Voting and Optimal Provision of a Public Good*

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Abstract

In this paper, we study the optimal provision of a costly public good using an average efficiency criterion. For every fixed cost, we identify a quota mechanism as the optimal mechanism among those that are dominant-incentive-compatible, deficit-free and kind. Moreover, we also consider the asymmetric case and demonstrate that a committee mechanism is optimal for a large class of mechanisms. In particular, this mechanism dominates all VCG (pivotal) mechanisms.

Keywords: Public good, Mechanisms, Dominant-incentive compatibility, VCG mechanisms, Quotas, Committees

1 Introduction

When agents' preferences are private information, it is difficult for the central planner to provide public goods efficiently. This well-known free-rider problem has challenged public officials and scholars for years. There are several related issues. First, how can one solicit agents' preferences to determine the proper level of public goods? Second, how can one design tax/transfer schemes that finance the provision of public goods? Third, how can

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one choose the most efficient method of providing of public goods from among the different methods?

Many scholars favor the use of VCG (Vickrey-Clarke-Groves) mechanisms (Vickrey, 1961; Clarke, 1971; Groves, 1973). By introducing proper taxes/transfers among agents, the celebrated VCG mechanisms induce agents to reveal private information truthfully, which, in turn, leads to the efficient allocation of public goods. However, VCG mechanisms do not solve the problem of efficient provision of public goods, as the aggregate tax revenues they collect often exceed the costs needed to finance the public goods.

In practice, decisions on public-good provision are often made through more straightforward mechanisms. For example, when a city decides whether to build a new public transportation system, it may put the issue to a public vote, and if the funding of the project is approved, it is usually in the form of an added tax. There might be variations in the voting rules: simple majority rule, unanimity rules, committee approval, etc. A common feature of these voting rules is that, should the project fail to pass, no money is wasted since no additional tax will be raised. However, there is no guarantee that voting results always correspond to efficient levels of the public good.

Clearly, neither VCG mechanisms nor voting schemes guarantee the efficient provision of public goods. This is inevitable since, as Green and Laffont (1977) demonstrate, there exists no dominant-incentive-compatible (DIC) mechanism that always yields both efficient levels of public goods and exact budget balance. Thus, to determine which mechanisms are better than others, one needs a criterion by which to evaluate their performance. But once a criterion is adopted, a more natural question would be: Which mechanism is *optimal* among all DIC mechanisms? That is the main question we address in this paper.

We focus on dominant-incentive-compatible mechanisms since they have the strongest incentive-compatibility property. Earlier scholars studied efficient Bayesian mechanisms that may provide partial solutions to the Green-Laffont conundrum (Arrow, 1979; d'Aspremont and Gerard-Varet, 1979). However, such solutions are highly sensitive to the specification of prior distributions of agents' types. If the priors are mis-specified, the proposed mechanism is not even incentive-compatible, let alone efficient. DIC mechanisms are the only mechanisms immune to this problem as a mechanism is dominant-incentive-compatible if and only if it is Bayesian-incentive-compatible for all prior distributions of agents' types. Interested readers can find more related discussion in Chung and Ely (2007) and, Bergemann and Morris (2005).

Although we consider only DIC mechanisms, we can use various criteria to evaluate their efficiencies. The first natural one is the dominance criterion: a DIC mechanism A is better than a DIC mechanism B if A is better than B for all realizations of agents' types. However, as the Green-Laffont classic result indicates, there is no optimal DIC mechanism according to the dominance criterion. Hence, we must work with some weaker criteria. Some scholars propose the minmax criterion (minimizing the maximal welfare loss): a DIC mechanism A is better than a DIC mechanism B if the worst outcome under A is better than the worst outcome under B. They use the minmax criterion to study both budget-balanced and VCG mechanisms in the public-good setting—e.g., Deb and Seo (1998) and Moulin and Shenker (2001). In this paper, we consider another criterion—the average criterion: a DIC mechanism A is better than a DIC mechanism B if A is better than B, on average. This idea of evaluating the efficiency of mechanisms by an average criterion dates back to Rae (1969). In more-recent papers, Shao and Zhou (2011), Drexl and Kleiner (2012, 2013) and Gershkov, Moldovanu and Shi (2014) all carry out the same type of exercise. Agents' types follow prior distributions. Although agents do not need or have such information under any DIC mechanism, the planner can and should use this information to evaluate the average efficiencies of various DIC mechanisms. We do not argue that the average criterion is necessarily better than the minmax criterion, but, rather, that both are worthy alternatives that merit serious investigation.

In this paper, we study a public-good model in which one must decide whether to build a public project—a binary social-choice problem. The public project can be built at a fixed cost of c. Each agent i's utility reservation value is zero in the absence of the public project. If the project is built, each i derives an additional utility of θ_i. θ_i is considered agent i's type and is privately known to agent i only. One uses the mechanism first to solicit all agents' types and then to decide whether or not to build the project and how much tax to impose on all agents.

Our main result identifies the most efficient DIC mechanisms according to the average criterion. More precisely, we find the best mechanism among all mechanisms that satisfy DIC, deficit-free, and one kindness condition. The optimal mechanism is a voting system with equal cost sharing. For each value of c, there is a quota $q(c)$; the public project is built if and only if the number of agents whose types are higher than the cost per capita c/n exceeds $q(c)$; and all agents equally share the total cost c. We further prove that the optimal quota mechanism outperforms all VCG mechanisms.

We want to make a couple of observations regarding our main result. First, we compare the voting system with VCG mechanisms. In the voting system, there might be under-provision or over-provision of the public good relative to the fully efficient outcome, but the budget is always balanced. In VCG mechanisms, the public good is always provided at the optimal level, but there are wasted funds at many type profiles. Many people who like VCG mechanisms tend to overlook their inability to balance the budget. Our result highlights the importance of budget-balancedness, as it is actually a consequence of optimality, albeit in our specific model.

Second, although the social choice literature has extensively studied voting systems very few studies have identified them as optimal mechanisms. An exception is a recent paper by Drexl and Kleiner (2013), which studies a case in which agents can have negative valuations, which corresponds roughly to our model with $c = 0$. With cost c as a parameter in our model, we obtain very interesting comparative statics about the structure of the optimal voting system. As the cost c increases, the minimum level of type for an agent to favor the project increases, and, at the same time, the quota $q(c)$ that is needed for approval of the project also increases. Undertaking a more costly public project requires more enthusiasm from more agents.

Our results can be extended to asymmetric cases in which agents' types may not follow the same distribution. We demonstrate two important points. First, every VCG mechanism is dominated by a committee mechanism. Second, some committee mechanism is optimal for a large class of asymmetric mechanisms that are DIC and satisfy a boundedness condition.

Before proceeding with the formal analysis, we briefly review several related results in the literature. As mentioned earlier, some recent papers have connected voting systems with optimal DIC mechanisms in public-good provision. Drexl and Kleiner (2013) study the case in which agents can have negative valuations with zero production cost and demonstrate that the optimal DIC mechanism is a voting system with zero transfers. But, as our result shows, this claim is not robust when the production cost becomes non-zero, which is most often the case in real situations. In several articles (for example, Schmitz and Tröger (2012) and Gershkov, Moldovanu and Shi (2014)), the authors consider optimal DIC mechanisms that maximize ex-ante utilities of agents when choosing from finite alternatives without monetary transfers. Gershkov, Goeree, Kushnir, Moldovanu and Shi (2013) establish an equivalence result between Bayesian mechanisms and dominant-strategy mechanisms. However, their result does not apply to our model, which imposes a budgetary constraint. Focusing on Bayesian mechanisms, Ledyard and Palfrey (2002) show that any optimal Bayesian incentive-compatible mechanisms can be approximated by a voting mechanism when the population grows large. In contrast, our results imply that the voting mechanism with a carefully chosen voting rule is optimal in the class of DIC mechanisms, regardless of the size of the population. Finally, Bierbrauer and Hellwig (2012) study a public-good model of a continuum of agents and prove that the mechanism that satisfies anonymity, robustness and coalition-proofness must take the form of a voting mechanism. Although our result and theirs are not comparable formally, both support the use of voting in public-good provision.

Our paper is organized as follows. In Section 2, we introduce the formal model and our main result. In Section 3, we generalize our model to deal with asymmetric cases. Section 4 concludes. We present all proofs in the appendix.

2 The Model and the Main Result

There is a society of n agents. A benevolent planner contemplates whether to provide a non-excludable public good—a bridge, a park, etc.—at a fixed cost $c \leq n$. If the public good is produced, the cost must be covered by taxes collected from the agents. Agent i's utility is $\theta_i + t_i$, in which θ_i is her benefit (her type) from the public good when it is provided, and t_i is the amount of tax she pays. If the public good is not provided, agents' benefits are zero. An agent's type is privately known only to herself, but agents' types are independently distributed on $[0, 1]$ according to a prior distribution function F with a density function f. We assume that the distribution function is regular:

Regularity. $(1 - F(\theta)) / f(\theta)$ is decreasing in θ; $F(\theta) / f(\theta)$ is increasing in θ.[1]

A (direct) mechanism M decides for each agent's type profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n) \in [0, 1]^n$ whether to provide the public good and how much the transfer for each agent should be. More formally, a mechanism $M = \{d(\boldsymbol{\theta}), t_i(\boldsymbol{\theta}) \;\; i = 1, ..., n\}$ consists of a collection of functions in which $d = 0$, or 1, and t_i are real values. And each agent's utility is quasi-linear—i.e.,

$$U_i(\theta_i) = \theta_i d(\theta_i, \boldsymbol{\theta}_{-i}) + t_i(\theta_i, \boldsymbol{\theta}_{-i}).$$

We consider mechanisms that satisfy three important properties. First, we require that mechanisms provide incentive for agents to reveal their private type truthfully. In this paper, as in many others in the literature, we consider the strong form of *dominant-incentive compatibility* (a.k.a., strategy-proofness condition):

$$\theta_i d(\theta_i, \boldsymbol{\theta}_{-i}) + t_i(\theta_i, \boldsymbol{\theta}_{-i}) \geq \theta_i d(\theta_i', \boldsymbol{\theta}_{-i}) + t_i(\theta_i', \boldsymbol{\theta}_{-i}) \text{ for any } \theta_i, \, \theta_i' \text{ and } \boldsymbol{\theta}_{-i}, \qquad \text{(DIC)}$$

where $\boldsymbol{\theta}_{-i} = (\theta_1, ..., \theta_{i-1}, \theta_{i+1}, ..., \theta_n)$.

Second, we require that mechanisms do not need outside funding for the public good; in other words, the mechanism is *deficit-free* (DF):

$$\sum_{i=1}^{n} t_i(\boldsymbol{\theta}) + cd(\boldsymbol{\theta}) \leq 0, \text{ for } \boldsymbol{\theta} \in [0, 1]^n. \qquad \text{(DF)}$$

[1] The first part of this condition is also known as the hazard-rate condition, and the second part means that F is logconcave. For example, all $F(\theta) = \theta^\alpha$ with $\alpha \geq 1$ satisfies these conditions. Both parts are commonly used in the literature. The regularity condition can also be implied by assuming that density f is logconcave. Bagnoli and Bergstrom (2005) have a nice discussion of these assumptions.

Note that this condition (DF) does not exclude the possibility of surplus when the total tax revenue exceeds the cost. Many DIC mechanisms, such as VCG mechanisms, do have surpluses at various type profiles. This is one source of inefficiency of a mechanism.

Third, we require that mechanisms treat agents fairly and equitably. As we are working with a quasi-linear model, it is sensible to make interpersonal utility comparisons. Differences in agents' types may lead to differences in agents' utilities. A "kind" mechanism should respect differences in agents' types but should not amplify such differences. Formally, a mechanism is *kind* if

$$0 \leq U_i\left(\theta_i, \theta_j, \boldsymbol{\theta}_{-i,j}\right) - U_j\left(\theta_i, \theta_j, \boldsymbol{\theta}_{-i,j}\right) \leq \theta_i - \theta_j \text{ for } \theta_i \geq \theta_j. \tag{K}$$

Let $\mathcal{M}$ denote the set of all mechanisms that satisfy (DIC), (DF), and (K). Our objective is to find the "most efficient" mechanism in $\mathcal{M}$.

Ideally, the most efficient mechanism would provide the highest sum of agents' utilities at all type profiles. Unfortunately, such a mechanism cannot satisfy (DIC) (Green-Laffont, 1977). Thus, we have to find a different criterion for efficiency. As mentioned in the introduction, we propose to evaluate the performance of a mechanism according to the average criterion, with the goal of finding a mechanism in $\mathcal{M}$ that maximizes the sum of agents' utilities *ex ante*. Specifically, we try to find an optimal solution for the following problem:

$$\max_{M \in \mathcal{M}} \int_{\boldsymbol{\theta}} \left[\left(\sum_{i=1}^{n} \theta_i \right) d\left(\boldsymbol{\theta}\right) + \sum_{i=1}^{n} t_i(\boldsymbol{\theta}) \right] \left(\prod dF\left(\theta_i\right) \right).$$

For any type profile, define index set $I = \left\{ i | \theta_i \geq \frac{c}{n} \right\}$ and index set $J = \left\{ j | \theta_j < \frac{c}{n} \right\}$. Sets I, J are the agents whose valuations are greater or smaller than the per-capita cost, respectively. We refer to agents in I as high-type agents.

We define a *quota mechanism*: The public good is built with equal cost-sharing if and only if the number of high-type agents exceeds quota q; otherwise, agents pay nothing—i.e., $t_i\left(\boldsymbol{\theta}\right) = -\frac{c}{n} d\left(\boldsymbol{\theta}\right)$, and $d\left(\boldsymbol{\theta}\right) = 1$ when $|I| \geq q$; otherwise, $d\left(\boldsymbol{\theta}\right) = 0$.

Our main result is:

Theorem 1. The optimal solution for the problem is a quota mechanism in which the quota $q^*\left(c\right)^+$ for the voting mechanism is determined by

$$q^*\left(c\right) = \frac{n \mathbb{E}\left[\left(\frac{c}{n} - \theta \right) | \theta \leq \frac{c}{n} \right]}{\mathbb{E}\left[\left(\theta - \frac{c}{n} \right) | \theta \geq \frac{c}{n} \right] + \mathbb{E}\left[\left(\frac{c}{n} - \theta \right) | \theta \leq \frac{c}{n} \right]},$$

and $q^*\left(c\right)^+$ is the smallest integer that is greater than $q^*\left(c\right) \in \left[0, n\right]$, which is weakly increasing in c.

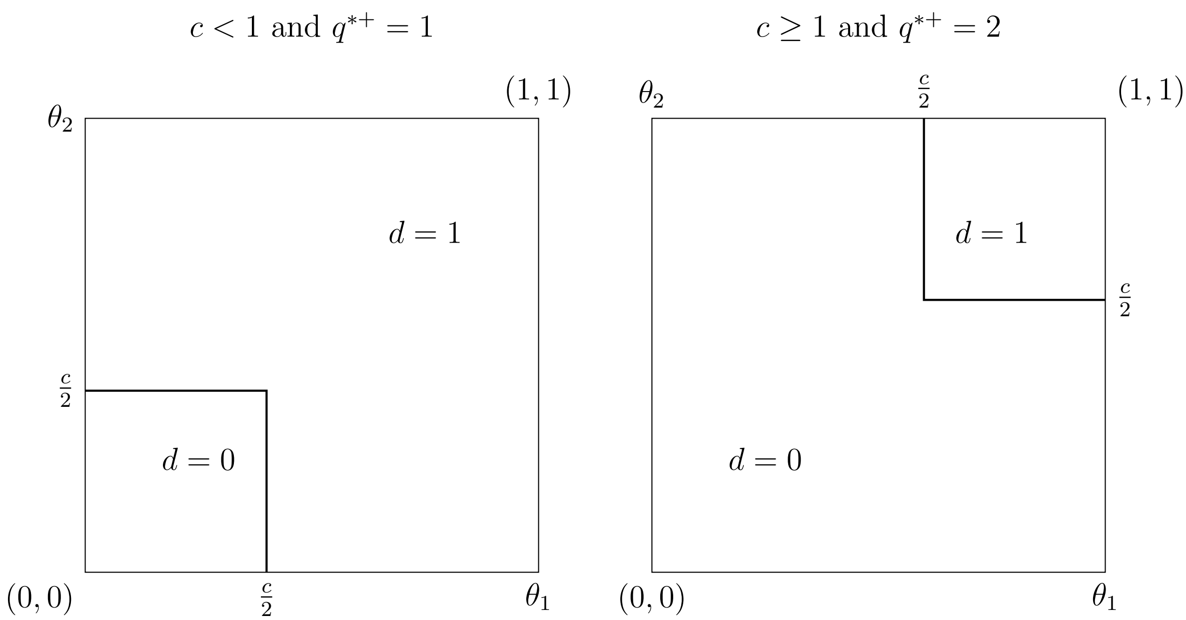

Figure 1: A graphic illustration of the mechanism for the two-agent case with different costs.

Let us make several important observations about our result. First, it implies immediately that one must keep the budget balanced at all profiles in order to achieve efficiency *ex ante*. This is in sharp contrast with VCG mechanisms and the like. The loss of efficiency of any mechanism can come from two sources: either the public good is not provided at the right level; or there is a budget surplus due to an over-collection of taxes. While VCG mechanisms cleverly prevent the first source of inefficiency, they have no effect on the second. Although one might reasonably conjecture that efficiency *ex ante* might require a compromise between these two sources of inefficiency, our result surprisingly asserts that keeping a balanced budget is the top priority.

Second, we should note that the quota for the optimal voting system is a weakly increasing function of c. This is quite intuitive. When the cost is small, the chance that the aggregate valuation of the public good exceeds the cost is quite high *ex ante*. If a certain number of agents have types greater than the per-capita cost, the public good is provided. The risk of over-provision of the public good is small, and the inefficiency of over-provision is also small. When the cost goes up, both the risk of over-provision and the inefficiency due to over-provision will be higher. Therefore, more high-type agents are needed to achieve efficiency *ex ante*. Depending on the cost c, the quota can be any number between 0 and n. For example, when agents' types are distributed uniformly, one can easily show that $q^* (c) = c$.

Third, for ease of exposition, we make the assumption that agents' type space is $[0,1]$. Our result is still valid for any bounded type space $[a,b]$, where a might be negative to allow for economic bads. In this sense, Drexl and Kleiner (2013) is a special case of ours. Their conclusion that there is no need for monetary transfers holds only when the cost of providing the public good is zero. Nevertheless, we have shown that the necessary monetary transfers in the optimal mechanism are quite simple, as they are merely equal-cost sharing if the public good is provided. Whether or not the public good is provided depends on the quota determined by the distribution of types, but the monetary transfers would be the same. This is much simpler than their counterparts in VCG mechanisms.

Fourth, in addition to (DIC) and (DF), we have imposed (K) in order to derive our main result. But we can relax (K) further to allow for a larger class of mechanisms. The optimality of the voting system still holds. A key implication of (K), in conjunction of (DIC) and (DF), is that all agents' transfers are properly bounded:

$$t_i \leq -\frac{c}{n}d, \text{ for all } i. \tag{B}$$

Condition (B) means that nobody gets any subsidy if the public good is not built, whereas everyone pays at least the per-capita cost if the public good is built. It is a strengthening of (DF), as we can derive (DF) when we sum inequalities in (B) over all agents. We actually will prove a result stronger than Theorem 1.

Theorem 2. The quota mechanism identified in Theorem 1 is optimal among all mechanisms that satisfy (DIC) and (B).

The advantage of Theorem 2 is that it allows us to demonstrate the superiority of the voting mechanism over many other mechanisms, as (B) can be derived from alternative sets of conditions in the literature. For example, Moulin (1986) considers two conditions – distribution axiom (DA) and no positive transfers (NPT). In our context, (DA) means that when one agent can potentially benefit more from the provision of the public good, this additional gain should be shared with all other agents—i.e.,

$$\text{If } d(\theta_i, \boldsymbol{\theta}_{-i}) = 1, \text{ then } \theta_i' > \theta_i \text{ implies that } U_j(\theta_i', \boldsymbol{\theta}_{-i}) \geq U_j(\theta_i, \boldsymbol{\theta}_{-i}) \text{ for all } j \neq i. \tag{DA}$$

We can easily show that (DA) and (NPT), together with (DF) and (DIC), imply (B). Another important application of Theorem 2 is that the optimal voting mechanism is more efficient than the "canonical" VCG mechanism, or the pivotal mechanism. We use the term "canonical" since there are several ways of setting up a pivotal mechanism in this model, depending on how the cost is distributed among agents. The canonical setup splits the cost equally among agents. In the canonical setting, our public-good model becomes a costless binary choice model: $d = 0$ or $d = 1$, in which $d = 0$ represents the decision that the public good is not built, and $d = 1$ represents the decision that the public good is built, and the cost is shared equally. (We discuss non-canonical cases fully in the next section.) In this binary model,

$$U_i = \left(\theta_i - \frac{c}{n}\right) d + T_i, \text{ for all } i.$$

Now, we consider the pivotal mechanism for this model. It is well known that it satisfies

$$T_i = \left(\sum_{j \neq i} \theta_j\right) d^* - \max_{d \in \{0,1\}} \left(\sum_{j \neq i} \theta_j\right) d, \text{ and}$$

$$d^* = 1 \text{ when } \sum_{j=1}^{n} \theta_j \geq c, \text{ otherwise } d^* = 0.$$

Therefore,

$$T_i \leq 0, \text{ for all } i.$$

Since $t_i = -\frac{c}{n} d + T_i$, $T_i \leq 0$ implies that $t_i \leq -\frac{c}{n} d$. Hence, we have:

Corollary 1. The optimal quota mechanism is more efficient than the "canonical" pivotal mechanism.

3 Asymmetric Mechanisms

In this section, we continue our investigation of asymmetric mechanisms. Asymmetry may arise for at least two reasons. First, the model may be intrinsically asymmetric. In what follows, we assume that agents have independent but non-identical type distributions that satisfy the regularity condition. Second, for some unspecified parameters (say incomes), we may also want to ask some agents to pay higher shares of the cost when the public good is built.

The most prominent examples of asymmetric mechanisms are "non-canonical" pivotal mechanisms in which agents share the cost unequally. Take any fixed division $\alpha = (\alpha_1, ..., \alpha_n)$ of c with $\sum_i \alpha_i = c$ and $\alpha_i \geq 0$. Suppose that agent i always contributes whenever the public good is built. (An additional incentive tax can be imposed separately.) Analogous to the construction in Section 2, we can construct a costless binary-choice model, in which the utility function of every agent is given by

$$U_i = (\theta_i - \alpha_i)\, d + T_i.$$

We call the pivotal mechanism of this model a non-canonical pivotal mechanism. It is dominant-incentive-compatible. It is also well known that

$$T_i \leq 0, \text{ for all } i.$$

Hence, a non-canonical pivotal mechanism satisfies both (DIC) and (DF).

Since a non-canonical pivotal mechanism is not symmetric, it is not guaranteed that the quota mechanism identified in Theorems 1 and 2 is more efficient. As symmetry is removed from the canonical pivotal mechanism, symmetry in quota mechanisms should also be removed in order to restore our earlier results. This leads us to consider committee mechanisms introduced by Barbera, Sonnenschein and Zhou (1991) in a different setup.

A *committee* is a pair $C = (N, W)$, where $N = \{1, ..., n\}$ is the set of agents, and W is a collection of coalitions (subsets) of N that satisfies monotonicity—i.e., $[M \in W$ and $M' \supseteq M] \rightarrow M' \in W$. Coalitions in W are called winning coalitions of C.

For any cost division $\alpha = (\alpha_1, ..., \alpha_n)$ and any committee $C = (N, W)$, we can define a *committee mechanism* $M(\alpha, C)$: The public good is built with cost division α if and only if the set of agents who value the public good more than their cost shares form a winning coalition—i.e., $\{i | \theta_i \geq \alpha_i\} \in W$.

Many decision mechanisms are special cases of committee mechanisms.

Example 1. A quota mechanism with quota q is a committee mechanism in which $N \in W$ if and only if $|N| \geq q$.

Example 2. A dictatorial mechanism with dictator i is a committee mechanism in which $N \in W$ if and only if $i \in N$.

Example 3. More generally, a legislature mechanism with L as the legislature is a committee mechanism in which $N \in W$ if and only if $L \subseteq N$.

Introducing these (asymmetric) committee mechanisms also allows us to deal with heterogeneous agents. In what follows, we assume that agents have independent but non-identical type distributions that satisfy the regularity condition.

Given any cost division α, the corresponding pivotal mechanism satisfies $T_i \leq 0$ for all i, or

$$\text{for all agent } i, \ t_i \leq -\alpha_i d. \qquad (\mathrm{B}_\alpha)$$

We say a mechanism satisfies (B') if there exists an α such that B_α holds. Apparently, (B) is a subclass of (B') corresponding to $\alpha_i = \frac{c}{n}$ for all i.

Applying the same technique for proving Theorems 1 and 2, we can prove that for every α, there is a committee mechanism $M(\alpha, C)$ that is more efficient than all the mechanisms that satisfy (DIC) and (B_α). Then, by optimizing over α, we can find an overall optimal committee mechanism. Hence, we obtain the next results, which generalize Theorem 2 and Corollary 1.

Theorem 3. There exists some committee mechanism $M(\alpha^*, C^*)$ that is optimal among all mechanisms that satisfy (DIC), (B').

Since all VCG (pivotal) mechanisms are defined through some cost division α, Theorem 3 implies that that there exists some committee mechanism that is more efficient than all VCG mechanisms.

Corollary 2. The optimal committee mechanism $M(\alpha^*, C^*)$ in Theorem 3 is more efficient than all pivotal mechanisms.

4 Conclusions

In this paper, we study the problem of optimal provision of a costly public good by dominant-incentive-compatible mechanisms when agents' valuations are private information. Since no first-best mechanism exists for this problem, we study it via a second-best approach. By introducing an average efficiency criterion, we identify a quota mechanism as the optimal one among all symmetric mechanisms that are dominant-incentive-compatible, deficit-free and kind. When symmetry is dropped, we identify a committee mechanism as the optimal mechanism for a large class of mechanisms. Since the class of mechanisms under consideration includes all VCG (or pivotal) mechanisms, we show that there is a committee mechanism that can outperform all VCG mechanisms.

Even though we focus on deterministic mechanisms only, the main results can be extended to include stochastic mechanisms—i.e., $d \in [0, 1]$. Specifically, Theorems 2 and 3 still hold without changes of proofs.

Finally, from a pure theoretical perspective, our results still fall short of a complete answer to the ambitious inquiry, which is to find optimal mechanisms among all mechanisms that satisfy dominant-incentive-compatibility and deficit-freeness without additional restrictions. In the two-agent case, we can prove through brute force that there is a committee mechanism that is optimal among all mechanisms satisfying (DIC) and (DF).[2] Whether the result is true in general remains a challenging open question.

Appendix

Proof of Theorem 1:

[2]The proof of this result is rather long, so we will not include it in this paper.

The strategy of the proof is as follows. We divide the interval $[0,1]$ into two subintervals—$[0,\frac{c}{n}]$ and $[\frac{c}{n},1]$—and subsequently partition the type space $[0,1]^n$ into 2^n "blocks." On each such block, we estimate an upper bound of the aggregate of agents' utilities of any mechanism in $\mathcal{M}$. It turns out that this upper bound depends only on the number of high-type agents on each block. We then construct a mechanism for which the aggregate of all agents' utilities achieves this upper bound on each block. Hence, this mechanism is an optimal one. The public good is provided on each block if and only if there are enough high-type agents. This cut-off number constitutes the quota of the optimal voting mechanism.

Before we proceed with the estimation, we first derive some useful lemmas. Some are well known for the general case, and some are specific to our model.

Following Myerson (1981), we can show that Lemma 1 holds for all M in $\mathcal{M}$:

Lemma 1.

$$U_i(\boldsymbol{\theta}) = \theta_i d(\boldsymbol{\theta}) + t_i(\boldsymbol{\theta}) = \int_0^{\theta_i} d(s, \boldsymbol{\theta}_{-i})ds + h_i(\boldsymbol{\theta}_{-i}), \text{ and} \tag{1}$$

$$d(\boldsymbol{\theta}) \text{ is increasing in all } \theta_i.$$

Next, we show:

Lemma 2.

$$t_i(\boldsymbol{\theta}) \leq -\frac{c}{n} d(\boldsymbol{\theta}).$$

First, consider $d(\boldsymbol{\theta}) = 0$. Suppose that $t_i(\boldsymbol{\theta}) > 0$ for some agent i. When we decrease the value of θ_i to zero, Lemma 1 implies that

$$t_i(\theta_i, \boldsymbol{\theta}_{-i}) = t_i(0, \boldsymbol{\theta}_{-i}) > 0.$$

By (K),

$$0 \leq U_j(0, \theta_j, \boldsymbol{\theta}_{-i,j}) - U_i(0, \theta_j, \boldsymbol{\theta}_{-i,j}) = t_j(0, \theta_j, \boldsymbol{\theta}_{-i,j}) - t_i(0, \theta_j, \boldsymbol{\theta}_{-i,j}), \text{ for all } j \neq i.$$

Thus, $t_i(0, \boldsymbol{\theta}_{-i}) \leq t_j(0, \boldsymbol{\theta}_{-i})$ for all $j \neq i$. Then, $\sum_{i=1}^n t_i(0, \boldsymbol{\theta}_{-i}) \geq n t_i(0, \boldsymbol{\theta}_{-i}) > 0$, contradicting (DF).

Now, consider the case $d(\boldsymbol{\theta}) = 1$. Suppose that $t_i(\boldsymbol{\theta}) > -\frac{c}{n}$ for some i. When we increase the value of θ_i to one, Lemma 1 implies that

$$t_i(1, \boldsymbol{\theta}_{-i}) = t_i(\boldsymbol{\theta}) > -\frac{c}{n}.$$

By (K),

$$
\begin{aligned}
&U_i\left(1,\theta_j,\boldsymbol{\theta}_{-i,j}\right)-U_j\left(1,\theta_j,\boldsymbol{\theta}_{-i,j}\right)\\
&=1+t_i\left(1,\theta_j,\boldsymbol{\theta}_{-i,j}\right)-\theta_j-t_j\left(1,\theta_j,\boldsymbol{\theta}_{-i,j}\right)\leq 1-\theta_j,\ \text{for all } j\neq i.
\end{aligned}
$$

Thus, $t_i\left(1,\theta_j,\boldsymbol{\theta}_{-i,j}\right)<t_j\left(1,\theta_j,\boldsymbol{\theta}_{-i,j}\right)$ for all $j\neq i$. Then, $\sum_{j=1}^{n} t_j\left(1,\boldsymbol{\theta}_{-i}\right)\geq nt_i\left(1,\theta_j,\boldsymbol{\theta}_{-i,j}\right)>-c$, contradicting (DF).

Finally, as Lemma 1 implies that $U_i\left(\theta_i,\boldsymbol{\theta}_{-i}\right)=U_i\left(\tilde{\theta}_i,\boldsymbol{\theta}_{-i}\right)+\int_{\tilde{\theta}_i}^{\theta_i} d(s,\boldsymbol{\theta}_{-i})ds$, and Lemma 2 implies that $U_i\left(\frac{c}{n},\boldsymbol{\theta}_{-i}\right)\leq 0$, we immediately obtain:

Lemma 3.

$$
U_i\left(\boldsymbol{\theta}\right)\leq\int_{\frac{c}{n}}^{\theta_i} d\left(s,\boldsymbol{\theta}_{-i}\right)ds.
$$

Lemma 3 allows us to estimate the upper bound of the aggregate of agents' utilities. We divide the type space into 2^n "blocks" determined by c/n—i.e., $\boldsymbol{\theta}_J\times\boldsymbol{\theta}_I=\left[0,\frac{c}{n}\right]^{|J|}\times\left[\frac{c}{n},1\right]^{|I|}$—and Lemma 3 on each block:

$$
\begin{aligned}
&\int_{\boldsymbol{\theta}_J}\int_{\boldsymbol{\theta}_I}\left(\sum\left(\theta_i d\left(\boldsymbol{\theta}\right)+t_i(\boldsymbol{\theta})\right)\right)\left(\prod dF\left(\theta_i\right)\right)\\
&\leq\int_{\boldsymbol{\theta}_J}\int_{\boldsymbol{\theta}_I}\left(\sum_{i\in I}\int_{\frac{c}{n}}^{\theta_i} d\left(s,\boldsymbol{\theta}_{-i}\right)ds-\sum_{j\in J}\int_{\theta_j}^{\frac{c}{n}} d\left(s,\boldsymbol{\theta}_{-i}\right)ds\right)\left(\prod dF\left(\theta_i\right)\right)\\
&=\int_{\boldsymbol{\theta}_J}\int_{\boldsymbol{\theta}_I}\left(\sum_{i\in I}\frac{1-F\left(\theta_i\right)}{f\left(\theta_i\right)}+\sum_{j\in J}\frac{-F\left(\theta_j\right)}{f\left(\theta_j\right)}\right)d\left(\theta_i,\boldsymbol{\theta}_{-i}\right)\left(\prod dF\left(\theta_i\right)\right).
\end{aligned}\tag{2}
$$

The last equality is obtained by integration by parts. We then apply Chebyshev's sum inequality[3] to the right-hand side of (2) to continue our estimation:

$$
\begin{aligned}
&\int_{\boldsymbol{\theta}_J}\int_{\boldsymbol{\theta}_I}\left(\sum_{i\in I}\frac{1-F\left(\theta_i\right)}{f\left(\theta_i\right)}+\sum_{j\in J}\frac{-F\left(\theta_j\right)}{f\left(\theta_j\right)}\right)d\left(\theta_i,\boldsymbol{\theta}_{-i}\right)\left(\prod dF\left(\theta_i\right)\right)\\
&\leq\left(\sum_{i\in I}\frac{1}{1-F\left(\frac{c}{n}\right)}\int_{\frac{c}{n}}^{1}\frac{1-F\left(\theta_i\right)}{f\left(\theta_i\right)}dF\left(\theta_i\right)+\sum_{j\in J}\frac{1}{F\left(\frac{c}{n}\right)}\int_{0}^{\frac{c}{n}}\frac{-F\left(\theta_j\right)}{f\left(\theta_j\right)}dF\left(\theta_j\right)\right)\\
&\times\int_{\theta_J}\int_{\theta_I} d\left(\theta_i,\boldsymbol{\theta}_{-i}\right)\left(\prod dF\left(\theta_i\right)\right).
\end{aligned}\tag{3}
$$

[3]If $\alpha\left(s\right)$ and $\beta\left(s\right)$ are increasing, then

$$
\int_a^b\alpha\left(s\right)\beta\left(s\right)dF\left(s\right)\geq\frac{1}{F\left(b\right)-F\left(a\right)}\int_a^b\alpha\left(s\right)dF\left(s\right)\int_a^b\beta\left(s\right)dF\left(s\right).
$$

See, also, Mitrinović, Pečarić and Fink (1993).

It is easy to check

$$\frac{1}{1-F\left(\frac{c}{n}\right)}\int_{\frac{c}{n}}^{1}\frac{1-F(\theta_i)}{f(\theta_i)}dF(\theta_i)=\mathbb{E}\left[\left(\theta-\frac{c}{n}\right)|\theta\geq\frac{c}{n}\right],\text{ and}$$
$$\frac{1}{F\left(\frac{c}{n}\right)}\int_{0}^{\frac{c}{n}}\frac{F(\theta_j)}{f(\theta_j)}dF(\theta_j)=\mathbb{E}\left[\left(\frac{c}{n}-\theta\right)|\theta\leq\frac{c}{n}\right].$$

Let q denote the cardinality of index set I. The cardinality of J is $n-q$. Together, inequalities (2) and (3) can be further written as

$$\begin{aligned}&\int_{\boldsymbol{\theta}_J}\int_{\boldsymbol{\theta}_I}\left(\sum\left(\theta_i d(\boldsymbol{\theta})+t_i(\boldsymbol{\theta})\right)\right)\left(\prod dF(\theta_i)\right)\\&\leq\left(q\mathbb{E}\left[\left(\theta-\frac{c}{n}\right)|\theta\geq\frac{c}{n}\right]-(n-q)\,\mathbb{E}\left[\left(\frac{c}{n}-\theta\right)|\theta\leq\frac{c}{n}\right]\right)\\&\times\int_{\theta_J}\int_{\theta_I}d(\theta_i,\boldsymbol{\theta}_{-i})\left(\prod dF(\theta_i)\right).\end{aligned}\tag{4}$$

The first term of the last expression is negative when $q=0$ and positive when $q=n$. The unique number

$$q^*=\frac{n\mathbb{E}\left[\left(\frac{c}{n}-\theta\right)|\theta\leq\frac{c}{n}\right]}{\mathbb{E}\left[\left(\theta-\frac{c}{n}\right)|\theta\geq\frac{c}{n}\right]+\mathbb{E}\left[\left(\frac{c}{n}-\theta\right)|\theta\leq\frac{c}{n}\right]}$$

solves

$$q^*\mathbb{E}\left[\left(\theta-\frac{c}{n}\right)|\theta\geq\frac{c}{n}\right]-(n-q^*)\,\mathbb{E}\left[\left(\frac{c}{n}-\theta\right)|\theta\leq\frac{c}{n}\right]=0.$$

It is easy to see that, for any integer $q<q^*$, the maximum of the last term in (4) is achieved when d is set to 0 on the block, and for any integer $q\geq q^*$, the maximum is achieved when d is set to 1 on the block. Hence, inequality (4) not only finds an upper bound for the aggregate utility on each block, but also identifies a mechanism that achieves these upper bounds, which is the simple voting mechanism with quota $q^*(c)^+$ and equal cost sharing (when the public good is built). We can easily see that this mechanism satisfies (DIC), (DF) and (K). We can also calculate its aggregate utility and verify that it achieves the upper bound on each block.

Proof of Theorem 3:

Lemma 1 still holds, as it is true for all DIC mechanisms. Since condition (B_α) directly assumes the inequality in Lemma 2, Lemma 3 also holds when α_i replaces $\frac{c}{n}$ as the lower bound in the integral:

$$U_i(\boldsymbol{\theta})\leq\int_{\alpha_i}^{\theta_i}d(s,\boldsymbol{\theta}_{-i})\,ds.$$

We divide the type space into 2^n "blocks" determined by α_i—i.e., $\boldsymbol{\theta}_J \times \boldsymbol{\theta}_I = \prod_{j \in J} [0, \alpha_j] \times \prod_{i \in I} [\alpha_i, 1]$—and Lemma 3 on each block:

$$\begin{aligned}
&\int_{\boldsymbol{\theta}_J} \int_{\boldsymbol{\theta}_I} \left(\sum \left(\theta_i d\left(\boldsymbol{\theta}\right) + t_i(\boldsymbol{\theta}) \right) \right) \left(\prod dF_i\left(\theta_i\right) \right) \\
&\leq \int_{\boldsymbol{\theta}_J} \int_{\boldsymbol{\theta}_I} \left(\sum_{i \in I} \int_{\alpha_i}^{\theta_i} d\left(s, \boldsymbol{\theta}_{-i}\right) ds - \sum_{j \in J} \int_{\theta_j}^{\alpha_j} d\left(s, \boldsymbol{\theta}_{-i}\right) ds \right) \left(\prod dF_i\left(\theta_i\right) \right) \\
&= \int_{\boldsymbol{\theta}_J} \int_{\boldsymbol{\theta}_I} \left(\sum_{i \in I} \frac{1 - F_i\left(\theta_i\right)}{f_i\left(\theta_i\right)} + \sum_{j \in J} \frac{-F_j\left(\theta_j\right)}{f_j\left(\theta_j\right)} \right) d\left(\theta_i, \boldsymbol{\theta}_{-i}\right) \left(\prod dF_i\left(\theta_i\right) \right). \qquad (5)
\end{aligned}$$

The last equality is obtained by integration by parts. We then apply Chebyshev's sum inequality to the right-hand side of (5) to continue our estimation:

$$\begin{aligned}
&\int_{\boldsymbol{\theta}_J} \int_{\boldsymbol{\theta}_I} \left(\sum_{i \in I} \frac{1 - F_i\left(\theta_i\right)}{f_i\left(\theta_i\right)} + \sum_{j \in J} \frac{-F_j\left(\theta_j\right)}{f_j\left(\theta_j\right)} \right) d\left(\theta_i, \boldsymbol{\theta}_{-i}\right) \left(\prod dF_i\left(\theta_i\right) \right) \\
&\leq \left(\sum_{i \in I} \frac{1}{1 - F_i\left(\alpha_i\right)} \int_{\alpha_i}^{1} \frac{1 - F_i\left(\theta_i\right)}{f_i\left(\theta_i\right)} dF_i\left(\theta_i\right) + \sum_{j \in J} \frac{1}{F_j\left(\alpha_j\right)} \int_0^{\alpha_j} \frac{-F_j\left(\theta_j\right)}{f_j\left(\theta_j\right)} dF_j\left(\theta_j\right) \right) \\
&\times \int_{\theta_J} \int_{\theta_I} d\left(\theta_i, \boldsymbol{\theta}_{-i}\right) \left(\prod dF_i\left(\theta_i\right) \right). \qquad (6)
\end{aligned}$$

It is easy to check

$$\begin{aligned}
\frac{1}{1 - F_i\left(\alpha_i\right)} \int_{\alpha_i}^{1} \frac{1 - F_i\left(\theta_i\right)}{f_i\left(\theta_i\right)} dF_i\left(\theta_i\right) &= \mathbb{E}_i\left[\left(\theta_i - \alpha_i\right) | \theta_i \geq \alpha_i\right], \text{ and} \\
\frac{1}{F_j\left(\alpha_j\right)} \int_0^{\alpha_j} \frac{F_j\left(\theta_j\right)}{f_j\left(\theta_j\right)} dF_j\left(\theta_j\right) &= \mathbb{E}_j\left[\left(\alpha_j - \theta_j\right) | \theta_j \leq \alpha_j\right].
\end{aligned}$$

Together, inequalities (5) and (6) can be further written as

$$\begin{aligned}
&\int_{\boldsymbol{\theta}_J} \int_{\boldsymbol{\theta}_I} \left(\sum \left(\theta_i d\left(\boldsymbol{\theta}\right) + t_i(\boldsymbol{\theta}) \right) \right) \left(\prod dF_i\left(\theta_i\right) \right) \\
&\leq \left(\sum_{i \in I} \mathbb{E}_i\left[\left(\theta_i - \alpha_i\right) | \theta_i \geq \alpha_i\right] - \sum_{j \in J} \mathbb{E}_j\left[\left(\alpha_j - \theta_j\right) | \theta_j \leq \alpha_j\right] \right) \qquad (7) \\
&\times \int_{\theta_J} \int_{\theta_I} d\left(\theta_i, \boldsymbol{\theta}_{-i}\right) \left(\prod dF_i\left(\theta_i\right) \right).
\end{aligned}$$

In the same spirit as the proof of Theorem 1, on the block $\boldsymbol{\theta}_J \times \boldsymbol{\theta}_I$, $d = 1$ whenever

$$\sum_{i \in I} \mathbb{E}_i \left[(\theta_i - \alpha_i) \left| \theta_i \geq \alpha_i \right. \right] - \sum_{j \in J} \mathbb{E}_j \left[(\alpha_j - \theta_j) \left| \theta_j \leq \alpha_j \right. \right] > 0;$$

otherwise, $d = 0$. As the above inequality determines if a coalition I is winning ($d = 1$ on the block), the collection of winning coalitions satisfies monotonicity and, therefore, constitutes a committee.

The total efficiency is a continuous function of α. Since all cost divisions constitute a compact set, we can find an optimal committee mechanism $M(\alpha^*, C^*)$ with the optimal cost division α^* over all α with its corresponding committee C^*.

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