrow \infty} \cup_{k \geq n} \mathcal{V}^{\epsilon_k}(q_k)$  there is a  $V^* \in \mathcal{V}(q)$  such that  $V \subset V^*$ .*

*Proof.* We prove the claim in its second form. Assume the contrary. Then there exists a velocity  $v_\infty$  and sequences  $q_n \rightarrow q$ ,  $v_n \rightarrow v_\infty$  and  $\epsilon_n \rightarrow \epsilon$  such that  $\tau^\epsilon(q, v_\infty) < \infty$  and  $\tau^{\epsilon_n}(q_n, v_n) = \infty$ . This contradicts Lemma 5.  $\square$

### 3.3 Finiteness of free subspaces

**Lemma 8.** *For any configuration point  $q \in \tilde{Q}$  the set of free subspaces is finite.*

*Proof.* The proof is inductive by codimension  $d - \dim(V)$ . If  $\dim V = d$ , then there are no scatterers at all and  $\mathbb{R}^d$  is the only free subspace. Assume we have proven the statement for dimensions larger than  $d' (< d)$ .

The induction step is indirect. We are going to show that, if the number of  $d'$  dimensional free subspaces is infinite, then for every positive  $\epsilon$  there exists a

free subspace of higher dimension in  $\tilde{Q}^\epsilon$ . We will apply the inductive condition to  $\tilde{Q}^\epsilon$  ( $\epsilon > 0$  sufficiently small) to derive a contradiction.

For any given  $\delta > 0$  there are only finitely many  $d'$  dimensional lattice subspaces, for which the lattice translates are  $\delta$ -separated. By the indirect condition we have a free subspace, for which the lattice translates are  $\delta$ -dense in a higher dimensional subspace. This higher dimensional subspace is therefore free in  $\tilde{Q}^\epsilon$  (as long as  $\epsilon < (1/7)\delta^2\kappa_{\max}$ ), but is not free in  $\tilde{Q}$  (free subspaces can not contain each other by maximality).

By the inductive assumption the number of higher (i. e.  $> d'$ ) dimensional free subspaces is finite. For each  $\epsilon$  we create a vector  $\vec{n}^\epsilon$  such that the first coordinate is the number of  $d$  dimensional free subspaces for  $q$  in  $\tilde{Q}^\epsilon$ , the second is the number of  $d - 1$  dimensional free subspaces for  $q$  in  $\tilde{Q}^\epsilon$ , and so on, the last coordinate is the number of  $d' + 1$  dimensional free subspaces for  $q$  in  $\tilde{Q}^\epsilon$ . We consider the lexicographical ordering on these vectors, so the biggest is  $(1, 0, \dots, 0)$ , and  $(0, 2, 3, 0, 1) > (0, 2, 2, 23, 11)$ . The set of possible vectors is not finite, but well ordered.

We claim that  $\vec{n}^\epsilon$  does not increase as  $\epsilon$  decreases, and that  $\vec{n}^\epsilon$  is right continuous in  $\epsilon$ . For the first claim, observe that new free subspaces can only appear, if they were covered by higher dimensional free subspaces for higher  $\epsilon$  values. So the first changing coordinate is decreasing. For the second claim, observe that  $\tilde{Q} = \bigcap_{\epsilon > 0} \tilde{Q}^\epsilon$ , so if a free subspace is present for all small enough  $\epsilon > 0$ , then it is also present for  $\epsilon = 0$ . Therefore

$$\lim_{\epsilon \searrow 0} \vec{n}^\epsilon = \min_{\epsilon > 0} \vec{n}^\epsilon = \vec{n}^0$$

the first equality follows from monotonicity and well-orderedness, the second from right continuity.

This is a contradiction with the previously proven statement:  $\vec{n}^\epsilon > \vec{n}^0$  for all  $\epsilon > 0$ . Indeed, by Lemma 3.3 for any point  $q \in \tilde{Q}$  there exists an  $\epsilon$  such that  $\vec{n}^\epsilon = \vec{n}^0$ , and therefore the free subspaces are the same. □

**Lemma 9.** *There are finitely many maximal horizons.*

*Proof.* (see also Lemma 1 in [D12] and Lemma A.2.2. of [Sz94]) For every  $q \in Q$  pick a stability neighborhood using Lemma 7. Since  $Q$  is compact, one can choose a finite cover of  $Q$  by such neighborhood. This yields that there are only finitely many maximal dimensional free subspaces. It remains to prove that for such a free subspace  $V$ , there are only finitely many corresponding horizons. For this, project the scatterer configuration to  $V^\perp$ . Note that there is no higher dimensional free subspace than  $V$ , thus the complement of the images of the scatterers is the union of the bases of horizons with free subspace  $V$ . Since the complement of finitely many convex sets has finitely many connected components, the statement follows. □

## 4 The proportionality lemma

The next lemma states that any long enough free flight has a fixed proportion of its time spent in a horizon without leaving it. The technical formulation is a little bit different, and formally we will use the statement below, where instead of a horizon we use the vicinity of a free subspace.

**Lemma 10** (Proportionality lemma). *For every  $\epsilon > 0$  there exist  $T > 0$  and  $c \in (0, 1)$ , such that for any  $x \in M$  if  $\infty > \tau(x) > T$  then there exist  $\tau(x) > s > t > 0$  with  $s - t > c\tau(x)$  and a free subspace  $p + V \subset \tilde{Q}$  such that the configuration component of  $\Phi_u(x)$  is  $\epsilon$  close to  $p + V$  in  $\tilde{Q}$  for every  $s > u > t$ .*

**Remark 1** As we will see in Section 5, this lemma is only used for handling the remainder term, i. e. the contribution of a countable union of smaller dimensional horizons in vicinities of maximal horizons. This is why it is sufficient to ensure that only a positive proportion of a long free path is close to the free subspace of a horizon.

**Remark 2** The analogous Lemma in the planar case is much simpler: for any long enough free flight (except for its two extreme parts of bounded length) is entirely spent in a horizon (see [B92]).

*Proof.* The proof is indirect. We are going to suppose, that there exists an  $\epsilon > 0$  such that for all  $T > 0$  and  $c \in (0, 1)$  there exists  $x \in M$  with  $\infty > \tau(x) > T$  such that for any free subspace  $p + V \subset \tilde{Q}$  and for any time segment  $\tau(x) > s > t > 0$  if the configuration component of  $\Phi_u(x)$  is  $\epsilon$  close to  $p + V$  for  $s > u > t$ , then  $\tau(x) > (s - t)/c$ .

**Choice of constants** Choose  $T_n \rightarrow \infty$ , and  $c_n \rightarrow 0$ , and choose  $(q_n, v_n) = x_n \in M$  according to the indirect assumption. By compactness of  $M$  we have an accumulation point  $x_\infty = (q_\infty, v_\infty)$ . Apply lemma 7 to get  $\xi$  as the stability fattening factor for  $q_\infty$ . We have an  $\epsilon$  from the indirect statement. Choose  $\eta$ , such that  $\frac{3}{2}\eta < \epsilon$ , and  $2^d\eta < \xi$ . For  $1 \leq k \leq d$  let us define

$$r_k = \frac{\ell_{k-1}}{\ell_d} \left(\frac{\eta}{2}\right)^{d-k} D^{d-k}, \quad (4)$$

where  $D^j$  is the  $j$ -dimensional volume of the  $j$ -dimensional unit ball, and  $D^0 = 1$ . Choose  $n$  such that  $|q_n - q_\infty| < \eta/2$  and

$$T_n > \frac{1}{r_1}, \quad (5)$$

$$c_n < 2^{-d}\eta \min_{1 \leq k \leq d} r_k. \quad (6)$$

**Inductive assumptions** We are going to prove the following statements in an inductive fashion for  $1 \leq k \leq d$ .

- We have linearly independent lattice vectors  $\{l_i\}_{i=1}^k$ , all from a free subspace for  $q_\infty$  in  $\tilde{Q}$ .
- We have  $0 < t_k < \tau(x_n)$ , such that the parallelepiped  $q_n + \sum_{i=1}^k \lambda_i l_i$ ,  $\lambda_i \in [0, 1]$  is contained in the  $(2^k - 1)\eta$  radius tubular neighborhood of the trajectory segment  $\Phi_{[0, t_k]} x_n$  (the  $\rho$  tubular neighborhood of a line segment  $[a, b]$  is the set of such points in  $\mathbb{R}^d$  which are  $\rho$ -close to the line segment  $[a, b]$ , and whose orthogonal projection to the line defined by  $a$  and  $b$  lies between  $a$  and  $b$ ).
- Denote by  $v_n^{\perp k}$  the component of  $v_n$  which is orthogonal to  $\text{span}\{l_i\}_{i=1}^{k-1}$ , this gives  $v_n$  for  $k = 1$ . We require that:

$$t_k < \sum_{i=1}^k 2^{k-i} \frac{1}{|v_n^{\perp i}| r_i} \quad (7)$$

The last statement is purely technical.

**Start of induction** By condition (5) the tubular  $\eta/2$  neighborhood of the free flight trajectory of  $x_n$  has a bigger volume than  $\text{vol}(\mathbb{R}^d/\mathcal{L})$ , therefore it has a self intersection in  $Q = \tilde{Q}/\mathcal{L}$ . This means that, in this tubular neighborhood, there are two points  $q'$ , and  $q' + l_1$  which are lattice translates with  $0 \neq l_1 \in \mathcal{L}$ . Moreover  $|q' - q_n| < \eta/2$  and  $|q' + l_1 - (q_n + t_1 v_n)| < \eta/2$  and  $0 < t_1 < \tau(x_n)$ . Consequently in the fattened space  $\tilde{Q}^{\eta/2}$  the line segment  $q', q' + l_1$  is collision free and periodic, hence  $\tau^{\eta/2}(q', l_1) = \infty$ . Applying the stability lemma we conclude that  $l_1$  is part of a free subspace for  $q_\infty$  in  $\tilde{Q}$ . We also note that the line segment  $q_n, q_n + l_1$  is in the tubular  $\eta$  neighborhood of the trajectory segment  $\Phi_{[0, t_1]} x_n$ .

Note that we only used  $\tau(x_n) > 1/r_1$  about the length of the free flight, so actually  $t_1 < 1/r_1$ , which gives equation (7) for  $k = 1$ .

**Inductive step** Suppose we have all the inductive statements for  $k - 1$ . For simplicity we denote the lattice subspace  $V = \text{span}\{l_i\}_{i=1}^{k-1}$ , and its orthocomplement  $V^\perp$ . Consider the orthogonal projection of the free flight  $\Phi_{[0; \tau(x_n)]}^\perp(x_n)$ . Since  $|q_n - q_\infty| < \eta/2$  and  $\epsilon > 3\eta/2$  the projection of the free flight lies in at least  $\eta$  length (and equivalently for at least  $\eta/|v_n^{\perp k}|$  time) in the  $\epsilon$  neighborhood of  $q_\infty^\perp$ , meaning that the non projected free flight spends the same  $\eta/|v_n^{\perp k}|$  time in the  $\epsilon$  neighborhood of the free subspace containing  $V$ . By the indirect condition, the complete length of the projection  $|v_n^{\perp k}| \tau(x_n)$  is at least  $\eta/c_n > 1/r_k$ , therefore

$$\tau(x_n) > \frac{\eta}{|v_n^{\perp k}| c_n} \quad (8)$$

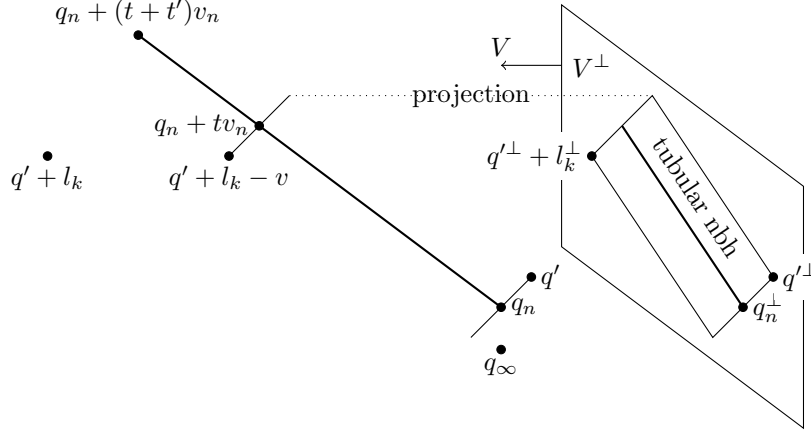


Figure 1: Constellation of vectors in the inductive step in the proof of lemma 10

By the definition of  $r_k$  we have that the  $((d-k+1)$ -dimensional) volume of the tubular  $\eta/2$  neighborhood of the projected free flight trajectory, is bigger than  $\ell_d/\ell_{k-1}$ , so in particular bigger than  $\text{vol}(\mathbb{R}^d/\mathcal{L})/\text{vol}(V/\mathcal{L} \cap V)$ , which is by Claim 4 the covolume of the projected lattice  $\mathcal{L}_V^\perp$ . Therefore this neighborhood contains a pair of points  $q'^\perp$ , and  $q'^\perp + l_k^\perp$ , with  $0 \neq l_k^\perp \in \mathcal{L}_V^\perp$ . The latter means that there is  $l_k \in \mathcal{L} \setminus V$ , such that  $l_k^\perp$  is its projection. We can choose  $q'$  such that  $|q' - q_n| < \eta/2$  and

$$|q' + l_k - (q_n + tv_n) - v| < \eta/2, \quad (9)$$

for some  $\tau(x_n) > t > 0$ , and some  $v \in V$ . We can suppose, that  $v$  is in the parallelepiped  $\sum_{i=1}^{k-1} \lambda_i l_i$ , since the lattice component can be added to  $l_k$ , it does not change the property, that  $l_k \in \mathcal{L} \setminus V$ . The inductive condition gives that  $v$  is in the tubular  $(2^{k-1} - 1)\eta$  neighborhood of the trajectory segment  $\Phi_{[0, t_{k-1}]} x_n$ , we have from equation 9 that

$$|q' + l_k - (q_n + (t + t')v_n)| < \left(2^{k-1} - \frac{1}{2}\right) \eta, \quad (10)$$

where  $0 < t' < t_{k-1}$ . The positivity of  $t'$  comes from the sign of  $v$  in equation 9 and the fact, that all  $l_i$  has positive scalar product with  $v_n$  by construction. It follows, that the line segment  $q_n, q_n + l_k$  is in the tubular  $2^{k-1}\eta$  neighborhood of  $\Phi_{[0, t+t']} x_n$ , and therefore the parallelepiped  $q_n + \sum_{i=1}^k \lambda_i l_i$  is in the  $(2^k - 1)\eta$  neighborhood of  $\Phi_{[0, t+t'+t_{k-1}]} x_n$ .

We declare  $t_k = t + t' + t_{k-1}$ , and note that in the construction of  $t$  we have only used  $\tau(x_n) > 1/r_k |v_n^{\perp k}|$  about the length of the free flight, so actually  $t < 1/r_k |v_n^{\perp k}|$ . Using  $t' < t_{k-1}$ , and equation 7 from the inductive condition

for  $k - 1$  we get

$$t_k < \frac{1}{r_k |v_n^{\perp k}|} + 2 \sum_{i=1}^{k-1} 2^{k-1-i} \frac{1}{|v_n^{\perp i}| r_i} = \sum_{i=1}^k 2^{k-i} \frac{1}{|v_n^{\perp i}| r_i},$$

which is equation 7 for  $k$ . To show  $t_k < \tau(x_n)$ , observe, that  $|v_n^{\perp i}|$  is decreasing with  $i$ , hence

$$t_k < \sum_{i=1}^k 2^{k-i} \frac{1}{|v_n^{\perp i}| r_i} < \sum_{i=1}^k 2^{k-i} \frac{1}{|v_n^{\perp k}| \min r_i} < 2^k \frac{1}{|v_n^{\perp k}| \min r_i} \leq \frac{\eta}{|v_n^{\perp k}| c_n}.$$

The last inequality follows from equation 6. The last expression in the row, and hence  $t_k$  is smaller than  $\tau(x_n)$  by equation 8.

In the fattened space  $\tilde{Q}^{(2^k-1)\eta}$  we have a  $k$  dimensional lattice parallelepiped, and (by  $\mathcal{L}$  periodicity) the generated lattice subspace free of scatterers. By the choice of  $\eta$  we can apply the stability lemma to conclude, that  $\{l_i\}_{i=1}^k$  are from a free subspace for  $q_\infty$  in  $\tilde{Q}$ .

**Contradiction** The last ( $k = d$ ) claim in the above induction states the existence of a  $d$  dimensional free subspace, which means that there are no scatterers. Even in that case the indirect condition states that the trajectory leaves this free subspace, which is the whole configuration space.  $\square$

## 5 Proof of Theorem 1

Here, we prove the generalization of Dettmann's first conjecture (i.e. Theorem 1).

### 5.1 Lower estimate

First, we prove the lower estimate, namely

$$\limsup_{t \rightarrow \infty} \sum_{H \in \mathbb{H}} F_H(t) / F(t) \leq 1. \quad (11)$$

Since  $\cup_H \{(q, v) \in M \mid q + sv \in H, \forall s \in [0, t]\} \subset \{(q, v) \in M \mid \tau(q, v) > t\}$ , (2) implies that (11) follows, whenever

$$\mu(\{(q, v) \in M \mid \exists H_1 \neq H_2 \in \mathbb{H}, \forall s \in [0, t], q + sv \in H_1 \cap H_2\}) = o(t^{d_H-d})$$

is established. Since there are finitely many maximal horizons, it suffices to prove that for every pair  $(H_1, H_2) \in \mathbb{H}^2$ ,

$$F_{H_1, H_2}(t) = \mu(\{(q, v) \in M \mid q + sv \in H_1 \cap H_2, \forall s \in [0, t]\}) = o(t^{d_H-d}).$$

Now assume that for fix  $(H_1, H_2)$  and for every  $n > 1$ , one can find  $x_n \in M$  such that the trajectory segment  $\Phi_{[0, n]}x_n$  lies entirely in  $H_1 \cap H_2$  (if not, then

obviously  $F_{H_1, H_2}(t) = 0$  for  $t$  large enough). Since maximal horizons are closed, there is an accumulation point  $x_\infty = (q_\infty, v_\infty)$  with  $\Phi_{[0, \infty]}x_\infty \in H_1 \cap H_2$ . Thus the set  $V_{H_1, H_2} = V_{H_1} \cap V_{H_2}$  is a non-empty subspace of  $V_{H_1}$ . Obviously it is strictly smaller than  $V_{H_1}$ , otherwise  $H_1$  and  $H_2$  would coincide. Now project the scatterer configuration to  $V_{H_1, H_2}^\perp$ . In this projection, the intersection of the images of  $\tilde{H}_1$  and  $\tilde{H}_2$  does not contain any subspace (indeed, if it contained a line, that could be added to  $V_{H_1, H_2}$ ). Then the same argument used to prove (2) provides  $F_{H_1, H_2}(t) = O(t^{d_H - 1 - d})$ .

## 5.2 Upper estimate

The estimate will work as an induction by dimension. If  $d = 1$  the claim is trivial, the  $d = 2$  case was proved in [SzV07].

The idea of the present proof is briefly the following. The measure of points for which the trajectory up to time  $t$  is spent in a horizon of dimension  $d'$  is of order  $t^{d' - d}$ . In order to prove the upper bound, one needs to overcome two difficulties. First, there are trajectories which travel from one horizon to another - this problem is solved by the Proportional lemma. The second problem is that although there are finitely many maximal horizons, but there are infinitely many lower dimensional “attached” horizons, thus the above naive estimation cannot be summed up. To solve this problem, we slightly extend the maximal horizons in the estimation - this way, they swallow all, but finitely many attached horizons, while their leading constant ( $C_H$ ) do not change a lot.

Formally, in the general  $d$  dimensional case, we prove the following statement. For every  $\delta > 0$  there exists a  $T < \infty$  such that for every  $t > T$ ,

$$F(t) \leq (1 + \delta) \sum_{H \in \mathbb{H}} C_H t^{d_{\max} - d}, \quad (12)$$

where  $d_{\max}$  is the dimension of the maximal horizons. To prove this, let us introduce the fattened version of the maximal horizons. Since  $\cap_{\varepsilon > 0} \tilde{Q}^\varepsilon = \tilde{Q}$ , for  $\varepsilon$  small enough, the maximal horizons of the fattened configuration space  $\tilde{Q}^\varepsilon$  are in one to one correspondence with those of  $\tilde{Q}$ , and are slightly thicker than those. Thus one can choose  $\varepsilon > 0$  such that

$$\sum_{H \in \mathbb{H}} C_{H^{3\varepsilon}} < (1 + \delta/4) \sum_{H \in \mathbb{H}} C_H,$$

where  $H^{3\varepsilon}$  is the fattened version of the horizon  $H$  - which can also be written as  $B_H^{3\varepsilon} \times V_H$  - and  $C_{H^{3\varepsilon}}$  is the corresponding constant defined in (2). Note that the  $3\varepsilon$  neighborhood of  $H$  ( $B_H$ , resp.) is a proper subset of  $H^{3\varepsilon}$  ( $B_H^{3\varepsilon}$ , resp.). Fix this  $\varepsilon$  for the rest of the proof.

**Estimator environments** Now, for any fixed  $\varepsilon > 0$ , we construct a finite net of environments, called estimator environments, which will be used by the estimate. In fact, this finiteness will have an essential role in our arguments so despite of its simplicity we formulate the statement in a lemma.



**Lemma 11.** *Given  $\varepsilon > 0$ , one can find a finite set of points  $q_1, \dots, q_{\bar{i}}$  with  $\mathcal{V}(q_i) = \{V_j(q_i) : 1 \leq j \leq j(q_i)\}$  such that for arbitrary  $q \in Q$  and any free subspace  $V \in \mathcal{V}(q)$ , there are some  $q_i$  and  $j : 1 \leq j \leq j(q_i)$  such that  $q$  is in the  $\varepsilon$  neighborhood of  $q_i$  and  $V \subset V_j(q_i)$ .*

*Consequently, the  $2\varepsilon$  neighborhood of  $q_i + V_{i,j}$  contains the  $\varepsilon$  neighborhood of  $q + V$ .*

*Proof.* Using Lemma 7, for every point  $q \in Q$  pick a stability neighborhood  $U(q)$  of radius  $\xi(q) < \varepsilon$ . By the compactness of  $Q$ , fix a finite subcover  $\cup_{i=1}^{\bar{i}} U(q_i)$  of  $Q$  from these environments and remember that by Lemma 8 each  $\mathcal{V}(q_i) = \{V_j(q_i) : 1 \leq j \leq j(q_i)\}$  is finite. Then by the definition of stability neighborhoods, we have that for arbitrary  $q \in Q$  and any free subspace  $V \in \mathcal{V}(q)$ , there are some  $q_i$  and  $j : 1 \leq j \leq j(q_i)$  such that  $q$  is in the  $\varepsilon$  neighborhood of  $q_i$  and  $V \subset V_j(q_i)$ .  $\square$

**Remark** Those  $q_i + V_{i,j}$ 's with  $\dim V_{i,j} = d_{\max}$  are necessarily subsets of some maximal horizons. Since  $H^{2\varepsilon}$  contains the  $2\varepsilon$  neighborhood of  $H$ , the  $2\varepsilon$  neighborhoods of these  $q_i + V_{i,j}$ 's are covered by the  $H^{2\varepsilon}$ 's. Thus we call the sets  $H^{2\varepsilon}$  for  $H \in \mathbb{H}$ , and the  $2\varepsilon$  neighborhoods of the remaining  $q_i + V_{i,j}$ 's ( $i \in I, j \in J_i$ ) *estimator environments*. Remind that  $\dim V_{i,j} < d_{\max}$  for all  $i \in I, j \in J_i$ , and that the  $\varepsilon$  neighborhood of any affine free subspace is covered by some estimator environment - thus the Proportionality lemma asserts that the  $c$  portion of a long enough free flight is spent in an estimator environment.

**Proof of (12)** First, with the already fixed  $\varepsilon$ , use the Proportionality lemma to obtain some  $c$  and  $T$ . From now on, we always assume  $t > T$ . For the estimation of  $\mu(\tau > t)$ , we distinguish three cases.

**Case 1** **Such points  $x = (q, v) \in M$  with  $\tau(x) > t$ , for which the time interval  $[s_1, s_2]$  with  $0 < s_1 < s_1 + c\tau(x) < s_2 < \tau(x)$  guaranteed by the Proportionality lemma is spent in the  $2\varepsilon$  neighborhood of  $q_i + V_{i,j}$  for some  $i \in I, j \in J_i$ .**

Since there is a line segment of length at least  $ct/2$  spent in the neighborhood of  $q_i + V_{i,j}$ , the angle of  $v$  and  $V_{i,j}$  is necessarily smaller than  $2/(ct)$ . As it was also used by the proof of (2), the  $d - 1$  dimensional Lebesgue measure on  $S^{d-1}$  of such velocity vectors  $v$  is asymptotically

$$\left(\frac{2}{ct}\right)^{\dim V_{i,j} - d}.$$

Since  $\dim V_{i,j} < d_{\max}$  and there are finitely many estimator environments, for  $t$  large enough the  $\mu$ -measure of points of Case 1 are smaller than

$$\delta/4 \sum_{H \in \mathbb{H}} C_H t^{d_{\max} - d}.$$

**Case 2 (Main term)** **Such points  $x \in M$  with  $\tau(x) > t$ , where the configuration component of  $\Phi_{[0,t]}x$  is a subset of  $H^{3\varepsilon}$  for some  $H \in \mathbb{H}$ .**

The same argument used to prove (2) implies that the  $\mu$ -measure of such points is asymptotically not larger than

$$\sum_{H \in \mathbb{H}} C_{H^{3\varepsilon}} t^{d_{\max} - d},$$

thus for  $t$  large enough, is smaller than

$$(1 + \delta/2) \sum_{H \in \mathbb{H}} C_H t^{d_{\max} - d}.$$

**Case 3 Such points  $x \in M$  with  $\tau(x) > t$  not treated in Case 2, for which the time interval  $[s_1, s_2]$  with  $0 < s_1 < s_1 + c\tau(x) < s_2 < \tau(x)$  guaranteed by the Proportionality lemma is spent in  $H^{2\varepsilon}$  for some  $H \in \mathbb{H}$ .** It is worth noting that one difficulty of this case comes from the fact that it covers an infinite number of lower dimensional "attached" horizons.

Note that  $\Pi_Q \Phi_{[0, \tau(x)]} x$  for such an  $x$  has a part of length at least  $c\tau(x)/2$  in the region  $H^{3\varepsilon} \setminus H^{2\varepsilon}$  and also crosses this region in the sense that intersects with both  $H^{2\varepsilon}$  and the complement of  $H^{3\varepsilon}$ . Thus there are some  $s_3, s_5$  with  $0 < s_3 < s_3 + c\tau(x)/2 < s_5 < \tau(x)$  such that  $\Pi_Q \Phi_{s_3} x$  is in  $\partial B_{H^{2\varepsilon}} \times V_H$  and  $\Pi_Q \Phi_{s_5} x$  is in  $\partial B_{H^{3\varepsilon}} \times V_H$  (or  $\Pi_Q \Phi_{s_3} x$  is in  $\partial B_{H^{3\varepsilon}} \times V_H$  and  $\Pi_Q \Phi_{s_5} x$  is in  $\partial B_{H^{2\varepsilon}} \times V_H$ , which case can be treated analogously). As a starting idea, one can think about this trajectory segment as a long free flight in a  $d_{\max}$  dimensional billiard, which guarantees that the Lebesgue measure of points of Case 3 are not large. More precisely, write

$$\Phi_{s_3} x = (q^\perp + q^\parallel, v^\perp + v^\parallel),$$

where  $q^\perp$  and  $v^\perp$  are in  $V_H^\perp$ , while  $q^\parallel$  and  $v^\parallel$  are in  $V_H$ . Note that  $q \in Q$  by definition, but the components  $q^\perp, q^\parallel$  are in  $\mathbb{R}^d$ . The projection of the trajectory segment  $\Pi_Q \Phi_{[s_3, s_5]} x$  to  $V_H^\perp$ , prescribed by  $q^\perp$  and  $v^\perp$ , is going to be used to construct the billiard table of dimension  $d_{\max}$ , while  $q^\parallel$  and  $v^\parallel$  are going to define the trajectory in this lower dimensional billiard table. There is a point  $z \in \partial B_H$  such that in the intersection point of  $z + V_H$  and  $\partial Q$  the  $d$  dimensional sphere of radius  $\kappa_{\max}^{-1}$  touching the appropriate scatterer from inside has a center, the projection of which to  $V_H^\perp$  is collinear with  $z$  and  $q^\perp$ . (See Figure 2.) Let us denote this sphere by  $S$ . Now, consider the affine subspace  $q^\perp + V_H$ . By definition, there exists a point  $q^\perp + p$  in this affine subspace such that the  $d$  dimensional ball of radius  $2\varepsilon$  and center  $q^\perp + p$  is contained completely in  $S$  and hence in a scatterer.

Now let us define a  $d_{\max}$  dimensional billiard configuration space: the periodicity is  $\mathcal{L} \cap V_H$ , there is one spherical scatterer of radius  $\varepsilon$  and the center of this spherical scatterer is  $p$  (when  $\mathbb{R}^{d_{\max}}$  is identified with  $V_H$ ). Denote its configuration space by  $\tilde{Q}_{d_{\max}}$ . Note that the intersection of  $\tilde{Q}$  and  $q^\perp + V_H$  is contained in  $\tilde{Q}_{d_{\max}}$  (again, with  $\mathbb{R}^{d_{\max}}$  being identified with  $V_H$ ). Further, we claim that with the notation

$$s_4 = s_5 \wedge \min\{s > s_3 : \text{dist}(q^\perp, q^\perp + (s - s_3)v^\perp) > \varepsilon\},$$

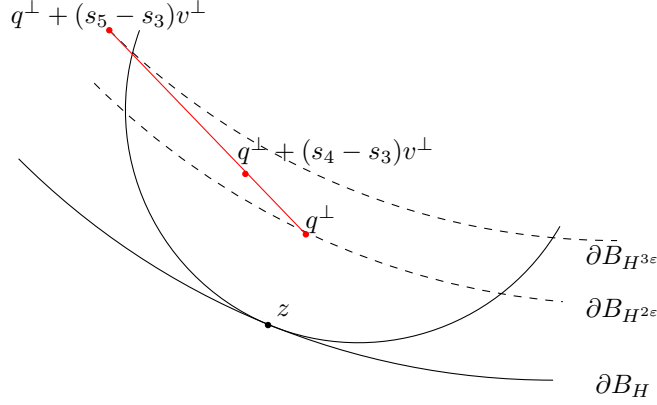


Figure 2: Construction of the  $d_{\max}$  dimensional billiard table - figure in  $V_H^\perp$

for every  $s_3 < s < s_4$ , the intersection of  $\tilde{Q}$  and  $q^\perp + (s - s_3)v^\perp + V_H$  is also contained in  $\tilde{Q}_{d_{\max}}$ . Indeed, since  $\text{dist}(q^\perp, q^\perp + (s - s_3)v^\perp) < \varepsilon$ , the  $d$  dimensional ball of radius  $\varepsilon$  and center  $q^\perp + (s - s_3)v^\perp + p$  is contained in the ball of radius  $2\varepsilon$  and center  $q^\perp + p$ . The latter statement is in general not true for  $s = s_5$ , since  $q^\perp + (s_5 - s_3)v^\perp$  can be outside of the projection of  $S$  (see Figure 2), that is why we needed to introduce  $s_4$ .

Now, we can easily map a long free flight in this  $d_{\max}$  dimensional billiard to our trajectory segment  $\Phi_{[s_3, s_5]}(x)$ . Namely, let us choose the free flight of the phase point  $(q^\parallel, v^\parallel)$  in  $Q_{d_{\max}}$ . Due to the construction, this free flight is longer than  $(s_4 - s_3)/2$ . We claim that this is longer than a universal constant (in the sense that does not depend on  $x$  but may depend on  $\varepsilon$  and also on  $H$  since there are finitely many of them) times  $t$ , i.e.

**Lemma 12.** *There is a constant  $c'(\varepsilon)$ , such that  $s_3 < s_3 + 2c'(\varepsilon)\tau(x) < s_4 \leq s_5$ .*

*Proof.* It is enough to prove that there exists some  $c''(\varepsilon)$  such that  $s_3 + c''(\varepsilon)(s_5 - s_3) < s_4$ . Since  $|(s_4 - s_3)v^\perp| > \varepsilon$ , it is enough to give an upper bound for  $|(s_5 - s_3)v^\perp|$ . Thus we need that the function

$$\Delta(y, z) = \max\{r \mid \exists l \in \mathcal{L} : \overline{y, z + l} \subset B_{H^{3\varepsilon}} \text{ and } \text{dist}(y, z + l) = r\} \quad (13)$$

on  $B_{H^{3\varepsilon}} \times B_{H^{3\varepsilon}}$  is bounded (then  $\varepsilon$  divided by this bound is an appropriate choice for  $c''(\varepsilon)$ ).

In order to see that (13) is bounded, first we prove that the set  $\{\Delta(y, z) \geq n\}$  is closed for any integer  $n$ . Choose any convergent sequence  $(y_i, z_i) \rightarrow (y_\infty, z_\infty)$  from the above set. There are corresponding  $l_i \in \mathcal{L}$  vectors by the definition of  $\Delta(y, z)$ . Then the set  $\{l_i : i \geq 1\}$  cannot be infinite, since if it was, then one could choose a convergent subsequence of  $l_i/|l_i|$  and the line with this direction containing  $y_\infty$  would be a subset of  $B_{H^{3\varepsilon}}$  which is a contradiction. Thus the set  $\{l_i : i \geq 1\}$  is finite. Hence one can choose a subset  $(y_{i_k}, z_{i_k}) \rightarrow (y_\infty, z_\infty)$  with

$l_{i_k} = l$ , yielding  $\Delta(y_\infty, z_\infty) \geq n$ . Whence  $\{\Delta(y, z) \geq n\}$  is closed. Now assume by contradiction that  $\Delta(y, z)$  is not bounded, thus the sets  $\{\Delta(y, z) \geq n\}$  for  $n \geq 1$  are closed subsets of each other. Thus there is a pair  $(y, z)$  such that  $\Delta(y, z) = \infty$ . Just like before, one can easily deduce the existence of an infinite line in  $B_{H^{3\varepsilon}}$  through  $y$  which is a contradiction. Thus we have proved that (13) is bounded and thus verified the existence of an appropriate  $c'(\varepsilon)$ .  $\square$

Now, we finish the proof of Case 3. Since at least  $ct$  time of the free flight is spent in  $H^{2\varepsilon}$ , the angle of  $v$  and  $V_H$  is smaller than  $C_1/t$  with some  $C_1$ . Thus the points  $x = (q, v)$  of Case 3 are elements of the set  $\mathbb{R}^d/\mathcal{L} \times V(t)$ , where

$$V(t) = \{v \in S^{d-1} : \angle(v, V_H) < C_1/t\}.$$

As before,  $\lambda_{d-1}(V(t)) < C_2 t^{d_{\max}-d}$ . Every point  $(q, v) \in \mathbb{R}^d/\mathcal{L} \times V(t)$  can uniquely be written in the form

$$(q, v) = (q_0^\perp + q_0^\parallel, v^\perp + v^\parallel)$$

with  $q_0^\perp, v^\perp \in V_H^\perp$ ,  $q_0^\parallel, v^\parallel \in V_H$ . The conditional measure of  $\lambda_d \times \lambda_{d-1}$  on  $\mathbb{R}^d/\mathcal{L} \times V(t)$  to such points where  $q_0^\perp, v^\perp$  are fixed, is also Lebesgue on the possible set of pairs  $(q_0^\parallel, v^\parallel)$ . Note that since  $|v^\perp|$  is small, the set of possible  $v^\parallel$ 's is a  $d_{\max}$  dimensional sphere of radius close to one. But the set of possible  $q_0^\parallel$ 's depends on  $q_0^\perp$ , since  $V_H^\perp$  is not necessarily generated by lattice vectors. Thus write

$$\mathfrak{q}(q) = \{\bar{q} \in \mathbb{R}^d/\mathcal{L} : \bar{q}_0^\perp = q_0^\perp\}.$$

One can easily prove that there exists some  $\eta > 0$  such that

$$\lambda_d\{q : \lambda_{d_{\max}}(\mathfrak{q}(q)) < \eta\} < \frac{C_H \delta}{8c_\mu C_2}.$$

Thus

$$\mu(\{q : \lambda_{d_{\max}}(\mathfrak{q}(q)) < \eta\} \times V(t)) < \frac{\delta}{8} C_H t^{d_{\max}-d}.$$

Now we can assume that

$$\lambda_{d_{\max}}(\mathfrak{q}(q)) > \eta. \tag{14}$$

Since the  $c$  portion of the line segment  $\Pi_{V_H^\perp} \Pi_Q \Phi_{[0, \tau(x)]} x$  is spent in  $H^{2\varepsilon}$ , once  $q_0^\perp, v^\perp$  are fixed, the number of possible  $q^\perp$ 's (that is, the projection of  $\Pi_Q \Phi_{s_3} x$  to  $V_H^\perp$ ) is bounded. This, the inductive hypothesis (used on the billiard table  $\bar{Q}_{d_{\max}}$ ), Lemma 12 and (14) imply that once  $q_0^\perp, v^\perp$  are fixed, the  $\lambda_{d_{\max}} \times \lambda_{d_{\max}-1}$  measure of such coordinates  $(q_0^\parallel, v^\parallel)$  with which the free flight is longer than  $t$  is bounded by some universal constant times  $t^{-1}$ . Consequently, for  $t$  large enough, the  $\mu$  measure of points in Case 3 are smaller than

$$\frac{\delta}{4} \sum_{H \in \mathbb{H}} C_H t^{d_{\max}-d}.$$

## 6 Proof of Theorem 2

### 6.1 Lorentz process with small scatterers

First, we recall the following result of Bourgain, Golse and Wennberg (see [BGW98] and [GW00]).

Consider a billiard table with periodicity  $\mathbb{Z}^D$  ( $D \geq 2$ ) and one spherical scatterer of radius  $r < 1/2$ . Define  $\mu_{\mathbb{Z},r}$  and  $\tau_{\mathbb{Z},r}$  for this billiard table as before. Then there exist  $c'(D)$  and  $C'(D)$  such that

$$\frac{c'(D)}{tr^{D-1}} \leq \mu_{\mathbb{Z},r}(\tau_{\mathbb{Z},r} > t) \leq \frac{C'(D)}{tr^{D-1}} \quad (15)$$

is true whenever

$$t > r^{1-D}. \quad (16)$$

In the case  $t \approx r^{D-1}$ , the so-called Boltzmann-Grad limit, much more is known than (15), see [MS10], or Remark 8.3.

In order to prove Theorem 2, we need a slightly extended version of the above estimation.

Let  $\mathcal{L}'$  be any  $D$ -dimensional lattice and let  $q_1, \dots, q_{n'} \in \mathbb{R}^D/\mathcal{L}'$ . Consider the billiard table with periodicity  $\mathcal{L}'$  and finitely many disjoint spherical scatterers of radius  $r$  centered at  $q_1, \dots, q_{n'}$ . Let  $Q'$ ,  $M'$ ,  $\mu'$  and  $\tau'$  be defined accordingly.

**Lemma 13.** *There exist  $c'(\mathcal{L}')$  and  $C'(\mathcal{L}')$  such that*

$$\frac{c'(\mathcal{L}')}{tr^{D-1}} \leq \mu'(\tau' > t) \leq \frac{C'(\mathcal{L}')}{tr^{D-1}} \quad (17)$$

is true whenever

$$t > r^{1-D}. \quad (18)$$

**Remark** Obviously, Lemma 14 also implies that for any fixed  $\eta > 0$ , (17) is true if  $t > \eta r^{1-D}$ , with some  $c'(\mathcal{L}')$  and  $C'(\mathcal{L}')$  depending also on  $\eta$ . Thus, whenever we refer to (18), it may be true only with some  $\eta$ , but in order to make the exposition simpler, we do not keep track of the  $\eta$ 's.

*Proof.* First, we prove the upper estimate. Pick a basis  $\{a_i\}_{i=1}^D$  of the lattice  $\mathcal{L}'$  and denote by  $A$  the matrix whose  $i$ -th column is  $a_i$ . Also write  $\sigma_i$  for the  $i$ -th smallest singular value of  $A^{-1}$ . Further, identify  $\mathbb{R}^D/\mathbb{Z}^D$  with the unit cube and  $\mathbb{R}^D/\mathcal{L}'$  with the parallelepiped  $(a_i)_{i=1}^D$ . Without loss of generality, we may assume that one of the spherical scatterers is centered at the origin (i.e.  $q_1 = 0$ ). Now assume that for some  $x' = (q', v') \in M'$ ,  $\tau'(x') > t$ . Then for the point

$$\phi(x') := x_{\mathbb{Z}} = \left( A^{-1}q', \frac{A^{-1}v'}{\|A^{-1}v'\|} \right),$$

we have

$$\tau_{\mathbb{Z},r\sigma_1}(x_{\mathbb{Z}}) > t\sigma_1.$$

Indeed, the image under  $A^{-1}$  of the sphere of radius  $r$  centered at the origin contains the sphere of radius  $r\sigma_1$  (the images of the possible other scatterers are simply omitted). The Lebesgue measure on  $Q'$  is transformed by  $\phi$  to  $\det(A^{-1})$  times the Lebesgue measure in  $\mathbb{R}^D/\mathbb{Z}^D$  minus an ellipse centered at the origin, which is dominated by the Lebesgue measure on  $\mathbb{R}^D/\mathbb{Z}^D \setminus B(0, r\sigma_1)$ . The image of the Lebesgue measure on  $S^{D-1}$  by  $\phi$  is  $\frac{1}{\|A^{-1}v'\|} dv'$ .

Thus, using (15), one can prove the second part of (17) with

$$C'(\mathcal{L}') = \det(A^{-1})\sigma_n\sigma_1^{-D-1}C'(D)$$

at least, for  $t > \sigma_1^{-D}r^{D-1}$ , but consequently for  $t > r^{D-1}$  too, possibly with a different  $C'(\mathcal{L}')$ .

Now, we prove the lower estimate. Observe that it is enough to prove the statement for the special case  $\mathcal{L}' = \mathbb{Z}^D$ . Indeed, once  $c(\mathbb{Z}^D)$  is found, one can prove the existence of  $c(\mathcal{L}')$  for any  $\mathcal{L}'$  the same way as in the upper estimation. Thus the statement we are going to prove is indeed a slight modification of the first part of (15): the difference is that we have  $n'$  spherical scatterers of radius  $r$  centered at arbitrary points  $q_1, \dots, q_{n'}$ , instead of just one scatterer. We claim that an obvious modification of the proof of Golse and Wennberg applies here. Indeed, if  $q$  is an integer vector with  $g.c.d.(q) = 1$  and one projects the scatterer configuration to the line with direction  $q$ , then observes a gap of length at least  $(1/|q| - 2n'r)/n'$  among the images of the scatterers, assuming of course that  $r < (2n'|q|)^{-1}$ . Hence there is a principal horizon perpendicular to  $q$  (or “sandwich layer”) whose middle third has width

$$a_{q,r} = \frac{1}{3} \left( \frac{1}{|q|} - 2n'r \right) \frac{1}{n'}.$$

Considering only those  $q$ 's for which  $|q| < q_{\max} = (4n'r)^{-1}$ , the density of the middle third layers is larger than  $(12n')^{-1}$  (instead of  $1/6$ , see page 1158 in [GW00] for more details). With these modifications, the proof of [GW00] yields the statement. □

## 6.2 Upper estimate

We assume that there is one principal incipient horizon, if there were more, an analogous proof would apply. As in Subsection 5.2, let us fix an  $\varepsilon$ , define the estimator environments - one of them is the  $2\varepsilon$  neighbourhood of the principal incipient horizon ( $H^{2\varepsilon}$ ), the others have dimension at most  $d - 2$ . The proportionality lemma implies that the  $c$  portion of a long enough flight is spent in one of the estimator environments. The  $\mu$ -measure of such points for which this is not  $H^{2\varepsilon}$  is  $O(t^{-2})$  as in Case 1 of Subsection 5.2.

The essence of the proof is the following statement:

**Lemma 14.** For a fixed  $\varepsilon$  small enough,

$$\begin{aligned} & \lambda_d \times \lambda_{d-1} \left( \{x = (q, v) | q \in H^{2\varepsilon}, \tau(x) > s, \Pi_Q \Phi_{[0, s]} x \subset H^{2\varepsilon}\} \right) \\ &= \begin{cases} O(s^{-2}), & 3 \leq d \leq 5 \\ O(s^{-2} \log s), & d = 6 \\ O\left(s^{\frac{2+d}{2-d}}\right), & d > 6. \end{cases} \end{aligned}$$

*Proof.* Denote by  $V$  the  $d - 1$  dimensional hyperplane defining the incipient horizon. Without loss of generality, we may assume that the origin is in this horizon, that is  $H = V$ . Since  $V$  is a lattice subspace, one can choose a lattice vector  $v_d \in \mathcal{L} \setminus V$  such that  $V \cap \mathcal{L}$  and  $v_d$  generate  $\mathcal{L}$ . Since  $\mathbb{R}/\mathcal{L}$  can be identified with a parallelepiped generated by  $v_1, \dots, v_d$  with  $v_1, \dots, v_{d-1} \in V$ , for every  $q \in Q \cap H^{2\varepsilon}$ , there is a unique decomposition

$$q = q_V + q_W$$

with  $q_V \in V$ ,  $q_W \parallel v_d$  and  $|q_W| < 2\varepsilon \cot \alpha$ , where  $\alpha$  is the angle of  $V$  and  $v_d$ . We also write

$$v = v^\parallel + v^\perp,$$

where  $v \in S^{d-1}$ ,  $v^\parallel \in V$ ,  $v^\perp \in V^\perp$ .

The idea of the proof is reminiscent to that of Case 3 in Subsection 5.2. If there is a long flight in  $H^{2\varepsilon}$ , then  $v$  is close to  $V$ . Thus we can think of this trajectory as a long free flight in a  $d - 1$  dimensional billiard. Note that here, the  $d - 1$  dimensional scatterer size can be arbitrary small, since the trajectory is close to  $V$ . Thus a delicate analysis of this scatterer size, and the upper estimation of (17) are needed.

Let us chop the set of possible  $q_W$ 's and  $v^\perp$ 's into the following pieces:

$$\begin{aligned} V_i &= \{v^\perp \in V^\perp | |v^\perp| \in [2^{-i}, 2^{-i+1}]\} & i > \log s - \log 2\varepsilon \\ Q_j &= \{av_d | |a| \in [2^{-j} \cot \alpha, 2^{-j+1} \cot \alpha]\} & j > -\log 2\varepsilon. \end{aligned}$$

Accordingly, we write

$$H_j = H^{2^{-j+1}} \setminus H^{2^{-j}}.$$

Here, and also in the sequel,  $\log$  always stands for  $\log_2$ .

Now assume that  $v^\perp \in V_i$  and  $q_W \in Q_j$  for some fixed  $i, j$ . We want to estimate the  $\lambda_{d-1} \times \lambda_{d-2}$  measure of parameters  $q_V$ ,  $v^\parallel$  with which  $(q, v)$  is an element of the set

$$Q_{long} = \{x = (q, v) | q \in H^{2\varepsilon}, \tau(x) > s, \Pi_Q \Phi_{[0, s]} x \subset H^{2\varepsilon}\}.$$

We can assume that the projection of  $q_W$  to  $V^\perp$  and  $v^\perp$  are oppositely oriented. If they are not, a simpler version of the forthcoming proof is applicable. From now, we distinguish four cases.

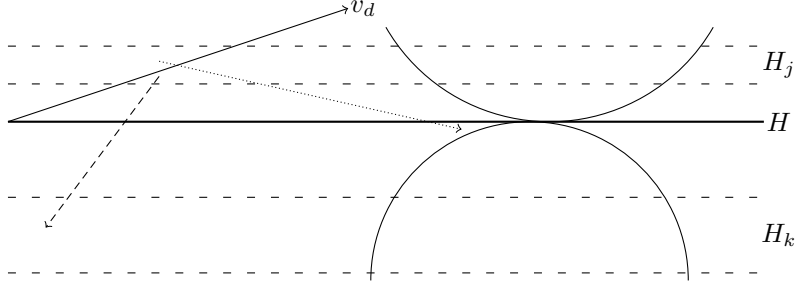


Figure 3:  $H^{2^\varepsilon}$  - a  $d$  dimensional picture. Densely dotted trajectory:  $j < i - \log s$ . Densely dashed trajectory:  $j \geq i - \log s$ .

- Case a**  $i < \frac{d}{d-2} \log s$  and  $j \leq i - \log s$ .  
 In this case, there is a line segment of  $\Pi_Q \Phi_{[0,s]} x$  of length at least  $s/5$  spent in the strip  $H_{j+1}$ .  
 Note that for every  $q \in H_{j+1}$ , the intersection of  $Q$  and  $q + V$  is a billiard configuration of dimension  $d - 1$ . Further, this billiard configuration is contained in a larger one, where there is only one spherical scatterer of radius approximately  $\sqrt{\kappa_{\max}^{-1} 2^{-j}}$ . Indeed, there is at least one  $d$  dimensional scatterer touching  $V$  from the appropriate side. If one takes the  $d$  dimensional ball of radius  $\kappa_{\max}^{-1}$  touching  $V$  in this point and considers the intersection of the ball and a close enough affine hyperplane, obtains a  $d - 1$  dimensional ball of the desired radius (which is roughly the square root of the distance of the hyperplanes). As in Case 3 of Subsection 5.2, by projecting the previously obtained trajectory segment of length  $s/5$  to the “lower boundary of  $H_{j+1}$ ” (i.e.  $\partial H^{2^{-j}}$ ) we obtain a free flight of length at least  $s/6$  (if  $\varepsilon$  is small enough) in a  $d - 1$  dimensional billiard table with periodicity  $\mathcal{L} \cap V$  and one spherical scatterer of radius  $\sqrt{\kappa_{\max}^{-1} 2^{-j}}$ . Note that this mapping to the lower dimensional billiard is simpler than that of Subsection 5.2, since  $V^\perp$  is one dimensional, thus the billiard configuration space in  $q + V$  is increasing as  $q$  moves from  $\partial H^{2^\varepsilon}$  to  $V$  (the issue of moving scatterers is simply absent). Observe that  $i < \frac{d}{d-2} \log s$  and  $j \leq i - \log s$  imply  $j \leq \frac{2}{d-2} \log s$  which yields that (18) is satisfied by  $t = s$ ,  $r = \kappa_{\max}^{-1/2} 2^{-j/2}$  and  $D = d - 1$ . Thus the second part of (17) implies that whenever  $v^\perp \in V_i$  and  $q_W \in Q_j$  are fixed, the  $\lambda_{d-1} \times \lambda_{d-2}$  measure of parameters  $q_V, v^\parallel$  with which  $(q, v) \in Q_{long}$  is  $O(s^{-1} 2^{j(d-2)/2})$ .
- Case b**  $\frac{d}{d-2} \log s \leq i < \frac{d+2}{d-2} \log s$  and  $j \leq \frac{2}{d-2} \log s$ .  
 The same estimation as in Case a yields that the  $\lambda_{d-1} \times \lambda_{d-2}$  measure of parameters  $q_V, v^\parallel$  with which  $(q, v) \in Q_{long}$  is  $O(s^{-1} 2^{j(d-2)/2})$ .
- Case c**  $i < \frac{d}{d-2} \log s$  and  $j > i - \log s$ .  
 Note that distance of  $\Pi_{V^\perp} \Pi_Q x$  and  $\Pi_{V^\perp} \Pi_Q \Phi_s x$  (here,  $\Pi_{V^\perp}$  is the orthogonal projection to  $V^\perp$ ) is at least  $s 2^{-i}$ , which is larger than  $2^{-j}$ . Hence



there is a  $k$  such that a line segment of  $\Pi_Q \Phi_{[0,s]} x$  of length at least  $s/8$  is spent in  $H_k$  and  $2^{-k}$  is larger than  $s2^{-i}/4$ . Now using the same estimation as in Case a in the strip  $H_k$ , one obtains that the  $\lambda_{d-1} \times \lambda_{d-2}$  measure of parameters  $q_V, v^\parallel$  with which  $(q, v) \in Q_{long}$  is  $O(s^{-1-\frac{d-2}{2}} 2^{i(d-2)/2})$ .

- **Case d**  $\frac{d}{d-2} \log s \leq i < \frac{d+2}{d-2} \log s$  and  $j > \frac{2}{d-2} \log s$ , or  $i \geq \frac{d+2}{d-2} \log s$ . In this case, we simply estimate the measure of the appropriate parameters  $q_V, v^\parallel$  by a constant.

Note that  $\lambda_1(V_i) \sim 2^{-i}$  and  $\lambda_1(Q_j) = 2^{-j}$ . Taking into account this fact and the estimations of Cases a-d, one obtains that  $\mu(Q_{long})$  is bounded from above by some constant times the following expression:

$$\begin{aligned} & \sum_{i=\log s - \log 2\varepsilon}^{\frac{d}{d-2} \log s} \left[ \left( \sum_{j=-\log 2\varepsilon}^{i-\log s} 2^{-i} 2^{-j} s^{-1} 2^{j(d-2)/2} \right) + 2^{-i} s 2^{-i} s^{-1-\frac{d-2}{2}} 2^{i(d-2)/2} \right] \\ & + \sum_{i=\frac{d}{d-2} \log s}^{\frac{d+2}{d-2} \log s} \left[ \left( \sum_{j=-\log 2\varepsilon}^{\frac{2}{d-2} \log s} 2^{-i} 2^{-j} s^{-1} 2^{j(d-2)/2} \right) + 2^{-i} s^{\frac{2}{2-d}} \right] + s^{\frac{2+d}{2-d}}. \end{aligned}$$

An elementary computation shows that this is the same order of magnitude as stated in the lemma.  $\square$

In order to finish the proof of the upper estimate, we need to bound the measure of points  $x = (q, v)$  for which  $\tau(x) > t$  and the proportionality lemma gives the estimator environment  $H^{2\varepsilon}$ . Observe that in this case, the angle of  $v$  and  $V$  is necessarily smaller than  $2\varepsilon/t$ . The Lebesgue measure of points for which  $q \in H^{2\varepsilon}$  is bounded by the desired order of magnitude due to Lemma 14. Thus assume that  $q \notin H^{2\varepsilon}$ . For every such point  $x = (q, v)$ , there is a point  $\phi(x) = x_b = (q_b, v)$ , which is the initial point of the free flight segment in  $H^{2\varepsilon}$  (i.e.  $\exists s < (1-c)\tau(x) : \Phi_s(x) = (q_b, v), q_b \in \partial H^{2\varepsilon}, \Pi_Q \Phi_{[s, s+c\tau(x)]} x \subset H^{2\varepsilon}$ ). The proportionality lemma also implies that for any such  $x_b$ ,

$$\lambda_1(\phi^{-1}(x_b)) < \frac{1}{c} \max\{s : s < \tau(x_b), \Pi_Q \Phi_{[0,s]} x_b \subset H^{2\varepsilon}\}.$$

Thus, also using Lemma 14 (with  $s = ct/2$ ), the integral

$$\int_{\partial H^{2\varepsilon} \times \{v : \angle(v, V) < 2\varepsilon/t\}} \sin(\angle(v, V)) \lambda_1(\phi^{-1}(x_b)) d\lambda_{d-1}(q_b) \times \lambda_{d-1}(v)$$

can be bounded by the desired order of magnitude which yields the upper estimate of Theorem 2.

### 6.3 Lower estimate

Now, we prove the second part of Theorem 2, which is a lower estimate in the dispersing case.

In dimension  $d \leq 5$ , the statement is straightforward, since obviously there are horizons of codimension 2 “attached” to the incipient horizon (indeed, a hyperplane parallel to the incipient horizon and close to it, intersects the scatterers in tiny convex bodies - approximate ellipsoids - which depend continuously on the distance of the hyperplanes). Then the same argument used to prove (2) provides a subset of the phase space of measure  $O(t^{-2})$  consisting of points having free flight longer than  $t$ .

In dimension  $d \geq 6$ , we use a simplified version of the proof of Lemma 14. The main observation is that due to the lower bound on the curvature, the scatterers touch the incipient horizon in finitely many points (in  $q_1, \dots, q_{n'}$ , say). Further, the intersection of the scatterers and a hyperplane parallel to the incipient horizon at distance  $h$  from it, is contained in  $n'$  spheres of radius  $\sqrt{\kappa_{\min}^{-1} h}$  centered at  $q_1, \dots, q_{n'}$ . Thus in Cases a-d of Lemma 14, by such a choice of  $i$  and  $j$ , where  $s2^{-i} \approx 2^{-j}$ , using the first part of (17) instead of the second, one easily obtains a lower bound of the same order of magnitude. In fact, for  $d > 6$ , only one pair of indices  $(i, j)$  is enough. Namely, choose

$$i = \lceil \frac{d}{d-2} \log s \rceil$$

and  $j = \lceil i - \log s \rceil$ . With this choice and the notation  $r = \sqrt{\kappa_{\min}^{-1} s 2^{-i}}$ ,  $s = t$ , (18) is fulfilled, hence the Lebesgue measure of points  $x = (q, v)$  with  $v^\perp \in V_i$  and  $q_W \in Q_j$  having free flight longer than  $s$  is at least some constant times  $2^{-i} 2^{-j}$ , thus another constant times  $s^{\frac{2+d}{2-d}}$ .

In dimension  $d = 6$ , one needs to consider all indices  $i$  with  $\log s - \log 2\varepsilon < i < 3/2 \log s - \log \kappa_{\min}$  and for a fix  $i$ , the index  $j = i - \log s$ . Similarly to the case  $d > 6$ , the lower estimation of order  $s^{-2} \log s$  follows.

## 7 Examples

Equ. (35) of [D12] provides the form of the limiting covariances for the superdiffusive limit of dispersing Lorentz processes assuming his Conjectures 1 and 3 hold. His derivation of Equ. (35) from the conjectures can be extended to the semi-dispersing case thus our Theorem 1 can be used. His Conjecture 3 is of dynamical nature and for clarity we briefly summarize what is known and what we expect in general. For brevity - beside [D12] - we rely here on the works [Y98, BT08] where, for instance, the complexity hypothesis is also used and the precise forms of exponential decay of correlations (EDC) and of the central limit theorem (CLT) are given.

- [BT08] For multidimensional ( $d > 2$ ) dispersing billiards with finite horizon satisfying the complexity hypothesis, EDC and CLT hold and the diffusivity covariance is given by Green-Kubo;

In formulating what we expect we do not pursue the highest generality and will be satisfied to restrict ourselves to ergodic cylindrical billiards (cf. [SSz00]).

- **Conjecture A** (Dynamical) For multidimensional ( $d > 2$ ) ergodic cylindrical billiards with strictly convex bases 1. without a principal horizon and 2. satisfying the complexity hypothesis, EDC and CLT hold and the diffusivity covariance is given by Green-Kubo;
- **Conjecture B** (Dynamical) For multidimensional ( $d > 2$ ) ergodic cylindrical billiards with strictly convex bases 1. with at least one principal horizon and 2. satisfying the complexity hypothesis, EDC and the super-diffusive limit statement with scaling  $\sqrt{n \log n}$  or  $\sqrt{t \log t}$  hold. (cf. [SzV07, ChD09] for  $d = 2$ ).

**Example 1: Cylindrical billiard on  $\mathbb{T}^3$ .** (We note that this was the first semi-dispersing billiard whose ergodicity had been established (cf. [KSSz89]).) We assume that on  $\mathbb{T}^3$  we are given two nonintersecting cylindrical scatterers  $C_1$  and  $C_2$  - for simplicity - of equal radii  $0 < r < 1/4$ . Suppose that the generator of  $C_i$  is parallel to the coordinate direction  $e_i$ ,  $i = 1, 2$  and the distances between the two cylinders - in the coordinate direction 3 - are  $z$  and  $w$ . In this case we have two principal horizons of widths  $z$  and  $w$  parallel to the coordinate plane  $(e_1, e_2)$  and super-diffusion is expected in the directions  $e_1, e_2$  whereas regular one in the direction of the axis  $e_3$ .

$$\mathcal{D}_{11} = \mathcal{D}_{22} = \frac{1}{4(1 - 2r^2\pi)}(z^2 + w^2)$$

$$\mathcal{D}_{33} = 0$$

Of course, if - in the direction of the axis  $e_3$  - we apply diffusive scaling, then the limiting covariance in that direction should again be given by the Green-Kubo formula.

**Example 2: Two hard balls of radii  $1/(4\sqrt{2}) < r < 1/4$  on  $\mathbb{T}^d$ .** Under the complexity hypothesis it follows from [BT08] and from Theorem 1 that for the super-diffusive limiting covariance  $\mathcal{D}$  of the system,  $\mathcal{D}_{ij} = \delta_{ij}\mathcal{D}$ , where

$$\begin{aligned} \mathcal{D} &= \sqrt{2} \frac{1}{1 - |B_d|(2r)^d} \frac{|B_{d-1}|}{|S_{d-1}|} (d-1)(1-4r)^2 \\ &= \sqrt{2} \frac{1}{1 - \frac{\pi^{d/2}}{\Gamma((d+2)/2)}(2r)^d} \frac{\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)} (1-4r)^2. \end{aligned}$$

Here  $B_d$  is the  $d$ -dimensional unit ball and  $S_{d-1}$  is its surface (cf. Equ. (37) of [D12]).

## 8 Concluding remarks

1. Our methods also make possible to obtain the asymptotics of the free path length for cases when the maximal, but not principal, horizon(s) are incipient. We omitted the discussion for brevity.

2. In order to prove the above Conjecture B, a first step could be determining the limiting joint distribution of  $\tau$  and the forthcoming free flight (i.e.  $\tau \circ \Phi_{\tau+}$ , where  $\Phi_{\tau+}$  means that the velocity is the post-collisional one), when  $\tau$  is large (see also Conjecture 3 in [D12] or in the planar case [B92] and [SzV07]). Thus we formulate another conjecture.

- **Conjecture C** (Geometric) In a  $d$  dimensional dispersing billiard with at least one principal, non-incipient horizon, if  $\tau$  is large, then  $\tau \circ \Phi_{\tau+}$  is typically of order  $\tau^{1/d}$ .

Now we explain why we expect Conjecture C to be true. Note that if  $\tau(x)$  is larger than some large  $t$ , then  $x = (q, v)$  - with probability close to one - is such that  $q$  is in a principal horizon  $H$ , and the angle of  $v$  and  $V_H$  is roughly  $1/t$ . Further, the component of  $v$  in  $V_H$  is uniformly distributed. After some time, the free flight from  $x$  reaches the boundary  $\partial B_H \times V_H$  of the horizon. Now we claim that the remaining time until the collision is typically  $t^{\frac{d-2}{d}}$ , or in other words, the distance of  $\Pi_{V_H^\perp} \Pi_Q \Phi_{\tau(x)} x$  and  $B_H$  is roughly  $t^{-2/d}$ . Indeed, in the hyperplane  $q_h + V_H$  at distance  $h$  from  $B_H + V_H$ , there are  $d-1$  dimensional scatterers (approximate ellipsoids of bounded eccentricity due to the dispersing assumption) of diameter  $\sqrt{h}$ . Thus (17) yields that in this hyperplane, a  $\lambda_{d-1} \times \lambda_{d-2}$ -typical phase point does not collide until time  $ht$  if and only if  $ht \ll h^{\frac{2-d}{2}}$ . Now a similar argument used to prove Lemma 14, implies that typically the distance of  $\Pi_{V_H^\perp} \Pi_Q \Phi_{\tau(x)} x$  and  $B_H$  is roughly  $h = t^{-2/d}$ . Denote the post collisional velocity by  $v'$ . We expect that the angle of  $v'$  and  $V_H$  is typically of order  $t^{-1/d}$  which would provide Conjecture C.

3. Consider a  $\mathbb{Z}^d$ -periodic ( $d \geq 2$ ) arrangement of balls of radii  $r < 1/2$ , and select a random direction and point (outside the balls). The dependence of the free flight function on  $r$  will be denoted by  $\tau_r$ . Bourgain, Golse and Wennberg, [BGW98] initiated the study of the asymptotic behavior of  $F_r(t) = \mu(\tau_r > t)$ , when  $r \searrow 0$ . Since  $\lim_{r \searrow 0} F_r(t) = 1$ , the question makes sense in an appropriate scaling, only. Their main result showed that limit is non-trivial in the scaling  $\mu(r^{d-1} \tau_r > t)$ , only (known in statistical physics as the Boltzmann-Grad scaling). Then Marklof and Strömbergsson, [MS10] could prove the existence of the limit for any lattice, any dimension, more general objects than spheres and also obtained further delicate results. In this limit dynamical questions can also be answered, and, in particular, Golse-Wennberg, [GW00] showed that the limiting equation is not the classical Boltzmann one. Finally, Marklof and Strömbergsson, [MS11] could prove that the limiting equation is a (second order Markov-) version of the linear Boltzmann equation. (As to a survey on these and related results see [M10].) Pólya's visibility problem is, in turn, related to the maximum of  $\tau_r$ , when one erases a ball and chooses the initial point to be its center (see [P18] and [K08]). On the other hand, for Lorentz processes with fixed configuration of scat-

terers, i. e. without the Boltzmann-Grad limit, Dettmann’s conjectures elaborated and made those of Sanders ([S05, S08]), based on computational observations, more precise.

Dettmann observed that the constant in the tail asymptotics of the free path length in the Boltzmann-Grad limit of  $\mathbb{Z}^d$ -periodic spherical scatterers of radii  $r \searrow 0$  (cf. Equ. (1.43) in [MS11a]) coincides with the constant arising in his heuristic computation (cf. Equ. (31) in [D12]) by taking the large time limit of Theorem 1 and the limit  $r \searrow 0$  in reversed order. Marklof has raised the intriguing question to prove this coincidence rigorously that would also require uniform estimate of remainder term in our Theorem 1.

4. [Sz08] also raised the problem of the limiting behavior of a quasi-periodic Lorentz process, for instance that of the Penrose-Lorentz one. As [W12] points out the tail distribution of the free path length is exponential in random Lorentz processes with non-intersecting scatterers whereas - as we have seen - it is algebraic in the presence of horizons. The simulations of the author suggest that for a 1-dimensional quasi-periodic paradigm of the Lorentz process, this tail behavior is not exponential. On the other hand, [KS12] stresses that for the random non-intersecting Lorentz process one has normal diffusion and observes computationally three different regions for a 2-dimensional quasi-periodic Lorentz process showing super-diffusion, diffusion and subdiffusion.

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